A Nonparametric Study of Dependence Between S&P 500 Index and Market Volatility Index (VIX)

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Abstract

This paper studies the relationship between the S&P 500 index returns and the returns on the market volatility index (VIX) via a nonparametric copula method. We further propose a conditional dependency index to investigate how the dependency between the two return series varies across different segments of market return distribution. We observe the following findings: (a) the two series exhibit strong, negative, extreme tail dependency; (b) the negative dependency is stronger in extreme bearish markets than in extreme bullish markets; (c) the dependency gradually weakens as the market return

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moves toward the center of its distribution, or in quiet markets. The unique dependence structure supports the VIX as a barometer of markets' mood in general. Lastly, we propose a simple model of the returns of VIX based on the S&P 500 returns that incorporates the asymmetry and tail dependence between the two series.

Keywords: Kendall's Tau; Nonparametric copula; Tail Dependence Index; Conditional Dependence Index

JEL Classification: C13; C22; G1

1 Introduction

Investors have seen severe downturn in stock markets since October 2008 and the mood of bearish market was often cited through a volatility index (VIX). The VIX is a trade mark held by the Chicago Board Options Exchange (CBOE), which is constructed from S&P 500 index option prices and is designed to retrieve market's aggregate expectations of near-term (30 days) S&P 500 index volatility. The frequently observed counter-movements between the market index and the market volatility index earn the VIX a reputation of market barometer of investors' fear.

Although the relation between market returns and volatilities has been extensively studied in both applied finance and applied econometrics literature, we noticed that most of the existing work employs parametric models such as variant versions of GARCH-in-Mean type of models combined with leverage and asymmetric features in the conditional heteroschedastic volatility equation. See, the leverage model of Back (1976), the volatility feedback model of Porterba and Summers (1986) and Campbell and Hentschel (1992), and behavioral models of Hibbert, Daigle and Dupoyet (2009), to name a few among many. To overcome possible model misspecification problem, this paper re-examines the relation between returns and volatilities via a model-free nonparametric copula method.

Specifically, we study the joint distribution of the S&P 500 index returns and the returns on the VIX index via a nonparametric copula method. We then propose a conditional dependence index (given in Section 4) to investigate how the dependency between the two return series varies across different segments of market return distribution. Consistent with Hibbert, et al.'s (2009) findings, we observe strong daily negative asymmetric relation between market return and the return of market volatility. Moreover, we observe these noteworthy results: (a) the two series exhibit strong, negative, extreme tail dependency; (b) the negative dependency is stronger in extreme bearish markets than in extreme bullish markets; (c) the dependency gradually weakens as the market return moves toward the center of its distribution, or in quiet markets.

Investigation of the dependence structure between the S&P 500 index and the VIX provides useful guidance on constructing predicatory models. For example, Whaley (2009) reports asymmetric dependence between these two series and proposes a simple model of the VIX returns based on the market returns and a dummy variable for downturn market. In this paper, based on our finding that the asymmetric dependence between these two series is largely driven by the different degrees of tail dependence in the extreme tails, we propose an alternative model for the VIX based on the market returns and two dummy variables indicating if the S&P 500 index is in its upper 5% or lower 5% tails. The simple model is demonstrated to capture the overall pattern between the two series well.

The rest of the paper is organized as follows. Section 2 presents the data and summary statistics. Section 3 constructs tail dependence index from the joint distribution of the returns of the S&P 500 index and of the VIX. In Section 4, we construct conditional dependence index, which measures the dependency between the two return series conditional on the return of the S&P 500 index falling into specific segment of its distribution. Section 5 presents a simple model of VIX returns based on the S&P 500 index returns. The last section concludes.

2 Data and Descriptive Statistics

The CBOE started publishing the implied volatility index of the S&P 100 index since 1993, which was constructed from the at-the-money S&P option prices using the Black-Scholes-Merton formula; see details in Whaley (1993, 2000). The VIX was introduced with two purposes in mind. First, it was intended to provide a benchmark of expected short-term market volatility. Second, it was intended to provide an index upon which futures and options contracts on volatility could be written.[1] In September 2003, the CBOE replaced the old volatility index with the current volatility index, which is constructed via a model-free formula developed by Demeterfi, Derman, Kamal and Zou (1999) and originated from the seminal work of Breeden and Litzenberger (1978). The current volatility index is extracted from both at-the-money and out-of–the-money S&P 500 index option prices. Detailed information can be found at http://www.cboe.com.

