Pricing Derivative Securities Using Cross-Entropy: An Economic Analysis

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JEL: C61, G12, G13

Key Words: Equivalent Martingale Measure, Stochastic Discount Factor, Cross-Entropy, Implied Distributions, Option Pricing

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1 Motivation

In a complete and arbitrage-free market, derivatives can be priced by constructing a replicating portfolio and applying the principle of no-arbitrage to conclude that the price of the contingent claim is equal to the price of the replicating portfolio. If the market is incomplete, there are non-redundant derivatives for which no replicating portfolio exists. For these derivatives, the principle of no arbitrage no longer results in a unique price, but in some upper and lower bounds for the price which can be determined by using super-hedging strategies (see e.g. El Karoui, Quenez [9]). The pricing function, that is the mapping of future payoffs to current prices, is not unique.

Starting from the price processes of the basis assets, the market is incomplete if the number of traded assets is too small compared to the number of risk factors. Examples include models with stochastic volatility (e.g. Heston [18]) or models with jumps (e.g. Bakshi, Cao, Chen [3]), but also short rate models for interest rate derivatives (e.g. Vasicek [25] or Cox, Ingersoll, Ross [7]). Instead of specifying the price processes of the basis assets, we can also start from observed market prices and calibrate the pricing function to these market prices. In general, the number of observed prices will be too small to obtain a unique pricing function, and we face the problems of market incompleteness here as well.

So in an incomplete market there is a whole set of candidate pricing functions which are consistent with the observable market information. This information includes the prices or price processes of basis assets and perhaps the market prices of some traded derivatives. Given only the principle of no arbitrage, there is no "most natural" pricing function in this set just like there is no "most natural" value in some set of real numbers.[1] Indeed, every pricing function in this set may be the true one used by the market.

Implied methods solve the problem of non-uniqueness by specifying criteria for choosing one pricing function out of the set of candidate pricing functions. Rubinstein [23] for example suggests to choose the equivalent martingale measure (EMM) having minimal quadratic distance from a given benchmark measure. Jackwerth, Rubinstein [20] maximize the smoothness of the risk-neutral distribution. Cont [6] and Jackwerth [19] review some classes of these implied methods.

The use of cross-entropy to choose an EMM is proposed by Rubinstein [23]. Buchen, Kelly [5] and Stutzer [24] provide an implementation. Gulko [14, 15, 16, 17] gives some intuition for this method and applies it to stock and bond options. Avellaneda et al. [1] combine cross-entropy with a Monte-Carlo simulation. Avellaneda [2] considers the sensitivity of this approach with respect to the given information, and Frittelli [10] analyzes the relation between the use of cross-entropy and a portfolio planing problem, assuming an exponential utility function.

[1] I thank Walter Schachermayer for this remark.

The methods proposed in these articles rely on the choice of some EMM. For this, a numeraire has to be chosen first. In the literature this choice of the numeraire is usually not discussed in detail, perhaps based on the implicit assumption that the choice of numeraire is irrelevant for the resulting pricing function. However, as will be shown below, this is not necessarily the case.

Consider for example the method proposed by Rubinstein [23]. He chooses the risk-neutral measure with minimal quadratic distance from a discrete counterpart of the log-normal distribution, where the use of the log-normal distribution is motivated by the model of Black-Scholes. Changing this method slightly by using the stock instead of the money market account as the numeraire results in a different pricing function. This impact of the numeraire can also be observed if the EMM with minimal cross-entropy relative to some benchmark measure is chosen, as we show in a simple one-period example in section 2.

In this paper, we explicitly analyze the role of the numeraire. We show that it has an impact on the resulting pricing function, and we also discuss the economic intuition for this result. Based on the findings of this analysis, we propose two new methods which avoid the dependence on the numeraire, and which have a sound economic motivation.

An impact of the numeraire is not consistent with the information-theoretical motivation of cross-entropy. It is claimed (e.g. by Gulko [14]) that the chosen pricing function is the least-prejudiced one, i.e. the pricing function which requires the fewest additional assumptions. For this to be true, the chosen pricing function should only depend on the given data and on an explicitly specified benchmark. However, due to the impact of the numeraire, the pricing function depends on a subjective and arbitrary element, and is indeed not the least-prejudiced one.

Further, we are able to provide an economic interpretation of this impact. Minimizing cross-entropy corresponds to maximizing expected exponential utility. The impact of the numeraire can be explained by noting that the utility function is not applied to wealth itself, but to wealth expressed in units of the numeraire. While this provides an economic explanation for the role of the numeraire, it does not give an economic *justification* for this role. Even worse, the fact that we have to work with normalized wealth explicitly shows that the role of the numeraire is economically counter-intuitive.

The second contribution of this paper is the development of two new methods which avoid this numeraire dependence. The main modification compared to existing approaches is that the criterion for the choice of a certain pricing function is not applied to EMMs, but to stochastic discount factors (henceforth SDFs) or to Arrow-Debreu (henceforth AD) prices. We show how to generalize the concept of cross-entropy, which can only be applied to probability measures, to the concept of extended cross-entropy, which can also be applied to random variables only satisfying a positivity restriction. We also show that it does not matter whether the new method is applied to SDFs or to AD prices, because the resulting pricing functions coincide.

