Analyzing Interactive Exercising Policies of Callable and (or) Convertible Bonds

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Abstract

Various options are designed to be embedded in corporate bonds to meet various requirements from investors and issuers. Analyzing how the exercise strategy of an embedded option and hence the evaluation of the bond are influenced by the presence of other embedded options, the issuers' creditworthiness, and market variables like prevailing interest rate levels has drawn much attention from academic studies and market participants. This paper analyzes the interactions of exercise strategies among call options, default options, and conversions options embedded in **c**on**v**ertible-**c**allable-**d**efaultable bonds (CVCDs) and compare these strategies with those of bonds with only call options or conversion ones. Based on the structural-form credit risk model, our theoretical analyses show that the presence of conversion (call) options would attract option issuers (bond holders) to precipitates call (conversion) decisions to maximize their values at the expense of options owned by their counter parties. In addition, the increment of the intrinsic value of an embedded option and the value of other coexisting options held by the counterparty also precipitate the exercise decision of the embedded option. Precipitations and delay of exercise decisions can also be visualized by observing the changes of embedded options' exercise boundaries generated by our quantitatively examinations. Empirical tests for recent twenty-year bond data gathered from FISD support both our theoretical analyses and quantitative examinations. Out-of-the-money calls (or early calls) phenomena for CVCDs that are found but are not well explained in past studies can now be satisfactory explained to preempt conversion due to our findings. Similarly, we can also explain that CVCDs' holders may convert early to preempt the issuer's redemption right.

Keywords: Callable Convertible Bond, Interaction Effect, Out-of-the-Money Call Strategy, Out-of-the-Money Conversion Strategy, Game Option

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1 Introduction

A corporate bond is becoming an important financial instrument for a firm to raise capital to fulfill operation expenses and finance business expansions. Securities Industry and Financial Markets Association (abbreviated as SIFMA) reports that the amount of outstanding in the US market grew from 337.4 billion in 1996 to 1,526.9 billion in 2016.[1] Many outstanding bonds contain embedded options to meet various requirements from issuers and investors. For example, the statistical records in SIFMA says that about 60% of U.S. corporate bonds are callable bonds; callable convertible bonds that simultaneously grant bond issuers call options and bond holders options to convert (the bonds into the issuer's stocks) are also widely traded in bond markets. How the presences of embedded option(s) and relevant market variables like the prevailing interest rate influence the exercise strategies of embedded option(s) and hence the evaluations of host bonds have drawn much attention in theoretical analyses and empirical studies. Take the most popular callable corporate bonds for example. Acharya and Carpenter (2002) (abbreviated as AC) view default decisions as a default option owned by the bond issuer and study how the presence of these two options embedded in a callable default bond influence the issuer decisions to exercise options. Unlike the typical prediction made by Brennan and Schwartz (1977) and Ingersoll (1977a) that a callable bond should be redeemed once its market value exceeds the effective call price, AC find the "interaction effect" between these two options; that is, an bond issuer would postpone exercise decisions for both options to avoid value destruction of the other unexercised option owned by the issuer. These theoretical analyses are then confirmed empirically and quantitatively by Jacoby and Shiller (2010) and Liu et al. (2016). This paper, on the other hand, analyzes how the exercise decisions of a bond issuer (or a owner) is influenced by the presence of another embedded option(s) owned by the counterparty by analyzing a **c**on**v**ertible-**c**allable-**d**efaultable bond (abbreviated as CVCD), a bond that grants the issuer and the owner the right to redeem and to convert the bond, respectively. We show that the presence of this embedded "game option" – both writers and buyers simultaneously own the option(s) to terminate the contract– would precipitate the exercise decisions for both parties at the expense of the option(s) owned by the counterparty. Our findings not only explain Bhattarcharya (2012)'s insight on early call decisions but provide another explanation for Jensen and Pedersen (2016)'s observation about the early conversion decision.[2]

To study the interaction effect among bond embedded options, we analyze how the coexistence of these options influence the exercise behaviors of embedded call options, default options, and conversion options. We follow AC framework that decomposes a bond with embedded option(s) into a default-free bullet bond plus a combination of embedded options, and use the structural credit risk model pioneered by Merton (1974) to price these bonds, bond options, equity values as contingent claims on the bond issuer's value and the stochastic short-term interest rate. In light of Grundy and Verwijmeren (2016)'s empirical findings that the perfect capital market assumption can fairly predict call strategies of CVCDs after the prevalence of the anti-dilution provisions since 2003, our model follows the setting in AC and Finnerty (2015) that do not consider the market frictions like transaction costs. Ceteris paribus, we theoretically show that the redemption of a CVCD tends to be earlier than the redemption of an otherwise identical callable bond without conversion options since the early redemption of a CVCD can further increase the equity holder's value at the expense of bond holders' conversion rights. This explains why an issuer may redeem a CVCD early as found empirically in King and Mauer (2014)

[1] See http://www.sifma.org/research/statistics.aspx.

[2] Jensen and Pedersen (2016) address that bond holders tend to convert their CVCDs early due to market frictions, such as short-sale costs, transaction costs, or funding costs.

and Bechmann et al. (2014). Similarly, we also show that the conversion decision of a `CCV` tends to be earlier than the decision of an otherwise identical convertible bond without call options since the early conversion of a `CVCD` can further increase the holder's value at the expense of the issuer's call rights. This explains the early conversion phenomenon discussed in Jensen and Pedersen (2016). The aforementioned theoretical analyses are also empirically confirmed by analyzing twenty-year bond redemption and conversion data gathered from `FISD` database.

The increment of the intrinsic value of a bond embedded option and the value of other coexisting option(s) held by counter-parties may precipitate the exercise decision of the embedded option. The relationships among exercise behaviors and the market variables that influence the values of embedded options can be theoretically analyzed by extending the `AC` framework and empirically verified as discussed in later sections. Recall that a call option grants a bond issuer the right to redeem the bond with a contract-specified call price. Thus the decrement of the interest rate level or the increment of the coupon rate would increase the bond value as well as the option's intrinsic value that precipitates call decisions. Our empirical studies not only confirm our above theoretical analyzing results, but suggest that a bond with a better credit rating tends to be called earlier. Note that a higher credit rating entails a better issuer's financial status, a higher stock price, and hence a higher conversion value; this suggests that a higher value of the conversion option owned by bond holders precipitate the issuer's call decisions. Next, a conversion option grants holders the right to convert their bonds into issuer's stocks. Thus a higher-rating bond entails a higher intrinsic value of the conversion option and hence an earlier conversion, which will be empirically confirmed later. The impacts of some market variables on conversion decisions can be conflict since the issuer can call a `CVCD` back to force bond holders to convert. For example, a higher coupon rate would increase the bond value and hence the intrinsic value of the call option to trigger the call that may force holders to convert their bonds. However, a lower coupon rate could increase the intrinsic value of the conversion option to trigger the conversion voluntarily. Our empirical studies reflect this conflict by showing that the level of coupon rates can not significantly explain early conversion phenomena. Besides, the interaction effect of default options that are introduced in `AC` framework is reexamined in this paper. When default options are worth to be exercised, the issuer's financial status must be poor and conversion options are valueless. Thus the presence of conversion options only slightly accelerates default decisions as analyzed later.

The remainder of this paper proceeds as follows. Section 2 reviews past studies on exercise behaviors of options embedded in `CVCD`s and theoretical analyses of interaction effects among coexisting options embedded in bonds. Section 3 lays out some baseline assumptions adopted in `AC` and in our theoretical models, including the dynamics of interest rates and the bond issuer's asset value. `AC` decompose an option-embedded bond into a default-free bullet bond plus the an embedded option that grants the issuer the right to call back the bond, to default, or both. Their analyses of exercise behaviors under the structural credit risk model that will be applied in this paper are briefed in this section and in the appendix. Section 4 extends `AC` framework by allowing embedded (game) option simultaneously grant bond holders the conversion right in addition to the aforementioned rights granted to the issuer. By comparing `CVCD`s with bonds that contain only conversion options, only call options, and etc., we analyze how the exercise behavior of an embedded option and hence the bond evaluation are influenced by the presence of other embedded options (if any) and the change of market variables. These analyses are empirically examined with twenty-year data of bonds with call options, conversion options, or both gathered from FISD as in Section 5. Section 6 concludes this paper.

2 Literature Reviews

Call strategies of CVCDs would significantly influence fund raising strategies, bond evaluations, as well as hedging, and they are widely studied theoretically and empirically in past literature. Ingersoll (1977a) and Brennan and Schwartz (1977) adopt the contingent claim evaluation approach pioneered by Black and Scholes (1973) and Merton (1974) to analyze CVCDs. They predict that a CVCDs is called optimally once its conversion value — the market value of the common stocks obtained by converting a CVCDs — exceeds its call price. This "at-the-money" call strategy minimize the CVCDs value, which is equivalent to maximizing the equity value under the perfect capital market assumption considered in the Modigliani-Miller capital structure irrelevance theorem (see Modigliani and Miller, 1958). Although the at-the-money call strategy is widely applied for evaluating CVCDs (see Hung and Wang, 2002; Chambers and Lu, 2007; **?**), in-the-money calls (late calls) and out-of-the-money calls (early calls) are more widely observed in past empirical studies.

Much literature explains the phenomenon of in-the-money calls with the properties of imperfect capital market assumptions, like the existences of information asymmetry (see Harris and Raviv, 1985), transaction costs (see Ingersoll, 1977b; Emery and Finnerty, 1989), and tax shield benefits due to corporate taxes (see Asquith and Mullins, 1991; Campbell et al., 1991; Sarkar, 2003; Liao and Huang, 2006; Chen et al., 2013). On the other hand, some works argue that this phenomenon is due to mitigation of agency conflicts (see Billett et al., 2007; King and Mauer, 2014), the existences of call notice periods (see Ingersoll, 1977b; Altintig and Butler, 2005), or call protections (see Asquith, 1995). However, Grundy and Verwijmeren (2016) show that the prevailing of dividend protection provisions[3] considerably increases the issuer's incentive to call and hence diminish the phenomena of in-the-money calls since the bond conversion value is protected against the dilution due to dividend payouts. Thus researching the rationales for out-the-money calls seems to be more important thereafter.

The phenomena of out-of-the-money calls are also widely observable (see Cowan et al., 1993; Grundy and Verwijmeren, 2012; King and Mauer, 2014; Bechmann et al., 2014); however, the researches for the rationales are limited and contradicted. Sarkar (2003) and Chen et al. (2013) propose that issuers trigger out-of-the-money calls to preempt default. On the other hand, in addition to the decrement of the interest rate level, Bhattarcharya (2012) argue that out-of-the-money calls are triggered to preempt conversions. Bechmann et al. (2014) confirm preemptions of conversions as they observe "avoid-dilution arguments" when out-of-the-money calls are announced.[4] To justify the arguments of preemptions of conversions or defaults, we take advantage of AC framework and the game option analysis pioneered by Kifer (2000) to study the conflict of interest between bond issuers and holders. Both our theoretical analyses and empirical results suggest that issuers call to preempt conversions rather than defaults. Besides, just like call decisions of callable bonds discussed in King and Mauer (2000), we also confirm that the decrement of the interest rate level would precipitate the call decisions of CVCDs.

Empirical studies of conversions strategies are relatively rare. Jensen and Pedersen (2016) observe

[3] Grundy and Verwijmeren (2016) find that more than 82% of convertible bonds issued between 2003 and 2006 are dividend-protected. Finnerty (2015) confirms that this prevailing has continued through 2013.

[4] Bechmann et al. (2014) state that

> Several out-of-the-money call announcements explicitly mention the main reason is 'avoid-dilution arguments' as the main reason for the call. For example, on November 5, 1997, BancTec Inc. made an out-of-the-money convertible bond call. In the call announcement, the firm said "the call will be funded with internal capital and existing lines of credit and should allow the company to avoid dilution of 1.5 million shares," which should be compared to the 21.1 million shares outstanding.

that bond holders tend to convert early when facing market frictions like transaction costs. This paper empirical compare CVCDs and convertible bonds without call options to show that the presence of calls precipitate the conversion options with rigorous theoretical analyses based on the AC framework.

Analyzing the interaction effects among multiple coexisting options embedded in a contingent claim can be challenging, since the exercise of one option would destroy the contingent claim and hence all other options embedded on that claim. AC evaluate non-perpetual bonds with call and/or default options owned by the issuer under stochastic bond issuer's asset value process and the short rate process. They decompose an option-embedded bond into an otherwise identical default-free bond minus a (combination) of option(s) to call and/or default. Then they analyze their values and exercise strategies with the risk-neutral valuation method. To maximize equity holders' value, AC show that a bond issuer would postpone its call/default decisions to avoid the value destruction of the other unexercised option owned by itself. Moreover, the more valuable the call (or default) option is, the more significant the default (or call) decision is delayed. For example, the deterioration of the issuer's creditworthiness would increase the value of the default option and make call delay phenomenon more salient. On the other hand, the improvement of the issuer's creditworthiness and the drop of the interest rate level would increase the value of the call option and further delay the default decision. In addition to analyzing the scenario that options are owned by the same participant, Kifer (2000) analyzes "game options" that grant both the issuer and the holder the right to terminate the contract. His constant interest rate assumption is adopted by Sirbu et al. (2004) and Sîrbu and Shreve (2006) to build a zero-sum two-person game for analyzing perpetual and non-perpetual CVCDs, respectively. Although their models suggest the possibilities of out-of-the-money calls, they fail to analyze how the interactions among call, default, and conversion options are influenced by the evolutions of interest rates. Bielecki et al. (2008) release the constant interest rate assumption but fail to model the dilution effect due to adoptions of reduced form credit risk model. Chen et al. (2013) extend the above works to nonzero sum game and argue that out-of-the-money calls are triggered only for preempting defaults. But this argument contradicts AC's argument that an issuer tends to postpone its call decision to avoid the value destruction of its default options. Bechmann et al. (2014) and this paper also argue that out-of-the money calls are triggered to preempt conversions rather than defaults.

3 Baseline Assumptions

Our paper adopts the perfect capital market assumption as adopted in AC due to Grundy and Verwijmeren (2016)'s empirical findings that this assumption can fairly predict call strategies after the prevalence of anti-dilution provisions since about 2003. The mathematical models and terminologies in AC that will be used in the paper are introduced as follows. All market participants have equal access to market information and trade continuously in a frictionless and complete market without arbitrage opportunity. The values of all contingent claims and embedded options are assessed under the risk neutral valuation method and exercise strategies of embedded options are determined to maximize the benefits of their holders. There exists a unique risk-neutral probability measure $\tilde{P}$ to make discounted price processes of all assets be martingale processes (see Harrison and Kreps, 1979). r_t follows the stochastic process

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)d\tilde{Z}_t, \tag{1}$$

where $\tilde{Z}_t$ denotes a standard Brownian motion under $\tilde{P}$. μ and σ are continuous functions satisfying Lipschitz and linear growth condition described in Equations (2) $\backsim$ (3) in AC. For convenience, we assume all bonds are issued at time 0 and mature at time T. The asset value of the bond issuing firm at time t, V_t, is assumed to follow a log-normal diffusion process

$$V_t = V_0 e^{\int_0^t r_s ds - \int_0^t \gamma_s ds - \frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \phi_s d\tilde{W}_s}, \tag{2}$$

where $\gamma_t \geq 0$ denotes the payout ratio for serving contractually-obligated debt repayments and dividend payout at time t, $\phi_t > 0$ represents the firm value volatility, and $\tilde{W}_t$ refers to a standard Brownian motion under $\tilde{P}$. Note that $\rho_t \in [-1, 1]$ denote the correlation of $\tilde{Z}_t$ and $\tilde{W}_t$. For ease of analyses, we follow AC by considering a firm with one outstanding non-perpetual bond. Protective bond covenants deter the firm from arbitrarily adjusting both the payout ratio γ_t and the investment policy proxied by the firm value volatility ϕ_t. The available cash payout $\gamma_t V_t$ is first used to fulfill interest payments, and the rest (if any) is then distributed to equity holders as dividends. On the other hand, if the payout fails to meet interest payments, we follow Chen (2010) by assuming that the firm will issue new equity to cover the shortfall. In addition, default decisions are viewed as options owned by the firm as defined in AC. In other words, we do not consider exogenous bond provisions like minimum net worth covenants mentioned in Leland (1994) or net cash flow covenants (see Fan and Sundaresan, 2000) that force the firm to default when its asset value V_t or cash flow $\gamma_t V_t$ fall below certain thresholds, respectively.

Contingent claims can be evaluated as the lump sum of the present values of future expected payoffs under the risk neutral valuation method. Under the stochastic interest rate setting in **Equation (1)**, the discount factor from time τ back to t can be expressed as

$$\beta_{t,\tau} \equiv e^{-\int_t^\tau r_s ds}. \tag{3}$$

Thus the value of a unit face value default-free straight bond at time t can be expressed as

$$P_t = \tilde{E}\left[c \int_t^T \beta_{t,s} ds + 1 \cdot \beta_{t,T} \middle| \mathcal{F}_t \right], \tag{4}$$

where c denotes a fixed continuous coupon rate, T denotes the maturity date, $\tilde{E}[\cdot]$ represents the expectation under the risk neutral measure $\tilde{P}$, and $\{\mathcal{F}_t\}$ denotes a filtration generated by the evolvements of the interest rate and the firm's asset value defined in **Equations (1)** and **(2)**. Under AC framework, a bond's embedded option, like a call or (and) a default option, parasitizes in the aforementioned straight bond, or the so-called the "host bond". The value of the embedded option f_X can be interpreted as a function of the host bond price p, the bond issuer's asset value v, and the current time t. Thus the value of an option-embedded bond can be expressed as the value of the host bond minus the value of the embedded option owned by the issuer as

$$p_X = p - f_X(p, v, t), \tag{5}$$

where the subscript X can be C for a callable default-free (i.e., pure callable) bond, D for a non-callable defaultable (i.e., pure defaultable) bond, and CD for a callable defaultable bond under the AC framework.

Both embedded option and hence the option embedded bond are evaluated by assuming that the bond issuer exercises his own option optimally to maximize his benefit. The optimal option exercise strategy to maximize the bond issuer's equity value is equivalent to the strategy that minimizing the bond value in a perfect capital market due to Modigliani-Miller theorem; therefore, to maximize the value of the embedded option, the issuer would optimally exercise his option at time τ_X to exchange the call price k_{τ_X} (i.e. to redeem the bond early) or the firm value V_{τ_X} (i.e. to bankrupt) for the host bond with value P_{τ_X} as follows:

$$f_X(p, v, t) = \sup_{t \leq \tau_X \leq T} \tilde{E} \left[\beta_{t,\tau_X} (P_{\tau_X} - \kappa (V_\tau, \tau_X))^+ \middle| \mathcal{F}_t \right], \tag{6}$$

where the strike price $\kappa(V_{\tau_X}, \tau_X) = k_{\tau_X}$, V_τ, and $k_\tau \wedge V_\tau$ for $X = C$, D, and CD, respectively. The stopping time τ_X adapted to the filtration $\{\mathcal{F}_t\}$ denotes the optimal option execution time that satisfies the condition

$$\tau_X = \inf\{t \in [0, T] : f_X(p, v, t) = (p - \kappa(v, t))^+\}.$$

Specifically, the embedded option value $f_X(p, v, t)$ is larger than or equal to the exercise value $(p - \kappa(v, t))$, and the option is exercised once the exercise value equals to the option value.

To analyze the impacts of the bond issuer's creditworthiness and the interest rate level on the interaction effect between call and default decisions, `AC` derive useful mathematical lemmas and properties for embedded call and default options to theoretically compare the exercise boundaries of these options embedded in pure callable, pure defaultable, and callable-convertible bonds. These lemmas and properties that will be used in this paper are introduced in **Appendix A**. These mathematical properties can be applied to derive more lemmas in **Appendix B** that can used to analyze exercise strategies of game options embedded in bonds in **Appendix C**.