In this study, we downloaded daily S&P 500 index prices from DataStream, and the daily implied volatilities (VIX) from the CBOE. The data spans from January 2, 1990 to December 31, 2008. The VIX is frequently cited as a barometer of investors' fear and markets' aggregate expectations of near-term market volatility. Such idea has found strong popularity among investor community since its first debut at the CBOE in 1993. A high VIX beyond 40 is usually linked to a strong bear market and a low VIX value is linked to a market with more confidence. The first time that VIX passed the value of 40 was on August 31, 1998, a year marked with Russia's currency devaluation and national debt moratorium, and the collapse of the

[1]The social benefits of trading volatility have long been recognized. The Chicago Board Options Exchange (CBOE) launched trading of VIX futures contracts in May 2004 and VIX option contracts in February 2006.

Long Term Capital Management in the United States. For the sample period under consideration, the number of transaction days with the VIX value exceeding 40 is 15, 4, 10, and 64 in 1998, 2001, 2002, and 2008, respectively. On November 20, 2008, the VIX reached its record high of 80.86, marking an unprecedented financial crisis faced by global stock markets. It is interesting to observe that the counter-movement between the VIX and the S&P 500 index did not become a dominant tone until August 1998 as shown in Figure 1. See, of 76.76 percent of the total 2,259 transaction days that the S&P 500 fell, the VIX gained; of 76.85 percent of the total 2,531 transaction days that the S&P 500 index gained, the VIX fell. In total, the two return series moved to opposite directions in 76.58 percent of transaction days considered in this paper, and the number increased to 84.92% and 88.93% in 1998 and 2008, respectively. These numbers indicate prominent counter-movements between the market index and market volatility index, especially in bearish markets.

Let P_t and VIX_t be the S&P 500 index price and the volatility index at date t, respectively. Next, we construct the daily (or monthly) returns from the daily (or monthly) S&P 500 index and VIX by

$$rsp_t = 100 \times \ln\left(P_t/P_{t-1}\right) \text{ and } rvix_t = 100 \times \ln\left(VIX_t/VIX_{t-1}\right).$$

Table 1 reports the summary statistics of daily and monthly returns of the two indexes. First, as the VIX has lower average return but significantly higher volatility than the S&P 500 index, investors having a concave utility function will prefer the S&P 500 index to the VIX as an investment vehicle. We then split the data according to the signs of the returns of the S&P 500 index, and calculate the upside and downside average returns and sample standard deviations for both series. Interestingly, we observe that both series exhibit stronger volatilities in the downturn markets than in the upturn markets. In the downturn markets, market index performed considerably worse than in the upturn markets, and the opposite holds true for the VIX index.[2]

Next, to study the counter-movement between the two returns series, we calculate three statistics such as Pearson's correlation coefficient, Kendall's tau, and the probability that the two return series move to opposite direction, which is defined as $\lambda = \Pr\left(\mathrm{rsp}_t \times \mathrm{rvix}_t < 0\right)$. The closer λ is to one, the stronger is the negative association between the two series. Kendall's tau is given by

$$\begin{aligned}\tau &= \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0] \\ &= 2\Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - 1,\end{aligned}$$

where (X_1, Y_1) and (X_2, Y_2) are continuous random vectors drawn from the joint and marginal distribution $F(x, y)$, $F_X(x)$ and $F_Y(y)$ respectively(see Chapter 5 of Nelsen, 1999). Apparently, Kendall's tau reveals a strong negative (or positive) association between the two return series if it is close to negative (or positive) one, and a weak association if it is close to zero. Kendall's tau equals zero, if rsp_t and $rvix_t$ are independent; but it may not hold vice versa.

Table 2 reports the sample correlation, Kendall's tau, λ, and the com-

[2] Some results are studied but not reported in the main text for brevity. We constructed two optimal portfolios based on minimum variance criterion and maximum Sharpe ratio criterion. The results show that the optimal portfolios enjoy much smaller volatility than the market index, but not much improvement on average returns. In addition, consistent with our intuition, the optimal portfolios allot higher percentage of investment to the VIX in bearish markets than in bullish markets.

pound returns of the S&P 500 index for the whole sample period and for each year from 1990 to 2008. We find that the sample correlation and Kendall's tau are both significantly negative, where the sample correlation ranges from -0.450 in 1995 to -0.850 in 2007 and Kendal's tau ranges from -0.295 in 1995 to -0.690 in 2008. Also, during the sample period, there are 76.6 percent of chances that the S&P 500 index and the VIX moved to opposite directions, and this figure even reached 88.9 percent in 2008 and bottomed at 63.9 percent in 1995.

Given the summary statistics presented in Tables 1 and 2, we observe strong negative association between the VIX and the S&P 500 index, and the negative dependence was more prominent in the first decade of the 21th century than in the 1990's. Also, the worse the market is, the stronger is the negative dependence. In the following section, we shall further explore the negative dependence by examining the joint distribution of the two return series.

