Fair pensions*

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Abstract

This paper examines the allocation of market risk in a general class of collective pension arrangements: Collective Defined Contribution (CDC) schemes. In a CDC scheme participants collectively share funding risk through benefit level adjustments. There is a concern that, if not well designed, CDC schemes are unfair and will lead to an unintended redistribution of wealth between participants and, in particular, between generations. We define a pension scheme as fair if all participants receive an arbitrage-free return on the market risk they bear. The fact that the participants' claim on the CDC schemes' collective assets is expressed in terms of a stochastic future benefit, makes the arbitrage-free allocation of market risk non-trivial. It depends crucially on the specification of the discount rate process in combination with the benefit adjustment process. We show that fair CDC schemes may use a default-free market interest rate in combination with a specific horizon-dependent benefit adjustment process. Alternative discount rates are also permissible, but require additional correction terms in the benefit adjustment process.

Key words: Pension; Retirement; Asset Pricing; Fair value; Intergenerational risk-sharing; Funded pension systems.

JEL Codes: H55, G13, G23, J26, J32

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1 Introduction

Defined Benefit (DB) pension schemes are under pressure. DB schemes guarantee pension benefits, regardless of the pension fund's actual investment returns, Bodie (1990). An external guarantor must ensure that the pension scheme can fulfill the benefits it has promised the scheme participants. In recent decades, falling interest rates and rising life expectancy have made the provision of such long-term guarantees more expensive. As a result, employers, labor unions, and policy makers seek alternatives to traditional DB schemes.

One alternative is to switch to an individual Defined Contribution scheme (DC). In a DC scheme, individuals accumulate retirement funds on an individual account basis. The level of future retirement income depends on the actual investment return and is no longer guaranteed, as is the case in a DB scheme.

Some employers, labor unions, and policy makers propose a different alternative: a Collective Defined Contribution scheme (CDC).[1] CDC schemes combine features of DB and DC schemes. The approach is to stick to the defined benefit administration framework, while reducing the risk burden on the schemes' sponsor by 'softening' the guarantee.[2] Participants still accrue a retirement benefit denoted in terms of a future income level, but this benefit level fluctuates as a function of the funding status of the pension scheme. The 'DC' in CDC points to the fact that, as in conventional individual DC, there is no guaranteed benefit level, while addition of the letter 'C' for 'Collective' points to the fact that, unlike the conventional individual DC scheme, there is no individual capital accumulation.

CDC schemes are a hybrid form of DB and DC schemes. Over time, hybrid pension schemes have become more popular, see Figure 1. This figure shows the growth of hybrid pension schemes in an unbalanced sample of OECD countries. The share of each pension scheme type is based on weighting countries equally. In 2005, 7 percent of the 19 countries in the OECD pension database had some form of hybrid pension plan. In 2017, this had grown to 16 percent across 18 countries.[3]

[1]Existing defined benefit schemes in both Canada and the Netherlands evolve into CDC pension schemes. At the time of writing, the UK government is consulting on the potential introduction of CDC pension schemes (Department for Work and Pensions, 2018). Novy-Marx and Rauh (2014) study the effect of linking benefits to investment performance in American state pension plans.

[2]Note that under rational behavior, employees will negotiate higher wages to compensate for a lower value of their pension guarantee, see, e.g., Sharpe (1976) and Treynor (1977). A move away from DB should therefore be seen primarily as a shift of risk from the sponsor to the individual pension scheme participant.

[3]Hybrid is defined as the sum of mixed DB and protected DC.

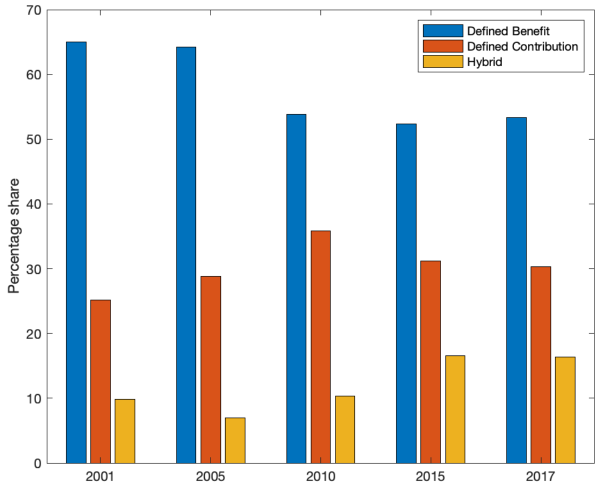


Figure 1. Growth of hybrid pension plans in selected OECD countries, source: https://stats.oecd.org/

As a CDC scheme pools assets of all participants, the scheme can potentially redistribute wealth. Especially because a CDC scheme allocates wealth over time, and hence across different generations, a complex economic problem emerges, namely, whether all participants are treated fairly. We define a pension scheme to be fair if all participants, at any point in time, make an arbitrage-free return on the market risk they bear. This definition implies that if a scheme is not fair, some participants could get a better risk-return trade-off outside the scheme. We argue that this definition of fairness is an important sustainability criterion. If violated, participants or groups of participants can benefit by leaving the scheme. This would make the pension scheme unstable. Pension scheme stability is, e.g., analysed in Chen, Beetsma, Broeders, and Pelsser (2017). Our definition of a fair pension scheme is in line with the concept of fair contracts, as referred to in insurance literature, see, e.g., Grosen and Jørgensen (2002), Chen and Suchanecki (2007), Døskeland and Nordahl (2008), and Orozco-Garcia and Schmeiser (2017).

CDC schemes distributes risk across participants by adjusting benefit levels. The benefit level adjustment is a function of the scheme's funding ratio, which is the ratio of the scheme's assets to the present value of benefits. The present value of benefits is obtained by discounting the outstanding benefits. Fairness will depend crucially on the combined specification of the benefit adjustment process and the discount rate process that are used. In this paper we focus on market risk only.

We show that the arbitrage-free allocation of market risk is non-trivial. Starting, for example, from a defined benefit context, introducing benefit adjustments that are equal for all participants (in percentage terms) is not arbitrage-free. This may be a surprising result because adjusting all benefit levels equally intuitively feels like a 'fair' arrangement. This is, however, not the case because of interaction between the benefit adjustment process and the discount rate process. In a world with stochastic interest rates, the covariance between discount rates and benefit level adjustments implies an arbitrage opportunity. Unless the pension scheme fully hedges its discount rate risk, homogeneous benefit adjustment implies a continuous transfer of wealth from young to old participants. We show how to make this contract arbitrage-free through the introduction of a horizon-dependent benefit adjustment rule.

An additional advantage of using a horizon-dependent benefit adjustment rule is that it allows for an arbitrage-free *smoothing* policy. In practice, policy makers prefer the idea that current retirement income is not too volatile on a year-to-year basis. We show how a specific horizon-dependent benefit adjustment process can reduce the volatility of current retirement income while preserving the arbitrage-free allocation of market risks.

We also address the widely debated topic of liability discounting. In a defined benefit setting the discount rate follows trivially from the nature of the liabilities. If benefits are default-free, the market consistent value of the pension liabilities is found by discounting benefits using a default-free market interest rate. In a CDC scheme, future benefit levels are stochastic. Therefore, it is less obvious how to obtain the present value of benefits. There is actually a circularity. Benefit levels are adjusted based on the funding ratio, which is itself a function of the present value of the benefits. We show that, in order to preserve a fair allocation of market risks, CDC benefits still have to be discounted using a default-free market rate.

Finally, an important disclaimer is in place. In this paper we only address the fair allocation of market risk in CDC. Fairness is important to ensure stability of the pension scheme as it prevents unintended wealth redistributions between participants. Fairness, however, should not be confused with optimality. Fair CDC pension schemes may be suboptimal in terms of the risk exposure of individual participants. The matter of optimality will need to be explored in more detail in future studies. However, the analysis in this paper does provide the building blocks for such an analysis by deriving the implied exposure of individual participants in the scheme to risk.

The remainder of the paper is structured as follows. In Section 2 we define the concepts used (fairness, CDC, etc.) and in Section 3 we discuss some of their implications. In Section 4 we introduce some technical assumptions and notation. In Section 5 we derive the return of an individual participant in a CDC scheme and show how this return can be market inconsistent. Section 6 explains how to achieve fairness through a horizon-dependent benefit adjustment and how this also allows for smoothing of benefit adjustments. In Section 7 we focus on the role of the discount factor and to what extent one could use alternative discount factors. Finally, we evaluate CDC in Section 10 and Section 11 draws a conclusion.

2 Definitions and technical assumptions

2.1 Market risk and fair pension schemes

Collective pension schemes pool and allocate risks of individual participants. We can subdivide these risks into two classes: traded risks and non-traded risks. Idiosyncratic longevity risk could, for example, be considered a non-traded risk, while equity risk and interest rate risk are traded risks. As equity risk and interest rate risk are priced in the market, we also refer to these risk factors as market risks. The fact that risk is traded on a financial market has an important consequence. If the pension scheme, internally, allocates traded risks on terms that are inconsistent with the market, some participants will have an incentive to leave the scheme and trade at the more favorable market terms outside the scheme. Put more formally, the return these participants make inside the scheme is not arbitrage-free. In line with the concept of 'fair value' we will define a pension scheme to be fair if all participants make an arbitrage-free return.

Definition 1 *A pension scheme is fair if all participants, at each point in time, make an arbitrage-free return.*

Put more technically, we define the pension scheme to be fair if the discounted market value of a participant's claim on the fund is a martingale under the risk-neutral measure.

If a pension scheme is fair, all participants will receive a market consistent return on any traded risks they bear. If not, some participants are forced to bear traded risks inside the CDC scheme at terms less favorable than those that prevail in the market. From the perspective of these individuals, the CDC scheme levies an implicit tax. They could have made a higher return in the market with certainty. Since the fund as a whole makes a market-consistent return, the implicit tax on one participant is by definition an implicit subsidy for the other.