4 Theoretical Analyses of Embedded Options Exercise Strategies

To theoretically analyze the exercise strategies of various options embedded in corporate bonds, we prove the existences and the shapes of exercise boundaries for these options based on the risk-neutral valuation method and the analyses of callable (and/or) defaultable bonds proposed in `AC`. These exercise boundaries can also be numerically estimated by the three-dimensional tree like **?** and Dai et al. (2013). Thus we can plot these boundaries like a computerized tomography to explain and to verify our proofs. This section will first explain the visualization of exercise boundaries. After that, we analyze a pure **con**vertible bond (abbreviated as `CV` hereafter), or the bond with only a conversion option, by decomposing it into an otherwise identical host bond plus a conversion option. Then we analyze how the change of the issuer's credit worthiness and the interest rate level influence the value of the conversion option, and hence the exercise boundary of the option (abbreviated as the "conversion boundary" hereafter). To analyze the interaction of exercise strategies simultaneously owned by issuers and bond holders, we extend the analyses of `CD` in `AC` and our former analyses for `CV` to analyse the game option embedded in `CVCD`. By comparing the exercise boundaries of options embedded in `CVCD`s, `CD`s, and `CV`s, we show that issuers (or bond holders) would precipitate their exercise decisions to destroy the options owned by the counterparty.

4.1 Visualized Exercise Boundaries

A three dimensional tree that discretely models the issuer's asset value and the short rate process at each time step as illustrated in **Figure 1** can be used to evaluate a defaultable option-embedded bond and to estimate the exercise strategy of embedded options for each node of the tree (see Liu et al., 2016). For each option, we draw the exercise boundary that separates the exercise nodes from

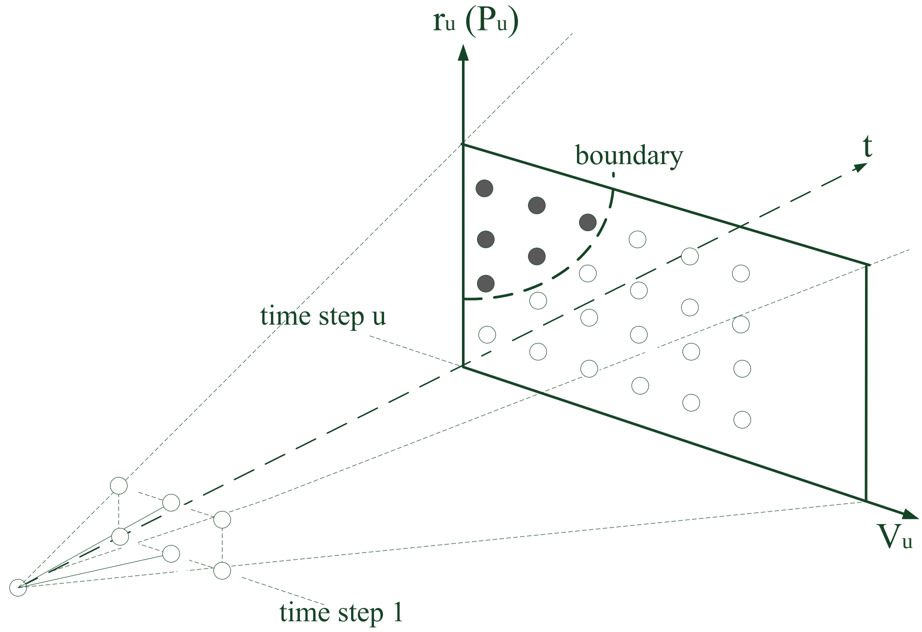

Figure 1: A Computerized Tomography of Exercise Boundaries in a Tree. We implement a three-dimensional tree that can discretely models the evolution of the firm value and the short rate at each time step. For each option-embedded bond, we can determine whether its embedded option is exercised or not at each node of the tree and hence evaluate the bond with the backward induction method. We can draw an exercise boundary (denoted by the dash curve) that separates exercise nodes (denoted by gray circles) and unexercised ones (by white circles) at an arbitrary time step u. Thus the exercise regions of different option-embedded bonds expressed in terms of the issuer's asset value V_u and the host bond price P_u (implied by the prevailing short rate r_u) can be visualized and compared as illustrated in **Figure 2**.

unexercise ones at an arbitrary time step u to obtain a "computerized tomography" of exercise regions as illustrated in **Figure 2**. For example, the dark gray dashed curve, which is partially overlapped with the dark solid one, denotes the call boundary of `CD`. A state (p',v',t) which denotes that the host bond price, the issuer's value, and the time are p', v', and t, respectively, may fall above or below the call boundary to denote the `CD` shall be called or not in this very "state", respectively. Indeed, this "state" may fall into one of the three following exercise regions that denotes whether the corresponding embedded option is exercised or not. The "default region" in the upper left corner denotes that bond issuers would default when the issuer's asset value (denoted by the x-axis) is relatively low and the host bond price (denoted by the y-axis) is relatively high. The "call region" in the upper right corner denotes that the a bond issuer would redeem the bond early when both the issuer's value and the host bond price are relatively high. The "conversion region" in the lower right corner denotes that a bond holder would convert the bond when the issuer's value is relatively high and the host bond price is low. Note that the call region and the conversion one for a `CVCD` may overlap; and this scenario denotes that bond conversions are forced by issuer's call decisions. If this state does not fall into one of above exercise region, then no embedded option is exercised and the bond survives at that time step. To explain the existence of exercise boundaries in later discussions, we may compare a state, says (p', v', t) with a "threshold" state that locates exactly on an exercise boundary. The host bond price of a threshold state can be viewed as a function of the issuer's asset value v and the prevailing

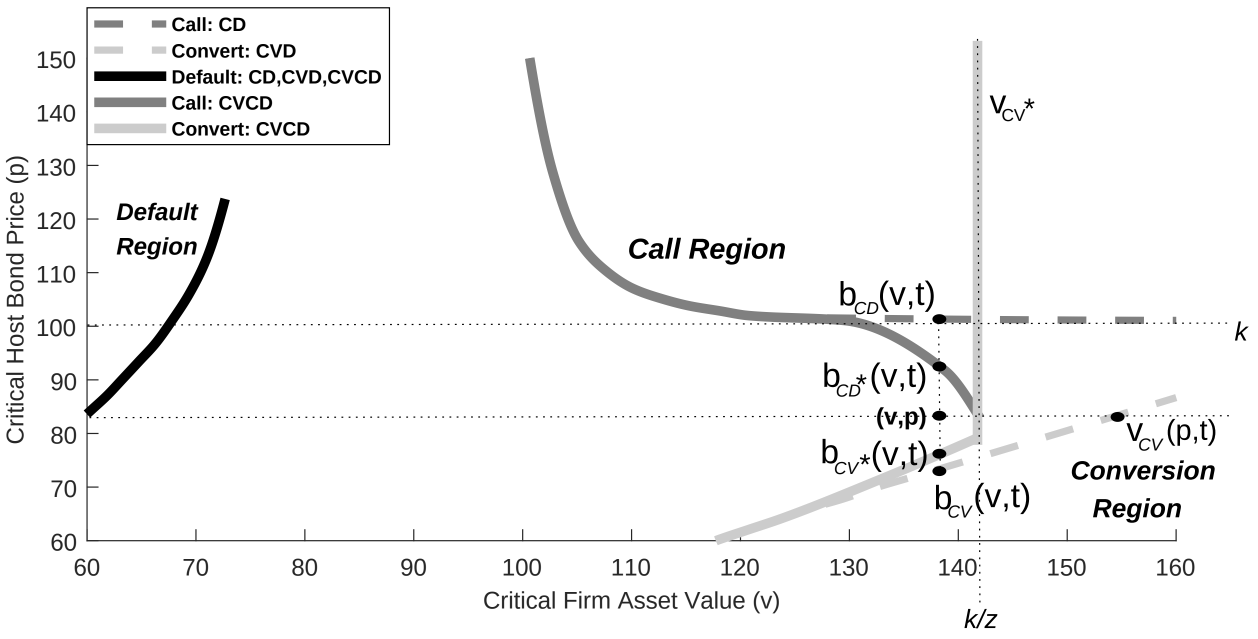

Figure 2: Optimal Default, Call, and Conversion Boundaries. Optimal default, call, and conversion boundaries for three otherwise identical 3-year CD, CV, and CVCD with coupon rates 3% and face value 100 are illustrated in a two-dimensional profile, the x and the y axes of which denote the issuer's asset value v and the host bond price p, respectively. The call prices k for both CD and CVCD are set to 100. The horizontal dot line $p = k$, which denotes the call boundary for a default-free callable bond, is plotted for comparing with the call boundaries of the CD (denoted by the dark dash curve) and the CVCD (the dark solid curve). The conversion boundaries for CVCD and CV are denoted by light solid and the light dash curves, respectively. The vertical dot line $v = k/z$ denotes the boundary that the CVCD is forced to be converted, where z denotes the proportion of the issuer's value obtained by converting CVCD. The default boundaries for CD and CVCD are almost overlapped and is denoted by the black sold curve. These boundaries divide the profile into the call region, the conversion region, the default region, and the remaining region that no embedded options are exercised. Given the issuer's value v, the corresponding host bond price located at the the call boundaries of CD and CVCD are $b_{CD}(v,t)$ and $b_{CD^*}(v,t)$, respectively; the bond price located at the conversion boundaries of CV and CVCD are $b_{CV}(v,t)$ and $b_{CV^*}(v,t)$, respectively. Similarly, given the host bond price p, the corresponding issuer's value located at the conversion boundary of CV is $V_{CV}(p,t)$. We adopt the Hull-White short rate model (see Hull and White, 1994) with the average interest rate level, the mean reversion rate, and the volatility being set to 6%, 0.5 and 0.1, respectively. The payout ratio γ and the volatility ϕ for the issuer's asset value process V_t in **Equation (2)** are 0.04 and 0.2, respectively. The correlation ρ between the asset value V_t and the short rate r_t is 0. z is set to 0.7.

time t. For example, $b_{CD}(v',t)$ and $b_{CV}(v',t)$ denote the "critical" host bond prices of two threshold states located at the call boundary of CD and the conversion boundary of CV, respectively. Similarly, the issuer's asset value of a threshold state can be viewed as a function of the host bond price p and the prevailing time t. For example, $v_{CV}(p',t)$ denotes the critical issuer's asset value of the threshold state located at the conversion boundary of CV.

The shape and the size of an exercise region expresses how the change of the issuer's asset value, the host bond price, and the presence of other embedded options influence the exercise decision. For example, the upward sloping of default boundaries for both CD and CVCD (denoted by a black solid curve) entail that a lower issuer asset value or a higher host bond price would encourage the bond issuer to default. The overlap of these two default boundaries entails that the presence of the conversion option does not have significant influence on the issuer's default decision. In addition, the size of an exercise region reflects the tendency to exercise the corresponding option. For example, the call region of a `CD` (whose call boundary is denoted by the dark dashed curve) is relatively smaller than that of a `CVCD` (whose boundary is denoted by the dark solid curve), which implies that presence of a conversion option would precipitate the decision to exercise the call option. Note that the latter region becomes wider than the former one with the increment of the issuer's asset value; this entails that the difference of call strategies becomes more significant with the increment of the issuer's value.

4.2 Pure Convertible Bonds

4.2.1 Evaluations of Bonds and Embedded Conversion Options

To reflect the prevailing of the anti-dilution covenant since 2003 as studied in Grundy and Verwijmeren (2016), we assume that an embedded conversion option allows a `CV` holder to convert the bond into the issuer's common stocks being worth a fixed fraction $z \in [0, 1]$ of the issuer's asset value.[5] By mimicking the analyses of `AC` in Equation (5), we decompose the value of a `CV` at time t into the value of an otherwise identical host bond p plus the value of a conversion option f_{CV} that can be treated as a function of the host bond price p, the issuer's value v, and the time t (or as the function of the state (p, v, t) in shorthand) as follows:

$$p_{CV} = p + f_{CV}(p, v, t). \tag{7}$$

A `CV` holder would maximize his benefit by exercising his conversion option optimally at time τ_{CV} and the option value can be expressed as

$$f_{CV}(p, v, t) = \sup_{t \le \tau_{CV} \le T} \tilde{E}\left[\beta_{t,\tau_{CV}}(zV_{\tau_{CV}} - P_{\tau_{CV}})^+ \middle| \mathcal{F}_t\right], \tag{8}$$

where $zV_{\tau_{CV}}$ is the conversion value at time τ_{CV}. The optimal conversion time is defined as

$$\tau_{CV} = \inf\{t \in [0, T] : f_{CV}(p, v, t) = (zv - p)^+\},$$

ensuring the condition

$$f_{CV}(p, v, t) \ge (zv - p)^+$$

for $(p, v, t) \in R^+ \times R^+ \times [0, T]$. Note that the moneyness of the conversion option is determined through the sign of the value $zv - p$; in other words, in (out of) the money conversion denotes that the conversion value exceeds (falls below) the host bond price when a `CV` is converted. The following theorem describes how the conversion option value $f_{CV}(p, v, t)$ is influenced by the change of the issuer's asset value (to reflect the issuer's creditworthiness) and the host bond price (to reflect the interest rate level).

[5] This anti-dilution setting can be found in Sarkar (2003), Liao and Huang (2006), Sîrbu and Shreve (2006) and Chen et al. (2013). Note that issuing new shares at time t for financing the shortage to repay the coupon payment with firm value payout ratio γ_t as in Equation (2) would dilute the value of existing shares. Allowing obtaining a constant fraction of the issuer's asset from converting `CV`s immunizes this dilution.

Theorem 1 *Denote two different host bond prices by $p^{(1)}$ and $p^{(2)}$ and two different firm's asset values by $v^{(1)}$ and $v^{(2)}$. The following properties hold for the $f_{CV}(p,v,t)$.*

1. $p^{(1)} > p^{(2)} \Rightarrow f_{CV}\left(p^{(1)},v,t\right) < f_{CV}\left(p^{(2)},v,t\right)$.

2. $v^{(1)} < v^{(2)} \Rightarrow f_{CV}\left(p,v^{(1)},t\right) < f_{CV}\left(p,v^{(2)},t\right)$.

3. $p^{(1)} \neq p^{(2)} \Rightarrow -1 \leq \frac{f_{CV}\left(p^{(2)},v,t\right)-f_{CV}\left(p^{(1)},v,t\right)}{p^{(2)}-p^{(1)}} < 0$. (Put delta inequality)

4. $v^{(1)} \neq v^{(2)} \Rightarrow 0 < \frac{f_{CV}\left(p,v^{(1)},t\right)-f_{CV}\left(p,v^{(2)},t\right)}{v^{(1)}-v^{(2)}} < z$. (Call delta inequality)

The proofs are similar to those in AC and we detail them in **Appendix C.1**. The first two parts of the this theorem denote that the conversion option value increases with the decrement of the host bond price or the increment of the bond issuer's asset value (that reflects the improvement of the issuer's creditworthiness). According to **Equation (8)**, an embedded conversion option can be viewed as a put option on the host bond price. The put delta inequality in part 3 entails that the value of the option increases at a slower rate than the decrement rate of the underlying host bond price. Since the value of CV, p_{CV}, can be decomposed into the value of the host bond plus f_{CV} as in **Equation (7)**, the put delta inequality entails that the price of p_{CV} decreases with the increment of the interest rate level. Similarly, an conversion option can be regarded as an call option on the conversion value, and the call delta inequality in part 4 entails that the value of f_{CV} increases at a slower rate than the increment rate of the issuer's asset value.

4.2.2 Conversion Boundaries

The properties of $f_{CV}(p,v,t)$ proved above can be used to infer the existence and the shape of the conversion boundary (i.e. the light dash curve in **Figure 2**) described in **Theorem 2** and **Theorem 3**, respectively, as follows.

Theorem 2 *The existence of the conversion boundary.*
1. If it is optimal to convert a CV at a state (p',v',t), then there exists a critical host bond price $b_{CV}(v',t) < zv'$. It is optimal to convert the CV if an only if the host bond price is lower than $b_{CV}(v',t)$ given that the issuer's asset value is v' at time t.
2. The aforementioned conversion state also entails the existence of a critical issuer's asset value $v_{CV}(p',t) > p'/z$. It is optimal to convert the CV if and only if the issuer's value is higher than $v_{CV}(p',t)$ given that the host bond price is p' at time t.

The proof of the theorem is detailed in page 32 of the **Appendix C.1**. It sketches the location of the conversion region is located in the lower right part of **Figure 2** by showing that bond holders tends to convert the bond when the host bond price is relatively low or when the issuer's asset value is relatively high. Besides, it implies that out-of-the-money conversion is always not the optimal strategy for a CV because $b_{CV}(v',t) < zv'$ and $zv_{CV}(p',t) > p'$. In other words, the conversion of a CV only occurs when its conversion value is larger than its host bond price.

Theorem 3 *The shape of the conversion boundary.*
For each $t \in [0,T)$,

1. $v^{(1)} < v^{(2)} \Rightarrow b_{CV}\left(v^{(1)},t\right) \leq b_{CV}\left(v^{(2)},t\right)$.

2. $p^{(1)} > p^{(2)} \Rightarrow v_{CV}\left(p^{(1)}, t\right) \geq v_{CV}\left(p^{(2)}, t\right)$.

The proof of the theorem is detailed in page 33 of the **Appendix C.1**. The first part of this theorem indicates that the critical host bond price decreases with the decrement of the issuer's asset value. This implies that it takes a lower host bond price (or a higher prevailing interest rate level) to trigger a conversion when the issuer's financial status (proxied by its asset value) is worse. The second part also show that the issuer's asset value increases with the increment of the host bond price. Both aforementioned results suggest that the conversion boundary denoted by the light gray dashed curve in **Figure 2** should be upward-slopping.

4.3 Bonds with Game Options

4.3.1 Evaluations of Bonds and Embedded Game Options

This section analyzes bonds with game options that simultaneously grants its holders the right to convert the bond into the issuer stock and its issuer the right to call bonds or to default. By analyzing these game-option-embedded bonds like a `CVCD`, a **con**vertible-**c**allable bond (abbreviated as `CVC`), and a **con**vertible-**d**efaultable bond (abbreviated as `CVD`), we can theoretically analyze the impact of one option, says a conversion option, on other coexisting option(s), like a call option. Now we decompose the value of a `CVCD` at time t into the value of an otherwise identical host bond p plus the value of a game option f_{CVCD} granting both the issuer and holders the rights to terminate the bond and the game options prematurely as follows.

$$p_{CVCD} = p + f_{CVCD}(p, v, t). \tag{9}$$

Note that each party would maximize its benefit at the expense of the other one. Specifically, to minimize the value of a `CVCD`, the issuer could either call the bond or default as the `CVCD` is still alive; on the other hand, bond holders would maximize the value by converting the bond. To express the interactions between the bond issuer and holders' decisions as well as their conflict of interests, we define the stopping time τ_{CV^*} as the bond holder's optimal conversion time subject to the issuer's optimal call or default time τ_{CD^*}, where the star sign here denotes the exercise time for the option embedded in `CVCD`. The `CVCD` and its embedded game option terminates if either party exercises his option at time $\tau_{CV^*} \wedge \tau_{CD^*}$; thus the payoff can be expressed as

$$f_{CVCD}(p, v, t) \quad = \quad \tilde{E}\left[\beta_{t,\tau_{CV^*}\wedge\tau_{CD^*}} G\left(\tau_{CV^*}, \tau_{CD^*}\right) \middle| \mathcal{F}_t\right].$$

The payoff of the option at the exercised date is denoted by the function $G\left(\tau_{CV^*}, \tau_{CD^*}\right)$ as

$$G\left(\tau_{CV^*}, \tau_{CD^*}\right) = \left(zV_{\tau_{CV^*}} - P_{\tau_{CV^*}}\right) \mathbf{I}_{\{\tau_{CV^*} \leq \tau_{CD^*}\}} - \left(P_{\tau_{CD^*}} - \kappa\left(V_{\tau_{CD^*}}, \tau_{CD^*}\right)\right) \mathbf{I}_{\{\tau_{CD^*} < \tau_{CV^*}\}}, \tag{10}$$

where the optimized times for both parties to exercise their options are

$$\begin{aligned}
\tau_{CV^*} &= \inf\{t \in [0, T] : f_{CVCD}(p, v, t) = (zv - p)\}, \\
\tau_{CD^*} &= \inf\{t \in [0, T] : f_{CVCD}(p, v, t) = -\left(p - \kappa\left(v, t\right)\right)\}.
\end{aligned}$$

Note that bond holders may either voluntarily convert the bonds or be forced to convert by the issuer's call announcements. The former case can be represented by $\tau_{CV^*} < \tau_{CD^*}$ while the latter one can be represented by $\tau_{CV^*} = \tau_{CD^*}$.[6] Note that **Equation (10)** also allows negative payoffs for exercising calls $\left(zV_{\tau_{CV^*}} - P_{\tau_{CV^*}}\right)$ and conversion options $\left(P_{\tau_{CD^*}} - \kappa\left(V_{\tau_{CD^*}}, \tau_{CD^*}\right)\right)$ to capture phenomena of out-of-the-money calls and conversions, respectively, on `CVCDs` found in empirical studies.