There is still one degree of freedom in the approach, since the pricing function depends on the benchmark model which gives the benchmark pricing function. Like the numeraire and the benchmark measure, this benchmark model can basically be chosen arbitrarily. As the market is incomplete, there is simply no way around making a subjective choice. The main difference between this method and the approaches discussed above is that the subjective choice now relates to a purely economic object, namely to a pricing function itself. There are many more economic arguments for the choice of a benchmark pricing function than there are economic arguments for the selection of a benchmark probability measure or a numeraire.

The remainder of this paper is organized as follows. In section 2 we specify the general model setup and discuss the use of cross-entropy. We show that the resulting pricing function depends on the numeraire and discuss the implications of this finding for the economic justification of the method. In section 3 we propose a new implied method which avoids this dependence on the numeraire by choosing a stochastic discount factor or by choosing Arrow-Debreu-Prices. Section 4 concludes.

2 EMM with Minimal Cross-Entropy

2.1 General Setup

We consider an economy with equally spaced trading dates $t = 0, 1, \ldots, T$ and a finite state space. There are n basis assets. The price processes of these assets are given exogeneously, the price of asset i at time t is denoted by $S^i(t)$. For ease of notation, we combine the asset prices in the vector $S(t) = (S^1(t), \ldots, S^n(t))'$ where the prime symbol denotes transposition.

A numeraire is a traded asset or a self-financing portfolio whose price is strictly positive at all points in time with probability one. We assume that there exists at least one such numeraire. If N is the price process of the numeraire, the normalized prices are defined via

$$S^i_N(t) = \frac{S^i(t)}{N(t)}.$$

Fixing a numeraire N, we can describe a self-financing trading strategy by the initial capital c and the predictable process $\{\gamma(t)\}_{t=1,\ldots,T}$ where $\gamma^i(t)$ denotes the number of units of asset i held in the portfolio from time $t-1$ to time t. The trading-strategy is self-financing by construction if we choose an appropriate position in the numeraire. The value of this portfolio at time t is

$$V(t) = N(t)\Big(\frac{c}{N(0)} + \sum_{u=1}^{t} \gamma(u)'\Delta S_N(u)\Big)$$

where $\Delta S_N(u) = S_N(u) - S_N(u-1)$. The set of all payoffs at time T that can be attained with an initial capital equal to c by following a self-financing trading strategy is denoted by $\mathcal{G}(0,T,c)$. The set of payoffs attainable with an arbitrary initial capital is given by $\mathcal{G}(0,T) = \bigcup_{c\in\mathbb{R}} \mathcal{G}(0,T,c)$.

In the following, we assume that the market is incomplete. In this case, there are non-redundant contingent claims whose payoff at time T is not in the space $\mathcal{G}(0,T)$ of attainable payoffs. For such a non-redundant contingent claim, there is no replicating portfolio, and the principle of no arbitrage does not give a unique price for the claim. The pricing function, that is the mapping of future payoffs on prices today, is not unique. Instead, there is a whole set of candidate pricing functions, each of which correctly prices the basis assets. None of these pricing functions is the "most natural one" when it comes to pricing, just like there is no "most natural" value in some set of real numbers. In order to choose one of these pricing functions, some additional criteria must be applied. In the following, we consider criteria that start from some benchmark and try to deviate as little as possible from this benchmark.

2.2 Applying Cross-Entropy to Equivalent Martingale Measures

We first discuss the choice of the EMM having minimal cross-entropy with respect to a given benchmark measure P. Each candidate pricing function can be represented by an EMM P^N for a numeraire N. The measure P^N is equivalent to P, denoted by $P^N \sim P$, meaning that the tow measures have identical null sets. Under P^N, the prices of all basis assets, normalized by the numeraire N, are martingales:

$$S^i_N(t) = E^{P^N}\big[S^i_N(u) \bigm| \mathcal{F}_t\big] \qquad 0 \le t \le u \le T, \qquad i = 1, \dots, N.$$

The set of all candidate EMMs is

$$\mathcal{P}(N,P) = \big\{Q \bigm| \{S_N(t)\}_{t=0,\dots,T} \text{ is a Q-martingale}, Q \sim P\big\}.$$

The benchmark measure P is in general not an element of this set, as it is in general not an EMM for the numeraire N.

The measures in $\mathcal{P}(N,P)$ price the given basis assets and all self-financing portfolios correctly. Each of these measures represents one candidate pricing function. The price that the candidate pricing function represented by some P^N assigns to a contingent claim C maturing at time t can be calculated as

$$C(0) = N(0)\, E^{P^N}\Big[\frac{C(t)}{N(t)}\Big]. \tag{1}$$

Choosing a pricing function is equivalent to choosing an EMM out of the set $\mathcal{P}(N, P)$. This set depends on the measure P (determining the states having non-zero probability) and on the numeraire N. We will discuss the choice of the numeraire N below.

The principle of no arbitrage and the price processes of the basis assets determine the set $\mathcal{P}(N, P)$. However, the principle of no arbitrage does not tell us which element from $\mathcal{P}(N, P)$ is the true one used by the market. In order to choose one of these measures, we need some additional criteria. One criterion proposed in the literature is to choose the measure that deviates as little as possible from a given benchmark measure P. This can for example be formalized by choosing the measure with minimal cross-entropy relative to P. If P is the uniform distribution, this approach is equivalent to choosing the EMM having maximal entropy, so that the criterion of maximal entropy is nested within the approach discussed here (see, e.g., Buchen, Kelly [5]).