3 Copula Function and Tail-Dependence Index

To further our understanding of the dependence relationship between the S&P 500 index and the VIX, we use the device of copula to decompose their joint PDF. According to the Skalar's theorem, the joint density of two continuous random variables X and Y can be written as

$$f(x,y) = f_X(x)\, f_Y(y)\, c\left(F_X(x), F_Y(y)\right),$$

where $f(x,y)$, $f_X(x)$, and $f_Y(y)$ are the joint and marginal PDFs respectively. The function $c(F_X(x), F_Y(y))$ is called the copula density function, which completely summarizes the dependence structure between X and Y. See Nelsen (1999) for a thorough treatment of the copula. Below, we estimate the copula density function of the two return series, using the Exponential Series Density Estimator (ESE) in Wu (2007). The estimator takes the form

$$c(u,v) = \exp\left(\sum_{0<i+j<m} \theta_{ij} u^i v^j + \theta_0\right), 0 \leq u, v \leq 1,$$

where m is a positive integer, and $\theta_0 = -\ln \int_0^1 \int_0^1 \exp\left(\sum_{0<i+j<m} \theta_{ij} u^i v^j\right) dudv$ such that $c(u,v)$ integrates to unity. The degree of the exponential polynomial m is selected according to the information criterion AIC. Unlike the kernel density estimator, the ESE does not suffer from boundary bias.

For our sample, since we do not observe $u \equiv F_{\text{rvix}}(x)$ and $v \equiv F_{\text{rsp}}(y)$, we replace them by their estimates $\hat{u} = 1/T \sum_{t=1}^{T} I(rvix_t \leq x)$ and $\hat{v} = 1/T \sum_{t=1}^{T} I(rsp_t \leq y)$ respectively. Serveral benefits could result from the one-to-one transformation of the return series via its cumulative distribution function: a) it can effectively mitigate potential outlier problems in the nonparametric estimation; b) as a measure of the likelihood of the occurrence of an event, probability provides a direct way of capturing market relative status than the raw return value does across time, which is upmost important in our study of the relation between the two indexes in a quick-changing market. On the other words, the study of the transformed data (u,v), instead of the raw return series, provides a key tool to consolidate historical study of silimiar issues.

The estimated copula density function of (u,v), with $m = 6$ selected by the AIC, is reported in Figure 2. The preliminary results in Section 2 indiate strong negative association between the two return series without identifying the sources of the observed relation. By visualizing the copula density function in Figure 2, we see that the negative dependence between the S&P 500 index and VIX is largely driven by the counter-movements of the tails, since the bulk of the copula density is along the anti-diagonal line and spikes up at the two corners. In other words, the co-movements of the opposite tails of two marginal distributions contribute significantly to the negative dependence between the S&P 500 index and the VIX. (More results on the sources of negative association will be given gradually below.) In addition, the density at the upper left corner in the figure, corresponding to the case of low market index returns and high VIX returns, is larger than its counterpart associated with high market index returns and low VIX returns. Except for the two tails along the anti-diagonal, the copula density appears to be rather symmetric.

Motivated by our observation in Figure 2, we calculate the Tail Dependence Index (TDI) between the daily returns on the S&P 500 index and the VIX at opposite tails. Generally speaking, the TDI captures the tail probability of one variable given that another variable is residing in its tail area. Taking clues from the estimated copula density reported above, we focus on the following two TDIs that capture the co-movements of opposite tails of the two series:

$$\mathrm{TDI}_1(\alpha) = \Pr\left(rvix < rvix_\alpha | rsp > rsp_{1-\alpha}\right) = \Pr\left(u < \alpha | v > 1-\alpha\right),$$
$$\mathrm{TDI}_2(\alpha) = \Pr\left(rvix > rvix_{1-\alpha} | rsp < rsp_\alpha\right) = \Pr\left(u > 1-\alpha | v < \alpha\right),$$

where $rvix_\alpha$ and rsp_α are the percentile of the return series $rvix$ and rsp, respectively. Taking $\alpha = 0.01$ and $\alpha = 0.05$ respectively, we calculate the TDIs based on the estimated copula function. We obtain $\text{TDI}_1(0.01) = 0.2292, \text{TDI}_2(0.01) = 0.3542, \text{TDI}_1(0.05) = 0.3687$, and $\text{TDI}_2(0.05) = 0.4234$. If the two series were independent, we would have obtained $\text{TDI}_j(\alpha) = \alpha$ for $j = 1, 2$. Therefore, the fact that $\text{TDI}_j(\alpha)$ is substantially larger than α indicates strong negative tail dependence between the two return series. In particular, our results suggest that extreme movements in the S&P 500 index are associated with extreme movements of the VIX to opposite direction with rather strong probability. These results are consistent with the notion that the VIX is a barometer of stock market's mood.