Note that the payoff of f_{CVCD} defined in **Equation (10)** is evaluated from bond holders's viewpoint. Define f_{CDCV} as the value of the same game option from the issuer's view point, and then we have

$$f_{CVCD}(p, v, t) \equiv -f_{CDCV}(p, v, t). \tag{11}$$

Note that the conversion option value of `CVCD` should be less than that of `CVD` since the former conversion right can be impaired by the issuer's call announcements. Similarly, the value that combines call and default options owned by the issuer can also be impaired by holders conversions. Thus the values of options embedded in `CV`, `CD`, and `CVCD` would follow

$$\begin{aligned} f_{CVCD}(p, v, t) &\leq f_{CVD}(p, v, t), \qquad (12) \\ f_{CDCV}(p, v, t) &\leq f_{CD}(p, v, t), \qquad (13) \end{aligned}$$

By comparing the value relationships of these embedded option, we can analyze the interaction effects between the issuer's and holder's exercise strategies, like why the conversion (call) option of `CVCD` is more likely to be exercised than the same option embedded in `CVD` (`CD`) as illustrated in **Figure 2**. Besides, to analyze how the presence of the call option and/or the default option influence aforementioned interaction effects, we compare exercising strategies of **c**on**v**ertible-**c**allable bonds (abbreviated as `CVC`) and **c**on**v**ertible-**d**efaultable bonds (abbreviated as `CVD`) with `CVCD`. By mimicking the aforementioned analyses, the value of a `CVC` at time t can be decomposed into the value of a host bond plus the value of a game option f_{CVC}

$$p_{CVC} = p + f_{CVC}(p, v, t). \tag{14}$$

Define the stopping time $\hat{\tau}_C$ as the bond issuer's optimal call time subject to holders' optimal conversion time $\hat{\tau}_{CV}$, where the overhead hat symbol denotes the exercise time for the option embedded in `CVC`. The payoff of f_{CVC} is defined by substituting $k_{\hat{\tau}_C}$ for $\kappa\left(V_{\hat{\tau}_C}, \hat{\tau}_C\right)$ in **Equation (10)** due to the absence of the default risk. Similarly, the value of a `CVD` at time t can be expressed as

$$p_{CVD} = p + f_{CVD}(p, v, t). \tag{15}$$

Again, we define the stopping time $\check{\tau}_D$ as the bond issuer's optimal default time subject to holders' optimal conversion time $\check{\tau}_{CV}$, where the overhead check symbol denotes the exercise time for the option embedded in `CVD`. The payoff of f_{CVD} is defined by substituting $V_{\check{\tau}_D}$ for $\kappa\left(V_{\tau_{CD^*}}, \tau_{CD^*}\right)$ in **Equation (10)** due to the absence of the call provision. The relations among the values of different game options f_{CV}, f_{CVC}, f_{CVD}, and f_{CVCD} can be expressed by the following theorem.

[6] However, given the conversion fraction $0 < z < 1$, it is always optimal for a bond holder to keep the bond unconverted once the issuer announces default. That is because the recovery value v is larger than the conversion value zv; therefore, default never forces conversion.

Theorem 4 *For each $t \in [0, T)$,*

$$f_{CVC}(p, v, t) \vee f_{CVD}(p, v, t) \quad \leq \quad f_{CV}(p, v, t), \tag{16}$$
$$f_{CVC}(p, v, t) \wedge f_{CVD}(p, v, t) \quad \geq \quad f_{CVCD}(p, v, t). \tag{17}$$

Inequality (16) is intuitive since granting the issuer either a call option or a default option would destroy the value of embedded conversion option owned by bond holders. The second part of this theorem, **Inequality (17)**, suggests that granting a bond issuer both the call and the default option would harm holders' conversion right the most than only granting the issuer either call or default options. Rigorous proofs are given in page 34 in Appendix C.2. The aforementioned theorem can be used to infer how the values of game options (i.e., $f_{CVCD}(p, v, t)$, $f_{CVC}(p, v, t)$ and $f_{CVD}(p, v, t)$) are influenced by the issuer's asset value and the host bond price as follows.

Theorem 5 *Denote two different host bond prices at time t by $p^{(1)}$ and $p^{(2)}$ and two different issuer's asset value at time t by $v^{(1)}$ and $v^{(2)}$. The following properties hold for the $f_{CVCD}(p, v, t)$, $f_{CVD}(p, v, t)$ and $f_{CVC}(p, v, t)$.*

1. $p^{(1)} > p^{(2)} \Rightarrow f_{CVCD}\left(p^{(1)}, v, t\right) \not< f_{CVCD}\left(p^{(2)}, v, t\right)$.
   $p^{(1)} > p^{(2)} \Rightarrow f_{CVD}\left(p^{(1)}, v, t\right) < f_{CVD}\left(p^{(2)}, v, t\right)$.
2. $v^{(1)} < v^{(2)} \Rightarrow f_{CVCD}\left(p, v^{(1)}, t\right) < f_{CVCD}\left(p, v^{(2)}, t\right)$.
   $v^{(1)} < v^{(2)} \Rightarrow f_{CVC}\left(p, v^{(1)}, t\right) < f_{CVC}\left(p, v^{(2)}, t\right)$.
3. $p^{(1)} \neq p^{(2)} \Rightarrow -1 \leq \frac{f_{CVCD}\left(p^{(2)},v,t\right) - f_{CVCD}\left(p^{(1)},v,t\right)}{p^{(2)}-p^{(1)}}$.
   $p^{(1)} \neq p^{(2)} \Rightarrow -1 \leq \frac{f_{CVD}\left(p^{(2)},v,t\right) - f_{CVD}\left(p^{(1)},v,t\right)}{p^{(2)}-p^{(1)}} < 0$. (Put delta Inequality)
4. $v^{(1)} \neq v^{(2)} \Rightarrow 0 \leq \frac{f_{CVCD}\left(p,v^{(1)},t\right) - f_{CVCD}\left(p,v^{(2)},t\right)}{v^{(1)}-v^{(2)}} < 1$.
   $v^{(1)} \neq v^{(2)} \Rightarrow 0 \leq \frac{f_{CVC}\left(p,v^{(1)},t\right) - f_{CVC}\left(p,v^{(2)},t\right)}{v^{(1)}-v^{(2)}} < z$. (Call delta Inequality)

The proof is detailed in page 35 in **Appendix C.2**. Parts 1 and 3 analyze how the change of the host bond values influence a bond holder's game option value. Note that the increment of the host bond price (or decrement of the interest rate level) would discourage a bond holder to convert the bond into equity but encourage the issuer to default. Thus the game option value f_{CVD} decreases with the increment of the host bond price (or the decrement of the interest rate level), which is similar to the property of the conversion option value f_{CV} described in part 1 of **Theorem 1**. On the other hand, the presence of the call option may invalidate the size relationship since its exercise can be triggered either to save the interest rate cost (by substituting new debts for old ones) in the low interest rate environment or to reduce bond holders' conversion value (i.e, the interaction effect) in the high interest rate environment. The size relationships in part 1 forms the right hand side put delta inequality of `CVD` in part 3 but no upper bound for `CVCD` due to the interaction effect. The lower bounds for put delta inequalities suggest that the decrement magnitude of holders' game option value can not be larger than the increment magnitude of the host bond value. Therefore, the value of a bond with embedded game option increases with the increment of the host bond price (or the decrement of the interest rate level) since the bond value can be decomposed into the host bond value plus the game option value as defined in **Equations (9)** and **(15)**.

Part 2 and 4 analyze how the change of the issuer's asset value influence a holder's game option value. Similar to part 2 in **Theorem 1**, inequalities in part 2 of this theorem suggests that the game option value increases with the increment of the issuer's asset value. These inequalities form the left hand side call delta inequality of part 4. The absent of the default right gives a tighter upper bound z for f_{CVC} (compared to 1 for CVCD) because the presence of the default right makes the corresponding game option more sensitive to the issuer's creditworthiness. The upper bounds of the call delta inequalities entail that the value of a bond with embedded game options increases at a slower rate than the increment of the issuer's asset value.

4.3.2 Exercise Boundaries for Game Options and Interaction Effects

We first analyze the the upper and the lower bound of the game option value f_{CVCD} in the following proposition. Grundy and Verwijmeren (2016) argues that in-the-money call phenomena for CVCD disappears after the prevailing of anti-dilution provisions; therefore, perfect capital market assumptions can fairly predict call strategies. Our following position shows that incorporating the anti-dilution policy[7] and perfect market assumptions into the risk neutral valuation model theoretically analyze the rationale of out-of-the-money and at-the-money calls.

Proposition 1 *The value range of the game option f_{CVCD} is*

$$zv - p \leq f_{CVCD}(p, v, t) \leq \kappa\,(v, t) - p. \tag{18}$$

This size relationship suggests the rationale of out-of-the-money and at-the-money calls.

Proof. From a bond holder's point of view, the value of the game option f_{CVCD} should be always larger than or equal to the payoff to exercise the embedded conversion option as follows:

$$f_{CVCD}(p, v, t) \geq zv - p.$$

A holder would convert his/her CVCD into equities when the equality holds. This inequality forms the left hand side of **Equation (18)**. On the other hand, from an issuer's view point, the game option value f_{CDCV} should be larger than or equal to the payoff to exercise the embedded call or default option as follows.

$$f_{CDCV}(p, v, t) \geq p - \kappa\,(v, t)\,.$$

The issuer exercises the call/ default option when the equality holds. This inequality forms the right hand side of **Equation (18)** due to the property $f_{CVCD}(p, v, t) \equiv -f_{CDCV}(p, v, t)$ defined in **Equation(11)**. **Equation (18)** defines the lower bound and the upper bound values for f_{CVCD} to make the embedded game option unexercised and hence CVCD alive. This value range becomes an empty set (i.e., the lower bound exceeds the upper one) when the issuer's value v exceeds the threshold $V_{CV^*} \equiv k_t/z$. Note that the issuer would not consider exercising default option at this scenario since the strike price κ defined in **Equation (6)** could be evaluated as $(v, t) = k_t \wedge v = k_t$. This entails that a CVCD should be optimally called prior to the issuer's value soaring to exceed V_{CV^*}. Above property is consistent with our quantitative analyses of exercise strategies in **Figure 2** by observing that the call boundary (denoted by the dark solid curve) for CVCD lies at or to the left of the vertical threshold $v = V_{CV^*}$. Note that exercising calls at this threshold implies at-the-money call phenomena proposed in Ingersoll (1977a) and Brennan and Schwartz (1977) since the conversion value zv is equal to the

[7] This policy is implemented by setting a bond can be converted into a constant fraction z of the issuer's value as defined in page 10.

call price k_t. Exercising calls to the left of the threshold implies out-of-the money calls phenomena. ■

The above analysis for the value range inequality also provides the rationale for converting a CVCD prior to the issuer's value soaring to exceed V_{CV^*}. This property consists with the property that the conversion boundary (denoted by the light solid curve in **Figure 2**) would lie at or to the left of $v = V_{CV^*}$. The overlap of the call and the conversion regions reflect the "call force to conversion" phenomenon widely studied in empirical literature. Note that removing the call provision from the embedded option of CVCD would remove the upper bound constraint of **Equation (18)**. The threshold constraint V_{CV^*} also disappears and conversion decisions are postponed by observing the shrinking conversion region defined by the conversion boundary (denoted by the light dash curve) of CVD. Similarly, removing the conversion provision would remove the lower bound constraint of **Equation (18)** and call decisions are postponed by observing the shrinking call region defined by the call boundary (denoted by the dark dash curve) of CD.

Since **Proposition 1** suggests that a CVCD must be called or converted prior to the issuer's value soaring to exceed v_{CV^*}, we analyze the optimal exercise strategies for embedded options given the range of the issuer's value located within the region $(0, v_{CV^*})$. The size relations of game options described in **Theorem 5** is used to infer the existences and the shapes of exercise boundaries by **Theorem 6** and **7** described as follows.

Theorem 6 *The existences of exercise boundaries.*
1. If it is optimal to convert (call) a CVCD at a state (p', v', t), then there exists a critical host bond price, $b_{CV^*}(v', t)$ $(b_{CD^*}(v', t))$. It is optimal to convert (call) the CVCD if and only if the corresponding host bond price is lower (higher) than $b_{CV^*}(v', t)$ $(b_{CD^*}(v', t))$ given that the issuer's asset value is v' at time t.
2. If it is optimal to call a CVCD at a state (p', v', t), then there exists a critical host bond price $\bar{v}_{CD^*}(p', t)$ to satisfy the constraint $k_t \le \bar{v}_{CD^*}(p, t) < v_{CV^*}$. It is optimal to call the CVCD if and only if the issuer's value is higher than $\bar{v}_{CD^*}(p', t)$ given that the corresponding host price is p' at time t. Similarly, if it is optimal for a CVCD issuer to declare default at a state (p', v', t), then there exists a critical host bond price $v_{CD^*}(p', t)$ to satisfy the constraint $0 \le v_{CD^*}(p, t) \le k_t$. It is optimal to declare default if and only if the issuer's value is lower than $v_{CD^*}(p', t)$ given that the corresponding host price is p' at time t.

The proof of this theorem is described in page 39 of **Appendix C.2**. It explains why the call region is located in the upper right part of **Figure 2** by showing that an issuer tends to call the bond when the host bond price or the issuer's value is relatively high. Similar arguments can be used to explain why the conversion and the default regions are located in the lower right and the upper left parts, respectively. The existence of $b_{CV^*}(v, t)$ explains the rationale of premature voluntary conversions of CVCD as found in Finnerty (2015). On the other hand, the existences of $b_{CD^*}(v, t)$ and $\bar{v}_{CD^*}(p, t)$ explain the reason of out-of-the-money calls studied in Cowan et al. (1993); Grundy and Verwijmeren (2012); King and Mauer (2014); Bechmann et al. (2014). Above theorems can be used to infer the shape of exercise boundaries and in consequence the interaction effects among coexisting options as follows.

Theorem 7 *Shapes of exercise boundaries. For each $t \in [0, T)$*

1. $v^{(1)} < v^{(2)} \le k_t < v_{CV^*} \Rightarrow b_{CD^*}\left(v^{(1)}, t\right) \le b_{CD^*}\left(v^{(2)}, t\right)$. (Default case)

2. $k_t < v^{(1)} < v^{(2)} < v_{CV^*} \Rightarrow b_{CD^*}\left(v^{(1)}, t\right) \ge b_{CD^*}\left(v^{(2)}, t\right)$. (Call case)

3. $v < v_{CV^*} \Rightarrow b_{CV^*}(v,t) \geq b_{CV}(v,t)$.

4. $v < v_{CV^*} \Rightarrow b_{CD^*}(v,t) \leq b_{CD}(v,t)$.

The proof of this theorem is sketched in page 40 of **Appendix C.2**. The first part indicates the positive slope of the default boundary as illustrated by the black curve in **Figure 2** by showing that the critical host bond price increases with the increment of the issuer's value. On the other hand, the second part indicates the negative slope of the call boundary as illustrated by the dark solid curve by showing that the critical host bond price decreases with the increment of the issuer's value. This negative-slope call boundary also explain why out-of-the-money calls are more likely triggered by lower interest rate levels (or higher host bond price) as observed in Bechmann et al. (2014). The shapes of exercise boundaries mentioned above are similar to the shapes of exercise strategies of `CD` discussed in **Theorem 9** in **Appendix A** to reflect the interaction effect of coexisting call and default options as proposed in Acharya and Carpenter (2002). Specifically, the issuer exercises one of the option would destroy the value of the other one, and the issuer would delay its exercise decision to alleviate the impact of value destruction. The delay becomes salient as the value destruction becomes significant. That is why the critical host bond price for the default (call) boundary increases when the issuer's asset value increases (decreases) since the value of call (default) option being destroyed also increases.

Part 3 and 4 analyze the interaction effect of a game option by comparing the exercise boundaries of `CVCD`, `CV`, and `CD`. Part 4 suggests that the presence of the conversion option would precipitates the call decision by observing that the call of a `CVCD` is more easily to be triggered than the call of an otherwise identical `CD` in **Figure 2**. Specifically, the call region of `CVCD` (defined by b_{CD^*}) is larger than the region of `CD` (defined by b_{CD}). This is because exercising call options of `CVCDs` can further destroy corresponding conversion options that may trigger equity dilution injurious to the existing equity holder as found in Bechmann et al. (2014). To maximize the bond value subject to this call strategy, a `CVCD` would be converted earlier as observed in Jensen and Pedersen (2016) to prevent loss of conversion value. This can be confirmed in part 3 that the conversion of a `CVCD` is more easily to be triggered than the conversion of an otherwise identical `CD`. Specifically, the conversion region of `CVCD` (defined by b_{CV^*}) is larger than the region of `CV` (defined by b_{CV}). Part 3 and 4 capture the interaction effect of a game option that reflect the conflict of interests between the bond issuer and holders.

5 Empirical Tests

5.1 Empirical Implications

Compared with the results in `AC` and those in Section 4.2, we theoretically document the interaction effect revealing the conflict of interest between bond issuers and bond holders behaving to maximize their own value: the presence of the conversion option precipitates call and, in response to this call strategy, the presence of the call provision precipitates conversion; the more valuable the the call and conversion options, the more salient the precipitation. If this interaction effect holds, out-of-the-money calls of convertible bonds is possible, and we can further infer that this type of call is for pre-empting conversion rather than for pre-empting default though either call or conversion can eliminate the possibility of default. In addition, this effect provides another tunnel to explain Jensen and Pedersen (2016)'s observation about early conversion decision. Based on the results illustrated in **Figure 2**, we summarize the empirical implications as follows.

Hypothesis 1. A callable bond would be called earlier in its call period due to the presence of the conversion option given the level of interest rate, its coupon rate, its issuer's credit quality, and its average call price during the call period. Note that the higher the coupon rate and the better the firm's credit quality, the earlier the call timing; the higher the level of interest rate and the average call price, the later the call timing.

Subject to the call strategy for callable convertible bonds addressed in Hypothesis 1, the conversion strategy would be developed as the following hypothesis.

Hypothesis 2. A convertible bond would be converted earlier in its conversion period due to the presence of the call option given the level of interest rate, its coupon rate, its issuer's credit quality, and the dividend per bond if converted. Note that the better the firm's credit quality, the earlier conversion timing. However, the level of interest rate, the coupon rate, and the dividend per bond if converted may be insensitive to the conversion timing.

We especially notice that higher coupon rates lead to higher possibility of call but result in less possibility to trigger conversion. That thus implies higher possibility of bond mature if firms want to force conversion. Similarly, higher dividend per bond if converted lead to less possibility of call but results in higher possibility to trigger conversion that dilutes existing equity holder's value. That thus implies higher possibility of bond mature if firms behave to maximize the equity holder's value.