In our opinion, a sound objective for a CDC scheme is to rule out such redistribution. Not because redistribution is generally unwanted, but there seem to be other entities and policy tools more suited for the purpose of redistribution of wealth between individuals and generations (i.e. the government and tax policy). Market inconsistencies in a CDC scheme introduce an unnecessary political dimension that may at some point threaten the sustainability of the scheme as some groups will have an incentive to leave the scheme.

If policy makers, by contrast, prefer to allow the CDC scheme to redistribute risk in a market inconsistent manner, our fairness criterion will still be useful as a benchmark. Our analysis will make the redistributional flows within the scheme transparent. This can help policy makers to decide whether or not the redistribution implied by a specific CDC scheme design is in line with their policy objective.

2.2 Defining Collective Defined Contribution Schemes

Next, we define the CDC pension scheme. CDC is a general type of pension scheme that is similar to traditional Defined Benefit (DB) schemes in that the scheme promises its participants a stream of cash flows from retirement date until death. CDC pension schemes, however, differ from traditional DB schemes as funding risk is born by the participants through benefit level adjustments. We work in continuous time and formally define the CDC pension scheme as follows:

Definition 2 *A CDC pension scheme is an institutional arrangement through which individuals ('participants') collectively save and invest to finance a retirement income. The CDC scheme has the following features:*

- *There is a single collective pool of wealth ('Assets'), with a market value at time t denoted $A(t)$.*
- *The pension schemes' administration contains, for each participant indexed i, a 'benefit level' denoted $b_i(t)$.*
- *Once a participant reaches the retirement age, he/she receives a continuous 'benefit payment' equal to $b_i(t)dt$. Payments stop once the participant dies.*
- *At each point in time, the pension fund determines the 'regulatory present value', denoted $L(t)$ (for 'Liabilities'), of all benefits currently present in its administration.*
- *The regulatory present value of benefits is determined by discounting all future payments assuming that benefit levels remain constant in the future and using a potentially horizon-dependent 'regulatory discount rate'.*
- *There is an 'adjustment rule' that describes how benefit levels change in response to changes in the value of the pension funds' assets and the regulatory value of benefits.*

In a DB setting, the benefit level would also be the eventual level of the benefit cash flow the participant receives during retirement. In a CDC setting, the interpretation is somewhat less clear. In a fair CDC scheme, which uses a term-structure of default-free interest rates to discount the benefits, the benefit level can be interpreted as the expected future benefit cash flow under the τ-forward measure.[4]

Note that the aggregate exposure to market risk in the CDC scheme is determined by the asset portfolio. So, aggregate exposure is selected at the collective level through a single asset allocation choice. The way the scheme subsequently allocates aggregate risk to participants depends on two choices that still need to be made: the choice of the discount factor process and the choice of the benefit adjustment process. Whether the pension scheme is fair depends on the combined choice for these two processes.

[4] Here, τ refers to the maturity date of the pension benefit. See i.e. Shreve (2004) Chapter 9.4 for a discussion on forward measures.

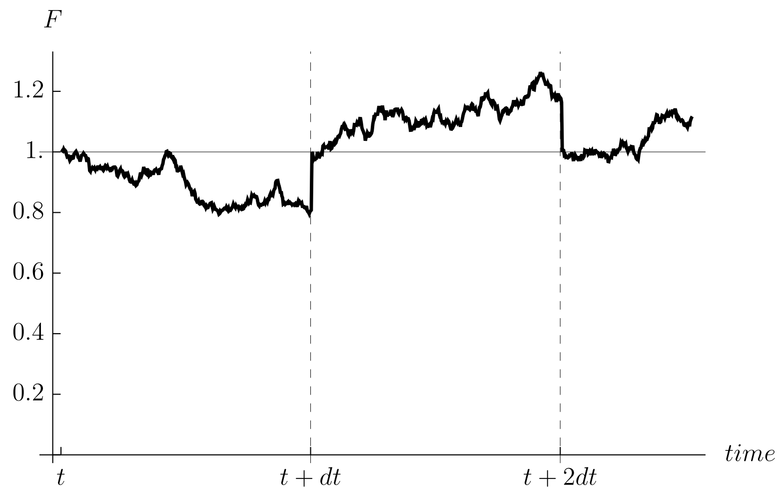


Figure 2. Periodic resetting of the regulatory funding ratio $F = \frac{A}{L}$ in a CDC scheme

2.3 Completeness

Before we are able to evaluate a CDC scheme's fairness, we have to impose one technical restriction: completeness.

Definition 3 *A CDC scheme is complete if the choice for the discount rate process and the benefit adjustment process implies that $A(t) = L(t)$ for all t.*

This definition imposes that all value in the CDC scheme is always explicitly allocated to the participants under the regulatory discounting regime. Completeness simplifies our evaluation of pension schemes' fairness. Clearly, if some risk were not explicitly allocated, we cannot determine if all participants receive appropriate compensation for the risk they bear. It makes sure that the funding status of the scheme will not be a state variable to our analysis.

Figure 2 shows a discrete time illustration of the benefit adjustment mechanism in a complete CDC scheme. The scheme adjusts benefit levels at each time step such that the regulatory funding ratio is reset to 1. Between the time steps the benefit levels $b_i(t)$ are constant and the regulatory funding ratio freely drifts as the market value of the assets, the discount rate, and the regulatory present value of benefits, change.

In practice, proposals for CDC schemes typically allow the scheme to have temporary imbalances between assets and the regulatory value of liabilities. The idea is that, by allowing temporary imbalances or buffers, the impact of shocks can be smoothed. Current retirement income does not fluctuate one-to-one with funding risk. We will allow for smoothing in Section 6.1. To ensure that the scheme is still complete, we need not keep track though of participants that eventually pick up surpluses and deficits. To do this we will 'expand the state variables'. Instead of administering a single benefit level for each participant, we will allow the scheme to administer a benefit level for each participant that is horizon-dependent. This allows us to explicitly allocate any surpluses or deficits to participants as future benefit increases or cuts. By doing so, we maintain completeness ($A(t) = L(t)$), while allowing the impact of funding shocks to be allocated smoothly over future horizons.

3 Implications of fairness and completeness

Now that we have defined fairness and completeness, a few propositions immediately follow.

Proposition 1 *In a fair CDC scheme, contributions paid must equal the regulatory value of newly accrued benefits.*

Proposition 2 *In a fair CDC scheme, a participant leaving the scheme receives a cash transfer value equal to the regulatory value of the surrendered benefits.*

The rationale behind these two propositions is straightforward. If the contributions paid and the transfer value were not equal to the regulatory value of the benefits, the assets and liabilities would no longer be in balance. This would imply a benefit level adjustment for all participants in the scheme. Such an adjustment would clearly imply an arbitrary transfer of value between participants and hence would imply a breach of fairness.

Proposition 3 *In a fair CDC scheme, the regulatory value of benefits equals the market consistent value of benefits.*

So far, we have not imposed that the regulatory value of a CDC benefit is equal to the market consistent value of this claim on the scheme. However, it follows directly from Proposition 1 that in a fair CDC scheme this is the case. In a fair CDC scheme, individuals pay contribution equal to the regulatory value of the accrued benefit. If this value were to be different from the actual market consistent value of this benefit, the individual would immediately incur a gain or loss in market value terms. This gain or loss implies a market value transfer to or from the other CDC scheme participants. With these definitions and propositions in place, we can now turn to technical specification of the financial market and CDC schemes.

4 Assumptions and relevant notation

4.1 Technical assumptions

We will keep our specification of the financial market general. By doing so, our evaluation of the fairness of the pension scheme does not rely on subjective assumptions about unobservable or potentially unobservable variables, such as risk premiums or asset price volatilities. We assume that an arbitrage-free financial market exists. For mathematical convenience, we follow Merton (1973) by assuming that the economy has a continuous time Markov structure and all variables follow Ito-processes.[5] So, all uncertainty is captured by $\boldsymbol{dz}$, which is an $n \times 1$ vector of independent Wiener processes. Investment opportunities at each point in time depending on m state variables that we denote by vector $\boldsymbol{X}(t)$. Vector $\boldsymbol{X}(t)$ could, for example, contain variables such as the current yield curve, and the dividend price ratio (see i.e. Campbell and Viceira (2005)). The instantaneous return on asset j can then be written as:

$$\frac{dS_j}{S_j} = \mu_j(\boldsymbol{X}(t), t)dt + \boldsymbol{\sigma_j}(\boldsymbol{X}(t), t)\boldsymbol{dz} \tag{4.1}$$

We assume that asset prices are observable, while state vector $\boldsymbol{X}(t)$, expected return vector $\mu_j(\boldsymbol{X}(t), t)$, and the variance-covariance matrix $\sigma_j(\boldsymbol{X}(t), t)$ may be unobserved. We will assume that default-free bonds of all maturities are traded or can otherwise be replicated. We will denote the price at time t of a zero-coupon bond that pays one unit of currency at time τ by $P(\boldsymbol{X}(t), t, \tau)$ and we will write the process of this price as:

$$\frac{dP(\boldsymbol{X}(t), t, \tau)}{P(\boldsymbol{X}(t), t, \tau)} = \mu_P(\boldsymbol{X}(t), t, \tau) + \boldsymbol{\sigma_P}(\boldsymbol{X}(t), t)\boldsymbol{dz} \tag{4.2}$$

We will denote the instantaneously risk-less rate of interest by $r(\boldsymbol{X}(t), t)$. Furthermore, we define the instantaneous discount process by $D_0 = e^{-\int_0^t r(\boldsymbol{X}(t),t)dt}$, such that:

$$\frac{dD_0(t)}{D_0(t)} = -r(\boldsymbol{X}(t), t)dt \tag{4.3}$$

As we will focus on the fair allocation of market risk, we will ignore longevity risk throughout

[5]This setup allows for a rich stochastic environment, although it does rule out jumps in the time-series variables. We rule out jumps for mathematical convenience. In appendix 11 we discuss fairness in a discrete time setting.

the paper. Each individual participant i retires at a known predetermined date R_i and passes away at known date T_i. One interpretation of this is that we assume all idiosyncratic mortality risk to be fully diversified through a large enough pool of homogeneous participants and that we assume systemic longevity risk to be uncorrelated with (other) traded risk factors.