The cross-entropy of a probability measure Q with respect to a probability measure P is defined as (see, e.g., Kapur, Kesavan [22] or Golan, Judge, Miller [12])

$$H(Q\,|\,P) \;=\; \begin{cases} \sum_{\omega\in\Omega} Q(\omega) \ln \frac{Q(\omega)}{P(\omega)} & Q \text{ is absolutely continuous w.r.t. } P \\ \infty & \text{otherwise} \end{cases}$$

Cross-entropy is equal to zero if $Q = P$, and it is strictly greater than zero otherwise. In an intuitive sense, it is the greater the more the measure Q deviates from the measure P. Interpreting the deviation of Q from P as the result of additional restrictions (which are met by Q, but not by P), cross-entropy is the greater the more new 'information' is contained in Q relative to P.

The problem of choosing an EMM can now be stated as

$$\begin{aligned} &\min H(P^N|P) \\ &\text{s.t. } P^N \in \mathcal{P}(N, P) \end{aligned} \tag{2}$$

Before solving this problem, we briefly pause to discuss the motivation for the use of cross-entropy. First note that if $P \in \mathcal{P}(N, P)$, the criterion obviously chooses P. In general, however, the benchmark measure will not be an EMM. To meet the restrictions imposed by the price processes of the basis assets, it is necessary to go from P to a different measure $P^N \in \mathcal{P}(N, P)$. A possible objective can then be to use only the information given in the data without adding any further information. Together with the interpretation of cross-entropy given above this results in choosing the measure with minimal cross-entropy relative to the benchmark measure P. Because this method does not impose any additional restrictions beyond that justified by the data, it is claimed to be the least prejudiced way to choose the EMM.

The solution to the optimization problem can be found using a Lagrange approach. It is given for example by Stutzer [24]:

Proposition 1 (Minimizing cross-entropy of the equivalent martingale measure) *Let $\mathcal{P}(N, P)$ denote the set of all candidate equivalent martingale measures for the numeraire N. The benchmark measure is P. Then the minimal cross-entropy measure $P^{N,MCE}$ solving the problem*

$$\begin{aligned} &\min H(P^N|P) \\ &\text{s.t. } P^N \in \mathcal{P}(N, P) \end{aligned}$$

can be calculated as

$$P^{N,MCE}(\omega) = \exp\big\{X_N(T,\omega)\big\}P(\omega)$$

where $X(T) \in \mathcal{G}(0,T)$ is an appropriately chosen payoff at time T and $X_N(T)$ is the normalized payoff.

2.3 Impact of the numeraire on the pricing function

When choosing a pricing function by choosing an EMM, one has to decide on the numeraire first. Normally, this happens in a rather ad-hoc way, without any economic motivation. This procedure is based on the implicit assumption that the numeraire has no impact on the resulting pricing function, just in analogy to the fact that changing the numeraire in calculating the price of a contingent claim has no impact on the price. While the latter proposition is certainly true and refers to the well-known change of numeraire (see e.g. Geman, El Karoui, Rochet (1995) [11]), the former implicit assumption is not met: It turns out that the numeraire indeed has an impact on the measure chosen, and that it also has an impact on the resulting pricing function.

In order to show the impact of the numeraire, we represent the chosen pricing function by a stochastic discount factor (SDF). Let $SDF(0,T)$ denote the SDF at time 0 for pricing contingent claims with payoff at time T. The pricing equation for a payoff $C(T)$ at time T is

$$C(0) = E^P\big[SDF(0,T)C(T)\big]. \tag{3}$$

For pricing a payoff $C(t)$ occurring at time t, we invest $C(t)$ from t to T into the numeraire, which gives a payoff equal to $C(t)\frac{N(T)}{N(t)}$ at time T. Plugging this into equation (3) yields

$$C(0) = E^P\left[SDF(0,T)C(t)\frac{N(T)}{N(t)}\right]. \tag{4}$$

Comparing equations (1) and (4), the relation between an EMM for the numeraire N and an SDF representing the same pricing function is

$$SDF(0,T) = \frac{P^N}{P} \frac{N(0)}{N(T)}.$$

When we use the numeraire N, the pricing function chosen by proposition 1 can be represented by the SDF

$$SDF^N(0,T) = \frac{N(0)}{N(T)} \exp\left\{X_N(T)\right\} \tag{5}$$

where the payoff $X \in \mathcal{G}(0,T)$ and possibly also the SDF depend on the numeraire N. When the numeraire M is used, the chosen pricing function can be represented by

$$SDF^M(0,T) = \frac{M(0)}{M(T)} \exp\left\{Y_M(T)\right\}$$

where $Y \in \mathcal{G}(0,T)$. The two pricing functions are identical when the SDFs are identical. For the numeraire not to have any impact, the following condition must therefore be true:

$$\frac{N(0)}{N(T)} \exp\left\{X_N(T)\right\} = \frac{M(0)}{M(T)} \exp\left\{Y_M(T)\right\}.$$