In addition, the fact that $\text{TDI}_1(\alpha) < \text{TDI}_2(\alpha)$ for both $\alpha = 0.01$ and $\alpha = 0.05$ reveals that the VIX asymmetrically responds to the extreme movements in the S&P 500 index. The probability that the VIX jumps up at rare high speed when the market index faces extreme free-fall is much higher than the probability that the VIX falls back at high speed when the market index enjoys strong rebound. The asymmetry is consistent with the stylized fact that *the markets tend to respond more to bad news than to good news.* The fac that the tail dependence is more pronounced when the market is in turmoil explains why the VIX is dubbed the Investor Fear Gauge.

Next, extending the concept of TDI, we estimate the conditional CDF curves $F(u|v > 0.95)$ and $F(u|v < 0.05)$, see the left panel in Figure 3. The conditional CDFs describe the stochastic behavior of the returns of the VIX while the market returns rose above its 5% right-tail distribution or fall below its 5% left-tail distribution. We make the following observations. First, the conditional CDF, $F(u|v < .05)$, shows that $rvix$ has a high probability to move to the right tail of its distribution when rsp is at the far left tail of its distribution, since our estimate gives $P\left(u \leq .6|v \leq .05\right) < 0.04$. Extreme down market is highly associated with strong market uncertainty on market risk. Second, The conditional CDF, $F(u|v > .95)$, shows that $rvix$ has a higher probability to move to the left tail of its distribution when rsp is at the far right tail of its distribution, as our estimation gives that the conditional probability that $u \leq .2$ given $v > .95$ is greater than 76%. Strong upturn market is highly associated with market's relief on potential market volatilities.

To summarize, we find strong left-right and right-left tail dependency between rsp and $rvix$. In the section below, we propose a conditional dependence index to formally measure the dependence between the two return series conditional on variant market status.

4 Conditional Dependence Index

Figure 3 plots several estimated conditional distribution functions of u given $v \in A$, where A is a subset of the interval $[0, 1]$. The left panel in Figure 3 shows that for a given $\alpha \in [0, 1]$, $F(u|v < \alpha) \leq F(u) \leq F(u|v > \alpha)$ for all $u \in [0, 1]$. $F(u|v < \alpha)$ is further below $F(u) = u$ as α is closer to zero, and that $F(u|v > \alpha)$ is further above the 45-degree line as α is closer to one. The right panel in Figure 3 shows that the conditional CDFs are closer to the 45-degree line as A is closer to the center of the interval $[0, 1]$.

If u and v, or rsp and $rvix$, are independent of each other, we have $F(u|v \in A) = F(u)$ for all $u \in [0, 1]$, and knowing the information set $\{v \in A\}$ does not help make better prediction of u. On the other hand, the further is the conditional CDF away from the 45-degree line, the higher is the dependency between u (or $rvix$) and the realization segment of v (or rsp). Therefore, it is natural to use the area between the conditional CDF and the

45-degree line as a proxy of the predictive power of v on u. In doing so, we are able to learn where u and v are most dependent. Consequently, we can make inference on the relation between the VIX and its underlying market index conditional market status.

Specifically, we propose a conditional dependence index which equals the area between the conditional CDF and the 45-degree line, conditional on the information set $\{v \in A\}$. Thus, the index is defined as a functional of A:

$$G(A) = 2\int_0^1 |F(u|v \in A) - u|\, du = 2E\left[|F(u|v \in A) - u|\right].$$

Apparently, for any given subset $A \subset [0,1]$, $0 \leq G(A) \leq 1$.[3] This can be considered as a conditional version of the Gini index.

Partitioning the [0,1] interval into twenty equal-width intervals, we calculate $G(A)$ for each interval based on the estimated copula density between $rvix$ and rsp. We also calculate their corresponding 95% confidence intervals using the block bootstrap method. The results are reported in Table 3. It is shown that there exists strong left-right and right-left tail dependency between $rvix$ and rsp, and much weaker but still significant dependency between the two series when rsp moves to the middle of its distribution. Also, we find that both $G(v \leq 0.05) = 0.8064$ and $G(v \leq 0.10) = 0.7346$ are larger than $G(v > 0.95) = 0.7004$. This shows asymmetric negative dependency between the two return series: the negative dependency is stronger for extreme down market periods than for extreme upturn market periods of similar magnitude. In addition, Table 3 also presents the estimated probability that the VIX falls given the return of the S&P 500 index falling into a specific inter-

[3] Following the definition of the Gini index, we multiply the area between the conditional CDF and the 45-degree line by two such that $0 \leq G(A) \leq 1$.

val of its distribution: on a given date, the probability that the VIX falls is around 6.22% if the S&P 500 return is in the $[-1.724\%, -1.183\%]$ range (or its return falls into 0.05th and 0.10th interval), while the probability that the VIX increases is around 11.3% if the S&P 500 index moves up by 1.168% to 1.644% (or its return falls into 0.90th and 0.95th interval), see the last three columns of Table 3.