5.2 Data

To examine the empirical implications derived from our theoretical framework, we collect the information about the features of dollar-denominated callable nonconvertible, noncallable convertible and callable convertible corporate bonds issued during the period January 1990 – December 2010 from *Mergent Fixed Income Securities Database*.That includes bond issue dates, maturity dates, principal amounts, coupon rates, ratings with corresponding rating dates, whether the bonds are called, converted and mature, call information (e.g., the first call dates, call effective dates, call price schedules and call frequency), conversion information (e.g., the first conversion dates, conversion effective dates, conversion prices, conversion commodities and quantity of conversion commodities) and other covenant details. We then search for the call announcement date through *ABI/INFORM Complete* in *ProQuest* system. Besides, we obtain the information on constant maturity Treasury rates based on Federal Reserve Board's H.15 Report from *Federal Reserve Bank Reports* and gather the details on dividends paid by convertible bond issuers from *Compustat*.

We focus on the fixed and non-resettable coupon-bearing bonds with $1000 par value and 30/360 day count convention. We eliminate puttable, exchangeable or pay-in-kind bonds (see Sarkar, 2003; King and Mauer, 2014) or bonds with credit enhancement. Besides, to avoid the call decision that may be influenced by bond covenants, we exclude the callable bonds with sinking fund, make whole, maintenance and replacement fund, and sudden death par provisions and those with indexed principal redemptions. In addition, to concentrate on the bonds convertible to the common stocks of bond issuers, we also exclude the bonds convertible to cash, preferred stocks of bond issuers, stocks of other firms, and others. Finally, we choose non-perpetual bonds that can be called or converted at any time within the stated call and conversion periods, and we select only those that are already mature, called and converted to construct our sample.

The remaining bond samples are separated into two groups: the callable and convertible samples.

The former sample includes 2687 callable nonconvertibles and 362 callable convertibles that are called or mature, whereas the latter one includes 34 noncallable convertibles and 223 callable convertibles that are converted and mature. We remove the bonds with incomplete information. For example, we eliminate the callable bonds missing valid call price schedules (i.e., the complete set of call prices and dates), and call announcement and effective dates from the callable sample. Specifically, the time spans between announcement and effective dates are limited to 90 days. In addition, we eliminate the convertible bonds missing the information about bond issuers' dividend payments, and conversion effective dates from the convertible sample. The final callable sample comprises 2432 callable nonconvertibles (399 mature and 2033 called bonds) and 301 callable convertibles (115 mature and 186 called bonds). The final convertible sample comprises 24 noncallable convertibles (23 mature and 1 converted bonds) and 141 callable convertibles (105 mature and 36 converted bonds; 29 of the 36 converted bonds are identified as forced conversion). Table 1 reports the distribution of the final sample by year issued.

	Callable Sample		Convertible Sample	
Year	Callable	Callable Convertible	Convertible	Callable Convertible
1990–1995	862	31	0	9
1996–2000	934	152	3	63
2001–2005	421	101	9	57
2006–2010	215	17	12	12
Total	2432	301	24	141

Table 1: Distribution of the bond sample by year issued during the period 1990–2010. The bond sample is separated into two subsamples: the callable sample and convertible sample. The callable sample consists of callable nonconvertible and callable convertible bonds issued during the period 1990–2010, and we include only the bond issues that are already called and mature. The convertible sample contains noncallable convertible and callable convertible bonds issued during the same period, and we include only the bond issues that are already converted and mature.

5.3 Variables and Descriptive Statistics

Notice that life spans and the lengths of call periods of callable nonconvertible bonds are usually longer than those of callable convertibles. To make the call or conversion time comparable, we standardize the option exercise time by defining the dependent variable, the ratio of time span (`RatioTS`), as follows:

$$\texttt{RatioTS} = \frac{\texttt{Maturity Date} - \texttt{Effective Date}}{\texttt{Maturity Date} - \texttt{First Date}},$$

where the `Effective Date` stands for call/conversion effective date or bond maturity date, and the *First Date* represents the first call or conversion date. Note that this ratio is bounded within $[0, 1]$; it equals 1 as the `Effective Date` equals the `First Date` and equals 0 as the `Effective Date` equals the `Maturity Date`. Thus, the greater the ratio, the earlier the bond is called or converted in the stated call or converted period.

The following explanatory variables are used in the empirical tests to proxy the parameters of our theoretical model. We choose 2-year Treasury rate as the proxy for the reference interest rate level, because most of our convertible bonds are short-term or middle-term. We then select the rates in month -5 given that the month 0 is the call, conversion or maturity effective month and treat them as the determinants of the outcomes of our bond samples (i.e., called, converted or mature).[8] The

[8] King and Mauer (2000) select the rates in month -2 or -3 given that the month 0 is the call announcement month,

coupon rate is the constant coupon rate on the bond. With our theoretical models, it can thus be predicted that greater `RatioTS` follows the lower interest rate level or the higher coupon rate for the callable sample (i.e., bonds called or mature). However, following the discussion in Section 4.3.2 and in Hypothesis 2, `RatioTS` is insensitive to the interest rate level and the coupon rate for the convertible sample.

Our bond samples include both rated and unrated bonds, and we identify them using the unrated indicator as in Kroszner and Rajan (1994): 0 if the bond is rated and 1 if otherwise. For the rated bonds, we use *S&P's*, *Moody's* or *Fitch's* bond ratings as proxies for bond issuers' credit quality,[9] and we imitate King and Mauer (2000) to cardinalize the bond ratings as AAA = 1, AA+ = 2... and D = 25. For the unrated bonds that are not rated by *S&P*, *Moody* and *Fitch*, we follows Lemmon and Roberts (2010)'s observation and regard them as the bonds with the worst rating in the junk bond rating category though this treatment may overly degrade their true credit quality.[10] With the cardinalized ratings and according to our model, it can thus be predicted that greater `RatioTS` follows the lower bond credit rating for both of the callable and convertible samples. Following this prediction, if treating unrated bonds as the worst rating junk bonds overly degrades their credit quality on average, the unrated indicator would be positively related to `RatioTS`.

Note that we select the callable bond samples that can be called at any time within the stated call periods. The average levels of call prices during the call periods are then determined by calculating the summation of each call price multiplied by the ratio of the corresponding call period to whole stated call period under 30/360 day count convention. Regarding the dividend per bond, we follow Sarkar (2003) to approximate it as the dividend income per convertible bond if converted for the most recent year prior to the conversion or maturity effective year. Notice that we focus only on the bond samples with \$1000 par value. Thus, our model predict that greater `RatioTS` follows the lower average levels of call price during the call period for the callable sample. However, similar to the coupon rate, `RatioTS` is insensitive to the dividend per bond if converted for the convertible sample as discussed in Hypothesis 2.

Finally, for the callable bond samples, we use the convertible indicator to identify whether the callable bond is convertible: 0 if the callable bond is nonconvertible and 1 if otherwise. Similarly, for the convertible bond samples, we use the callable indicator to identify whether the convertible bond is callable: 0 if the convertible bond is noncallable and 1 if otherwise. With the indicators, the interaction effect implied by our model then predicts that the coefficient estimates on them is positively related to `RatioTS` for both of the callable and convertible samples given the aforementioned exploratory variables. That is, compared with callable nonconvertible bonds, callable convertible bonds are prone to be called by bond issuers in stated call periods. Likewise, compared with noncallable convertible bonds, callable convertible bonds are prone to be converted by bond holders in stated conversion periods. Table 2 summarizes the sample characteristics.

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and we limit the time spans between announcement and effective dates to 90 days and choose the rates in month -5 given that the month 0 is the call, conversion or maturity effective month.

[9] That is on the premise that the three rating agencies use similar rating criteria.

[10] According to Molyneux and Shamroukh (1996), the junk bond market consists of bonds rated Ba1 or lower by *Moody's*, BB+ or lower by *S&P's*, or unrated. Also, Lemmon and Roberts (2010) empirically identify that speculative grade firms are more profitable than unrated firms "on average" and that they are significantly financially healthier than unrated firms as indicated by higher Altman Z-scores.

2.A. Callable Sample

	Callable			Callable Convertible		
	Mean	Median	SD	Mean	Median	SD
Maturity (year)	13.27	9.84	9.44	6.74	6.70	2.59
Call period (year)	8.40	5.00	7.20	3.73	3.98	2.38
Ratio of time span	0.64	0.75	0.36	0.40	0.24	0.41
Treasury rate (%)	2.76	2.53	1.67	3.13	2.76	1.72
Coupon rate (%)	8.32	8.00	1.93	5.21	5.25	1.57
Bond rating	12.11	13.00	5.42	18.57	18.00	5.96
Average call price (%) (% of par value)	101.01	100.95	1.12	101.28	101.42	1.64

2.B. Convertible Sample

	Convertible			Callable Convertible		
	Mean	Median	SD	Mean	Median	SD
Maturity (year)	5.64	4.99	1.87	6.50	6.63	2.80
Conversion period (year)	5.60	4.98	1.87	6.38	6.45	2.81
Ratio of time span	0.00	0.00	0.02	0.15	0.00	0.28
Treasury rate (%)	0.97	0.56	1.13	3.23	3.22	1.62
Coupon rate (%)	4.26	5.00	2.40	5.40	5.25	1.79
Bond rating	20.33	25.00	6.02	21.51	25.00	4.83
Dividend per bond ($ divindned per $1000 bond)	4.53	0.00	15.15	6.85	0.00	26.09

Table 2: Descriptive statistics of variables used in the regressions. This table displays the descriptive statistics of variables that would be used in the regression, including mean, median and standard deviation (SD). The panel A is for the callable sample and B is for the convertible sample. The maturity is the bond initial maturity in years. The call period is the length of the period in years during which a callable bond may be called. The conversion period is the length of period in years during which a convertible bond may be converted. The ratio of time span defined for the callable sample is the ratio of time span between the call effective (maturity) date and maturity date to the length of the call period, and that for the convertible sample is the ratio of time span between the bond conversion effective (maturity) date and maturity date to the length of the conversion period. The bond rating is the bond issuer's cardinalized *S&P's* (or *Moody's* or *Fitch's*) bond rating, and AAA = 1,..., D = 25. The Treasury rate in percent per annum is the 2-year Treasury rate in month −5 given that the month 0 is the call/conversion effective month or the bond maturity month. The coupon rate is the constant coupon rate on a bond issue. The average call price is the weighting average of the stated call prices in percentage of bond par value. The dividend per bond is the dividend income per bond (if converted) for the most recent year before the conversion effective month or the bond maturity month.

5.4 Regression Results

Table 3 displays the predicted signs following the aforementioned section and the regressions of `RatioTS` on Convertible/Callable indicator, Treasury rate, Coupon rate, Bond rating, Average call price/Dividend per bond, and Unrated indicator for the callable sample of 2733 bonds and the convertible sample of 165 bonds. In regression (2) and (4), we further take the financial indicator as the control variable to identify whether the bond is issued by a highly regulated financial company.

For the callable sample, as predicted, regression (1) indicates that `RatioTS` is significantly negatively related to Treasury rate, Bond rating, and Average call price but is significantly positively related to Coupon rate. Though the sample statistics in Table 2 shows the ratio of time span for the callable convertible bonds is less than that of the callable nonconvertible bonds on average, it can observed through this regression that the Convertible indicator is significantly positively related to `RatioTS` by taking the aforementioned exploratory variables as given. That implies callable convertible bonds are prone to be called by a bond issuer in the stated call period. This pattern is the same even if we take the financial indicator as the control variable in regression (2). On the other hand, for the convertible sample, the regression (3) indicates that the `RatioTS` is significantly negatively related to Bond rating. Taking the relation as given, we find that the Callable indicator is significantly positively

Independent Variables	Predicted Sign	Callable Sample (1)	(2)	Predicted Sign	Convertible Sample (3)	(4)
Intercept		1.762	1.908		0.158	0.163
		(2.90)***	(3.15)***		(0.98)	(1.01)
Convertible indicator	+	0.212	0.201			
		(7.78)***	(7.40)***			
Callable indicator				+	0.130	0.133
					(2.00)**	(2.03)**
Treasury rate	−	−0.895	−1.016	?	0.256	0.278
		(−2.39)**	(−2.72)***		(0.20)	(0.21)
Coupon rate	+	7.654	7.442	?	0.800	0.931
		(17.58)***	(17.12)***		(0.74)	(0.85)
Bond rating	−	−0.044	−0.046	−	−0.016	−0.017
		(−26.29)***	(−26.97)***		(−1.69)*	(−1.74)*
Average call price	−	−1.203	−1.288			
		(−1.95)**	(−2.09)**			
Dividend per bond				?	−0.001	−0.001
					(−1.56)	(−1.34)
Unrated indicator	+	0.230	0.260	+	0.274	0.278
		(8.02)***	(8.99)***		(2.80)***	(2.83)***
Financial indicator	−		−0.093	−		−0.05
			(−5.42)***			(−0.70)
Adjusted R^2		0.28	0.29		0.08	0.08
F-statistic		180.40***	160.43***		3.51***	3.07***
No. of observations		2733	2733		165	165

Table 3: Regression of ratio of time span on explanatory variables. The dependent variable is the ratio of time span. For the callable sample, it is the ratio of time span between the call effective (maturity) date and maturity date to the length of the call period. For the convertible sample, it is the ratio of time span between the bond conversion effective (maturity) date and maturity date to the length of the conversion period. The exploratory variables are defined as follows. The convertible indicator equals to one once the callable bond is convertible, and zero otherwise. The callable indicator equals to one once the convertible bond is callable, and zero otherwise. The Treasury rate is the 2-year rate in month -5 given that the month 0 is the call, conversion or maturity effective month. The coupon rate is the constant coupon rate on the bond. The bond rating is the bond issuer's cardinalized *S&P's* (or *Moody's* or *Fitch's*) bond rating, and AAA = 1,..., D = 25. The average call price is the weighting average of the stated call prices in percentage of bond par value. The dividend per bond is the dividend income per bond if converted for the most recent year before the conversion or maturity effective month. The unrated indicator equals to one if the bond is unrated, and zero otherwise. The financial indicator equals to one if the bond is issued by a financial firm, and zero otherwise. T-statistics are listed in parentheses. Asterisks indicate significance levels: *, **, *** signify the 10%, 5% and 1% levels using a two-tailed test.

related to `RatioTS`, suggesting callable convertible bonds are prone to be converted by bond holders in the stated conversion period. This pattern is again the same even if we take the financial indicator as the control variable in regression (4). Notice especially that the conversion timing is thus insensitive to the level of interest rate once the interaction effect holds (i.e., both of the Convertible and Callable indicators are significantly positively related to `RatioTS`).

6 Conclusion

This paper constructs a valuation framework based on a structural model of credit risk for a callable convertible bond associated with the option execution decisions revealing the conflict of interest between bond issuers and holders. We then document that this conflict stimulates the interaction precipitating call and conversion decision as the two option holders behave to maximize their own value at expense of the other. The precipitation would be more salient as the values of the two options appreciate. Out-of-the-money call or conversion is even triggered following this interaction effect, and it is the effect that makes the conversion strategy insensitive to the level of interest rate. We thus address that a bond issuer would call early to pre-empt conversion and the corresponding bond holder converts early to pre-empt redemption rather than pre-empting default though either call or conversion can eliminates the possibility of default. These empirical implications can better capture Bhattarcharya (2012)'s insight into the observable early call decision and can provide another tunnel to explain Jensen and Pedersen (2016)'s observation about the early conversion decision. We finally consolidate these implications by carrying out empirical tests with twenty years of callable nonconvertible, noncallable convertible, and callable convertible bond data.

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Appendix A A Brief Review of Acharya and Carpenter (2002)(AC)

This section provides a quick review of AC's mathematical properties and analyses of bonds with embedded default and/or call options that will facilitate our analyses. They first characterize the interest rates by a continuous-time Markov process as in **Equation (1)** and use the no-crossing property in Karatzas and Shreve (1987) to derive useful corollaries as follows. Let $\left\{r_\tau^{(1)}\right\}_{\tau\geq t}$ and $\left\{r_\tau^{(2)}\right\}_{\tau\geq t}$ denote two otherwise identical processes of short-term interest rates with initial values $r_t^{(1)} \leq r_t^{(2)}$, then the no-crossing property implies

$$\tilde{P}\left[r_t^{(1)} \leq r_t^{(2)}, 0 \leq t < \infty\right] = 1.$$

AC apply this property to prove the no-crossing properties for $\{\beta_{t,\tau}\}_{\tau\geq t}$, $\{P_\tau\}_{\tau\geq t}$, $\{\beta_{t,\tau}P_\tau\}_{\tau\geq t}$ and $\{V_\tau\}_{\tau\geq t}$ given that $t \geq 0$ as follows.

Corollary 1 *Let $\left\{\beta_{t,\tau}^{(1)}\right\}_{\tau\geq t}$ and $\left\{\beta_{t,\tau}^{(2)}\right\}_{\tau\geq t}$ be two discount factors processes defined in* **Equation (3)** *that are corresponding to two different initial short-term interest rates $r_t^{(1)}$ and $r_t^{(2)}$, respectively. Then*

$$r_t^{(1)} < r_t^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} \geq \beta_{t,\tau}^{(2)}, \tilde{P} - a.s.\ \forall\tau \in [t,\infty).$$

Note that $\tilde{P} - a.s.$ is the abbreviation of $\tilde{P}-$*almost surely*. Similarly, let $\left\{P_\tau^{(1)}\right\}$ and $\left\{P_\tau^{(2)}\right\}$ be two host bond price processes defined in **Equation (4)** that are corresponding to $r_t^{(1)}$ and $r_t^{(2)}$, respectively. Then **Corollary 1** implies that

Corollary 2 $r_t^{(1)} \leq r_t^{(2)} \Rightarrow P_\tau^{(1)} \geq P_\tau^{(2)},\ \tilde{P} - a.s.\ \forall\tau \in [t,T].$

Associating **Corollary 1** with **Corollary 2**, we have

Corollary 3 $r_t^{(1)} < r_t^{(2)} \Rightarrow \beta_{t,\tau}^{(1)}P_\tau^{(1)} > \beta_{t,\tau}^{(2)}P_\tau^{(2)},\ \tilde{P} - a.s.\ \forall\tau \in [t,T].$

Again, let $\left\{V_\tau^{(1)}\right\}$ and $\left\{V_\tau^{(2)}\right\}$ be two bond issuer's asset value processes defined in **Equation (2)** that are corresponding to $r_t^{(1)}$ and $r_t^{(2)}$, respectively. We derive the no-crossing property for the issuer's asset value described as

Corollary 4 $r_t^{(1)} < r_t^{(2)} \Rightarrow V_\tau^{(1)} \leq V_\tau^{(2)},\ \tilde{P} - a.s.\ \forall\tau \in [t,T].$

Now we suppress the subscript t to make host bond price $P_t \equiv p$. Aforementioned corollaries are then applied to characterize the relations of host bond prices as follows.

Lemma 1 $r_t^{(1)} \leq r_t^{(2)} \Rightarrow \tilde{E}\left[\beta_{t,\tau}^{(2)}P_\tau^{(2)} - \beta_{t,\tau}^{(1)}P_\tau^{(1)}\middle|\mathcal{F}_t\right] \geq p^{(2)} - p^{(1)},\ \tilde{P} - a.s.\ \forall\tau \in [t,T].$

To analyze a bond issuer's strategy to exercise bond options, AC first analyze the size relationship of the options f_X to **c**all (i.e. $X = C$ for a pure callable bond), to **d**efault (i.e. $X = D$ for a pure defaultable bond), and to **c**all or **d**efault (i.e. $X = CD$ for a callable and defaultable bond) the host bond. The following theorem holds for aforementioned three embedded options.

Theorem 8 *Denote two different host bond prices at time t by $p^{(1)}$ and $p^{(2)}$ and two different firm's asset value at time t by $v^{(1)}$ and $v^{(2)}$. Then we have*

1. $p^{(1)} > p^{(2)} \Rightarrow f_X\left(p^{(1)}, v, t\right) > f_X\left(p^{(2)}, v, t\right).$

2. $v^{(1)} < v^{(2)} \Rightarrow f_X\left(p, v^{(1)}, t\right) \geq f_X\left(p, v^{(2)}, t\right)$.