4.2 Relevant notation

The individual participants' return and exposure to market risk in the CDC scheme depends on four processes: the asset process, the discount rate process, the benefit adjustment process, and the liability process. We introduce notation for these processes here.

4.2.1 The asset process

For the asset process we write:

$$dA(t) = A(t)\left[\mu_A(\boldsymbol{X}(t), t)dt + \boldsymbol{\sigma_A}(\boldsymbol{X}(t), t)\boldsymbol{dz}\right] + (Y(t) - b(t,t))dt \tag{4.4}$$

The values of μ_A and $\boldsymbol{\sigma_A}$ depend on 'the state of nature' (captured by $\boldsymbol{X}(t)$) and the asset allocation chosen by the pension scheme trustees.

4.2.2 The discount rate process

Although we speak of a discount rate, it will typically be easier to work with the discount factor. If we denote the discount rate at time t applied to benefits due at time τ by $d(t,\tau)$, the corresponding discount factor is $D(t,\tau) = e^{-\int_t^\tau d(t,s)ds}$. We write the stochastic process for the discount factor as follows:

$$\frac{dD(t,\tau)}{D(t,\tau)} = \mu_D(\boldsymbol{X}(t), t, \tau)dt + \boldsymbol{\sigma_D}(\boldsymbol{X}(t), t, \tau)'\boldsymbol{dz}, \tag{4.5}$$

with boundary condition $D(t,t) = 1$[6].

[6] We further discuss this boundary condition in 7

4.2.3 The benefit adjustment process

Comparable to the discount rate process, we write the benefit adjustment process as:

$$\frac{db_i(t)}{b_i(t)} = \mu_{b_i}(\boldsymbol{X}(t), t)dt + \boldsymbol{\sigma}_{\boldsymbol{b_i}}(\boldsymbol{X}(t), t)'\boldsymbol{dz} \tag{4.6}$$

Note that we allow the benefit adjustment rule to be participant-specific for now. We will restrict this later when we consider specific implementations of the CDC scheme.

4.2.4 The liability process

We will denote the regulatory present value of all CDC benefits by $L(t)$ and typically refer to $L(t)$ as the schemes' liabilities, a term inherited from the DB setting. By completeness of the CDC scheme, the process for $L(t)$ by construct mirrors $A(t)$. A more meaningful role in the analysis is played by the change in the regulatory present value of benefits *caused by discount rate changes only* (i.e. excluding benefit adjustments). Let $\tilde{L}(t)$ denote the index that tracks the change in liabilities caused by discount rate changes only, then we write:

$$\frac{d\tilde{L}(t)}{\tilde{L}(t)} = \mu_L(\boldsymbol{X}(t), t)dt + \boldsymbol{\sigma}_{\boldsymbol{L}}(\boldsymbol{X}(t), t)'\boldsymbol{dz}, \tag{4.7}$$

An exact specification of μ_L and $\boldsymbol{\sigma}_{\boldsymbol{L}}$ can be found in the appendix. The difference between $\frac{dA}{A}$ and $\frac{d\tilde{L}}{\tilde{L}}$ is the gap that needs to be closed through benefit level adjustments.

All drift and volatility terms may depend on the state of the economy ($\boldsymbol{X}(t)$) and the calendar date (t). In what follows we drop the dependence on $\boldsymbol{X}(t)$ and t in our notation for convenience. With this setting we now can turn to the return on pension benefits.

5 Return on CDC benefits

The fairness criterion applies to the return that an individual participant in the pension scheme makes. The CDC scheme may feature participants of all age groups and hence of all investment horizons. Therefore, in order for the CDC scheme to be fair, the return must be fair on benefits of all horizons. Since it is easier to analyze the return of a benefit cash flow with a specific horizon, we focus on the value of a future payment to individual i at time τ. We denote the value of this benefit payment by $V_i(t,\tau)$. Since the regulatory value is defined to be the discounted value of b_i under the assumption that b_i does not change in the future, we have:

$$V_i(\tau) = b_i(t)D(\tau) \tag{5.1}$$

By Proposition (3), if the CDC scheme is fair, $V_i(\tau)$ is also the market consistent value of the benefit cash flow. Using Ito's lemma we find that the return on this τ-horizon benefit cash flow to individual i is:

$$\frac{dV_i(\tau)}{V_i(\tau)} = \frac{db_i}{b_i} + \frac{dD(\tau)}{D(\tau)} + \frac{db_i dD(\tau)}{b_i D(\tau)} \tag{5.2}$$

The return has three distinct elements: the benefit adjustment effect, the change in discount rate effect, and the covariance between these two processes. In order for this return to be market consistent, we need the mean return to be a market consistent compensation for the given level of market risk. In technical terms, we need $D_0 V_i(\tau)$, the discounted value process, to be a martingale under the risk-neutral measure. So, fairness requires:

$$\mathbb{E}_t^Q\left[\frac{dD_0}{D_0}\frac{dV_i(\tau)}{V_i(\tau)}\right] = 0 \qquad \forall \tau \tag{5.3}$$

Because the return $\frac{dV_i(\tau)}{V_i(\tau)}$ depends on the combination of the discount rate and benefit process, we cannot say that a discount rate or benefit adjustment process is fair in isolation. A discount rate process can be fair in combination with a particular benefit adjustment process and unfair in combination with another. Therefore we always have to evaluate the two processes jointly.

5.1 Traditional DB as a special case of CDC

A traditional defined benefit scheme can be seen as a CDC scheme where benefit adjustment is always zero. In this case, the return on the claim to the time τ benefit payment is reduced to the discount rate effect only:

$$\frac{dV_i(\tau)}{V_i(\tau)} = \frac{dD(\tau)}{D(\tau)} = \mu_D(\tau)dt + \boldsymbol{\sigma_D}(\tau)'\boldsymbol{dz} \tag{5.4}$$

Since a default-free zero-coupon bond that matures at time τ is traded or can be replicated, no-arbitrage implies that the discount rate change must be equal to the return on this bond: $\frac{dP(\tau)}{P(\tau)}$. Hence, a fair DB pension scheme uses a default-free market interest rate as its discount rate and all participants earn a risk-free return on their participation in the DB scheme. Balance between $A(t)$ and $L(t)$ is maintained in this case by the fact that $A(t)$ includes the value of the claim on the external guarantor (e.g. the shareholders of the corporation or the government).

5.2 Example of an unfair CDC scheme

Fairness of CDC schemes is less obvious. To highlight that CDC schemes can easily be unfair, consider the following 'naive' introduction of CDC. Coming from a traditional DB setup, the benefit guarantee is abolished and the schemes' asset-liability mismatch risk is no longer absorbed by an outside sponsor. Instead, it is decided that mismatch risk is absorbed by adjusting benefit levels of all participants by the same percentage. The discount factor will still be the price of a default-free zero-coupon bond that is traded in the market. Since the benefit adjustments are the same for all participants, it is tempting to think this pension scheme will be fair. It turns out, though, that this is not the case. The return on the CDC benefit due at time τ in this case is (see appendix B for the derivation):

$$\begin{aligned}\frac{dV(\tau)}{V(\tau)} = &\underbrace{\mu_D(\tau)dt + \boldsymbol{\sigma_D}(\tau)'\boldsymbol{dz}}_{\text{Return on zero-coupon bond}} \quad + \quad \underbrace{\mu_A dt + \boldsymbol{\sigma}'_{\boldsymbol{A}}\boldsymbol{dz}}_{\text{Return on assets}} \quad - \quad \underbrace{\left(\mu_L dt + \boldsymbol{\sigma}'_{\boldsymbol{L}}\boldsymbol{dz}\right)}_{\text{Liability matching portfolio return}} \\ &+ \underbrace{\left(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\right)'\left(\boldsymbol{\sigma_D}(\tau) - \boldsymbol{\sigma_L}\right)dt}_{\text{Market inconsistent return component}}\end{aligned} \tag{5.5}$$

In the defined benefit setting, the participant only receives the return on the zero-coupon bond. In the general CDC setting there are three additional terms. The participant now also shares in the return on assets, the liability matching portfolio return, and finally there is a covariance term. Note that the first three elements in (5.5) together form a market consistent return. These three terms together are equale to the arbitrage-free market return on a portfolio which is long the default-free zero-coupon bond with maturity date τ, long the CDC schemes asset portfolio and short the portfolio of default-free bonds that replicates the discount factor return on the aggregate CDC benefits[7]. The fourth term, on the second row, is a non-stochastic term. So, the overall return is a market consistent return plus a deterministic term. Since the final term is deterministic, it cannot be a market consistent compensation for risk. Put more formally, the expected change in the discounted value process is given by:

$$\mathbb{E}_t^Q\left[\frac{dD_0}{D_0}\frac{dV_i(\tau)}{V_i(\tau)}\right] = \big(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\big)'\big(\boldsymbol{\sigma_D}(\tau) - \boldsymbol{\sigma_L}\big)dt \qquad \forall\tau \tag{5.6}$$

This result follows trivially, since the discounted value process for all traded asset are martingales under the risk-neutral measure. Now, for the CDC scheme to be fair, the right-hand side has to be zero conform (5.3).

There are two special cases in which (5.3) indeed holds. The right-hand side value in (5.6) is a covariance between two variables. This covariance is the inner product of two volatility vectors. The first vector, $\big(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\big)$, is the schemes' asset-liability mismatch. It is the volatility of the return difference between the assets and the liability matching portfolio. The second variable, $\big(\boldsymbol{\sigma_D}(\tau) - \boldsymbol{\sigma_L}\big)$, is what we will call the horizon mismatch. It is the difference between the discount factor volatility for the specific horizon τ and the discount factor volatility of the CDC scheme liabilities on aggregate. Fairness requires that these two vectors are orthogonal. Firstly, this is the case if the CDC scheme as a whole hedges all discount factor risk. So, the CDC scheme would have to choose an asset mix that fully matches the discount factor risk of all CDC benefits. This implies that the pension fund invests in the replicating portfolio of bonds and effectively becomes a DB scheme as discussed in Section 5.1. Secondly, this is the case if the exposure of the individual participant matches the exposure of the aggregate benefits in the CDC scheme. This condition is only met in the hypothetical case that the CDC scheme has

[7] For brevity we will refer to this as the 'liability matching portfolio'.

participants with identical retirement dates and hence discount rate exposures. This implies that the pension fund effectively becomes a DC scheme.[8] To illustrate the potential size of the unfair returns, we display a simple numerical example in the next section. We also further discuss the economic intuition.