To sum up, we find strong dependency between $rvix$ and rsp and the dependency is stronger in volatile market periods than in relatively quiet market periods. As the VIX reveals market's expectation on the future 30-day volatility, our results indicate that investors made sharp revision on their belief of market risks during extreme volatile market periods, and that the revision is less noticeable during tranquil market periods.

Moreover, the conditional dependence index proposed indicates that the dependence between the returns on the S&P 500 index and the VIX exhibits a U-shape curve as the returns on the S&P 500 index move from tails to the middle of its distribution, see Table 3.[4] It again confirms that the negative association between the S&P 500 index and the VIX mainly come from tail events.

5 Regression Analysis

Whaley (2009) notes that the VIX spikes during periods of market turmoil, which explains why it has become known as the investors' fear gauge. If expected market volatility increases (decreases), investors demand higher

[4] The U-shape relation also holds true when we calculate the conditional independence index conditional on rvix; i.e., $G(A) = E[|F(v|u \in A) - v|]$ for any subset A of the interval $[0, 1]$. The results are omitted for brevity.

(lower) rates of return on stocks, so stock prices fall (rise). This suggests the relation between rate of change in the VIX should be proportional to the rate of return on the S&P 500 index. But, the relation is more complicated. As we argued and documented above, the relation between the VIX and the S&P 500 index is asymmetric: the change in the VIX rises at a higher absolute rate when the market index falls than when the market index rises. In addition, the asymmetry is largely driven by different degrees of tail dependence. Guided by the findings we reported above, we consider the following three models:

Model 1: Regressing $rvix$ on rsp gives

$$rvix_t = \underset{\substack{(.0616) \\ <.1610>}}{.0863} - \underset{\substack{(.0542) \\ <.0000>}}{3.5761} \; rsp_t + error_t, \bar{R}^2 = .4765 \tag{1.1}$$

Model 2: Regressing $rvix$ on rsp and $rsp^+ = rsp \times I(rsp \geq 0)$ gives

$$rvix_t = \underset{\substack{(.0829) \\ <.0009>}}{.2768} - \underset{\substack{(.0928) \\ <.0000>}}{3.0853} \; rsp_t - \underset{\substack{(.1459) \\ <.0000>}}{.9485} \; rsp_t^+ + error_t, \bar{R}^2 = .4810 \tag{1.2}$$

Model 3: Regressing $rvix$ on rsp, rsp^U and rsp^L gives

$$rvix_t = \underset{\substack{(.0632) \\ <.9500>}}{.0040} - \underset{\substack{(.0911) \\ <.0000>}}{4.4263} \; rsp_t - \underset{\substack{(.1327) \\ <.0000>}}{1.7136} \; rsp_t^U + \underset{\substack{(.1305) \\ <.0000>}}{.9116} \; rsp_t^L + error_t, \bar{R}^2 = .4941 \tag{1.3}$$

where $\bar{R}^2$ is the adjusted R^2, standard errors are included in the parentheses, and the p-values are included in the brackets. In addition, in model (1.3), we define $rsp_t^U = rsp_t$ if the value of rsp_t is greater than the .95th empirical quantile of $\{rsp_t\}$ and zero otherwise; we define $rsp_t^L = rsp_t$ if the value of rsp_t is less than the .05th empirical quantile of $\{rsp_t\}$ and zero otherwise. Therefore, adding rsp_t^U and rsp_t^L allows us to explicitly investigate the impacts of stock market returns at extreme tails on the changes of market volatilities.

In model (1.1) and (1.3), the intercept term is not significantly different from zero at the 5% significance level. This is consistent with the prediction that the expected VIX remains constant when there is no change in market return. On the other hand, all the slope parameters are significantly different from zero at the 5% level. We also estimated a fourth model with the market returns divided into four intervals: the market return is less than its lower .05th quantile, between its lower .05th quantile and zero, between 0 and its upper .95th quantile, and beyond its upper .95th quantile. However, the F test does not reject model (1.3) against the fourth model at the 5% significance level. We therefore take model (1.3) as our final model.