3. $p^{(1)} \neq p^{(2)} \Rightarrow 0 < \frac{f_X\left(p^{(1)}, v, t\right) - f_X\left(p^{(2)}, v, t\right)}{p^{(1)} - p^{(2)}} \leq 1$. (Call delta inequality)

4. $v^{(1)} \neq v^{(2)} \Rightarrow -1 \leq \frac{f_X\left(p, v^{(1)}, t\right) - f_X\left(p, v^{(2)}, t\right)}{v^{(1)} - v^{(2)}} < 0$. (Put delta inequality)

According to **Equation (6)**, the aforementioned three embedded options can be treated as the call options on the host bond price, and the call delta inequality entails that the values of these options increase slower than the increment of the underlying host bond price. Similarly, these three embedded options can be regarded as put options on the issuing firm's asset value, and the put delta inequality entails that the values of these options decrease slower than the increment of the underlying firm's asset value. `AC` then analyze the value range of the combined option $f_{CD}\left(p, v, t\right)$ as follows:

Proposition 2

$$f_C\left(p, v, t\right) \vee f_D\left(p, v, t\right) \leq f_{CD}\left(p, v, t\right) \leq f_C\left(p, v, t\right) + f_D\left(p, v, t\right).$$

The first inequality describes that the value of the combined option is greater than that of either the constituent call or the default option, since the combined option has a lower strike price than the strike price of the other two options as in **Equation (6)**. However, the value of the combined option is less than the sum of these two constituent options due to the interaction effect (see Kim et al., 1993); that is, the loss of destroying one constituent option due exercising another constituent option lowers the value of the combined option. `AC` then argues how the existence of the call (or default) option postpones the execution decision of the default (or call) option.

`AC` characterizes the bond issuer's call and default strategies by analyzing the existences and the shapes of exercise boundaries. Given that the issuer's asset value is v at time t, they show that it is optimal to default or call the bond once the host bond price p is higher than the critical bond price $b_X\left(v, t\right)$ for $X = C$, D, or CD as illustrated by black and dray gray curves in **Figure 2**. On the other hand, given that the host bond price is p at time t, they show that it is optimal to call (default) the bond once the issuer's asset value is higher (lower) than the critical issuer's asset value $v_X\left(p, t\right)$ for $X = CD$ or D. The above findings not only confirm the existence of exercise boundaries, but can be used to sketch the shapes of option execution boundaries as follows:

Theorem 9 *For each* $t \in [0, T)$

1. $v^{(1)} < v^{(2)} \Rightarrow b_D\left(v^{(1)}, t\right) \leq b_D\left(v^{(2)}, t\right)$.

2. $p^{(1)} > p^{(2)} \Rightarrow v_D\left(p^{(1)}, t\right) \geq v_D\left(p^{(2)}, t\right)$.

3. $v^{(1)} < v^{(2)} \leq k_t \Rightarrow b_{CD}\left(v^{(1)}, t\right) \leq b_{CD}\left(v^{(2)}, t\right)$. (Default case)

4. $k_t < v^{(1)} < v^{(2)} \Rightarrow b_{CD}\left(v^{(1)}, t\right) \geq b_{CD}\left(v^{(2)}, t\right)$. (Call case)

5. $v \leq k_t \Rightarrow b_{CD}\left(v, t\right) \geq b_D\left(v, t\right)$.

6. $v > k_t \Rightarrow b_{CD}\left(v, t\right) \geq b_C\left(v, t\right)$.

The first two parts describe the change of the critical value and hence the default boundary (i.e., the default strategy) for a pure defaultable bond. The part one suggests that the stronger the issuer financial status (proxied by a higher issuer's asset value), the harder the host bond price p (can be treated as a proxy of the issuer's prevailing debt obligation level) would exceed the default critical bond price b_D. In other words, a healthier firm is less likely to default unless the prevailing market interest rate level drops lower enough to make the host bond price increases above b_D. On the other hand, the part two suggests that the higher debt obligation level (proxied by the host bond price), the easier the issuer's asset value fails to meet the default threshold v_D. The above two properties imply a upward sloping default boundary as illustrated by the black curve in **Figure 2**. They also entail that a bond issuer is less likely to default in the high interest rate (or the low bond price) environment, which is consistent with Duffee (1998) empirical observations that the bond credit spreads are negatively related to the level of interest rate. The default and the call boundaries for a `CD` are analyzed in Part 3 (given the issuer's asset value at time t, v_t, is smaller than the call price k_t) and in part 4 ($v_t > k_t$), respectively. The default boundary analysis in part 3 is analogous to that in part 1 and the boundary can also be expressed by the upward sloping black curve[11] in **Figure 2**. On the other hand, the call strategy analysis in part 4 suggests that the critical bond price b_{CD} decreases with the increment of the issuer asset value. This entails that a healthier issuer is more likely to redeem its `CD` once p exceeds b_{CD}. This implies a downward-sloping dark dash curve illustrated in **Figure 2**. The coexisting call and default option would postpone the exercise decision of the embedded default and call option as in part 5 and 6, respectively. This can be confirmed by the presence of the coexisting options makes the critical bond price b_{CD} higher than b_D or b_C; thus a lower interest rate level (i.e., a higher host bond price) is required to trigger exercise for bonds with coexisting options. `AC` argue that `CD` issuers would postpone their exercise decisions to avoid the value destruction of another unexercised option, and this phenomenon is empirically confirmed by Jacoby and Shiller (2010).

Appendix B Mathematical Properties Extended from `AC`

To analyze embedded game options in `CVCD`, we derive new mathematical properties derived from `AC` described in Appendix A as follows.

Corollary 5 *Let $\left(\beta_{t,\tau}^{(1)}\right)_{\tau\geq t}$ and $\left(\beta_{t,\tau}^{(2)}\right)_{\tau\geq t}$ be two processes of the discount factors corresponding to two different initial short-term interest rates $r_t^{(1)}$ and $r_t^{(2)}$, respectively. Then, for an arbitrary time $t \in [0,T]$,*

$$r_t^{(1)} \leq r_t^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} V_\tau^{(1)} = \beta_{t,\tau}^{(2)} V_\tau^{(2)}\ , \tilde{P} - a.s.\ \ \forall \tau \in [t,T].$$

Proof. According to **Equation (2)**, $V_\tau^{(1)}$ can be written as

$$V_\tau^{(1)} = V_t e^{\int_t^\tau r_s^{(1)} ds - \int_t^\tau \gamma_s ds - \frac{1}{2}\int_t^\tau \phi_s^2 ds + \int_t^\tau \phi_s d\tilde{W}_s}.$$

[11] The default boundaries for `D` and `CD` are almost overlapped.

Multiplying $\beta^{(1)}_{t,\tau}$ defined in **Equation (3)** on both sides of the above equation, we have

$$\begin{aligned}\beta^{(1)}_{t,\tau}V^{(1)}_{\tau} &= e^{-\int_t^\tau r^{(1)}_s ds}\left(V_t e^{\int_t^\tau r^{(1)}_s ds-\int_t^\tau \gamma_s ds-\frac{1}{2}\int_t^\tau \phi_s^2 ds+\int_t^\tau \phi_s d\tilde{W}_s}\right)\\ &= V_t e^{-\int_t^\tau \gamma_s ds-\frac{1}{2}\int_t^\tau \phi_s^2 ds+\int_t^\tau \phi_s d\tilde{W}_s}\\ &= e^{-\int_t^\tau r^{(2)}_s ds}\left(V_t e^{\int_t^\tau r^{(2)}_s ds-\int_t^\tau \gamma_s ds-\frac{1}{2}\int_t^\tau \phi_s^2 ds+\int_t^\tau \phi_s d\tilde{W}_s}\right)\\ &= \beta^{(2)}_{t,\tau}V^{(2)}_{\tau}\ .\end{aligned} \tag{19}$$

■

Corollary 6 *For an arbitrary time $t \in [0,T]$, the following statements hold given that $\kappa(V_\tau,\tau)$ is set as k_τ, V_τ, or $k_\tau \wedge V_\tau$.*

$$r^{(1)}_t \le r^{(2)}_t \Rightarrow \beta^{(1)}_{t,\tau}\kappa\left(V^{(1)}_\tau,\tau^{(1)}\right) \ge \beta^{(2)}_{t,\tau}\kappa\left(V^{(2)}_\tau,\tau^{(2)}\right), \tilde{P}-a.s.\ \ \forall\tau\in[t,T].$$

Proof. We first consider the case that $\kappa(V_\tau,\tau)\equiv k_\tau$. **Corollary 1** yields that

$$\beta^{(1)}_{t,\tau}\kappa\left(V^{(1)}_\tau,\tau\right) = \beta^{(1)}_{t,\tau}k_\tau \ge \beta^{(2)}_{t,\tau}k_\tau = \beta^{(2)}_{t,\tau}\kappa\left(V^{(2)}_\tau,\tau\right). \tag{20}$$

Then we consider the case $\kappa(V_\tau,\tau)\equiv V_\tau$. **Corollary 5** yields that

$$\beta^{(1)}_{t,\tau}\kappa\left(V^{(1)}_\tau,\tau\right) = \beta^{(1)}_{t,\tau}V^{(1)}_\tau = \beta^{(2)}_{t,\tau}V^{(2)}_\tau = \beta^{(2)}_{t,\tau}\kappa\left(V^{(2)}_\tau,\tau\right). \tag{21}$$

Finally, we consider the case $\kappa(V_\tau,\tau)\equiv k_\tau\wedge V_\tau$. For the case $k_\tau \le V^{(1)}_\tau < V^{(2)}_\tau$, this corollary holds due to **Equation (20)**. For the case $V^{(1)}_\tau < V^{(2)}_\tau \le k_\tau$, this corollary holds due to **Equation (21)**. If $V^{(1)}_\tau < k_\tau < V^{(2)}_\tau$, then

$$\beta^{(1)}_{t,\tau}\kappa\left(V^{(1)}_\tau,\tau\right) = \beta^{(1)}_{t,\tau}V^{(1)}_\tau = \beta^{(2)}_{t,\tau}V^{(2)}_\tau > \beta^{(2)}_{t,\tau}k_\tau = \beta^{(2)}_{t,\tau}\kappa\left(V^{(2)}_\tau,\tau\right).$$

■

Next, we calculate the conditional expectation of the discounted issuer's asset value $\beta_{t,\tau}V_\tau$.

Corollary 7 $\tilde{E}[\beta_{t,\tau}V_\tau|\mathcal{F}_t] = e^{-\int_t^\tau \gamma_s ds}V_t$, $\forall\tau\in[t,T]$, $t\ge 0$.

Proof. **Equation (19)** in the proof of **Corollary 5** can be rearranged as

$$\beta_{t,\tau}V_\tau = V_t e^{-\int_t^\tau \gamma_s ds}\left(e^{-\frac{1}{2}\int_t^\tau \phi_s^2 du+\int_t^\tau \phi_s d\tilde{W}_s}\right). \tag{22}$$

Take conditional expectation given $\mathcal{F}_t$ on both sides of above equation, we have

$$\begin{aligned}\tilde{E}[\beta_{t,\tau}V_\tau|\mathcal{F}_t] &= V_t e^{-\int_t^\tau \gamma_s ds}\left\{\tilde{E}\left[e^{-\frac{1}{2}\int_t^\tau \phi_s^2 ds+\int_t^\tau \phi_s d\tilde{W}_s}\middle|\mathcal{F}_t\right]\right\}\\ &= e^{-\int_t^\tau \gamma_s ds}V_t\left(e^{-\frac{1}{2}\int_t^t \phi_s^2 ds+\int_t^t \phi_s d\tilde{W}_s}\right)\\ &= e^{-\int_t^\tau \gamma_s ds}V_t\ .\end{aligned} \tag{23}$$

Because $e^{-\int_t^\tau \gamma_s ds}$ is deterministic and V_t is $(\mathcal{F}_t)$-*measurable*, they can be taken out from expectation as in **Equation (23)**. The property of a exponential martingale process (see Shreve, 2004) leads to **Equation (23)**, and $e^{-\frac{1}{2}\int_t^t \phi_s^2 ds+\int_t^t \phi_s d\tilde{W}_s} = e^0 = 1$ derives the last equality. ■

Corollary 8 *For an arbitrary time* $\tau \in [t,T]$, $V_t^{(1)} < V_t^{(2)} \rightarrow \beta_{t,\tau} V_\tau^{(1)} < \beta_{t,\tau} V_\tau^{(2)}$

Proof. By substituting $V_\tau^{(1)}$ and $V_\tau^{(2)}$ into **Equation (22)**, we have

$$\begin{aligned}\beta_{t,\tau} V_\tau^{(1)} &= V_t^{(1)} e^{-\int_t^\tau \gamma_s ds} \left(e^{-\frac{1}{2}\int_t^\tau \phi_s^2 du + \int_t^\tau \phi_s d\tilde{W}_s}\right) \\ &< V_t^{(2)} e^{-\int_t^\tau \gamma_s ds} \left(e^{-\frac{1}{2}\int_t^\tau \phi_s^2 du + \int_t^\tau \phi_s d\tilde{W}_s}\right) = \beta_{t,\tau} V_\tau^{(2)}.\end{aligned}$$

■

Corollary 9 *For an arbitrary time* $\tau \in [t,T]$, $V_t^{(1)} < V_t^{(2)} \rightarrow \beta_{t,\tau}\kappa\left(V_\tau^{(1)}, \tau\right) \le \beta_{t,\tau}\kappa\left(V_\tau^{(2)}, \tau\right)$.

Proof.

$$\beta_{t,\tau}\kappa\left(V_\tau^{(1)}, \tau\right) = \beta_{t,\tau} V_\tau^{(1)} \wedge k_\tau \le \beta_{t,\tau} V_\tau^{(2)} \wedge k_\tau = \beta_{t,\tau}\kappa\left(V_\tau^{(2)}, \tau\right).$$

The definition of κ can be found in **Equation(6)**. The above inequality is due to **Corollary 8**. ■

Corollary 10 *For an arbitrary time* $\tau \in [t,T]$, $V_t^{(1)} < V_t^{(2)} \rightarrow$

$$\beta_{t,\tau}\kappa\left(V_\tau^{(1)}, \tau\right) - \beta_{t,\tau}\kappa\left(V_\tau^{(2)}, \tau\right) \ge \beta_{t,\tau} V_\tau^{(1)} - \beta_{t,\tau} V_\tau^{(2)}.$$

Proof.

$$\beta_{t,\tau}\kappa\left(V_\tau^{(1)}, \tau\right) - \beta_{t,\tau}\kappa\left(V_\tau^{(2)}, \tau\right) = \beta_{t,\tau} V_\tau^{(1)} \wedge k_\tau - \beta_{t,\tau} V_\tau^{(2)} \wedge k_\tau. \tag{24}$$

Note that $\beta_{t,\tau} V_\tau^{(1)} < \beta_{t,\tau} V_\tau^{(2)}$ due to **Corollary 8**. We enumerate all possible scenarios for k_τ to show that **Equation (24)** should be larger than or equal to $\beta_{t,\tau} V_\tau^{(1)} - \beta_{t,\tau} V_\tau^{(2)}$ under these scenarios. Indeed, under the scenarios $k_\tau \le \beta_{t,\tau} V_\tau^{(1)}$, $\beta_{t,\tau} V_\tau^{(1)} \le k_\tau \le \beta_{t,\tau} V_\tau^{(2)}$, and $\beta_{t,\tau} V_\tau^{(2)} \le k_\tau$, **Equation (24)** becomes 0, $\beta_{t,\tau} V_\tau^{(1)} - k_\tau$, and $\beta_{t,\tau} V_\tau^{(1)} - \beta_{t,\tau} V_\tau^{(2)}$, respectively, which are all larger than or equal to $\beta_{t,\tau} V_\tau^{(1)} - \beta_{t,\tau} V_\tau^{(2)}$. ■

Appendix C Theoretical Analyses of Exercise Boundaries of Embedded Options

C.1 Pure Convertible Bonds

Proof of Theorem 1. The following proofs describe the impacts of the the host bond price and the issuer's asset value on the value of a `CV`'s embedded option f_{CV} defined in page 10.

1. Let the stopping time $\tau \equiv \tau_{CV}^{(1)} \in [t,T]$ be the optimal conversion time for a `CV` given the prevailing market state $\left(p^{(1)}, v, t\right)$. Since τ can be a feasible but not the optimal conversion time for another state $\left(p^{(2)}, v, t\right)$, we have

$$f_{CV}\left(p^{(1)}, v, t\right) - f_{CV}\left(p^{(2)}, v, t\right) \le \tilde{E}\left[\beta_{t,\tau}^{(1)}\left(zV_\tau^{(1)} - P_\tau^{(1)}\right)^+ - \beta_{t,\tau}^{(2)}\left(zV_\tau^{(2)} - P_\tau^{(2)}\right)^+ \middle| \mathcal{F}_t\right].$$

Note that the premise $p^{(1)} > p^{(2)}$ at time t implies their corresponding short rates $r_t^{(1)} < r_t^{(2)}$. **Corollary 3** and **5** can be applied to entail that $\beta_{t,\tau}^{(1)} P_\tau^{(1)} > \beta_{t,\tau}^{(2)} P_\tau^{(2)}$ and $\beta_{t,\tau}^{(1)} V_\tau^{(1)} = \beta_{t,\tau}^{(2)} V_\tau^{(2)}$. Thus we have

$$\begin{aligned}&\beta_{t,\tau}^{(1)}\left(zV_\tau^{(1)} - P_\tau^{(1)}\right)^+ - \beta_{t,\tau}^{(2)}\left(zV_\tau^{(2)} - P_\tau^{(2)}\right)^+ \\ =\ &\left(z\beta_{t,\tau}^{(1)} V_\tau^{(1)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}\right)^+ - \left(z\beta_{t,\tau}^{(2)} V_\tau^{(2)} - \beta_{t,\tau}^{(2)} P_\tau^{(2)}\right)^+ \le 0 \quad a.s.,\end{aligned}$$

and $\left(z\beta^{(1)}_{t,\tau}V^{(1)}_{\tau}-\beta^{(1)}_{t,\tau}P^{(1)}_{\tau}\right)^{+}-\left(z\beta^{(2)}_{t,\tau}V^{(2)}_{\tau}-\beta^{(2)}_{t,\tau}P^{(2)}_{\tau}\right)^{+}<0$ when the conversion option is exercised with a nonzero probability. This ensures

$$\tilde{E}\left[\beta^{(1)}_{t,\tau}\left(zV^{(1)}_{\tau}-P^{(1)}_{\tau}\right)^{+}-\beta^{(2)}_{t,\tau}\left(zV^{(2)}_{\tau}-P^{(2)}_{\tau}\right)^{+}\middle|\mathcal{F}_t\right]<0$$

and confirms $f_{CV}\left(p^{(1)},v,t\right)-f_{CV}\left(p^{(2)},v,t\right)<0$.

2. The premise $v^{(1)}<v^{(2)}$ at time t entails that $V^{(1)}_u<V^{(2)}_u$, $\forall u\in[t,T]$ due to **Equation (2)**. Let the stopping time $\tau\equiv\tau^{(1)}_{CV}\in[t,T]$ be the optimal conversion time for a CV given the prevailing market state $\left(p,v^{(1)},t\right)$. Since τ can be a feasible but not the optimal conversion time for another state $\left(p,v^{(2)},t\right)$, we have

$$f_{CV}\left(p,v^{(1)},t\right)-f_{CV}\left(p,v^{(2)},t\right)\leq\tilde{E}\left[\beta_{t,\tau}\left(zV^{(1)}_{\tau}-P_{\tau}\right)^{+}-\beta_{t,\tau}\left(zV^{(2)}_{\tau}-P_{\tau}\right)^{+}\middle|\mathcal{F}_t\right]. \tag{25}$$

Note that $\left(zV^{(1)}_{\tau}-P_{\tau}\right)^{+}-\left(zV^{(2)}_{\tau}-P_{\tau}\right)^{+}\leq 0$ due to the premise $V^{(1)}_{\tau}<V^{(2)}_{\tau}$ and this inequality is strictly negative when the conversion option is exercised with a positive probability. That ensures that **Inequality (25)** and hence $f_{CV}\left(p,v^{(1)},t\right)-f_{CV}\left(p,v^{(2)},t\right)$ are negative.