Solving this condition for $Y(T)$ gives

$$Y(T) = M(T)X_N(T) \ln \frac{N(0)M(T)}{N(T)M(0)},$$

so that the two pricing functions can only be identical if the term on the right hand side is a traded payoff. This condition is certainly true in a complete market where the space $\mathcal{G}(0,T)$ of attainable payoffs coincides with the space of $\mathcal{F}_T$-measurable random variables and where the pricing function is unique. It is not necessarily true in an incomplete market, as the following example will show:

Example: Consider a one-period model with three states $\omega_1, \omega_2, \omega_3$. There are two basis assets, N and M, whose payoffs at time 1 are given as follows:

$$\begin{array}{lll} N(1,\omega_1) = 2 & N(1,\omega_2) = 1 & N(1,\omega_3) = 0.5 \\ M(1,\omega_1) = 1.25 & M(1,\omega_2) = 1.25 & M(1,\omega_3) = 1.25. \end{array}$$

$N(1,\omega)$ denotes the payoff of asset N at time 1 in state ω. The prices of both basis assets at time $t = 0$ are equal to one:

$$N(0) = M(0) = 1$$

Both N and M can serve as a numeraire. The benchmark probability measure P is

$$P(\omega_1) = 0.25 \qquad P(\omega_2) = 0.5 \qquad P(\omega_3) = 0.25$$

The market is incomplete, since there are three states of nature, but only two traded assets. Using N as a numeraire, the pricing function chosen by minimizing cross-entropy is given by

$$
\begin{aligned}
SDF^N(0,1,\omega_1) &= 1.104671 \\
SDF^N(0,1,\omega_2) &= 0.742994 \\
SDF^N(0,1,\omega_3) &= 0.609342.
\end{aligned}
$$

Using M as a numeraire, we obtain the pricing function

$$
\begin{aligned}
SDF^M(0,1,\omega_1) &= 0.992303 \\
SDF^M(0,1,\omega_2) &= 0.911545 \\
SDF^M(0,1,\omega_3) &= 0.384606.
\end{aligned}
$$

Since the SDFs are not identical, the pricing functions are also different. □

This result has several important consequences. First, the pre-selection of the numeraire should not happen in an ad-hoc manner, but the numeraire should be chosen on the basis of a carefully developed criterion. The problem is that at this point there is simply no economic argument telling us what numeraire to use. Second, the use of cross-entropy is motivated by considering cross-entropy as the least-prejudiced way to choose an EMM. It would therefore also seem the least prejudiced way to choose a pricing function. The dependence of the resulting pricing function on the numeraire shows that this is not true: Minimizing cross-entropy of the EMM is not the least-prejudiced way to choose a pricing function, but the pre-selection of the numeraire introduces an element of arbitrariness.

The last argument against the selection of an EMM points towards the main problem of this approach, the necessity to choose a numeraire. This necessity is a consequence of representing pricing functions by EMMs. We seem to apply a sound criterion (least prejudiced choice) to the wrong object (equivalent martingale measure).

2.4 An Equivalent Portfolio Planning Problem

The use of cross-entropy to choose an EMM is often motivated by the information-theoretical argument of being the least prejudiced way to choose a measure. Besides this argument, there is also an economic motivation for the use of cross-entropy as a selection criterion. Indeed, the pricing function chosen by minimizing cross-entropy coincides with the pricing function resulting from the solution of a portfolio planning problem. The analysis of this problem will help to clarify the role of the numeraire, and it will show that this role of the numeraire is rather counter-intuitive.

The relationship between the choice of an EMM and portfolio planing problems is also considered, among others, by Frittelli [10], Kallsen [21] and Goll, Rüschendorf [13]. Different from the analysis here, they do not discuss the role of the numeraire, but start directly from discounted prices without saying anything about the choice of the numeraire.

For the portfolio planning problem, consider an investor whose initial wealth is equal to $W(0)$ and who chooses the portfolio that maximizes his expected utility. Non-redundant derivatives can then be priced by an indifference condition (see e.g. Davis [8]): the prices of these derivatives are set in such a way that the investor will neither want to buy nor to sell them. Put differently, the non-redundant derivatives are priced in such a way that the utility of the investor does not increase when they are introduced at these prices.

In order to obtain the same pricing function as by minimizing cross-entropy, we have to make several assumptions. We assume that expected utility is calculated under the benchmark measure P. We consider an exponential utility function with constant absolute risk aversion equal to one. Furthermore, we assume that this utility function is not applied to terminal wealth, but to terminal wealth expressed in units of the numeraire. The portfolio planing problem therefore is

$$\begin{aligned} &\max E^P[-e^{-W_N(T)}] \\ &\text{s. t. } W_N(T) = W_N(0) + \sum_{t=1}^{T} \gamma(t)' \Delta S_N(t). \end{aligned}$$

The resulting pricing function can be derived from the first order conditions. This yields

$$C(0) = E^P\Big[\frac{e^{-W_N^\star(T)}}{E^P[e^{-W_N^\star(T)}]}\,\frac{C(T)}{N(T)}\,N(0)\Big]$$

where $W^\star(T)$ denotes the optimal terminal wealth. Representing this pricing function by an SDF gives

$$SDF^N(0,T) = \frac{e^{-W_N^\star(T)}}{E^P[e^{-W_N^\star(T)}]}\,\frac{N(0)}{N(T)} \tag{6}$$

where the superscript N again denotes dependence on the numeraire. Comparing (6) to (5) shows that the pricing functions coincide for

$$W_N^\star(T) = W_N(0) + X_N(0) - X_N(T).$$

Note that there is no restriction on initial wealth $W(0)$. The pricing function chosen by minimizing cross-entropy can also be derived within a portfolio planning problem. This provides an economic motivation for the criterion "minimal cross-entropy" and yields some economic interpretation to the assumptions this criterion makes on the market pricing process.