Model (1.1) implies that a 1% increase in market return reduces market volatility by around 3.6% and that a 1% drop in market return pushes up market volatility by around 3.6%. Accounting for the asymmetry in the tail dependence between rsp and $rvix$, model (1.3) implies that

$$\begin{aligned} rvix &= .0040 - 3.5147\, rsp + \text{error, if } rsp < rsp_{.05} \\ rvix &= .0040 - 4.4263\, rsp + \text{error, if } rsp \in [rsp_{.05}, rsp_{.95}] \\ rvix &= .0040 - 2.7127\, rsp + \text{error, if } rsp > rsp_{.95} \end{aligned} \tag{1.4}$$

where rsp_α is the αth quantile of $\{rsp_t\}$. Therefore, when the market returns fall between its .05th and .95th quantiles, a 1% increase (or drop) in the market return reduces (pushes up) the market volatility by around 4.4%.

When the market returns fall below its .05th quantile in an extreme down-market, a 1% drop of the market return pushes up the market volatility by around 3.5%. If the market returns grow beyond its .95th quantile in an extreme up-market, a 1% increase of the market return reduces the market volatility by around 2.7%.

The results from model (1.4) tell us that the marginal effect of 1% change in market returns on the change of the market volatilities is the highest for stable market and the lowest for extremely upbeat market. This seems to be a surprise to investors who experienced higher market volatilities in bear markets than in bull markets. The importance is that readers should not confuse the marginal effects of market returns on market volatilities with the total changes of market volatilities. Extreme movements in market returns drive up market volatilities, although changes in market volatilities per unit change of market returns may not be high. Take one numerical example. The estimated expected change in market volatility is .004% when the market return is zero, and it is 6.06% when the market return is -1.72 (the .05th quantile of $\{rsp_t\}$).

Moreover, equation (1.4) reveals asymmetric marginal effects: the marginal effects are higher at the lower 5% tail of the market returns than at the upper 5% tail of the market returns in absolute values. From (1.4), we find that the estimated expected changes in market volatilities are no less than 6.06% conditional on the fact that market return lies at the lower 5% tail of its distribution, and that the estimated expected changes in market volatilities are at most -4.46% conditional on the fact that market return lies at the upper 5% tail of its distribution.

6 Conclusions

The VIX, a trade-mark held by the CBOE, is commonly cited as a barometer of stock market volatility. Applying nonparametric copula method and constructing conditional dependence indices, we find that the dependence between the returns on the S&P 500 index and the VIX exhibits a U-shape curve as the returns on the S&P 500 index move from tails to the middle of its distribution, and that extreme market downturns are involved with market's correction on near term expectation of market risks at very high probability. Lastly, we present a simple model to predict changes in the VIX, based on market returns, that accounts for the asymmetric and tail dependence between these two series.

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Appendix: Tables and Figures

Table 1: Summary Statistics (01/02/1990-12/31/2008)

Data Frequency	Variable	$\bar{x}$	$\bar{x}_-$	$\bar{x}_+$	$\hat{\sigma}$	$\hat{\sigma}_-$	$\hat{\sigma}_+$
Daily	$rvix$	0.018	3.410	-3.010	5.888	5.432	4.454
	rsp	0.019	-0.791	0.742	1.137	0.882	0.802
Monthly	$rvix$	0.201	11.961	-6.911	16.851	16.276	12.771
	rsp	0.445	-3.689	2.966	4.223	3.376	2.241

a. $\bar{x}$=average return,$\bar{x}_-$=downside average return over times with $rsp < 0$, $\bar{x}_+$=upside average return over times with rsp ≥ 0;
b. $\hat{\sigma}$=sample standard deviation, $\hat{\sigma}_-$=downside sample standard deviation over times with $rsp < 0$, $\hat{\sigma}_+$=upside sample standard deviation over times with $rsp \geq 0$.

Table 2: Sample Correlation, Kendall's τ, λ, and Compound Return of the S&P 500 Index

Year	Sample Correlation	Kendall's τ	λ	Compound Return of the S&P 500 Index
ALL	-0.690	-0.5165	0.766	92.077
1990	-0.537	-0.353	0.710	-8.548
1991	-0.557	-0.362	0.727	24.503
1992	-0.547	-0.351	0.673	4.327
1993	-0.510	-0.362	0.672	6.893
1994	-0.724	-0.496	0.750	-1.334
1995	-0.450	-0.295	0.639	29.384
1996	-0.687	-0.457	0.713	17.675
1997	-0.701	-0.531	0.771	27.514
1998	-0.819	-0.641	0.849	23.166
1999	-0.799	-0.600	0.829	17.928
2000	-0.784	-0.571	0.810	-9.731
2001	-0.820	-0.600	0.794	-11.132
2002	-0.818	-0.647	0.810	-27.185
2003	-0.642	-0.462	0.746	20.147
2004	-0.759	-0.539	0.806	8.922
2005	-0.831	-0.621	0.814	3.772
2006	-0.822	-0.564	0.737	11.139
2007	-0.850	-0.672	0.813	3.589
2008	-0.847	-0.690	0.889	-47.136

The compound return of the S&P500 index is the log-difference of market indexes observed at the ending and starting date of the period under consideration multiplied by 100; λ gives the relative frequency that the market index and market volatility index moved to opposite directions for the period of time under consideration.