3. Note that the part 1 of **Theorem 1** confirms the right hand side of the put delta inequality. Thus we proceed to prove the left hand side inequality by showing that $f_{CV}\left(p^{(1)},v,t\right)-f_{CV}\left(p^{(2)},v,t\right)\geq p^{(2)}-p^{(1)}$. Without loss of generality, we consider the case $p^{(1)}>p^{(2)}$ at time t, which implies $r^{(1)}_t<r^{(2)}_t$. Let the stopping time $\tau\equiv\tau^{(2)}_{CV}\in[t,T]$ be the optimal conversion time for a CV given the prevailing market state $\left(p^{(2)},v,t\right)$. Since replacing the optimal conversion time for the state $\left(p^{(1)},v,t\right)$ with the non-optimal conversion time τ would reduce the conversion option value $f_{CV}\left(p^{(1)},v,t\right)$, thus we have the first inequality as follows.

$$\begin{aligned}
& f_{CV}\left(p^{(1)},v,t\right)-f_{CV}\left(p^{(2)},v,t\right)\\
\geq\ & \tilde{E}\left[\beta^{(1)}_{t,\tau}\left(zV^{(1)}_{\tau}-P^{(1)}_{\tau}\right)^{+}-\beta^{(2)}_{t,\tau}\left(zV^{(2)}_{\tau}-P^{(2)}_{\tau}\right)^{+}\middle|\mathcal{F}_t\right]\\
=\ & \tilde{E}\left[\left(\beta^{(1)}_{t,\tau}\left(zV^{(1)}_{\tau}-P^{(1)}_{\tau}\right)^{+}-\beta^{(2)}_{t,\tau}\left(zV^{(2)}_{\tau}-P^{(2)}_{\tau}\right)\right)\cdot\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}\middle|\mathcal{F}_t\right] && (26)\\
\geq\ & \tilde{E}\left[\left(\beta^{(1)}_{t,\tau}\left(zV^{(1)}_{\tau}-P^{(1)}_{\tau}\right)-\beta^{(2)}_{t,\tau}\left(zV^{(2)}_{\tau}-P^{(2)}_{\tau}\right)\right)\cdot\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}\middle|\mathcal{F}_t\right] && (27)\\
=\ & \tilde{E}\left[\left(\left(z\beta^{(1)}_{t,\tau}V^{(1)}_{\tau}-\beta^{(1)}_{t,\tau}P^{(1)}_{\tau}\right)-\left(z\beta^{(2)}_{t,\tau}V^{(2)}_{\tau}-\beta^{(2)}_{t,\tau}P^{(2)}_{\tau}\right)\right)\cdot\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}\middle|\mathcal{F}_t\right] && (28)\\
=\ & \tilde{E}\left[\left(\beta^{(2)}_{t,\tau}P^{(2)}_{\tau}-\beta^{(1)}_{t,\tau}P^{(1)}_{\tau}\right)\cdot\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}\middle|\mathcal{F}_t\right] && (29)\\
\geq\ & \tilde{E}\left[\beta^{(2)}_{t,\tau}P^{(2)}_{\tau}-\beta^{(1)}_{t,\tau}P^{(1)}_{\tau}\middle|\mathcal{F}_t\right] && (30)\\
\geq\ & p^{(2)}-p^{(1)}. && (31)
\end{aligned}$$

Equation (26) replaces "+" in $\left(zV^{(2)}_{\tau}-P^{(2)}_{\tau}\right)^{+}$ with the indicator function $\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}$. It will be larger than **Inequality (27)** since $zV^{(1)}_{\tau}-P^{(1)}_{\tau}$ can still be non-positive given $zV^{(2)}_{\tau}-P^{(2)}_{\tau}>0$. By substituting the equality $\beta^{(1)}_{t,\tau}V^{(1)}_{\tau}=\beta^{(2)}_{t,\tau}V^{(2)}_{\tau}$ derived in **Corollary 5** into **Equation (28)**, we obtain **Equation (29)**. The premise $p^{(1)}>p^{(2)}$ at time t implies $r^{(1)}_t<r^{(2)}_t$ and hence $\beta^{(2)}_{t,\tau}P^{(2)}_{\tau}-\beta^{(1)}_{t,\tau}P^{(1)}_{\tau}\leq 0$ by **Corollary 3**. Thus dropping the indicator function $\mathbf{I}_{\left\{zV^{(2)}_{\tau}>P^{(2)}_{\tau}\right\}}$ would reduces the expectation in **Inequality (30)**. Finally, **Inequality (31)** is derived by taking advantage of **Lemma 1**.

4. Note that part 2 of **Theorem 1** confirms the left hand side of the call delta inequality. Thus we proceed to prove the right hand side inequality by showing that $f_{CV}\left(p,v^{(1)},t\right)-f_{CV}\left(p,v^{(2)},t\right)>z\left(v^{(1)}-v^{(2)}\right)$ by mimicking the aforementioned proof for part 3 as follows. Without loss of generality, we consider the case $v^{(1)} < v^{(2)}$ at time t; thus **Equation (2)** suggests that $V_u^{(1)} < V_u^{(2)}$, $\forall u \in [t, T]$. Let the stopping time $\tau \equiv \tau_{CV}^{(2)} \in [t, T]$ be the optimal conversion time for a `CV` given the prevailing market state $\left(p, v^{(2)}, t\right)$. Since replacing the optimal conversion time for the state $\left(p, v^{(1)}, t\right)$ with the non-optimal conversion time τ would reduce the conversion option value $f_{CV}\left(p, v^{(1)}, t\right)$, thus we have the first inequality as follows.

$$
\begin{aligned}
& f_{CV}\left(p, v^{(1)}, t\right) - f_{CV}\left(p, v^{(2)}, t\right) \\
\geq \;& \tilde{E}\left[\beta_{t,\tau}\left(zV_\tau^{(1)} - P_\tau\right)^+ - \beta_{t,\tau}\left(zV_\tau^{(2)} - P_\tau\right)^+ \middle| \mathcal{F}_t\right] \\
= \;& \tilde{E}\left[\left(\beta_{t,\tau}\left(zV_\tau^{(1)} - P_\tau\right)^+ - \beta_{t,\tau}\left(zV_\tau^{(2)} - P_\tau\right)\right) \cdot \mathbf{I}_{\left\{zV_\tau^{(2)} > P_\tau\right\}} \middle| \mathcal{F}_t\right] && (32) \\
\geq \;& \tilde{E}\left[\left(\beta_{t,\tau}\left(zV_\tau^{(1)} - P_\tau\right) - \beta_{t,\tau}\left(zV_\tau^{(2)} - P_\tau\right)\right) \cdot \mathbf{I}_{\left\{zV_\tau^{(2)} > P_\tau\right\}} \middle| \mathcal{F}_t\right] && (33) \\
= \;& z\tilde{E}\left[\left(\beta_{t,\tau}\left(V_\tau^{(1)} - V_\tau^{(2)}\right)\right) \cdot \mathbf{I}_{\left\{zV_\tau^{(2)} > P_\tau\right\}} \middle| \mathcal{F}_t\right] \\
> \;& z\tilde{E}\left[\beta_{t,\tau}\left(V_\tau^{(1)} - V_\tau^{(2)}\right) \middle| \mathcal{F}_t\right] && (34) \\
= \;& ze^{-\int_t^\tau \gamma_s ds}\left(v^{(1)} - v^{(2)}\right) && (35) \\
> \;& z\left(v^{(1)} - v^{(2)}\right). && (36)
\end{aligned}
$$

Equation (32) replaces "+" in $\left(zV_\tau^{(2)} - P_\tau\right)^+$ with the indicator function $\mathbf{I}_{\left\{zV_\tau^{(2)} > P_\tau\right\}}$. It will be larger than **Inequality (33)** since $zV_\tau^{(1)} - P_\tau$ can still be non-positive given $zV_\tau^{(2)} - P_\tau > 0$. The premise $v^{(1)} < v^{(2)}$ at time t implies $V_\tau^{(1)} < V_\tau^{(2)}$ and hence dropping the indicator function would reduce the value of the expectation in **Inequality (34)**. Then we derive **Equation (35)** by taking advantage of **Corollary 7**. Finally, **Inequality (36)** is derived due to the fact that $0 < ze^{-\int_t^\tau \gamma_s ds} < 1$ and the premise $v^{(1)} - v^{(2)} < 0$. ∎

With the properties of f_{CV} listed in **Theorem 1**, we prove the existence of the conversion boundary described in **Theorem 2** in page 11 as follows.

Proof of Theorem 2. Define the continuation region for the conversion option (i.e. the region that will keep `CV` unconverted) $U \equiv \left\{(p, v, t) \in R^+ \times R^+ \times [0, T] : f_{CV}(p, v, t) > (zv - p)^+\right\}$.

1. Without loss of generality, we consider two states $(p^{(1)}, v, t)$ and $(p^{(2)}, v, t)$ given that $p^{(1)} > p^{(2)}$. Now we prove that it is optimal to continue (i.e., not to convert the `CV`) at the former state under the premise that it is optimal to continue at the latter one. The put delta inequality in **Theorem 1** yields

$$
\begin{aligned}
& \frac{f_{CV}\left(p^{(2)}, v, t\right) - f_{CV}\left(p^{(1)}, v, t\right)}{p^{(2)} - p^{(1)}} \geq -1 \\
\Rightarrow \;& f_{CV}\left(p^{(1)}, v, t\right) \geq f_{CV}\left(p^{(2)}, v, t\right) + p^{(2)} - p^{(1)}.
\end{aligned}
$$

Since it is optimal to continue at $(p^{(2)}, v, t)$, the value of the conversion option $f_{CV}\left(p^{(2)}, v, t\right)$ should be larger than the value to convert the `CV` immediately $\left(zv - p^{(2)}\right)^+$. Thus we have

$$
\begin{aligned}
f_{CV}\left(p^{(1)}, v, t\right) &\geq f_{CV}\left(p^{(2)}, v, t\right) + p^{(2)} - p^{(1)} \\
&> \left(zv - p^{(2)}\right)^+ + p^{(2)} - p^{(1)} \\
&\geq \left(zv - p^{(2)}\right) + p^{(2)} - p^{(1)} \\
&= zv - p^{(1)}. && (37)
\end{aligned}
$$

Above inequality entails that it is optimal to continue at the state $(p^{(1)}, v, t)$ since its conversion option value $f_{CV}\left(p^{(1)}, v, t\right)$ is larger than the immediate conversion value $zv - p^{(1)}$. Given a fixed asset value v at time t, let $b_{CV}(v,t)$ be the infimum of the host bond price p to satisfy the condition $(p,v,t) \in U$. **Equation (37)** entails that a state (p', v, t) would belong to U if $p' > b_{CV}(v,t)$. Since $b_{CV}(v,t)$ is the infimum, (p', v, t) would not belong to U if $p' < b_{CV}(v,t)$. Above properties make $b_{CV}(v,t)$ the critical host bond price the separate the continuation region from the conversion one. In addition, the value of the conversion option $f_{CV}\left(b_{CV}(v,t), v, t\right)$ should be equal to the value to immediately convert the `CV` $zv - b_{CV}(v,t)$, which should be positive. This implies that $b_{CV}(v,t) < zv$.

2. Without loss of generality, we consider two states $\left(p, v^{(1)}, t\right)$ and $\left(p, v^{(2)}, t\right)$ given that $v^{(1)} < v^{(2)}$. We prove that it is optimal to continue at the former state under the premise that it is optimal to continue at the latter one. The call delta inequality in **Theorem 1** yields

$$
\begin{aligned}
& \frac{f_{CV}\left(p, v^{(1)}, t\right) - f_{CV}\left(p, v^{(2)}, t\right)}{v^{(1)} - v^{(2)}} < z \\
\Rightarrow \quad & f_{CV}\left(p, v^{(1)}, t\right) > f_{CV}\left(p, v^{(2)}, t\right) + z\left(v^{(1)} - v^{(2)}\right).
\end{aligned}
$$

Since it is optimal to continue at $\left(p, v^{(2)}, t\right)$, the value of the conversion option $f_{CV}\left(p, v^{(2)}, t\right)$ should be larger than the value to convert the `CV` immediately $\left(zv^{(2)} - p\right)^{+}$. Thus we have

$$
\begin{aligned}
f_{CV}\left(p, v^{(1)}, t\right) &\geq f_{CV}\left(p, v^{(2)}, t\right) + zv^{(1)} - zv^{(2)} \\
&> \left(zv^{(2)} - p\right)^{+} + zv^{(1)} - zv^{(2)} \\
&\geq \left(zv^{(2)} - p\right) + zv^{(1)} - zv^{(2)} \\
&= zv^{(1)} - p. \qquad (38)
\end{aligned}
$$

Above inequality entails that it is optimal to continue at the state $\left(p, v^{(1)}, t\right)$ since its conversion option value $f_{CV}\left(p, v^{(1)}, t\right)$ is larger than the immediate conversion value $zv^{(1)} - p$. Given a fixed host bond price p at time t, let $v_{CV}(p,t)$ be the supremum of the issuer's asset value v to satisfy the condition $(p,v,t) \in U$. **Equation (38)** entails that a state (p, v', t) would belong to U if $v' < v_{CV}(p,t)$. Since $v_{CV}(p,t)$ is the supremum, (p, v', t) would not belong to U if $v' > v_{CV}(p,t)$. Above properties make $v_{CV}(p,t)$ the critical issuer's asset price the separate the continuation region form the conversion region. In addition, the value of the conversion option $f_{CV}\left(p, v_{CV}(p,t), t\right)$ should be equal to the value to immediately convert the `CV` $zv_{CV}(p,t) - p$, which should be positive. This implies that $v_{CV}(p,t) > p/z$. ■

With the above two theorems, we can mathematically analyze the shape of the conversion boundary described in **Theorem 3** as follows.

Proof of Theorem 3.

1. To show that the critical host bond price increases with the increment of the prevailing issuer asset value, it suffices to show that the premise $p > b_{CV}\left(v^{(2)}, t\right)$ (i.e., $\left(p, v^{(2)}, t\right)$ is belong to the continuation region U) entails $p > b_{CV}\left(v^{(1)}, t\right)$ given $v^{(1)} < v^{(2)}$. The call delta inequality in **Theorem 1** yields

$$\begin{aligned}
f_{CV}\left(p, v^{(1)}, t\right) &\geq f_{CV}\left(p, v^{(2)}, t\right) + zv^{(1)} - zv^{(2)} \\
&> \left(zv^{(2)} - p\right)^{+} + zv^{(1)} - zv^{(2)} \\
&\geq \left(zv^{(2)} - p\right) + zv^{(1)} - zv^{(2)} \\
&= zv^{(1)} - p.
\end{aligned}$$

This entails that $p > b_{CV}\left(v^{(1)}, t\right)$ since the conversion option value $f_{CV}\left(p, v^{(1)}, t\right)$ is larger than the value to immediately convert the `CV` $zv^{(1)} - p$.

2. To show that the critical asset value increases with the increment of the prevailing host bond price, it suffices to show the premise $v < v_{CV}\left(p^{(2)}, t\right)$ (i.e., $(p^{(2)}, v, t)$ is belong to the continuation region U) entails $v < v_{CV}\left(p^{(1)}, t\right)$ given $p^{(1)} > p^{(2)}$. The put delta inequality in **Theorem 1** yields

$$\begin{aligned}
f_{CV}\left(p^{(1)}, v, t\right) &\geq f_{CV}\left(p^{(2)}, v, t\right) + p^{(2)} - p^{(1)} \\
&> \left(zv - p^{(2)}\right)^{+} + p^{(2)} - p^{(1)} \\
&\geq \left(zv - p^{(2)}\right) + p^{(2)} - p^{(1)} \\
&= zv - p^{(1)}.
\end{aligned}$$

This entails that $v < v_{CV}\left(p^{(1)}, t\right)$ since the conversion option value $f_{CV}\left(p^{(1)}, v, t\right)$ is larger than the value to immediately convert the `CV` $zv - p^{(1)}$. ■

C.2 Bonds with Game Options

The proofs for **Theorem 4** that analyzes the size relationship for f_{CV}, and game options f_{CVC}, f_{CVD}, and f_{CVCD} are as follows.

Proof of Theorem 4.
To prove **Inequality (17)**, it suffices to show both $f_{CVC}(p, v, t) - f_{CVCD}(p, v, t) \geq 0$ and $f_{CVD}(p, v, t) - f_{CVCD}(p, v, t) \geq 0$. The proof for the latter inequality is similar to that for the former one so we only prove the former one for simplicity. Recall that $\hat{\tau}_C$ and τ_{CV^*} are the optimal call time for `CVC` and the optimal conversion time for `CVCD` for the state (p, v, t), respectively. If we apply the call strategy $\hat{\tau}_C$ and the conversion strategy τ_{CV^*} for game options embedded in `CVC` and `CVCD`, we have

$$\begin{aligned}
& f_{CVC}(p, v, t) - f_{CVCD}(p, v, t) \\
\geq\ & \underbrace{\tilde{E}\left[\beta_{t,\tau_{CV^*}}\left(zV_{\tau_{CV^*}} - P_{\tau_{CV^*}}\right)\mathbf{I}_{\{\tau_{CV^*} \leq \hat{\tau}_C\}} - \beta_{t,\hat{\tau}_C}\left(P_{\hat{\tau}_C} - k_{\hat{\tau}_C}\right)\mathbf{I}_{\{\hat{\tau}_C < \tau_{CV^*}\}}\middle|\mathcal{F}_t\right]}_{\mathbf{A}} \\
& - \underbrace{\tilde{E}\left[\beta_{t,\tau_{CV^*}}\left(zV_{\tau_{CV^*}} - P_{\tau_{CV^*}}\right)\mathbf{I}_{\{\tau_{CV^*} \leq \hat{\tau}_C\}} - \beta_{t,\hat{\tau}_C}\left(P_{\hat{\tau}_C} - \kappa\left(V_{\hat{\tau}_C}, \hat{\tau}_C\right)\right)\mathbf{I}_{\{\hat{\tau}_C < \tau_{CV^*}\}}\middle|\mathcal{F}_t\right]}_{\mathbf{B}} \qquad (39) \\
=\ & \tilde{E}\left[\left(\beta_{t,\hat{\tau}_C}\left(P_{\hat{\tau}_C} - \kappa\left(V_{\hat{\tau}_C}, \hat{\tau}_C\right)\right) - \beta_{t,\hat{\tau}_C}\left(P_{\hat{\tau}_C} - k_{\hat{\tau}_C}\right)\right)\mathbf{I}_{\{\hat{\tau}_C < \tau_{CV^*}\}}\middle|\mathcal{F}_t\right] \qquad (40) \\
=\ & \tilde{E}\left[\left(\beta_{t,\hat{\tau}_C}\left(k_{\hat{\tau}_C} - \kappa\left(V_{\hat{\tau}_C}, \hat{\tau}_C\right)\right)\right)\mathbf{I}_{\{\hat{\tau}_C < \tau_{CV^*}\}}\middle|\mathcal{F}_t\right] \qquad (41) \\
\geq\ & 0.
\end{aligned}$$

Part A of **Inequality (39)** is less than $f_{CVC}(p, v, t)$ since the non-optimal conversion strategy τ_{CV^*} accompanied by the optimal call strategy $\hat{\tau}_C$ ruins the benefits of `CVC` holders; on the other hand, part B is greater than $f_{CVCD}(p, v, t)$ since the non-optimal call strategy $\hat{\tau}_C$ accompanied by the optimal conversion strategy τ_{CV^*} harms the benefits of the CVCD issuer and, in inconsequence, increases the benefits of CVCD holders. **Equation (40)** is obtained after rearranging **Inequality (39)**. The value $\beta_{t,\hat{\tau}_C}\left(k_{\hat{\tau}_C} - \kappa\left(V_{\hat{\tau}_C}, \hat{\tau}_C\right)\right)$ in **Equation (41)** is nonnegative since $\kappa\left(V_{\tau_{C^*}}, \tau_{C^*}\right)$ is the minimum of $V_{\hat{\tau}_C}$ and $k_{\hat{\tau}_C}$. Therefore we have $f_{CVC}(p,v,t) - f_{CVCD}(p,v,t) \geq 0$.