5.2.1 Numerical example: Vasicek interest rate model

We consider a numerical example of the CDC scheme that applies a homogeneous benefit adjustment mechanism and uses a term structure of default-free interest rates to discount the regulatory benefits. We assume that the instantaneous interest rate follows:

$$dr(t) = -\kappa_r(r(t) - \bar{r})dt + \sigma_r dz_r \tag{5.7}$$

where κ_r is the speed of mean reversion and $\bar{r}$ the long-term mean interest rate and dz_r is a standard Brownian motion. Assume that bond prices are consistent with the following pricing kernel, $M(t)$:

$$\frac{dM(t)}{M(t)} = -r(t)dt + \phi_r dz_r \tag{5.8}$$

where ϕ_r denotes the (constant) price of interest rate risk. Since the CDC scheme uses the risk-free interest rate to discount the regulatory benefits, the discount rate is equal to the price of a zero-coupon bond, whose price can be derived from the pricing kernel (see, i.e., Brennan and Xia (2002)).

We calibrate the CDC scheme such that the retirement benefits have an aggregate duration of approximately 20 years, while the assets have a duration of 10 years.[9] In addition, we use the parameter values for the financial market model as given in Table 1.

Figure 3 shows the market inconsistent excess return on the benefits as a function of the duration of the benefits. If the CDC scheme were to be fair, the market inconsistent excess return would be nil and the line would lie flat at zero. Instead, we see that the holders of

[8]To be precise, the horizon-specific return will still be market inconsistent in this case ($\boldsymbol{\sigma_D}(\tau) \neq \boldsymbol{\sigma_L}$). However, if we consider the total return on all benefit payments over all horizons jointly, the market inconsistent components will cancel out. Intuitively, all individuals have exactly overlapping investment horizons and hence have the same exposure and make the same return, which is simply the return on the funds' assets.

[9]These figures roughly match an average Dutch pension fund, both in terms of demographics and asset allocation.

Parameter	Value
$\bar{r} = r_0$	0.02
κ_r	0.0347
σ_r	0.01

Table 1. Financial market calibration. These are the parameter values used in our numerical illustration of CDC unfairness. Our choice of κ_r implies that the half-time of interest rate shocks is 20 years. Note that the bond risk-premium (ϕ_r) has no impact on the market inconsistent excess return.

benefits with a short duration (the elderly) receive above-market returns of up to 120 basis points, while the holders of benefits with a long duration (the young) receive returns that are well below market consistent levels. The CDC scheme with a homogeneous benefit adjustment mechanism that uses a traded discount rate redistributes wealth from young to old participants.

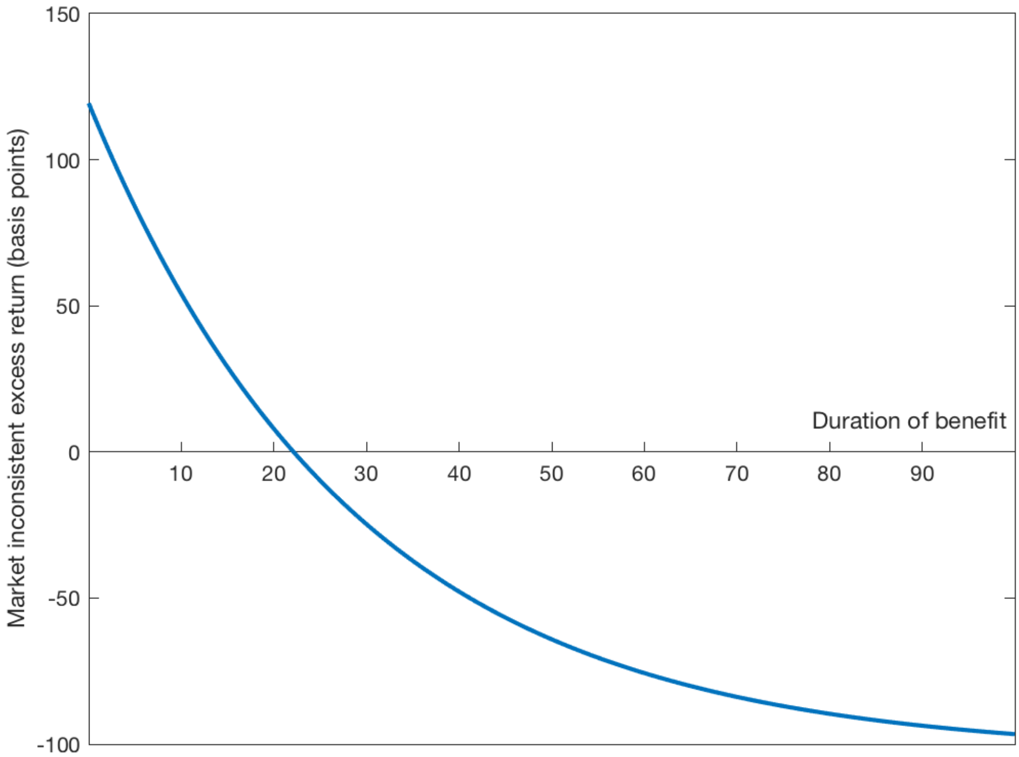


Figure 3. Illustration of market inconsistent return in CDC scheme This graph illustrates the difference between the actual expected return and the market consistent expected return benefits purchased at their regulatory value. The parameter values used to calculate these figures are given in Table 1.

The economic intuition behind this result is as follows. The CDC scheme chooses an interest rate sensitivity for its assets that is well below the interest rate sensitivity of the present value of the retirement benefits. This implies that in scenarios with declining interest rates, benefit levels will be reduced. When interest rates rise benefit levels will be increased. The change in interest rates will affect the regulatory value of participants differently, based on their investment horizon. The relative claim of young and old participants on the CDC scheme changes. When interest rates fall, the benefits of young participants become more valuable compared to the value of old participants' benefits. Vice-versa, when interest rates rise, the benefits of young participants become relatively less valuable. A consequence of this is that in bad times, when benefits are reduced due to interest rate shocks, young participants lose relatively valuable benefits. Conversely, in good times, when participants receive extra benefits, the young gain benefits that are relatively cheap. So, in good times a relatively small share of the surplus goes to the young, while in bad times the young absorb a relatively big share of the deficit. This asymmetry generates the unfair returns we see in Figure 3.

6 Fair CDC with horizon-dependent benefit adjustment

In the previous section we saw that fairness is not trivial. We saw that a traditional DB scheme can be viewed as a special case of the more general CDC concept. The DB scheme was fair as long as we discount benefits using default-free market interest rates. Once we introduce risk sharing with participants through homogeneous benefit adjustments, we saw that the CDC scheme turned unfair. Due to correlation between the discount factor and the benefit adjustments, the CDC scheme redistributes wealth from young to old participants. In this section we show how this problem can be corrected by introducing a horizon-dependent benefit adjustment process. Horizon-dependent adjustments do require the CDC scheme to keep a more general administration. Instead of administering a single benefit level $b_i(t)$ for each participant, we will now assume that the administration holds, for each individual, a potentially different benefit level for each horizon: $b_i(t,\tau)$. This allows us to eliminate the unfairness previously found. Additionally, it allows us to introduce a feature that policy makers often propose: smoothing. By smoothing, we mean that current shocks to the funding ratio of the scheme do not translate one-to-one into benefit level variation, but only affect benefit levels gradually over time. The concept of smoothing is decribed by Gollier (2008)

6.1 Fair horizon-dependent benefit adjustment

In Section 5.2 we saw that it is the covariance between the discount factor and benefit adjustment process that causes the return on CDC benefits with different horizons to be unfair. This can be fixed by adding an appropriate correction term to the benefit adjustment process. Since the covariance is horizon-dependent, this implies that the benefit adjustment process will also have to be horizon-dependent. For the moment, let us stick to the idea that we want all participants to share equally in the mismatch risk of the scheme. The fair adjustment process with fairness correction will then be:

$$\frac{db(\tau)}{b(\tau)} = \underbrace{\mu_A dt + \boldsymbol{\sigma}_{\boldsymbol{A}}' \boldsymbol{dz}}_{\text{Return on assets}} - \underbrace{(\mu_L dt + \boldsymbol{\sigma}_{\boldsymbol{L}}' \boldsymbol{dz})}_{\text{Liability matching portfolio return}} - \underbrace{(\boldsymbol{\sigma}_{\boldsymbol{A}} - \boldsymbol{\sigma}_{\boldsymbol{L}})' \boldsymbol{\sigma}_{\boldsymbol{D}}(\boldsymbol{\tau}) dt}_{\text{Horizon specific fairness correction}} \tag{6.1}$$

This benefit adjustment process is both complete and fair.[10] The return on the value of a CDC benefit with maturity date τ becomes:

$$\frac{dV(\tau)}{V(\tau)} = \underbrace{\mu_D(\tau)dt + \boldsymbol{\sigma_D}(\tau)'\boldsymbol{dz}}_{\text{Return on zero-coupon bond}} + \underbrace{\mu_A dt + \boldsymbol{\sigma'_A dz}}_{\text{Return on assets}} - \underbrace{\left(\mu_L dt + \boldsymbol{\sigma'_L dz}\right)}_{\text{Liability matching portfolio return}} \tag{6.2}$$

and hence,

$$\mathbb{E}_t^Q\left[\frac{dD_0}{D_0}\frac{dV_i(\tau)}{V_i(\tau)}\right] = 0$$

For the individual participant the exposure to market risk is now equivalent to holding a default-free zero-coupon bond with maturity date τ plus a long position in the schemes' assets and a short position in the schemes' liability matching portfolio.