The impact of the numeraire on the pricing function can now be explained via the solution of the portfolio planing problem. Note that we are not maximizing utility of terminal wealth, but utility of terminal wealth expressed in units of the numeraire. It is exactly this modification of the standard portfolio planning problem that is the economic reason for the dependence on the numeraire. This modification, however, does not provide any economic *justification* for this dependence. It only shows that the dependence is the result of some quite unusual specification of the portfolio problem. It is difficult to come up with a sound economic reason for normalizing wealth before calculating utility. And again, the choice of the numeraire is completely ad-hoc. Putting arguments together, the choice of the EMM having minimal cross-entropy with respect to a given benchmark measure P suffers from the dependence on the numeraire.

It also suffers from the choice of the benchmark probability measure P. This measure is the measure used to calculate expected utility. Again, the question is how to choose the benchmark measure. For example, the measure could be based on a historical probability distribution. In this case, we choose the EMM that is as close as possible to the historical distribution. Besides this seemingly natural choice, it is again difficult to give a sound economic argument for the objective of distance minimization.

Note that we could alternatively start from a benchmark pricing function. In this case, the first step is to represent this benchmark pricing function by a numeraire and the corresponding probability measure. The measure is then used as the benchmark, and the algorithm proceeds just as described above. Details are given in Branger [4]. The approach provides an economic argument for the choice of the benchmark measure and may thus be considered superior to the previously described method. Unfortunately, the resulting pricing function again depends on the numeraire.

3 Criteria Relying on the Use of Extended Cross-Entropy

In an incomplete market, some additional criterion is needed for choosing one out of many pricing functions. As we have seen the choice of the EMM with minimal cross-entropy relative to some benchmark probability measure does not solve the problem.

We suggest two modifications of this method. The first relates to the observation that the use of EMMs is only one out of several possibilities to represent a pricing function. As this possibility suffers from the necessity to pre-select a numeraire, we apply the selection criterion "minimal cross-entropy" to two other possible representations of the pricing function, namely to AD prices and to SDFs.

The second modification relates to the choice of the benchmark. Since we are interested in choosing a pricing function, it seems quite natural to start from a benchmark pricing function instead of starting from a probability measure and a numeraire that is chosen in an ad-hoc way.

In order to implement these modifications, we have to use a different objective function. Cross-entropy can only be applied to probability measures. We therefore generalize the concept of cross-entropy to what we call extended cross-entropy, which can additionally be applied to random variables which just satisfy a positivity constraint.

3.1 Extended Cross-Entropy

The extended cross-entropy of a positive random variable X with respect to some other positive random variable Y, using a probability measure P, is derived from the cross-entropy of some artificial probability measure P^X with respect to some artificial probability measure P^Y. We define

$$\widetilde{H}_P(X \mid Y) := H(P^X \mid P^Y).$$

$\widetilde{H}_P$ denotes extended cross-entropy calculated using some measure P, and H denotes the cross-entropy of P^X with respect to P^Y. The artificial probability measures P^X and P^Y are defined by the Radon-Nikodym derivatives

$$\frac{dP^X}{dP} := \frac{X}{E^P[X]}, \qquad \frac{dP^Y}{dP} := \frac{Y}{E^P[Y]}.$$

The definition of these artificial probability measures not only depends on the random variables X and Y, but also on an auxiliary probability measure P, and so does extended cross-entropy. We will discuss below whether this dependence causes problems similar to those created by the dependence on the numeraire for the methods discussed in section 2.

3.2 Applying Extended Cross-Entropy to Arrow-Debreu Prices

We first apply the selection criterion to AD prices instead of EMMs. Using AD prices, the pricing equation for a contingent claim with maturity u becomes

$$C(0) = \sum_{\omega \in \Omega} AD(0, T, \omega) C(u, \omega) \frac{N(T, \omega)}{N(u, \omega)}$$

where $AD(0, T, \omega)$ denotes the AD price at time $t = 0$ for state ω at time $t = T$. As in equation (4), the intuition is to invest a payoff occurring at time u up to time T into the numeraire, and then to use the AD prices to price the resulting payoff at time T.

The set of all candidate AD prices which price the basis assets correctly is denoted by

$$\mathcal{AD} = \Big\{ AD(0,T,\cdot) \;\Big|\; \begin{array}{l} S^i(0) = \sum_{\omega \in \Omega} AD(0,T,\omega) S^i(t,\omega) \frac{N(T,\omega)}{N(t,\omega)}, t = 1, \dots, T, i = 1, \dots, n, \\ N(0) = \sum_{\omega \in \Omega} AD(0,T,\omega) N(T,\omega), \\ AD(0,T,\omega) > 0 \Big\}. \end{array}$$

To choose one out of these admissible pricing function, we start from a benchmark pricing function which is represented by $AD^{prior}(0,T)$. We then choose the AD from $\mathcal{AD}$ which have minimal extended cross-entropy relative to $AD^{prior}(0,T)$. Extended cross-entropy is calculated using the uniform distribution U as the auxiliary measure, a choice that can again be motivated by the objective to select the pricing function in the least prejudiced way: Without any information that could restrict the auxiliary measure, there is no reason to use any other distribution than the uniform distribution.