■

Then we analyze how an issuer's asset value and the host bond price influence the values of game options described in **Theorem 5** as follows.

Proof of Theorem 5.

1. This part explains how the change of the host bond price (or the interest rate level) influences the values of game options. We first show that the presence of the call option makes the impact of changing the host bond price on a f_{CVCD} undetermined. This proof is then slightly modified to analyze the impact on a f_{CVD}. Let the stopping time $\tau^{(1)}_{CV^*} \in [t,T]$ be the optimal conversion strategy for a CVCD holder given the prevailing market state $\left(p^{(1)}, v, t\right)$. The stopping time $\tau^{(2)}_{CD^*} \in [t,T]$ be the optimal call/default strategy for the CVCD issuer given the market state $\left(p^{(2)}, v, t\right)$. Thus, we have

$$
\begin{aligned}
& f_{CVCD}\left(p^{(1)}, v, t\right) - f_{CVCD}\left(p^{(2)}, v, t\right) \\
\leq\; & \underbrace{\tilde{E}\left[\beta^{(1)}_{t,\tau^{(1)}_{CV^*}}\left(zV^{(1)}_{\tau^{(1)}_{CV^*}} - P^{(1)}_{\tau^{(1)}_{CV^*}}\right)\mathbf{I}_{\left\{\tau^{(1)}_{CV^*}\leq\tau^{(2)}_{CD^*}\right\}} - \beta^{(1)}_{t,\tau^{(2)}_{CD^*}}\left(P^{(1)}_{\tau^{(2)}_{CD^*}} - \kappa\left(V^{(1)}_{\tau^{(2)}_{CD^*}}, \tau^{(2)}_{CD^*}\right)\right)\mathbf{I}_{\left\{\tau^{(2)}_{CD^*}<\tau^{(1)}_{CV^*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{A}} \\
& - \underbrace{\tilde{E}\left[\beta^{(2)}_{t,\tau^{(1)}_{CV^*}}\left(zV^{(2)}_{\tau^{(1)}_{CV^*}} - P^{(2)}_{\tau^{(1)}_{CV^*}}\right)\mathbf{I}_{\left\{\tau^{(1)}_{CV^*}\leq\tau^{(2)}_{CD^*}\right\}} - \beta^{(2)}_{t,\tau^{(2)}_{CD^*}}\left(P^{(2)}_{\tau^{(2)}_{CD^*}} - \kappa\left(V^{(2)}_{\tau^{(2)}_{CD^*}}, \tau^{(2)}_{CD^*}\right)\right)\mathbf{I}_{\left\{\tau^{(2)}_{CD^*}<\tau^{(1)}_{CV^*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{B}}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
=\; & \tilde{E}\left[\underbrace{\left(\beta^{(1)}_{t,\tau^{(1)}_{CV^*}}\left(zV^{(1)}_{\tau^{(1)}_{CV^*}} - P^{(1)}_{\tau^{(1)}_{CV^*}}\right) - \beta^{(2)}_{t,\tau^{(1)}_{CV^*}}\left(zV^{(2)}_{\tau^{(1)}_{CV^*}} - P^{(2)}_{\tau^{(1)}_{CV^*}}\right)\right)}_{\mathbf{C}}\mathbf{I}_{\left\{\tau^{(1)}_{CV^*}\leq\tau^{(2)}_{CD^*}\right\}}\middle|\mathcal{F}_t\right] \\
& + \tilde{E}\left[\underbrace{\left(\beta^{(2)}_{t,\tau^{(2)}_{CD^*}}\left(P^{(2)}_{\tau^{(2)}_{CD^*}} - \kappa\left(V^{(2)}_{\tau^{(2)}_{CD^*}}, \tau^{(2)}_{CD^*}\right)\right) - \beta^{(1)}_{t,\tau^{(2)}_{CD^*}}\left(P^{(1)}_{\tau^{(2)}_{CD^*}} - \kappa\left(V^{(1)}_{\tau^{(2)}_{CD^*}}, \tau^{(2)}_{CD^*}\right)\right)\right)}_{\mathbf{D}}\mathbf{I}_{\left\{\tau^{(2)}_{CD^*}<\tau^{(1)}_{CV^*}\right\}}\middle|\mathcal{F}_t\right]
\end{aligned} \tag{43}
$$

$$
\not\leq\; 0.
$$

Since the optimal bond holders' conversion strategy $\tau^{(1)}_{CV^*}$ accompanied by the non-optimal issuer's call/default strategy $\tau^{(2)}_{CD^*}$ would be beneficial for holders given the market state $\left(p^{(1)}, v, t\right)$, part A of **Inequality (42)** should be greater than $f_{CVCD}\left(p^{(1)}, v, t\right)$. Similarly, the optimal issuer strategy $\tau^{(2)}_{CD^*}$ accompanied by the non-optimal holders' strategy $\tau^{(1)}_{CV^*}$ would ruin holders' benefits given the market state $\left(p^{(2)}, v, t\right)$, part B should be less than $f_{CVCD}\left(p^{(2)}, v, t\right)$. Combining above two size relationships establish the **Inequality (42)**. Rearranging **Inequality (42)** yields **Equation (43)**. Note that the premise the relation of time t bond prices $p^{(1)} > p^{(2)}$ entails $r^{(1)}_t < r^{(2)}_t$. Thus we can apply **Corollary 3** and **5** listed below

$$
\begin{aligned}
-\beta^{(1)}_{t,\tau^{(1)}_{CV^*}} P^{(1)}_{\tau^{(1)}_{CV^*}} + \beta^{(2)}_{t,\tau^{(1)}_{CV^*}} P^{(2)}_{\tau^{(1)}_{CV^*}} &< 0 \\
\beta^{(1)}_{t,\tau^{(1)}_{CV^*}} V^{(1)}_{\tau^{(1)}_{CV^*}} - \beta^{(2)}_{t,\tau^{(1)}_{CV^*}} V^{(2)}_{\tau^{(1)}_{CV^*}} &= 0,
\end{aligned}
$$

to entail that part C is negative. In addition, adding **Corollary 3** to **Corollary 6** listed below would yield

$$\begin{aligned} &\beta^{(2)}_{t,\tau^{(2)}_{CD*}} P^{(2)}_{\tau^{(2)}_{CD*}} - \beta^{(1)}_{t,\tau^{(1)}_{CD*}} P^{(1)}_{\tau^{(1)}_{CD*}} < 0, \\ &\beta^{(1)}_{t,\tau^{(2)}_{CD*}} \kappa\left(V^{(1)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right) - \beta^{(2)}_{t,\tau^{(2)}_{CD*}} \kappa\left(V^{(2)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right) \geq 0, \end{aligned} \tag{44}$$

part D. But we can not determine whether part D is positive or negative; thus the impact of changing the host bond price (or the interest rate level) on f_{CVCD} is undetermined. Indeed, our later empirical studies in **Table 3** also suggests that the conversion decisions of `CVCD` holders are insensitive to the Treasury rates.

The aforementioned undetermined phenomenon disappears when the embedded call option is absent; that is, we analyze `CVD` instead. The proof is similar as above except that the optimal stopping time for `CVCD` (marked by star signs) should be replaced by the stopping times for `CVD` (marked by overhead checks). Besides, $\kappa(V_\tau, \tau)$ defined in **Equation (6)** is replaced by issuer's asset value as default occurs since a `CVD` issuer only owns the default option. Therefore, **Inequality (44)** is changed to

$$\beta^{(1)}_{t,\check{\tau}^{(2)}_D} V^{(1)}_{\check{\tau}^{(2)}_D} - \beta^{(2)}_{t,\check{\tau}^{(2)}_D} V^{(2)}_{\check{\tau}^{(2)}_D} = 0$$

due to **Corollary 5**. By mimicking the derivations in **Equations (42)** and **(43)**, we have $f_{CVD}\left(p^{(1)}, v, t\right) - f_{CVD}\left(p^{(2)}, v, t\right) < 0$.

2. Let the stopping time $\tau^{(1)}_{CV*} \in [t, T]$ be the optimal conversion strategy for a `CVCD` holder given the prevailing market state $\left(p, v^{(1)}, t\right)$. The stopping time $\tau^{(2)}_{CD*} \in [t, T]$ be the optimal call/default strategy for the `CVCD` issuer given the market state $\left(p, v^{(2)}, t\right)$. Thus we have

$$\begin{aligned} & f_{CVCD}\left(p, v^{(1)}, t\right) - f_{CVCD}\left(p, v^{(2)}, t\right) \\ \leq\ & \underbrace{\tilde{E}\left[\beta_{t,\tau^{(1)}_{CV*}}\left(zV^{(1)}_{\tau^{(1)}_{CV*}} - P_{\tau^{(1)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(1)}_{CV*} \leq \tau^{(2)}_{CD*}\right\}} - \beta_{t,\tau^{(2)}_{CD*}}\left(P_{\tau^{(2)}_{CD*}} - \kappa\left(V^{(1)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(2)}_{CD*} < \tau^{(1)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{E}} \\ & - \underbrace{\tilde{E}\left[\beta_{t,\tau^{(1)}_{CV*}}\left(zV^{(2)}_{\tau^{(1)}_{CV*}} - P_{\tau^{(1)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(1)}_{CV*} \leq \tau^{(2)}_{CD*}\right\}} - \beta_{t,\tau^{(2)}_{CD*}}\left(P_{\tau^{(2)}_{CD*}} - \kappa\left(V^{(2)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(2)}_{CD*} < \tau^{(1)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{F}} \end{aligned} \tag{45}$$

$$\begin{aligned} =\ & \tilde{E}\left[\underbrace{\left(\beta_{t,\tau^{(1)}_{CV*}}\left(zV^{(1)}_{\tau^{(1)}_{CV*}} - P_{\tau^{(1)}_{CV*}}\right) - \beta_{t,\tau^{(1)}_{CV*}}\left(zV^{(2)}_{\tau^{(1)}_{CV*}} - P_{\tau^{(1)}_{CV*}}\right)\right)}_{\mathbf{G}}\mathbf{I}_{\left\{\tau^{(1)}_{CV*} \leq \tau^{(2)}_{CD*}\right\}}\middle|\mathcal{F}_t\right] \\ & + \tilde{E}\left[\underbrace{\left(\beta_{t,\tau^{(2)}_{CD*}}\left(P_{\tau^{(2)}_{CD*}} - \kappa\left(V^{(2)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)\right) - \beta_{t,\tau^{(2)}_{CD*}}\left(P_{\tau^{(2)}_{CD*}} - \kappa\left(V^{(1)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)\right)\right)}_{\mathbf{H}}\mathbf{I}_{\left\{\tau^{(2)}_{CD*} < \tau^{(1)}_{CV*}\right\}}\middle|\mathcal{F}_t\right] \end{aligned} \tag{46}$$

$$< \ 0.$$

Since the optimal bond holders' conversion strategy $\tau^{(1)}_{CV*}$ accompanied by the non-optimal issuer's call/default strategy $\tau^{(2)}_{CD*}$ would be beneficial for holders given the market state $\left(p, v^{(1)}, t\right)$, part E in **Inequality (45)** should be greater than $f_{CVCD}\left(p, v^{(1)}, t\right)$. Similarly, the optimal issuer strategy $\tau^{(2)}_{CD*}$ accompanied by the non-optimal holders' strategy would ruin holders' benefits given the market state $\left(p, v^{(2)}, t\right)$, part F should be less than $f_{CVCD}\left(p, v^{(2)}, t\right)$. Combining above two size relationships establish the **Inequality (45)**. Rearranging **Inequality (45)** yields **Equation (46)**. Under the premise $v^{(1)} < v^{(2)}$, **Corollary 8** entails that$\beta_{t,\tau^{(1)}_{CV*}} V^{(1)}_{\tau^{(1)}_{CV*}} - \beta_{t,\tau^{(1)}_{CV*}} V^{(2)}_{\tau^{(1)}_{CV*}} < 0$, which ensures part G is also negative. Similarly, **Corollary 9** entails that $\beta_{t,\tau^{(2)}_{CD*}} \kappa\left(V^{(2)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right) - \beta_{t,\tau^{(2)}_{CD*}} \kappa\left(V^{(1)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right) \leq 0$ and that part H is non-positive. Thus, we confirm $f_{CVCD}\left(p, v^{(1)}, t\right) - f_{CVCD}\left(p, v^{(2)}, t\right) < 0$. Note that the above proof can be easily extended to show f_{CVC} have the same inequality by replacing $\kappa\left(V^{(2)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)$ and $\kappa\left(V^{(1)}_{\tau^{(2)}_{CD*}}, \tau^{(2)}_{CD*}\right)$ with $V^{(2)}_{\tau^{(2)}_{CD*}}$ and $V^{(1)}_{\tau^{(2)}_{CD*}}$, respectively. Thus we skip the proof for simplicity.

3. The relation of CVD in part 1 of **Theorem 5** confirms the right hand side of the put delta inequality. Thus we proceed to prove the left hand side inequality by showing that $f_{CVCD}\left(p^{(1)}, v, t\right) - f_{CVCD}\left(p^{(2)}, v, t\right)$ (or $f_{CVD}\left(p^{(1)}, v, t\right) - f_{CVD}\left(p^{(2)}, v, t\right)$) is larger than $p^{(2)} - p^{(1)}$. We first focus on the former proof for CVCD, and it can then be slightly modified for CVD. Without loss of generality, we consider the case $p^{(1)} > p^{(2)}$ at time t, which implies $r^{(1)}_t < r^{(2)}_t$. Let the stopping time $\tau^{(1)}_{CD*}$ be the optimal call or default strategy for the CVCD issuer given the market state $\left(p^{(1)}, v, t\right)$. The stopping time $\tau^{(2)}_{CV*}$ be the optimal conversion strategy for a CVCD holder given the market state $\left(p^{(2)}, v, t\right)$. Thus, we have

$$
\begin{aligned}
& f_{CVCD}\left(p^{(1)}, v, t\right) - f_{CVCD}\left(p^{(2)}, v, t\right) \\
\geq\ & \underbrace{\tilde{E}\left[\beta^{(1)}_{t,\tau^{(2)}_{CV*}}\left(zV^{(1)}_{\tau^{(2)}_{CV*}} - P^{(1)}_{\tau^{(2)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(2)}_{CV*} \leq \tau^{(1)}_{CD*}\right\}} - \beta^{(1)}_{t,\tau^{(1)}_{CD*}}\left(P^{(1)}_{\tau^{(1)}_{CD*}} - \kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(1)}_{CD*} < \tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{J}} \\
& - \underbrace{\tilde{E}\left[\beta^{(2)}_{t,\tau^{(2)}_{CV*}}\left(zV^{(2)}_{\tau^{(2)}_{CV*}} - P^{(2)}_{\tau^{(2)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(2)}_{CV*} \leq \tau^{(1)}_{CD*}\right\}} - \beta^{(2)}_{t,\tau^{(1)}_{CD*}}\left(P^{(2)}_{\tau^{(1)}_{CD*}} - \kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(1)}_{CD*} < \tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{K}} \\
=\ & \tilde{E}\left[\underbrace{\left(\beta^{(1)}_{t,\tau^{(2)}_{CV*}}\left(zV^{(1)}_{\tau^{(2)}_{CV*}} - P^{(1)}_{\tau^{(2)}_{CV*}}\right) - \beta^{(2)}_{t,\tau^{(2)}_{CV*}}\left(zV^{(2)}_{\tau^{(2)}_{CV*}} - P^{(2)}_{\tau^{(2)}_{CV*}}\right)\right)}_{\mathbf{L}}\mathbf{I}_{\left\{\tau^{(2)}_{CV*} \leq \tau^{(1)}_{CD*}\right\}}\middle|\mathcal{F}_t\right] \\
& + \tilde{E}\left[\underbrace{\left(\beta^{(2)}_{t,\tau^{(1)}_{CD*}}\left(P^{(2)}_{\tau^{(1)}_{CD*}} - \kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right)\right) - \beta^{(1)}_{t,\tau^{(1)}_{CD*}}\left(P^{(1)}_{\tau^{(1)}_{CD*}} - \kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right)\right)\right)}_{\mathbf{M}}\mathbf{I}_{\left\{\tau^{(1)}_{CD*} < \tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right] \quad (47) \\
\geq\ & \tilde{E}\left[\underbrace{\left(\beta^{(2)}_{t,\tau^{(2)}_{CV*}} P^{(2)}_{\tau^{(2)}_{CV*}} - \beta^{(1)}_{t,\tau^{(2)}_{CV*}} P^{(1)}_{\tau^{(2)}_{CV*}}\right)}_{\mathbf{N}}\mathbf{I}_{\left\{\tau^{(2)}_{CV*} \leq \tau^{(1)}_{CD*}\right\}} + \underbrace{\left(\beta^{(2)}_{t,\tau^{(2)}_{CD*}} P^{(2)}_{\tau^{(2)}_{CD*}} - \beta^{(1)}_{t,\tau^{(2)}_{CD*}} P^{(1)}_{\tau^{(2)}_{CD*}}\right)}_{\mathbf{O}}\mathbf{I}_{\left\{\tau^{(1)}_{CD*} < \tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right] \quad (48) \\
=\ & \tilde{E}\left[\left(\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau}\right)\mathbf{I}_{\left\{\tau = \tau^{(2)}_{CV*} \wedge \tau^{(1)}_{CD*}\right\}}\middle|\mathcal{F}_t\right] \quad (49) \\
\geq\ & \tilde{E}\left[\beta^{(2)}_{t,\tau^{(2)}} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau}\middle|\mathcal{F}_t\right] \quad (50) \\
\geq\ & p^{(2)} - p^{(1)}. \quad (51)
\end{aligned}
$$

Since the optimal bond issuer's call/default strategy $\tau^{(1)}_{CD*}$ accompanied by the non-optimal holders' conversion strategy $\tau^{(2)}_{CV*}$ would ruin holders' benefits given the market state $\left(p^{(1)}, v, t\right)$, part J should be smaller than $f_{CVCD}\left(p^{(1)}, v, t\right)$. Similarly, the optimal holders' conversion strategy $\tau^{(2)}_{CV*}$ accompanied by the non-optimal issuer strategy would increase holders' benefits given the market state $\left(p^{(2)}, v, t\right)$, part K should be larger than $f_{CVCD}\left(p^{(2)}, v, t\right)$. Rearranging part J and K would yield **Equation (47)**. Part L can be simplified to Part N since $\beta^{(1)}_{t,\tau^{(2)}_{CV*}} V^{(1)}_{\tau^{(2)}_{CV*}} - \beta^{(2)}_{t,\tau^{(2)}_{CV*}} V^{(2)}_{\tau^{(2)}_{CV*}} = 0$ due to **Corollary 5**. Part O is less than Part M since $\beta^{(1)}_{t,\tau^{(1)}_{CD*}} \kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right) - \beta^{(2)}_{t,\tau^{(1)}_{CD*}} \kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}}, \tau^{(1)}_{CD*}\right) \geq 0$ due to **Corollary 6**. The above two relations establish **Inequality (48)**. Combining the indicator functions in **Inequality (48)** yields **Equation (49)**. **Inequality (50)** is established because $\beta^{(2)}_{t,\tau} P^{(2)}_{\tau} - \beta^{(1)}_{t,\tau} P^{(1)}_{\tau} \leq 0$ due to **Corollary 3**.

Inequality (51) is established due to **Lemma 1**.