6.2 Fair smoothing

Policy makers seem to prefer the idea that pension income is not too volatile on a year-to-year basis. It is therefore common that policy makers propose some kind of smoothing procedure. In practice, there are many types of smoothing procedures. Smoothing may be applied to the valuation of assets and liabilities themselves or the adjustment process may be smooth. Whatever tools are used in practice, the goal is to make sure that shocks to the pension schemes financial position only gradually affect benefit levels. Our generalized setup, where benefit levels are horizon-dependent, allows for easy implementation of such a policy. We can weigh the exposure of the benefit level to mismatch risk with a horizon-dependent smoothing parameter. The benefit adjustment process then becomes:

$$\frac{db(\tau)}{b(\tau)} = \underbrace{\alpha(\tau)}_{\text{Smoothing parameter}} \left[\mu_A dt + \boldsymbol{\sigma'_A dz} - \left(\mu_L dt + \boldsymbol{\sigma'_L dz}\right) - \left(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\right)'\boldsymbol{\sigma_D(\tau)}dt\right] \tag{6.3}$$

where $\alpha(\tau)$ is the smoothing parameter. This parameter controls to what extent benefits with a certain maturity date share in the CDC schemes' mismatch.

[10]One may wonder if implementation of the horizon-specific correction is feasible in a real world setting where adjustments must occur at discrete points in time. In appendix C we show that this is indeed feasible and the adjustment process takes a relatively simple form: $b_{\tau,t+1} = b_{\tau,t} \times \left[1 + \left(\frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t}\right)\frac{P_{\tau,t}}{P_{\tau,t+1}}\right]$.

The return on the value of a CDC benefit with maturity date τ becomes:

$$\frac{dV(\tau)}{V(\tau)} = \underbrace{\mu_D(\tau)dt + \boldsymbol{\sigma_D}(\tau)'\boldsymbol{dz}}_{\text{Return on zero-coupon bond}} + \underbrace{\alpha(\tau)}_{\text{Smoothing}} \left\{ \underbrace{\mu_A dt + \boldsymbol{\sigma_A'}\boldsymbol{dz}}_{\text{Asset return}} - \underbrace{\left(\mu_L dt + \boldsymbol{\sigma_L'}\boldsymbol{dz}\right)}_{\text{Matching portfolio return}} \right\} \tag{6.4}$$

The only difference with the return we saw in Section 6.1, is that the exposure to mismatch risk is now horizon-dependent and can be explicitly controlled by $\alpha(\tau)$. Policy makers could pick a low level for $\alpha(\tau)$ for short horizons and a higher level for long horizons. Completeness does imply the following restriction on the choice of $\alpha(\tau)$:

$$\int_t^\infty \alpha(t,\tau)L(t,\tau)d\tau = L(t) \tag{6.5}$$

An example of a ten-year linear smoothing policy that is complete, could be, for example[11]:

$$\alpha(t,\tau) = \begin{cases} \frac{\tau-t}{10} & \text{if } (\tau-t) < 10 \\ \frac{L(t)-\int_t^{t+10}\frac{\tau-t}{10}L(t,\tau)d\tau}{\int_{t+10}^{\infty}L(t,\tau)d\tau} & \text{if } (\tau-t) \geq 10 \end{cases}$$

We indexed $\alpha(t,\tau)$ explicitly with t to highlight not only that the smoothing policy may be changed over time, but also that it must be changed over time to maintain completeness. It is also important to note that $\alpha(t,\tau)$ is fixed before the shock occurs to which $\alpha(t,\tau)$ applies. $\alpha(t,\tau)$ may depend on past shocks though. So, we could update future levels of $\alpha(t,\tau)$ as shocks occur. By doing so, we could potentially create option-like exposures to mismatch risk that are both fair and complete.

[11] In case of discrete time benefit adjustments, the fair horizon-dependent adjustment rule becomes: $b_{\tau,t+1} = b_{\tau,t} \times \left[1 + \alpha(\tau)\left(\frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t}\right)\frac{P_{\tau,t}}{P_{\tau,t+1}}\right]$

7 CDC with alternative discount rates

In the previous section we saw that we can achieve fairness in the CDC scheme by making benefit adjustments horizon-dependent. We did so while sticking to the assumption that the CDC scheme discounts its benefits using a default-free market interest rate. This approach implies a specific interpretation of the benefit level. $b(t, \tau)$ is the expected future benefit cash flow under the τ-*forward measure*. Loosely speaking, it is the expected benefit level excluding any risk premia that will be earned in expectation. In practice, policy makers may like the benefit level to coincide with some the expected benefit payment or some kind of 'target benefit level'.[12] The idea is that if $b(\tau)$ is interpreted as a target level or the expected benefit payment, we should be applying a discount rate that does not ignore risk premia and that hence lies above the default-free yield curve.

In this section we discuss whether the CDC scheme could be fair if it uses a discount rate that deviates from the default-free market interest rate. Our point of departure will be the CDC scheme with smoothing, introduced in the previous section. So the benefit adjustment process will be as given in (6.3).

7.1 Non-default-free market-based discount rates

First, we will discuss why a fair CDC scheme cannot use a discount rate that trades in the market but is *not* default-free, i.e., the yield on a corporate bond.[13] In order to see that a default-free interest rate is the only potentially fair discount rate, note that, the regulatory value of a benefit at time of payment is equal to $b_i(\tau)D(\tau, \tau)dt$. The scheme, however, is designed such that $b_i(\tau)dt$ is paid out. This implies that we need $D(\tau, \tau) = 1$ with certainty, to make sure that there is no redistribution of value upon distribution of the benefit. If the scheme were to use a non-default-free discount factor, $D(t, \tau)$ may be lower than one at time τ. Consequently, whenever $D(\tau, \tau) < 1$, the retired individuals receive more than the actual value of their benefits. If a fair CDC scheme does want to work with a discount factor derived from assets subject to default, the CDC scheme should be generalized by setting the payment to retirees

[12] The Department for Work and Pensions for example speaks of such a 'target benefit level' in its consultation document (Department for Work and Pensions, 2018)

[13] Of course, a genuinely default-free asset only exists in theory. So, consistent with the argument in this paragraph, one could argue that the CDC scheme is never truly arbitrage-free, unless the payment to current retirees is set equal to $b_i(t)D(t, t)dt$ instead of $b_i(t)dt$.

equal to $b_i(t)D(t,t)dt$. So, participants would not only be subject to benefit adjustments, but also to discount factor default risk shocks. The consequence would be that generations would now also implicitly trade credit default derivatives through the CDC scheme. While technically possible, it is not clear why this would be a desirable feature of a CDC scheme.

7.2 Default-free discount rate with deterministic spread

In the previous section we argued that fair CDC schemes cannot use discount rates that are derived from assets that feature default risk. Such discount rates would violate the restriction that $D(\tau,\tau)$ has to be equal to 1 with certainty. This does not rule out that a deterministic spread may be added on top of the default-free market interest rate.[14] If we allow the spread to be both time and horizon dependent, the discount factor can be written as:

$$D(t,\tau) = P(t,\tau) \times e^{-\int_t^\tau \lambda(s,\tau)ds} \tag{7.1}$$

where $P(t,\tau)$ is the market price at time t of a default-free zero-coupon bond that pays one at time τ and $\lambda(s,\tau)$ is the spread on the default-free interest rate. The discount factor process becomes:

$$\frac{dD(t,\tau)}{D(t,\tau)} = \frac{dP(t,\tau)}{P(t,\tau)} + \lambda(t,\tau)dt \tag{7.2}$$

and the change in the present value of the liabilities due to discount factor change is

$$\mu_L dt + \boldsymbol{\sigma}_{\boldsymbol{L}}' \boldsymbol{dz} = \mu_{P_L} dt + \boldsymbol{\sigma}_{\boldsymbol{P_L}}' \boldsymbol{dz} + \int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau dt \tag{7.3}$$

where $L(t,\tau)$ is the present value of all benefits maturing at date τ and $\mu_{P_L} dt + \boldsymbol{\sigma}_{\boldsymbol{P_L}}' \boldsymbol{dz}$ is the return on the bond portfolio that replicates the discount rate risk of the liabilities, so:

$$\mu_{P_L} = \int_0^\infty \mu_P(t,\tau) \frac{L(t,\tau)}{L(t)} d\tau$$

$$\boldsymbol{\sigma}_{P_L} = \int_0^\infty \boldsymbol{\sigma}_P(t,\tau) \frac{L(t,\tau)}{L(t)} d\tau$$

[14] This idea was for example part of a policy proposal by the Dutch government.

The return on the CDC benefit maturing at date τ becomes:

$$\begin{aligned}\frac{dV(\tau)}{V(\tau)} =& \underbrace{\mu_P(\tau)dt + \boldsymbol{\sigma_P}(\tau)'\boldsymbol{dz}}_{\text{Return on ZCB}} \; + \; \lambda(t,\tau)dt \\ &+ \alpha(t,\tau)\left\{ \underbrace{\mu_A dt + \boldsymbol{\sigma}'_{\boldsymbol{A}}\boldsymbol{dz}}_{\text{Asset return}} - \underbrace{\left(\mu_{P_L} dt + \boldsymbol{\sigma}'_{\boldsymbol{P_L}}\boldsymbol{dz}\right)}_{\text{Matching portfolio return}} - \int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau dt \right\}\end{aligned} \tag{7.4}$$

and the expectation under the risk-neutral measure of the change in the discounted value process is:

$$\mathbb{E}_t^Q \left[\frac{dD_0}{D_0} \frac{dV_i(\tau)}{V_i(\tau)} \right] = \lambda(t,\tau) - \alpha(t,\tau) \int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau \tag{7.5}$$

So, fairness requires that:

$$\lambda(t,\tau) - \alpha(t,\tau) \int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau = 0$$

This condition implies that $\lambda(t,\tau)$ will, generally speaking, depend on $\alpha(t,\tau)$. For example: $\lambda(t,\tau) = \bar{\lambda}\alpha(t,\tau)$ would satisfy (7.5), with $\bar{\lambda}$ being any arbitrary number. There is a problem, however. The *current* discount factor $D(t,\tau)$ depends on $\int_t^\tau \lambda(s,\tau)ds$, which, through $\lambda(s,\tau)$, depends on all future values of $\alpha(t,\tau)$. As pointed out earlier, future values of $\alpha(t,\tau)$, depend on the future distribution of benefits, which is currently unknown. Consequently, fair CDC schemes with smoothing cannot apply a deterministic spread on top of the default-free term structure.