To calculate extended cross-entropy, we convert the AD prices into artificial probabilities. It is important to note that these are indeed artificial probability measures. Above all, they must not be confused with EMMs. The method and the resulting pricing function are summarized in

Proposition 2 (Minimizing extended cross-entropy of AD prices) *Let $\mathcal{AD}$ denote the set of all candidate AD prices. Then the minimal extended cross-entropy AD prices AD^{MECE} solving the problem*

$$\begin{aligned}
&\min \widetilde{H}_U(AD(0,T) \mid AD^{prior}(0,T)) \\
&\text{s.t. } AD(0,T) \in \mathcal{AD}
\end{aligned}$$

with U representing the uniform distribution are given by

$$AD^{MECE}(0,T) = AD^{prior}(0,T) \exp\left\{\lambda(0) + X(T)\right\}.$$

Here $\lambda(0)$ is an appropriately chosen constant, and $X(T) \in \mathcal{G}(0,T,0)$ is an appropriately chosen attainable payoff.

The proof is similar to the proof of the next proposition. Thus, only the latter proof will be given in the appendix.

The resulting AD prices do not depend on any numeraire but only on the benchmark pricing function. So the choice of the AD prices with minimal extended cross-entropy relative to the benchmark AD prices avoids the severe problems shown above for the choice of EMMs.

3.3 Applying Extended Cross-Entropy to Stochastic Discount Factors

The candidate pricing functions can not only be represented by AD prices, but also by SDFs. The set of all candidate SDFs is

$$\begin{aligned}
\mathcal{SDF}(P) = \Big\{SDF(0,T) \;\Big|\; & S^i(0) = E^P\Big[SDF(0,T)S^i(t)\tfrac{N(T)}{N(t)}\Big], t = 1,\ldots,T, i = 1,\ldots,n, \\
& N(0) = E^P[SDF(0,T)N(T)], \\
& SDF(0,T,\omega) > 0\Big\}.
\end{aligned}$$

This set again depends on the auxiliary measure P.

As above, we start from a benchmark pricing function that is now represented by the benchmark stochastic discount factor $SDF^{P,prior}(0,T)$. We choose the SDF out of $\mathcal{SDF}(P)$ which has minimal extended cross-entropy relative to the benchmark SDF. Extended cross-entropy is calculated using the auxiliary distribution P which is also used in the definition of SDFs. Again, the motivation for the use of this distribution is to choose the pricing function in the least prejudiced way. So we do not introduce any new distribution here, but stick with the one already used for the SDFs. The method and the resulting pricing function are summarized in

Proposition 3 (Minimizing extended cross-entropy of SDFs) *Let $\mathcal{SDF}(P)$ denote the set of all candidate SDFs. The minimial extended cross-entropy stochastic discount factor $SDF^{P,MECE}$ solving the problem*

$$\begin{aligned} &\min \widetilde{H}_P(SDF(0,T) \mid SDF^{P,prior}(0,T)) \\ &\text{s.t. } SDF(0,T) \in \mathcal{SDF}(P) \end{aligned}$$

is given by

$$SDF^{P,MECE}(0,T) = SDF^{P,prior}(0,T) \exp\left\{\lambda(0) + X(T)\right\}$$

where $\lambda(0)$ is an appropriately chosen constant and $X(T) \in \mathcal{G}(0,T,0)$ is an appropriately chosen attainable payoff.

The proof is given in appendix A.

The chosen stochastic discount factor $SDF^{P,MECE}(0,T)$ does not depend on any numeraire. However, the SDF depends on a probability measure that has to be pre-specified. So it could be that we have avoided the dependence on the numeraire at the cost of having introduced a dependence on an auxiliary probability measure. Fortunately, it turns out that the ad-hoc choice of P only has an impact on the SDF, but has no impact on the resulting pricing function which is exactly in line with what we wanted to achieve. Indeed, the pricing function coincides with the pricing function we obtain when applying the new method introduced above in proposition 2 to AD prices.

Proposition 4 (Equivalence of choosing AD prices and SDFs) *Assume that some benchmark pricing function is given. It can be represented by the Arrow-Debreu prices $AD^{prior}(0,T)$ or by the stochastic discount factor $SDF^{P,prior}(0,T)$ together with the measure P. Then, the pricing function chosen by solving the optimization problem*

$$\begin{aligned} &\min \widetilde{H}_U(AD(0,T) \mid AD^{prior}(0,T)) \\ &\text{s.t. } AD(0,T) \in \mathcal{AD} \end{aligned} \tag{7}$$

coincides with the pricing function chosen by solving the optimization problem

$$\begin{aligned} &\min \widetilde{H}_P(SDF(0,T) \,|\, SDF^{P,prior}(0,T)) \\ &\text{s.t. } SDF(0,T) \in \mathcal{SDF}(P) \end{aligned} \tag{8}$$

In order to prove this proposition, we represent the pricing function solving problem (8) by AD prices. As the SDF is just the AD price per unit of probability, we obtain from proposition 3

$$AD^{P,MECE}(0,T) = AD^{prior}(0,T) \exp\big\{\lambda(0) + X(T)\big\}. \tag{9}$$

As proposition 2 shows, this is exactly the pricing function chosen by solving (7).