The relation $f_{CVD}\left(p^{(1)},v,t\right)-f_{CVD}\left(p^{(2)},v,t\right)\geq p^{(2)}-p^{(1)}$ can be similarly derived except that $\kappa\left(V^{(1)}_{\check{\tau}^{(1)}_{CD*}},\check{\tau}^{(1)}_{CD*}\right)$ and $\kappa\left(V^{(2)}_{\check{\tau}^{(1)}_{CD*}},\check{\tau}^{(1)}_{CD}\right)$ are replaced by $V^{(1)}_{\check{\tau}^{(1)}_{CD*}}$ and $V^{(2)}_{\check{\tau}^{(1)}_{CD*}}$ in Part M, respectively. Since part M now equals to Part O due to **Corollary 5**, the size relationship in above derivations remains unchanged.

4. Note that part 2 of **Theorem 5** already confirms the left hand side of the call delta inequalities for both `CVCD` and `CVD`. Thus we proceed to prove the right hand side inequality for `CVCD` by showing that $f_{CVCD}\left(p,v^{(1)},t\right)-f_{CVCD}\left(p,v^{(2)},t\right)>v^{(1)}-v^{(2)}$. Then we modify this proof to show that `CVC` has a tighter upper bound. Without loss of generality, we consider the case $v^{(1)}<v^{(2)}$ at time t; thus **Equation (2)** suggests $V^{(1)}_u<V^{(2)}_u$, $\forall u\in[t,T]$. Let the stopping time $\tau^{(1)}_{CD*}$ be the optimal call and default strategy for the issuer given the prevailing market state $\left(p,v^{(1)},t\right)$, The stopping time $\tau^{(2)}_{CV*}$ be the optimal conversion strategy for bond holders given the market state $\left(p,v^{(2)},t\right)$. Thus we have

$$
\begin{aligned}
& f_{CVCD}\left(p,v^{(1)},t\right)-f_{CVCD}\left(p,v^{(2)},t\right)\\
\geq\;& \underbrace{\tilde{E}\left[\beta_{t,\tau^{(2)}_{CV*}}\left(zV^{(1)}_{\tau^{(2)}_{CV*}}-P_{\tau^{(2)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(2)}_{CV*}\leq\tau^{(1)}_{CD*}\right\}}-\beta_{t,\tau^{(1)}_{CD*}}\left(P_{\tau^{(1)}_{CD*}}-\kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(1)}_{CD*}<\tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{P}}\\
& -\underbrace{\tilde{E}\left[\beta_{t,\tau^{(2)}_{CV*}}\left(zV^{(2)}_{\tau^{(2)}_{CV*}}-P_{\tau^{(2)}_{CV*}}\right)\mathbf{I}_{\left\{\tau^{(2)}_{CV*}\leq\tau^{(1)}_{CD*}\right\}}-\beta_{t,\tau^{(1)}_{CD*}}\left(P_{\tau^{(1)}_{CD*}}-\kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)\right)\mathbf{I}_{\left\{\tau^{(1)}_{CD*}<\tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]}_{\mathbf{Q}}\\
=\;& \tilde{E}\left[\left(\beta_{t,\tau^{(2)}_{CV*}}\left(zV^{(1)}_{\tau^{(2)}_{CV*}}-P_{\tau^{(2)}_{CV*}}\right)-\beta_{t,\tau^{(2)}_{CV*}}\left(zV^{(2)}_{\tau^{(2)}_{CV*}}-P_{\tau^{(2)}_{CV*}}\right)\right)\mathbf{I}_{\left\{\tau^{(2)}_{CV*}\leq\tau^{(1)}_{CD*}\right\}}\middle|\mathcal{F}_t\right]\\
& +\tilde{E}\left[\left(\beta_{t,\tau^{(1)}_{CD*}}\left(P_{\tau^{(1)}_{CD*}}-\kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)\right)-\beta_{t,\tau^{(1)}_{CD*}}\left(P_{\tau^{(1)}_{CD*}}-\kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)\right)\right)\mathbf{I}_{\left\{\tau^{(1)}_{CD*}<\tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]\\
=\;& \tilde{E}\left[\underbrace{\left(z\beta_{t,\tau^{(2)}_{CV*}}V^{(1)}_{\tau^{(2)}_{CV*}}-z\beta_{t,\tau^{(2)}_{CV*}}V^{(2)}_{\tau^{(2)}_{CV*}}\right)}_{\mathbf{R}}\mathbf{I}_{\left\{\tau^{(2)}_{CV*}\leq\tau^{(1)}_{CD*}\right\}}\middle|\mathcal{F}_t\right]\\
& +\tilde{E}\left[\underbrace{\left(\beta_{t,\tau^{(1)}_{CD*}}\kappa\left(V^{(1)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)-\beta_{t,\tau^{(1)}_{CD*}}\kappa\left(V^{(2)}_{\tau^{(1)}_{CD*}},\tau^{(1)}_{CD*}\right)\right)}_{\mathbf{S}}\mathbf{I}_{\left\{\tau^{(1)}_{CD*}<\tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right] \qquad (52)\\
>\;& \tilde{E}\left[\underbrace{\left(\beta_{t,\tau^{(2)}_{CV*}}V^{(1)}_{\tau^{(2)}_{CV*}}-\beta_{t,\tau^{(2)}_{CV*}}V^{(2)}_{\tau^{(2)}_{CV*}}\right)}_{\mathbf{T}}\mathbf{I}_{\left\{\tau^{(2)}_{CV*}\leq\tau^{(1)}_{CD*}\right\}}+\underbrace{\left(\beta_{t,\tau^{(1)}_{CD*}}V^{(1)}_{\tau^{(1)}_{CD*}}-\beta_{t,\tau^{(1)}_{CD*}}V^{(2)}_{\tau^{(1)}_{CD*}}\right)}_{\mathbf{U}}\mathbf{I}_{\left\{\tau^{(1)}_{CD*}<\tau^{(2)}_{CV*}\right\}}\middle|\mathcal{F}_t\right]\\
& \qquad (53)\\
=\;& \tilde{E}\left[\left(\beta_{t,\tau}V^{(1)}_{\tau}-\beta_{t,\tau}V^{(2)}_{\tau}\right)\mathbf{I}_{\left\{\tau=\tau^{(2)}_{CV*}\wedge\tau^{(1)}_{CD*}\right\}}\middle|\mathcal{F}_t\right] \qquad (54)\\
\geq\;& \tilde{E}\left[\beta_{t,\tau}V^{(1)}_{\tau}-\beta_{t,\tau}V^{(2)}_{\tau}\middle|\mathcal{F}_t\right] \qquad (55)\\
=\;& e^{-\int_t^\tau\gamma_s ds}\left(v^{(1)}-v^{(2)}\right) \qquad (56)\\
\geq\;& v^{(1)}-v^{(2)}. \qquad (57)
\end{aligned}
$$

Since the optimal bond issuer's call/default strategy $\tau^{(1)}_{CD*}$ accompanied by the non-optimal holders' conversion strategy $\tau^{(2)}_{CV*}$ would ruin holders' benefits given the market state $\left(p,v^{(1)},t\right)$, part P should be smaller than $f_{CVCD}\left(p,v^{(1)},t\right)$. Similarly, given another market state $\left(p,v^{(2)},t\right)$, the optimal holders' conversion strategy $\tau^{(2)}_{CV*}$ accompanied by the non-optimal issuer strategy $\tau^{(1)}_{CD*}$ would increase holders' benefits; therefore, part Q should be larger than $f_{CVCD}\left(p,v^{(2)},t\right)$. Rearranging part P and Q would yield the next equation and can be further simplified as **Equation (52)**. Part R is larger than Part T in **Inequality (53)** because $0<z<1$ and $\beta_{t,\tau^{(2)}_{CV*}}V^{(1)}_{\tau^{(2)}_{CV*}}-\beta_{t,\tau^{(2)}_{CV*}}V^{(2)}_{\tau^{(2)}_{CV*}}<0$ due to **Corollary 8**.

Part S is greater than or equal to part U due to **Corollary 10**. Thus **Inequality (53)** is yielded by combining the above two size relationships. Combining the indicator functions in parts T and U yields **Equation (54)**. **Inequality (55)** is established because $\beta_{t,\tau}V_{\tau}^{(1)} - \beta_{t,\tau}V_{\tau}^{(2)} \leq 0$ due to **Corollary 8**. **Equation (56)** follows from **Corollary 7**. The relationships $0 < e^{-\int_t^{\tau}\gamma_s ds} < 1$, and $v^{(1)} - v^{(2)} < 0$ implies **Inequality (57)**. The proof of the right hand side call delta inequality of CVC can be straightforward derived by replacing $\kappa\left(V_{\tau_{CD^*}^{(1)}}^{(1)}, \tau_{CD^*}^{(1)}\right)$ and $\kappa\left(V_{\tau_{CD^*}^{(1)}}^{(2)}, \tau_{CD^*}^{(1)}\right)$ with $k_{\tau_{CD^*}^{(1)}}$ in above derivations. So we skip the proof for simplicity.

■

In the following proof of **Theorem 6**, we note that the continuation region at time t for the game option is the open set

$$U^* \equiv \left\{(p, v, t) \in R^+ \times R^+ \times [0, T] : zv - p < f_{CVCD}\,(p, v, t) < \kappa\,(v, t) - p\right\}$$

for $zv < \kappa\,(v, t)$.

Proof of Theorem 6.

1. Let $p^{(1)}$ and $p^{(2)}$ be two possible host bond prices at time t and we consider the case $p^{(1)} > p^{(2)}$ with $f_{CVCD}\left(p^{(1)}, v, t\right) < \kappa\,(v, t) - p^{(1)}$ and $f_{CVCD}\left(p^{(2)}, v, t\right) < \kappa\,(v, t) - p^{(2)}$ (i.e., the case the bond issuer does not exercise the call or default option). Suppose it is optimal to continue at $p^{(2)}$ given that the firm's asset value at time t is $0 < v < v_{CV^*}$, we show that it is then optimal to continue at $p^{(1)}$. According to part 3 of **Theorem 5**, we have

$$\begin{aligned} &\frac{f_{CVCD}\left(p^{(2)}, v, t\right) - f_{CVCD}\left(p^{(1)}, v, t\right)}{p^{(2)} - p^{(1)}} \geq -1 \\ \Rightarrow\quad & f_{CVCD}\left(p^{(1)}, v, t\right) \geq f_{CVCD}\left(p^{(2)}, v, t\right) + p^{(2)} - p^{(1)}. \end{aligned}$$

Because it is optimal to continue at $p^{(2)}$, we further have

$$\begin{aligned} f_{CVCD}\left(p^{(1)}, v, t\right) &\geq f_{CVCD}\left(p^{(2)}, v, t\right) + p^{(2)} - p^{(1)} \\ &> \left(zv - p^{(2)}\right) + p^{(2)} - p^{(1)} \\ &= zv - p^{(1)}. \end{aligned}$$

This confirms $f_{CVCD}\left(p^{(1)}, v, t\right) > zv - p^{(1)}$ and ensures that it is also optimal to continue at $p^{(1)}$. Let $b_{CV^*}(v, t)$ be the infimum of the host bond price p such that $(p, v, t) \in U^*$. The point $(b_{CV^*}(v, t), v, t)$ is not in continuation region U^* because the region is open. Thus, $f_{CVCD}\,(b_{CV^*}(v, t), v, t) = zv - b_{CV^*}(v, t)$.

On the other hand, we consider the case $p^{(1)} > p^{(2)}$ with $f_{CDCV}\left(p^{(1)}, v, t\right) < p^{(1)} - zv$ and $f_{CDCV}\left(p^{(2)}, v, t\right) < p^{(2)} - zv$ (i.e., the case the bond holder does not exercise the conversion option). Suppose it is optimal to continue at $p^{(1)}$ given that the firm's asset value at time t is $0 < v < v_{CV^*}$, we show that it is then optimal to continue at $p^{(2)}$. According to **Equation (11)** and part 3 of **Theorem 5**, we have

$$\begin{aligned} &\frac{f_{CDCV}\left(p^{(2)}, v, t\right) - f_{CDCV}\left(p^{(1)}, v, t\right)}{p^{(2)} - p^{(1)}} \leq 1 \\ \Rightarrow\quad & f_{CDCV}\left(p^{(2)}, v, t\right) \geq f_{CDCV}\left(p^{(1)}, v, t\right) + p^{(2)} - p^{(1)}. \end{aligned}$$

Because it is optimal to continue at $p^{(1)}$, we further have

$$\begin{array}{rcl} f_{CDCV}\left(p^{(2)},v,t\right) & \geq & f_{CDCV}\left(p^{(1)},v,t\right)+p^{(2)}-p^{(1)} \\ & > & \left(p^{(1)}-\kappa\left(v,t\right)\right)+p^{(2)}-p^{(1)} \\ & = & p^{(2)}-\kappa\left(v,t\right). \end{array}$$

This confirms $f_{CDCV}\left(p^{(2)},v,t\right) > p^{(2)}-\kappa\left(v,t\right)$ and ensures that it is also optimal to continue at $p^{(2)}$. Let $b_{CD^*}(v,t)$ be the supremum of the host bond price p such that $(p,v,t)\in U^*$. The point $(b_{CD^*}(v,t),v,t)$ is not in continuation region U^* because the region is open. Thus, $f_{CDCV}\left(b_{CD^*}(v,t),v,t\right) = b_{CD^*}(v,t)-\kappa\left(v,t\right)$.

2. Let $v^{(1)}$ and $v^{(2)}$ be two possible firm's asset values at time t and we consider the case $k_t \leq v^{(1)} < v^{(2)} < v_{CV^*}$ with $f_{CDCV}\left(p,v^{(1)},t\right) < p-zv^{(1)}$ and $f_{CDCV}\left(p,v^{(2)},t\right) < p-zv^{(2)}$ (i.e., the case the bond holder does not exercise the conversion option). Suppose it is optimal to exercise the call option at $v^{(1)}$, we show that it is then optimal to exercise the call option at $v^{(2)}$. According to **Equation (11)** and part 2 of **Theorem 5**, we have

$$f_{CDCV}\left(p,v^{(2)},t\right) \leq f_{CDCV}\left(p,v^{(1)},t\right) = p-\kappa\left(v^{(1)},t\right) = p-k_t.$$

Besides, $f_{CDCV}\left(p,v^{(2)},t\right) \geq p-\kappa\left(v^{(2)},t\right) = p-k_t$. Then, we obtain $f_{CDCV}\left(p,v^{(2)},t\right) = p-k_t$ and ensures that it is also optimal to exercise the call option at $v^{(2)}$. Let $\bar{v}_{CD^*}(p,t)$ be the minimum of the firm's asset value at time t, $k_t \leq \bar{v}_{CD^*}(p,t) < v_{CV^*}$, such that it is optimal to call at $(p.v,t)$.

On the other hand, we consider the case $0 \leq v^{(1)} < v^{(2)} \leq k_t$ with $f_{CDCV}\left(p,v^{(1)},t\right) < p-zv^{(1)}$ and $f_{CDCV}\left(p,v^{(2)},t\right) < p-zv^{(2)}$. Suppose it is optimal to continue at $v^{(1)}$, we show that it is then optimal to continue at $v^{(2)}$. According to **Equation (11)** and part 4 of **Theorem 5**, we have

$$\begin{array}{rl} & \dfrac{f_{CDCV}\left(p,v^{(1)},t\right)-f_{CDCV}\left(p,v^{(2)},t\right)}{v^{(1)}-v^{(2)}} > -1 \\ \Rightarrow & f_{CDCV}\left(p,v^{(2)},t\right) > f_{CDCV}\left(p,v^{(1)},t\right)+v^{(1)}-v^{(2)}. \end{array}$$

Because it is optimal to continue at $v^{(1)}$, we then have

$$\begin{array}{rcl} f_{CDCV}\left(p,v^{(2)},t\right) & \geq & f_{CDCV}\left(p,v^{(1)},t\right)+v^{(1)}-v^{(2)} \\ & > & \left(p-v^{(1)}\right)+v^{(1)}-v^{(2)} \\ & = & p-v^{(2)}. \end{array}$$

This confirms $f_{CDCV}\left(p,v^{(2)},t\right) > p-v^{(2)}$ and ensures that it is also optimal to continue at $v^{(2)}$. Let $v_{CD^*}(p,t)$ be the infimum of the firm's asset value v such that $(p,v,t)\in U^*$. The point $(p,v_{CD^*}(p,t),t)$ is not in continuation region U^* because U^* is open. Thus, $f_{CDCV}\left(p,v_{CD^*}(p,t),t\right) = p-v_{CD^*}(p,t)$ and $0 \leq v_{CD^*}(p,t) \leq k_t$. ∎

Proof of Theorem 7.

1. Consider the host bond price at time t is p and the case $v^{(1)} < v^{(2)} < v_{CV^*}$ with $f_{CDCV}\left(p,v^{(1)},t\right) < p-zv^{(1)}$ and $f_{CDCV}\left(p,v^{(2)},t\right) < p-zv^{(2)}$. For the scenario $v^{(1)} < v^{(2)} \leq k_t$, we want to confirm that if $0 < p < b_{CD^*}\left(v^{(1)},t\right)$ (i.e., the firm does not exercise its default option), then $p < b_{CD^*}\left(v^{(2)},t\right)$ as well. According to **Equation (11)** and part 4 of **Theorem 5** given $0 < p < b_{CD^*}\left(v^{(1)}, t\right)$, we have

$$\begin{aligned} f_{CDCV}\left(p, v^{(2)}, t\right) &\geq f_{CDCV}\left(p, v^{(1)}, t\right) + v^{(1)} - v^{(2)} \\ &> \left(p - v^{(1)}\right) + v^{(1)} - v^{(2)} \\ &= p - v^{(2)}. \end{aligned}$$

Thus, $f_{CDCV}\left(p, v^{(2)}, t\right) > p - v^{(2)}$, ensuring $p < b_{CD^*}\left(v^{(2)}, t\right)$.

2. Consider the case $v^{(1)} < v^{(2)} < v_{CV^*}$ with $f_{CDCV}\left(p, v^{(1)}, t\right) < p - zv^{(1)}$ and $f_{CDCV}\left(p, v^{(2)}, t\right) < p - zv^{(2)}$. We want to confirm that if $0 < p < b_{CD^*}\left(v^{(2)}, t\right)$ (i.e., the firm does not exercise its call option), then $p < b_{CD^*}\left(v^{(1)}, t\right)$ as well. According to **Equation (11)** and part 2 of **Theorem 5** given $0 < p < b_{CD^*}\left(v^{(2)}, t\right)$, we have

$$f_{CDCV}\left(p, v^{(1)}, t\right) \geq f_{CDCV}\left(p, v^{(2)}, t\right) > p - k_t.$$

Thus, $f_{CDCV}\left(p, v^{(1)}, t\right) > p - k_t$, ensuring $p < b_{CD^*}\left(v^{(1)}, t\right)$.

3. Consider the case $f_{CVCD}\left(p, v, t\right) < \kappa\left(v, t\right) - p$. We want to confirm that if $p > b_{CV^*}\left(v, t\right)$ (i.e., the callable convertible bond holder does not exercise the conversion option), then $p > b_{CV}\left(v, t\right)$ as well. According to **Inequality (12)** given $p > b_{CV^*}\left(v, t\right)$, we have

$$f_{CV}\left(p, v, t\right) \geq f_{CVCD}\left(p, v, t\right) > zv - p.$$

Thus, $f_{CV}\left(p, v, t\right) > zv - p$, ensuring $p > b_{CV}\left(v, t\right)$.

4. Consider the case $f_{CDCV}\left(p, v, t\right) < p - zv$. We want to confirm that if $p < b_{CD^*}\left(v, t\right)$ (i.e., the firm issuing callable convertible bond does not exercise its option), then $p < b_{CD}\left(v, t\right)$ as well. According to **Inequality (13)** given $p < b_{CD^*}\left(v, t\right)$, we have

$$f_{CD}\left(p, v, t\right) \geq f_{CDCV}\left(p, v, t\right) > p - \kappa\left(v, t\right).$$

Thus, $f_{CD}\left(p, v, t\right) > p - \kappa\left(v, t\right)$, ensuring $p < b_{CD}\left(v, t\right)$. ■