If the CDC scheme does not apply a smoothing policy, it could apply a deterministic spread without violating fairness. In this special case, it is still necessary that $\lambda(t,\tau)$ does not depend on τ. Otherwise (7.5) still implies that the current discount factor depends on the future maturity distribution. So, the fund could apply a spread $\lambda(t,\tau) = \bar{\lambda}(t)$.

Note that the path of $\bar{\lambda}(t)$ over time must be pre-determined and cannot freely be changed. Changing $\bar{\lambda}(t)$ would shift the value of $D(t,\tau)$, which would imply an arbitrary jump in the value of benefits.

7.2.1 Restoring fairness through a benefit adjustment process correction

In the analysis above, we considered a CDC scheme that uses the benefit adjustment process with smoothing, as given by (6.3). As pointed out earlier, fairness always depends on the combination of the benefit adjustment process and discount rate process. So, technically, if the CDC scheme wants to use a discount rate with deterministic spread, it could maintain fairness by adding a correction term to the benefit adjustment process. Instead of using the adjustment process given by (6.3) the CDC scheme could apply the following adjustment rule with correction:

$$\begin{aligned}\frac{db(\tau)}{b(\tau)} = \alpha(\tau)\Big[\mu_A dt + \boldsymbol{\sigma_A'}\boldsymbol{dz} - (\mu_L dt + \boldsymbol{\sigma_L'}\boldsymbol{dz}) \quad - (\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L})'\boldsymbol{\sigma_D}(\boldsymbol{\tau})dt\Big] \\ - \underbrace{\left[\lambda(t,\tau) - \alpha(t,\tau)\int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)}d\tau\right]dt}_{\text{discount rate spread correction term}}\end{aligned} \tag{7.6}$$

which simplifies to

$$\frac{db(\tau)}{b(\tau)} = \alpha(\tau)\Big[\mu_A dt + \boldsymbol{\sigma_A'}\boldsymbol{dz} - (\mu_{P_L} dt + \boldsymbol{\sigma_{P_L}'}\boldsymbol{dz}) \quad - (\boldsymbol{\sigma_A} - \boldsymbol{\sigma_{P_L}})'\boldsymbol{\sigma_P}(\boldsymbol{\tau})dt\Big] - \lambda(t,\tau) \tag{7.7}$$

If the scheme chooses this approach, the effective return an individual participant makes, only is given by

$$\frac{dV(\tau)}{V(\tau)} = \underbrace{\mu_P(\tau)dt + \boldsymbol{\sigma_P}(\tau)'\boldsymbol{dz}}_{\text{Return on zero-coupon bond}} + \underbrace{\alpha(\tau)}_{\text{Smoothing}}\left\{\underbrace{\mu_A dt + \boldsymbol{\sigma_A'}\boldsymbol{dz}}_{\text{Asset return}} - \underbrace{(\mu_{P_L} dt + \boldsymbol{\sigma_{P_L}'}\boldsymbol{dz})}_{\text{Matching portfolio return}}\right\} \tag{7.8}$$

Note that this return is almost identical to the case without deterministic spread that we saw in (6.4). The only difference is that the matching portfolio now uses a different weighting of different maturities. The yield spread changes the level of $L(t,\tau)$ which enters as a weight in the definition of μ_{P_L} and $\boldsymbol{\sigma_{P_L}}$. A positive spread implies that the duration of the matching portfolio is lowered, while a negative spread implies that the duration of the matching portfolio is increased. Since the specification of the matching portfolio does not affect the aggregate exposure to market risk, this change implies that discount rate risk is allocated differently across the plans participants.

Summarizing, if the scheme applies benefit smoothing, adding a spread on the default-free market interest rate would require an additional correction term in the benefit adjustment process. The main implication of adding a spread will be that it changes the allocation of discount rate risk. Additionally, the interpretation of $b(\tau)$ is no longer the expectation under the τ-forward measure and it adds complexity to the scheme for no obvious reason.

7.3 Non-stochastic discount rates

In the previous section we looked at the possibility of adding a deterministic spread on top of the market-based default-free discount rate. Here, we consider abolishing a market-based discount factor altogether and analyse the case in which the scheme uses a non-market-based discount rate.[15] Again, we will start by assuming that the CDC scheme applies the benefit adjustment rule with smoothing, given by (6.3). Consider the non-stochastic discount factor given by:

$$D(t,\tau) = e^{-\int_t^\tau \lambda(s,\tau)ds} \tag{7.9}$$

The return on the CDC benefit with horizon τ in this case will be:

$$\frac{dV(\tau)}{V(\tau)} = \lambda(t,\tau)dt + \alpha(t,\tau)\left\{ \underbrace{\mu_A dt + \boldsymbol{\sigma}_A' \boldsymbol{dz}}_{\text{Asset return}} - \int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau dt \right\} \tag{7.10}$$

The expectation under the risk-neutral measure of the change in the discounted value process is:

$$\mathbb{E}_t^Q\left[\frac{dD_0}{D_0}\frac{dV_i(\tau)}{V_i(\tau)}\right] = \left[\lambda(t,\tau) - \alpha(t,\tau)\int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)}d\tau - (1-\alpha(t,\tau))r(t)\right]dt \tag{7.11}$$

Fairness therefore requires that:

$$\lambda(t,\tau) - \alpha(t,\tau)\int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)}d\tau - (1-\alpha(t,\tau))r(t) = 0 \tag{7.12}$$

[15] This is, for example, the approach taken in Canada and the approach by the CDC pioneer in the UK, Royal Mail. Royal Mail writes ‘Pension liabilities would be valued by the Plan Actuary using central (or best) estimate assumptions, i.e. with no intended bias for prudence or optimism.’ So, the scheme uses a discount rate which is presumably non-stochastic, at least for a certain period of time.

This condition cannot hold generally since $r(t)$ is time-variant and cannot be known in advance. The condition only holds if the scheme does not apply smoothing ($\alpha(t,\tau) = 1 \quad \forall \tau$) and if $\lambda(t,\tau)$ is chosen to be independent of τ. In this special case, the return simplifies to:

$$\frac{dV(\tau)}{V(\tau)} = \underbrace{\mu_A dt + \boldsymbol{\sigma_A' dz}}_{\text{Asset return}} \tag{7.13}$$

The CDC scheme degenerates into a standard mutual fund in which all participants simply receive the return on assets.

7.3.1 Restoring fairness through a benefit adjustment process correction

Again, fairness can be achieved by implementing an appropriate correction to the benefit adjustment process. If policymakers wish to design a fair CDC scheme with smoothing *and* a non-stochastic discount rate, the fair benefit adjustment process will be as follows:

$$\begin{aligned} \frac{db(\tau)}{b(\tau)} = \alpha(\tau)\Big[\mu_A dt + \boldsymbol{\sigma_A' dz} - \big(\mu_L dt + \boldsymbol{\sigma_L' dz}\big) \quad - \big(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\big)'\boldsymbol{\sigma_D(\tau)} dt\Big] \\ - \underbrace{\left[\lambda(t,\tau) - \alpha(t,\tau)\int_0^\infty \lambda(t,\tau) \times \frac{L(t,\tau)}{L(t)} d\tau - (1-\alpha(t,\tau))r(t)\right] dt}_{\text{non-stochastic discount rate correction term}} \end{aligned} \tag{7.14}$$

which simplifies to:

$$\frac{db(\tau)}{b(\tau)} = (1-\alpha(t,\tau))r(t) + \alpha(\tau)\big(\mu_A dt + \boldsymbol{\sigma_A' dz}\big) - \lambda(t,\tau) \tag{7.15}$$

Consequently, the return made by individual participants will be:

$$\frac{dV(\tau)}{V(\tau)} = (1-\alpha(\tau))r(t) + \alpha(\tau)\left(\mu_A dt + \boldsymbol{\sigma_A' dz}\right) \tag{7.16}$$

So, participation in the CDC scheme is equivalent to investing an $\alpha(t,\tau)$ share in the pension scheme's assets and a $(1-\alpha(t,\tau))$ share in an instantaneously riskless asset.

	Discount rate				
	Market based			Deterministic	
Benefit adjustment	No spread	Constant spread	Horizon dependent	Constant	Horizon dependent
Homogenous	No	No	No	Yes	No
With horizon correction					
No smoothing	Yes	Yes	No	Yes	No
No smoothing + spread correction	Yes	Yes	Yes	Yes	Yes
Smoothing	Yes	No	No	No	No
Smoothing + spread correction	Yes	Yes	Yes	No	No

Table 2. Overview of CDC pension scheme fairness. This table shows the fair and unfair combinations of the benefit adjustment process and the discount rate process in Collective Defined Contribution pension schemes that were discussed so far.

7.4 Overview

Table 2 summarizes all combinations of the benefit adjustment process and the discount rates process we have discussed so far. It follows from the table that a market based discount rate without a spread is fair for all benefit adjustment processes, as long as the benefit adjustment process is horizon dependent. Furthermore, the table shows that without smoothing, a spread correction can make any discount rate process fair.

8 Index linked CDC

So far we have assumed that pension benefits are not explicitly inflation protected. Policy makers typically argue that a pension scheme should ideally provide inflation-linked benefits. The discussion in Section 7.1 shows how this could be done. Instead of defining the payment to retirees at time t to be $b_i(t)$, we could generalize this to be $b_i(t)D(t,t)$. We could then choose $D(t,\tau)$ to be the value of a default-free index-linked bond that pays one real unit of consumption at maturity date τ. The interpretation of $b_i(t)$ now changes from a nominal benefit to an index-linked benefit and the actual level of the payment at time t to retirees will be $b_i(t)\Pi(t)$, where $\Pi(t)$ is the level of the consumption price index underlying the index-linked bond. Of course, an issue in practice may be that the availability of inflation-linked bonds may be limited. In that case, there may not be a price available for all relevant horizons.