Table 3: Conditional Dependence Index with 95% Confidence Intervals

α	$Q_{.025}$	$G(v \leq \alpha)$	$Q_{.975}$	$Q_{.025}$	$G(v > \alpha)$	$Q_{.975}$	$\tilde{A}=(a,b]$	$Q_{.025}$	$G(\tilde{A})$	$Q_{.975}$	$Q_{.025}$	$F(0\|\tilde{A})$	$Q_{.975}$
0.05	0.787	0.806	0.833	0.043	0.044	0.046							
0.1	0.713	0.735	0.758	0.080	0.083	0.085	[.05,.1)	0.633	0.660	0.685	0.043	0.062	0.085
0.15	0.642	0.663	0.683	0.115	0.119	0.122	[.1,.15)	0.491	0.518	0.543	0.092	0.119	0.142
0.2	0.583	0.600	0.618	0.148	0.152	0.156	[.15,.2)	0.383	0.412	0.434	0.148	0.174	0.205
0.25	0.531	0.547	0.563	0.179	0.184	0.190	[.2,.25)	0.312	0.337	0.358	0.198	0.224	0.260
0.3	0.484	0.501	0.518	0.210	0.217	0.224	[.25,.3)	0.264	0.281	0.303	0.248	0.274	0.308
0.35	0.445	0.459	0.476	0.241	0.249	0.258	[.3,.35)	0.219	0.232	0.256	0.296	0.328	0.359
0.4	0.408	0.420	0.436	0.273	0.282	0.292	[.35,.4)	0.175	0.186	0.208	0.352	0.389	0.418
0.45	0.371	0.382	0.396	0.305	0.314	0.325	[.4,.45)	0.134	0.146	0.162	0.426	0.458	0.483
0.5	0.334	0.344	0.357	0.335	0.345	0.359	[.45,.5)	0.113	0.127	0.143	0.502	0.531	0.555
0.55	0.297	0.306	0.319	0.364	0.375	0.391	[.5,.55)	0.126	0.141	0.155	0.578	0.602	0.624
0.6	0.260	0.268	0.280	0.392	0.404	0.422	[.55,.6)	0.162	0.179	0.196	0.644	0.666	0.692
0.65	0.223	0.231	0.241	0.418	0.432	0.451	[.6,.65)	0.209	0.225	0.244	0.699	0.722	0.749
0.7	0.188	0.196	0.204	0.443	0.461	0.480	[.65,.7)	0.253	0.269	0.290	0.746	0.765	0.792
0.75	0.156	0.163	0.169	0.471	0.492	0.511	[.7,.75)	0.295	0.309	0.331	0.777	0.796	0.820
0.8	0.126	0.132	0.137	0.504	0.528	0.548	[.75,.8)	0.329	0.347	0.372	0.799	0.819	0.840
0.85	0.095	0.100	0.104	0.544	0.572	0.596	[.8,.85)	0.369	0.394	0.420	0.815	0.837	0.858
0.9	0.067	0.070	0.073	0.602	0.629	0.665	[.85,.9)	0.428	0.458	0.487	0.835	0.857	0.880
0.95	0.034	0.036	0.038	0.669	0.700	0.747	[.9,.95)	0.527	0.558	0.586	0.867	0.887	0.907

a. The columns below $Q_{.025}$ and $Q_{.975}$ refer to the starting and end values of the 95% confidence intervals obtained from block bootstrap method;
b. $G(v \in A) = E[|F(u|v \in A) - u|]$ measures the conditional dependence between $u = F_{rvix}(rvix)$ and $v = F_{rsp}(rsp)$.