Thus, the two criteria introduced in proposition 2 and proposition 3 yield the same pricing function. Therefore, the pricing function chosen by minimizing the extended cross-entropy of the SDF does not depend on the auxiliary measure P. Comparing this to the choice of the EMM by minimizing cross-entropy we can summarize the results: The ad-hoc choice of the numeraire has an impact on the resulting pricing function when choosing the EMM as in proposition 1, while the ad-hoc choice of the auxiliary measure has no impact when choosing the SDF as in proposition 3. The method introduced in proposition 3 is therefore less prejudiced than the method discussed in proposition 1 when choosing a pricing function. So, instead of choosing an EMM, we should choose an SDF.

It is also worth noting that it does not matter whether we apply the concept of extended cross-entropy to AD prices or to SDFs. The resulting pricing functions coincide. So we do not introduce an element of arbitrariness by deciding whether to represent the candidate pricing functions via AD prices or via SDFs.

3.4 Impact of the benchmark model on the pricing function

In summary, the resulting pricing function chosen either by proposition 2 or by proposition 3 neither depends on a numeraire nor on an auxiliary probability measure. For the two methods just discussed, the pricing function only depends on the benchmark model. In contrast to the choice of a numeraire, the choice of the benchmark model can be motivated using economic arguments.

The first possibility is to use a theoretical option pricing model and to calibrate this model to a given data set. In most cases, the model will not price all basis correctly. The method proposed here can then be interpreted as a way to modify the pricing function such that it prices all basis assets correctly, but deviates as little as possible from the theoretical option pricing model (Avellaneda et al. [1]). The second possibility arises in a setup where the pricing function is calibrated to market prices every day. Here, a natural choice for the actual benchmark pricing function is the pricing function used the day before.

From an economic point of view, the dependence on a benchmark pricing function is quite different from the dependence on a numeraire. The key point is that markets are incomplete. Therefore, the principle of no arbitrage does not suffice to give a unique price for all contingent claims. In order to derive a unique price for non-redundant claims also, there is simply no way around making some additional assumptions. Of course, there are quite different ways to make these assumptions, and that is the point where the methods discussed in sections 2 and 3 differ from each other: The method from section 2 makes an assumption about a benchmark measure and additionally about the numeraire to use. Both assumptions are at least difficult to interpret economically, if not counter-intuitive. The methods presented in this section make an assumption about the pricing process on the market. The impact of this assumption on the resulting pricing function is transparent, and we can control in a certain sense for the subjectivity of the procedure.

4 Conclusion

When the market is incomplete, there is a whole set of candidate pricing functions. Applying the principle of no-arbitrage only, there is no natural pricing function one should use to price non-redundant contingent claims. In order to select one of these pricing functions, some additional criteria have to be used. In this paper, we discuss criteria based on the principle of minimum distance from a benchmark.

In the literature this idea is applied by starting from a benchmark probability measure and choosing the EMM having minimal cross-entropy with respect to this benchmark measure. As we have shown, this method suffers from several shortcomings. The main difficulty is the dependence of the resulting pricing function on the numeraire, which is usually chosen without a sound economic reason. Also, the economic intuition relying on the portfolio planning problem does not help to solve this problem, but even shows that the role of the numeraire is quite counter-intuitive.

These findings create the necessity for developing new methods. We no longer use EMMs for representing pricing functions, but AD prices and SDFs. Thus, instead of starting from a benchmark measure, we start directly from a benchmark pricing function which can be seen as an economic benchmark for our problem. We introduce the criterion of minimal extended cross-entropy for choosing one pricing function. The two new methods are equivalent as the resulting pricing functions coincide. Indeed, the new methods choose a pricing function that only depends on the benchmark pricing function, but not on any weakly motivated ad-hoc choices.

The dependence on the benchmark pricing function cannot be avoided. It is no shortcoming of the method, but simply a characteristic of an incomplete market. Without any additional assumptions, it is not possible to choose a unique pricing function. However, there are many more economic arguments for the choice of a benchmark pricing function than there are economic arguments for the selection of a benchmark probability measure or a numeraire.

There are some open questions that are left for further research. First, one could think of implementing the different methods empirically in order to quantify the impact of the choice of the numeraire or the impact of the benchmark model. Second, the new method could be applied to interest rate models, where the choice of the numeraire is discussed much more extensively than in models for pricing equity derivatives.

Appendix

A Proof of Proposition 3

We have to solve the problem

$$\begin{aligned}&\min EH_P(SDF(0,T) \mid SDF^{P,prior}(0,T))\\&\text{s. t. } SDF(0,T) \in \mathcal{SDF}(P)\end{aligned}$$

The condition $SDF(0,T) \in \mathcal{SDF}(P)$ can be written as

$$\begin{aligned}&SDF(0,T,\omega) > 0 \qquad \forall \omega \in \Omega\\&E_P\left[SDF(0,T)N(T)\right] = N(0)\\&E_P\left[SDF(0,T)\Delta S_N^i(t)N(T)\right] = 0 \qquad i = 1,\dots,n, t = 1,\dots,T\end{aligned}$$

If a discount bond with maturity T is traded, the expectation of $SDF(0,T)$ has to be equal to the price of this bond. Otherwise, we add the restriction that this expectation is equal to some parameter a, and solve the problem under this additional restriction. The value of the objective function then depends on a, and in a second step, we minimize the objective function over a.