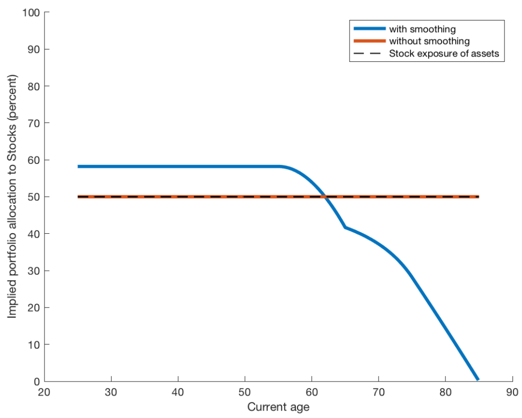
9 Implied exposures

In the previous sections we derived fair CDC schemes. What does a fair CDC scheme imply for the allocation of its participants to market risk? A CDC scheme sets a single asset allocation for its collective pool of assets. Yet, through the benefit adjustment process and discount rate process, each participant has a different exposure to risk. This difference can be seen by looking at the volatility of the individual CDC returns we derived (see (6.4), for example). Obviously, also smoohting will impact the allocation to market risk. In this section we illustrate how a CDC scheme with and without smoothing allocates risk over the course of the participants' life.

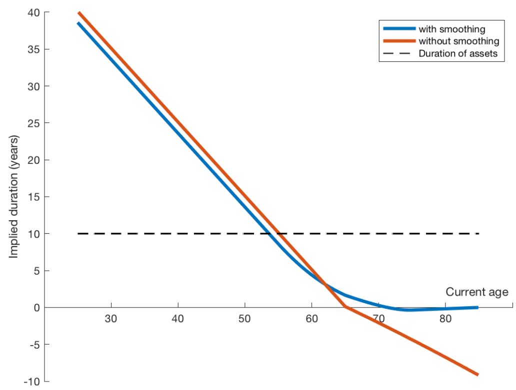
To illustrate the impact of smoothing we will consider the following smoothing policy. We set $\alpha(\tau)$ equal to zero at $\tau = t$. From there $\alpha(\tau)$ increases linearly until $\tau = t + 10$ and then remains flat for all $\tau > t + 10$. Roughly speaking this implies that shocks are smoothed over a 10 year period.

We assume in our example that the pension fund invests 50 percent of its wealth in equity and 50 percent in bonds. We assume that the duration of assets is 10 years. Figure 4, Panel A, shows the implied allocation to equities as a function of the age of the participants. The red line shows the implied exposure if the scheme did not apply smoothing. For equity risk, no smoothing implies that all participants have the same exposure to equity risk. The blue line shows the implied exposure under the smoothing policy. The smoothing policy lowers the exposure for old participants and increases the exposure for young participants. The exposure to equities of participants up to 55 years of age is elevated from 50 to almost 60 percent. From the age of 55, the effect of the smoothing policy gradually starts to decrease the participants exposure to equity risk. For old participants the exposure to equities drops (almost) linearly. For participants close to the end of their life, the exposure to equities falls to zero.

Figure 4, Panel (b), shows the implied duration as a function of age. Different from the equity risk case, the implied interest rate risk exposure, as measured by the implied duration, is not equal for everyone in the absence of smoothing. The red line shows that, in the absence of smoothing, the CDC scheme acts as an intergenerational interest rate swap. Young people, holding long-term benefits, have a very high implied duration. Since the schemes aggregate duration in this example is equal to ten, the high duration for the young implies that for the old the implied duration even goes negative. For the oldest cohort, the implied duration is



(a) Implied stock exposure



(b) Implied duration

Figure 4. Implied portfolio allocation in CDC scheme Panel (a) illustrates the implied portfolio share invested in stocks by age cohort. Panel (b) shows the implied duration by age cohort. The maturity profile of benefits is calibrated to the average Dutch pension scheme, which implies an aggregate duration of benefits of 20 years. The smoothing policy is such that $\alpha(\tau)$ is zero at $\tau = 0$, increases linearly until $\tau = 10$ and then remains flat for all $\tau > 10$.

exactly minus 10, which is exactly the duration mismatch of the scheme.

The impact of the smoothing policy (the blue line) is primarily visible for the retired cohorts. Without smoothing, participants aged 65 and older had a negative duration. The present value of their benefits increases, for example, when interest rates increase. With the smoothing policy in place the implied duration effectively becomes negligible. Changes in interest rates no longer affect the present value of the benefits of retirees. Compared to the situation without smoothing, the old generations now share less in the scheme's duration mismatch. Understanding the implied exposures is necessary for the next section in which we discuss the relation between fairness and optimality.

10 Fairness and optimality

In the previous sections we saw that fair CDC design is non-trivial. Only specific combinations of the benefit adjustment process and discount rates process lead to an arbitrage-free allocation of market risks. A specific horizon-dependent benefit adjustment process combined with a default-free discount rate leads to an arbitrage-free allocation of market risks. Within this configuration, benefit smoothing is possible while preserving the arbitrage-free condition. Under an arbitrage-free configuration, the pension scheme does not redistribute value across participants.

In addition to the no-arbitrage condition, policy makers should also be concerned about efficiency. A fair CDC scheme is not per se an optimal pension scheme. In the context of market risk, the question is whether CDC achieves an optimal exposure to market risk for all scheme participants. Although we did not analyze optimality of the risk exposure, our analysis can be used as input for this analysis because we derive the implied exposures of individual participants within the collective scheme.

Without deeper analysis we can however conclude that CDC acts as a constraint on the exposure of individuals to market risk. In the CDC scheme, the individual exposure is a function of the collective asset allocation, the benefit adjustment process, the discount rate process and the choice of the smoothing regime. This means that within a CDC scheme, policy makers are limited in the types of (implied) life-cycles they can achieve. In particular, interest rate risk is allocated over generations in a very specific way, with extremely long durations for young participants and very low durations for old participants (see Figure 4). One may wonder whether this is optimal. In particular the very low duration for old participants.[16] We will leave this issue for future research however.

[16] see i.e. Van Bilsen, Boelaars, and Bovenberg (2019) for a discussion of optimal exposure to interest rate risk over the life-cycle.

11 Conclusions

In this paper we examine the allocation of market risks in a general class of collective defined contribution schemes. We define a collective defined contribution scheme to be fair if all participants receive an arbitrage-free return at each point in time.

We firstly show that simply moving from a defined benefit setup to a CDC setup through the introduction of homogeneous benefit adjustments is typically not fair. Since the benefit levels in collective DC are stochastic, the actual return an individual makes has three components: the benefit adjustment effect, a discount rate effect, and, since these two components are correlated, a third covariance effect. It is this last component that makes it non-trivial to achieve fairness in a CDC scheme.

We secondly show how this problem can be solved by making benefit adjustments horizon-dependent. Horizon-dependent benefit adjustment also creates the opportunity for fair and smoothing the impact of shocks. This means that current retirement income can be made less sensitive to current shocks while keeping the scheme as a whole arbitrage-free.

Thirdly, we show to what extent a fair CDC scheme could deviate from the term-structure of default-free interest rates observed in the market as its discount rate. We show that deviating from a default-free market rate requires the scheme to add additional correction terms to its benefit adjustment process.

While we show that it is possible to design a fair CDC scheme, we do also stress that fairness is not the same as optimality. A CDC scheme can be seen as a rather specific constraint on the exposure to risk of different individuals and generations. Instead of having the opportunity to freely pick an exposure to risk for each individual or generation separately, a CDC scheme only allows its board to pick a single collective asset allocation. Through benefit smoothing the board can still differentiate the exposure to risk over the life-cycle a bit, but its degrees of freedom are limited compared to a system based on individual or generational accounts.

References

Bodie, Z. 1990. Pensions as Retirement Income Insurance. *Journal of Economic Literature* 28:28–49.

Brennan, M. J., and Y. Xia. 2002. Dynamic Asset Allocation under Inflation. *Journal of Finance* 57:1201–1238.

Campbell, J. Y., and L. M. Viceira. 2005. The term structure of the risk-return tradeoff. *Financial Analysts Journal* 61:34–44.

Chen, A., and M. Suchanecki. 2007. Default risk, bankruptcy procedures and the market value of life insurance liabilities. *Insurance: Mathematics and Economics* 40:231–255.

Chen, D. H. J., R. M. W. J. Beetsma, D. W. G. A. Broeders, and A. A. J. Pelsser. 2017. Sustainability of participation in collective pension schemes: An option pricing approach. *Insurance: Mathematics and Economics* 74:182–196.

Døskeland, T., and H. Nordahl. 2008. Optimal pension insurance design. *Journal of Banking and Finance* 32:382–392.

Gollier, C. 2008. Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics* 92:1463–1485.

Grosen, A., and P. L. Jørgensen. 2002. Life Insurance Liabilities at Market Value: An Analysis of Insolvency Risk, Bonus Policy, and Regulatory Intervention Rules in a Barrier Option Framework. *The Journal of Risk and Insurance* 69:63–91.

Merton, R. C. 1973. An Intertemporal Capital Asset Pricing Model. *Econometrica* 41:867–887.

Novy-Marx, R., and J. D. Rauh. 2014. Linking Benefits to Investment Performance in US Public Pension Systems. *Journal of Public Economics* 116:47–61.

Orozco-Garcia, C., and H. Schmeiser. 2017. Is Fair Pricing Possible? An Analysis of Participating Life Insurance Portfolios. *Journal of Risk and Insurance* 20:1–40.

Sharpe, W. F. 1976. Corporate pension funding policy. *Journal of Financial Economics* 3:183–193.

Shreve, S. E. 2004. *Stochastic Calculus for Finance II.* Springer.

Department for Work and Pensions. 2018. *Delivering Collective Defined Contribution Pension Schemes.* ISBN: 978-1-78659-105-0.

Treynor, J. L. 1977. The Principles of Corporate Pension Finance. *The Journal of Finance* 32:627–638.