Figure 1: Raw Data Plot (01/02/1990-12/31/2008)
(Black line: S&P 500 Index; red line: the VIX)

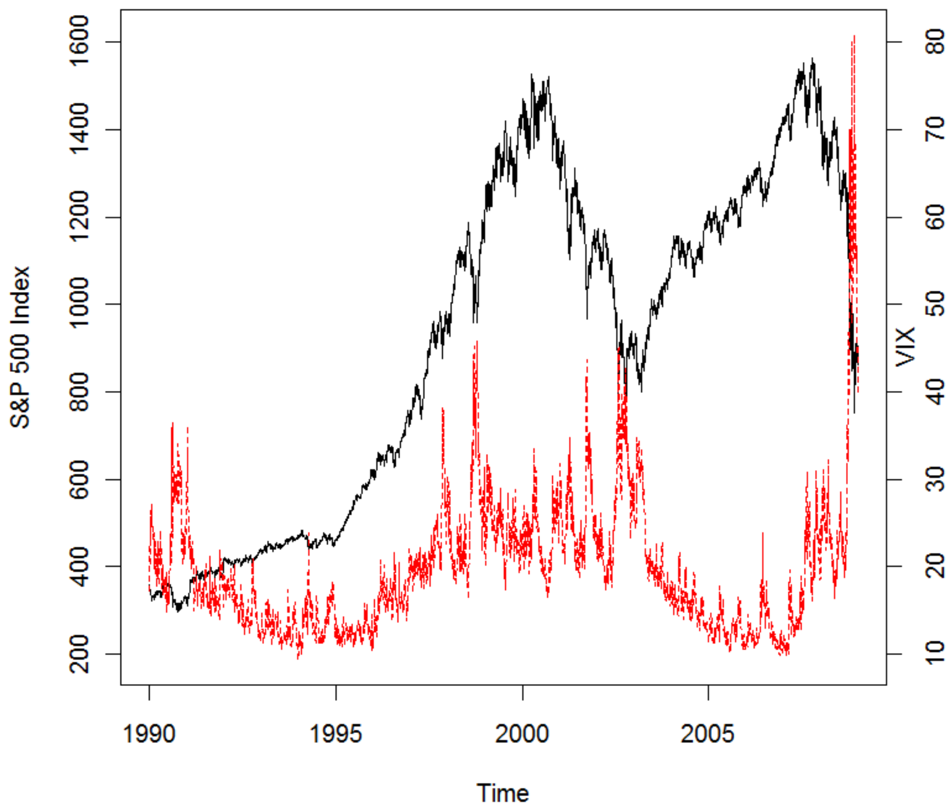
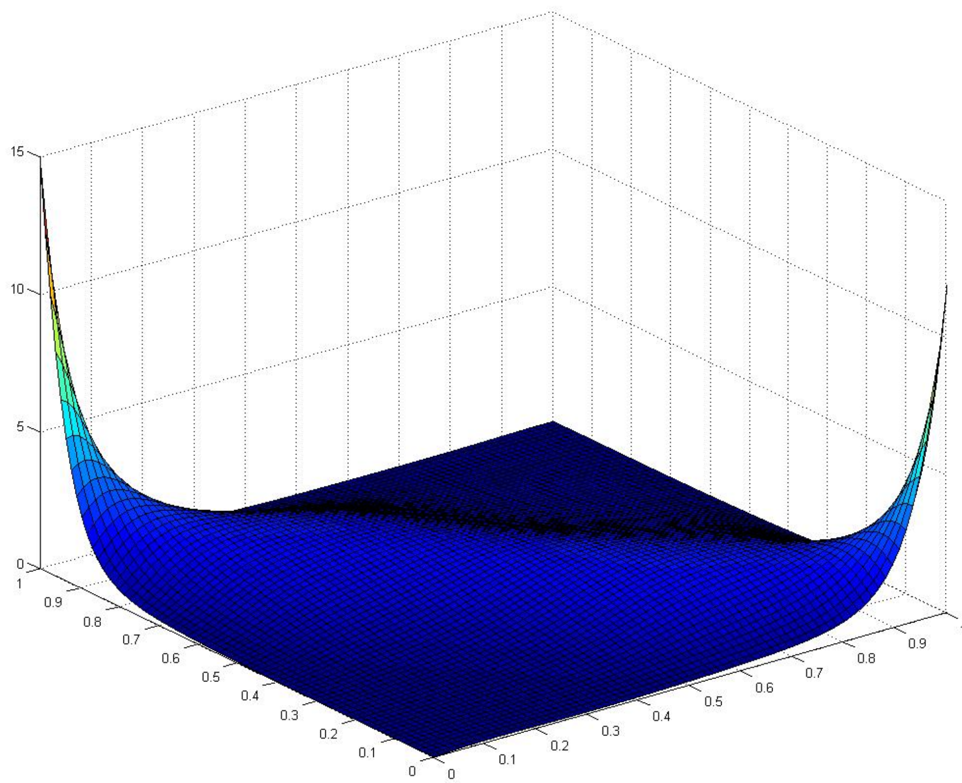


Figure 2: Estimated Joint Copula Density Function: $c(u, v)$



The joint copula density humps up along the u+v=1 diagonal of the [0,1]x[0,1] space

Figure 3: Empirical Conditional Probability Curves of u Given $v \in A$