We start with the modified optimization problem where we have added the restriction

$$E^P[SDF(0,T)] = a$$

with $a > 0$.

The minimization problem that has to be solved first is

$$\min_{SDF(0,T)} \quad E^P\Big[\frac{SDF(0,T)}{a} \ln\Big(\frac{SDF(0,T)}{a}\,\frac{E^P[SDF^{prior}(0,T)]}{SDF^{prior}(0,T)}\Big)\Big] \tag{A.1}$$

$$s.t. \quad SDF(0,T) \geq 0 \tag{A.2}$$

$$E_P\left[SDF(0,T)N(T)\right] = N(0) \tag{A.3}$$

$$E^P\left[SDF(0,T)\right] = a \tag{A.4}$$

$$E^P\left[SDF(0,T)\Delta S_N^i(t)N(T)\right] = 0 \qquad i = 1,\dots,n, t = 1,\dots,T \tag{A.5}$$

Note that $E^P[SDF(0,T)] = a$ does not imply $E^P[SDF^{prior}(0,T)] = a$.

This problem can be solved using the Lagrange approach. The solution is

$$SDF(0,T) \;=\; a\,\frac{SDF^{prior}(0,T)\exp\left\{N(T)\left(\frac{\gamma_0}{N(0)} + \sum_{t=1}^{T}\lambda_t \Delta S_N(t)\right)\right\}}{E_P\left[SDF^{prior}(0,T)\exp\left\{N(T)\left(\frac{\gamma_0}{N(0)} + \sum_{t=1}^{T}\lambda_t \Delta S_N(t)\right)\right\}\right]} \tag{A.6}$$

Note that the parameter γ_0 and the predictable process $\{\lambda_t\}$ both depend on a.

The optimal value $Z(a)$ of the objective function also depends on a. Plugging (A.6) into (A.1) gives

$$\begin{aligned} Z(a) &= \frac{1}{a}\frac{\gamma_0}{N(0)} + \ln E^P\Big[SDF^{prior}(0,T)\Big] \\ &\quad - \ln E^P\Big[SDF^{prior}(0,T)\exp\big\{N(T)\big(\frac{\gamma_0}{N(0)} + \sum_{t=1}^{T}\lambda_t \Delta S_N(t)\big)\big\}\Big] \end{aligned}$$

In the second step, we minimize over a. Noting that λ and γ_0 depend on a, the first order conditions are

$$\begin{aligned} &\frac{\partial Z(a)}{\partial a} \\ &= \frac{a\,\frac{\partial \gamma_0}{\partial a}\,\frac{1}{N(0)} - \frac{\gamma_0}{N(0)}}{a^2} \\ &\quad - \frac{E^P\Big[SDF^{prior}(0,T)e^{N(T)\big(\frac{\gamma_0}{N(0)}+\sum_{t=1}^T \lambda_t \Delta S_N(t)\big)}\,N(T)\big(\frac{\partial\gamma_0}{\partial a}\,\frac{1}{N(0)} + \sum_{t=1}^T \frac{\partial \lambda_t}{\partial a}\Delta S_N(t)\big)\Big]}{E^P\Big[SDF^{prior}(0,T)e^{N(T)\big(\frac{\gamma_0}{N(0)}+\sum_{t=1}^T \lambda_t \Delta S_N(t)\big)}\Big]} \\ &= \frac{1}{N(0)}\Big(\frac{1}{a}\frac{\partial\gamma_0}{\partial a} - \frac{1}{a^2}\gamma_0 - \frac{1}{a}\frac{\partial \gamma_0}{\partial a}\Big) \\ &= -\frac{1}{N(0)a^2}\gamma_0 \end{aligned}$$

This term is zero for $\gamma_0 = 0$ so that the solution for the stochastic discount factor is

$$SDF(0,T) \;=\; a\,\frac{SDF^{prior}(0,T)\exp\big\{N(T)\sum_{t=1}^T \lambda_t \Delta S_N(t)\big\}}{E_P\big[SDF^{prior}(0,T)\exp\big\{N(T)\sum_{t=1}^T \lambda_t \Delta S_N(t)\big\}\big]}$$

Noting that the exponent is a payoff that is attainable starting with zero initial wealth, we finally get

$$SDF(0,T) \;=\; a\,\frac{SDF^{prior}(0,T)\exp\big\{X(T)\big\}}{E_P\big[SDF^{prior}(0,T)\exp\big\{X(T)\big\}\big]}$$

where $X(T) \in \mathcal{G}(0,T,0)$. We can now combine a and the denominator into the factor $\exp\{\lambda_0\}$. This yields

$$SDF(0,T) \;=\; SDF^{prior}(0,T)\exp\big\{\lambda_0 + X(T)\big\}$$

Using the second derivative, we can show that it is indeed a minimum.

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