Van Bilsen, S., I. A. Boelaars, and A. L. Bovenberg. 2019. The Duration Puzzle in Life-Cycle Investment. *Available at SSRN: https://ssrn.com/abstract=3308174* .

Appendix A: Liabilities

We will here describe the process for the regulatory present value of liabilities in more detail. Let $b(t,\tau)$ denote the aggregate level of all benefits that belong to participants that are retired at future date τ. In other words: if benefit levels were not to change, this is the aggregate benefit payment the CDC scheme will make at date τ to retirees. So,

$$b(t,\tau) \equiv \sum_i b_i(t) I\{\tau > R_i\} I\{\tau \leq T_i\} \tag{A1}$$

The regulatory value of the aggregate payment due at time τ, assuming the benefit levels do not change in the future, is then given by:

$$L(t,\tau) \equiv b(t,\tau) D(t,\tau) \tag{A2}$$

The total regulatory value of the pension schemes liabilities follows from aggregation over all future horizons τ:

$$L(t) = \int_0^\infty L(t,\tau) d\tau \tag{A3}$$

Next, let us describe the change in the regulatory value $L(t)$ over time. We derive the instantaneous change in the pension funds liability by applying Ito's Lemma to (A3). Furthermore note that in a dynamic setting the liabilities will be reduced by payments made today, $b(t,t)dt$, and increased by aggregate contributions, denoted by $Y(t)$.

$$dL(t) = \int_0^\infty \left[\frac{db(t,\tau)}{b(t,\tau)} + \frac{dD(t,\tau)}{D(t,\tau)} + \frac{db(t,\tau)}{b(t,\tau)} \frac{dD(t,\tau)}{D(t,\tau)} \right] \times L(t,\tau) d\tau - (Y(t) - b(t,t))dt \tag{A4}$$

The first part of this expression shows that the change in present value of the regulatory liabilities is driven by a change in the benefit level, a change in the discount rate and a covariance term. The second term is the net inflow or net outflow of the pension scheme. It will be useful to define a process that describes the change in liabilities caused by discount rate variation only. The reason is that this is the part of the present value change that will enter the benefit adjustment process.

We will write the total change in the regulatory value of the liabilities due to discount rate changes as:

$$\frac{d\tilde{L}(t)}{\tilde{L}(t)} = \int_0^\infty \left[\frac{dD(t,\tau)}{D(t,\tau)}\right] \times \frac{L(t,\tau)}{L(t)} d\tau \equiv \mu_L(\boldsymbol{X}(t),t)dt + \boldsymbol{\sigma_L}(\boldsymbol{X}(t),t)'\boldsymbol{dz} \tag{A5}$$

where

$$\mu_L(t) = \int_0^\infty \frac{L(t,\tau)}{L(t)} \mu_D(\boldsymbol{X}(t),t,\tau)d\tau \tag{A6}$$

is the instantaneous return on liabilities, and

$$\sigma_L(t) = \int_0^\infty \frac{L(t,\tau)}{L(t)} \sigma_D(\boldsymbol{X}(t),t,\tau)'d\tau \tag{A7}$$

the instantaneous volatility of the return on liabilities.

Appendix B: Homogeneous benefit adjustment

Homogeneous benefit adjustment implies that we can write the benefit adjustment process as:

$$\frac{db(\tau)}{b(\tau)} = \frac{db_i}{b_i} = \mu_b dt + \boldsymbol{\sigma_b'} \boldsymbol{dz} \qquad \forall \tau, \forall i \tag{B1}$$

So, μ_b and $\boldsymbol{\sigma_b}$ are not horizon or individual dependent. Remember that we suppress the dependency on t and $\boldsymbol{X}(t)$. Combining this with the requirement that the scheme is complete ($A(t) = L(t)$) allows us to pin down what μ_b and $\boldsymbol{\sigma_b}$ are. Setting the change in value of the assets dA in (4.4) equal to the change in the value of liabilities dL in (A4) gives:

$$A(t)\left[\mu_A dt + \boldsymbol{\sigma_A'} \boldsymbol{dz}\right] = \int_0^\infty \left[\frac{db(\tau)}{b(\tau)} + \frac{dD(\tau)}{D(\tau)} + \frac{db(\tau)}{b(\tau)} \frac{dD(\tau)}{D(\tau)}\right] \times L(\tau) d\tau \tag{B2}$$

Using the fact that $A(t) = L(t)$ at all times and using (B1), we can re-write this as:

$$\mu_A dt + \boldsymbol{\sigma_A'} \boldsymbol{dz} = \mu_b dt + \boldsymbol{\sigma_b'} \boldsymbol{dz} + \mu_L dt + \boldsymbol{\sigma_L'} \boldsymbol{dz} + \boldsymbol{\sigma_b'} \boldsymbol{\sigma_L} dt \tag{B3}$$

This equality needs to hold for all t and for all potential stochastic paths, so we can solve the drift term and volatility term separately. Doing so, we find that:

$$\boldsymbol{\sigma_b'} = \underbrace{\boldsymbol{\sigma_A'} - \boldsymbol{\sigma_L'}}_{\text{A-L mismatch risk}} \tag{B4}$$

and (B5)

$$\mu_b = \underbrace{\mu_A - \mu_L}_{\text{expected return on A-L mismatch}} - \underbrace{\left(\boldsymbol{\sigma_A} - \boldsymbol{\sigma_L}\right)' \boldsymbol{\sigma_L}}_{\text{A-L mismatch risk / liability discount rate risk correlation}} \tag{B6}$$

Note that the unexpected change ($\boldsymbol{\sigma_b'} \boldsymbol{dz}$) in the benefit level is equal to the unexpected return on the schemes' asset-liability mismatch. The mean benefit adjustment is equal to the mean return on the asset-liability mismatch risk minus a term representing the correlation between asset-liability mismatch risk and the volatility of the (pre-adjustment) value of the liabilities caused by discount rate variation.

Now, substituting the benefit adjustment process into the return equation (5.2) gives us the return on the participants τ-horizon payment.

Appendix C: Fair benefit adjustment in discrete time

In this appendix we will consider the situation in which benefit payment and adjustment takes place in discrete time. We will consider the case in which discounting is done using a market consistent default-free interest rate. First, let us derive what the benefit adjustment process looks like in case the fund applies a homogeneous adjustment rule without smoothing. The regulatory present value of the benefits is given by:

$$L_t = \sum_{\tau > t} b_{\tau,t} D_{\tau,t} \tag{C1}$$

where $b_{\tau,t}$ is the sum of all benefits at time t that are due at time τ and $D_{\tau,t}$ is the discount factor applicable to these benefits. Homogeneous adjustment implies that all benefits increase by the same percentage. Let Δb_{t+1} denote the percentage change in the benefit levels at the start of period $t+1$. Completeness requires that, at the start of time $t+1$:

$$\sum_{\tau \geq t} b_{\tau,t}(1 + \Delta b_{t+1}) D_{\tau,t+1} = A_{t+1} \tag{C2}$$

Which implies that:

$$\Delta b_{t+1} = \frac{A_{t+1}}{\tilde{L}_{t+1}} - 1 \qquad \forall \tau \tag{C3}$$

where $\tilde{L}$ is the present value of the benefits before benefit adjustment.

The return on the benefit cash-flow becomes:

$$\frac{V_{t+1}}{V_t} = \frac{(1 + \Delta b_{t+1}) D_{\tau,t+1}}{D_{\tau,t}} = \frac{A_{t+1}}{\tilde{L}_{t+1}} \frac{D_{\tau,t+1}}{D_{\tau,t}} \tag{C4}$$

The right-hand side is not a return on a traded portfolio. It is the return on the discount factor times the return on assets divided by the return on the liability matching portfolio. Intuitively it is clear that this is a complex non-linear pay-off and we saw in the continuous time case, that this tends to result in a transfer of market value from young to old.

We can correct this unfairness, while preserving the idea that every participant shares in the funds mismatch return, by making Δb_{t+1} horizon depended in the following way:

$$\Delta b_{\tau,t+1} = \left(\frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t} \right) \frac{D_{\tau,t}}{D_{\tau,t+1}} \tag{C5}$$

Instead of making the benefit adjustment depend on the realized funding ratio it is now a function of the return mismatch. This gets rid of the non-linearity implied by using the funding ratio. Additionally, there is now a horizon-dependent correction term: $\frac{D_t(\tau)}{D_{t+1}(\tau)}$. If the fund uses a default-free interest rate, this term is the inverse of the period return on a zero-coupon bond that matures at time τ. Implementing this benefit adjustment rule results in a return on the CDC benefit of:

$$\frac{V_{t+1}}{V_t} = \frac{D_{t+1}(\tau)}{D_t(\tau)} + \frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t} \tag{C6}$$

which is the return on a zero-coupon bond plus the excess return on the funds mismatch. This is a market consistent return, since all three terms are returns on traded assets. Note that (C6) is indeed the discrete time equivalent of (6.2). Note that in order to establish fairness of the discrete time return in (C6) we do not need to assume that returns follow an Ito-process or have a Markov structure.

As we did in the continuous time case, we can also add a smoothing parameter to this equation. In this case we get:

$$\Delta b_{\tau,t+1} = \alpha_{\tau,t} \left(\frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t} \right) \frac{D_{\tau,t}}{D_{\tau,t+1}} \tag{C7}$$

$$\frac{V_{t+1}}{V_t} = \frac{D_{t+1}(\tau)}{D_t(\tau)} + \alpha_{\tau,t} \left(\frac{A_{t+1}}{A_t} - \frac{\tilde{L}_{t+1}}{L_t} \right) \tag{C8}$$

Notice that (C8) is exactly the discrete time equivalent of (6.4). The return is equal to a long position in the zero-coupon bond and an exposure to the funds mismatch return that is weighted by $\alpha_{\tau,t}$. Completeness in this case implies that $\alpha_{\tau,t}$ is chosen such that:

$$L_t = \sum_{\tau > t} \alpha_{\tau,t} b_{\tau,t} D_{\tau,t} \tag{C9}$$