

Tranching and Rating

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Abstract

In this paper we analyse the source and magnitude of marketing gains from selling structured debt securities at yields that reflect only their credit ratings, or specifically at yields on equivalently rated corporate bonds. We distinguish between credit ratings that are based on probabilities of default and ratings that are based on expected default losses. We show that subdividing a bond issued against given collateral into subordinated tranches can yield significant profits under the hypothesised pricing system. Increasing the systematic risk or reducing the total risk of the bond collateral increases the profits further. The marketing gain is generally increasing in the number of tranches and decreasing in the rating of the lowest rated tranche.

Keywords: credit ratings, collateralised debt obligations, expected loss rate, default probability, systemic risk

JEL classification: G12, G13, G14, G21, G24

1. Introduction

Approximately \$471 billion of the \$550 billion of collateralised debt obligations (CDOs) that were issued in 2006 were classified by the Securities Industry and Financial Markets Association (SIFMA) as ‘Arbitrage CDOs’,¹ which are defined by SIFMA as an ‘attempt

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¹ The remaining issuance is classified as ‘Balance Sheet’ CDOs which ‘remove assets or the risk of the assets off the balance sheet of the originator’.

to capture the mismatch between the yields of assets (CDO collateral) and the financing costs of the generally higher rated liabilities (CDO tranches).² In the simple world of Modigliani and Miller (1958) such arbitrage opportunities would not exist, which raises the issue of the source of the arbitrage gains in the markets for CDO's and other structured bonds.³ In this paper we present a simple theory of the effect of collateral diversification and the structuring or tranching of debt contracts on the prices at which debt securities can be marketed. The theory can account for the apparent arbitrage opportunities that were offered by the market for CDOs and the explosive growth of that market in the recent past. Of course, the long-term existence of untranching securitisations such as mortgage backed securities suggests that there are other sources of the marketing gains than the one we consider, such as liquidity enhancement.⁴

Our theory rests on the assumption that some investors are not able to assess for themselves the value of the debt securities issued by the special purpose vehicles holding collateral, but must rely instead on credit ratings provided by third parties. We shall make the extreme assumption that securities can be sold in the primary market at yields that reflect only their ratings. This is not to say that all investors rely only on credit ratings – but that at least some do, and that if ratings based valuations exceed fundamental values, then the investment banker will be able to sell to these investors in the primary market at prices that depend only on ratings. Our assumption is motivated by several considerations. First is the attention that has been focused on the role of ratings in the marketing of securities.⁵ Second is the concern expressed by the Securities and Exchange Commission⁶ 'that certain investors assumed the risk characteristics for structured finance products, particularly highly rated instruments, were the same as for other types of similarly rated instruments' and 'that some investors may not have performed internal risk analysis on structured finance products before purchasing them'.⁷ Third, this assumption provides a baseline for assessing the need for differentiated ratings for structured products as has been suggested by the SEC under Proposed New Rule 17g-7. Anecdotal evidence of reliance on ratings for risk assessment is provided by the *Financial Times* of 6 December 2007 which reports that 'for many investors ratings have served as a universally accepted benchmark' and 'some funds have rued their heavy dependence on ratings'.⁸ Regulators also rely on the reports of

² SIFMA, January 2008. http://archives1.sifma.org/assets/files/SIFMA_CDOIssuanceData2007q1.pdf

³ We generally follow the terminology of SIFMA and use the generic term CDO to refer to credit instruments issued against a portfolio of other credit instruments. There is a wide variety of CDO types which is discussed in more detail below.

⁴ See Subrahmanyam (1991) for a formal model.

⁵ The Treasurer of the State of California recently claimed that 'If the state of California received the triple-A rating it deserved, we could reduce taxpayers' borrowing costs by hundreds of millions of dollars over the 30-year term of the still-to-be issued bonds...'. Reuters, 12 March 2008. Moody's has agreed to provide municipalities with the equivalent of a corporate bond rating from May 2008; prior to this date default losses for municipal bonds were significantly below those of equivalently rated corporate bonds.

⁶ Federal Register, Vol 73, No. 123 page 36235, June 25, 2008.

⁷ Similar concerns were expressed by the Bank of England in its *Financial Stability Report of October 2007*.

⁸ Gorton (2009) also argues that 'investors purchased tranches of RMBS, CDOs, SIV liabilities, money market funds, and so on, and did so without knowing everything known

the rating agencies: 'As regulators, we just have to trust that rating agencies are going to monitor CDOs and find the subprime', said Kevin Fry, chairman of the Invested Asset Working Group of the US National Association of Insurance Commissioners. 'We can't get there. We don't have the resources to get our arms around it.' (*International Herald Tribune*, 1 June 2007). Even UBS, an investment bank that originated large numbers of structured securities and incurred substantial losses in the subprime crisis, refers to its own 'over-reliance on ratings' as a factor in its losses.⁹ Our assumption is consistent with the evidence of Cuchra (2005) who reports that ratings explain 70–80% of launch spreads on structured bonds in Europe, which he interprets as support for the 'theoretical prediction that some investors might base their pricing decisions almost exclusively on ratings'. He also finds that the importance of credit ratings in structured finance (yields) seems to be far greater than in the case of straight (corporate) bonds.

We do not argue that the marketing story we tell is the only explanation for the tranching of debt contracts.¹⁰ Previous contributions rely on asymmetric information and either the ability of the issuer to signal the quality of the underlying assets by the mix of securities sold,¹¹ or the differential ability of investors to assess complex risky securities. Thus, in Boot and Thakor (1993) cash flow streams are marketed by dividing them and allocating the resulting components to information insensitive and sensitive (intensive) securities. The former are marketed to uninformed investors, and the latter to information gathering specialist firms who face an exogenously specified deadweight cost of borrowing.¹² Other explanations include what Ross (1989) refers to as the 'old canard' of spanning.¹³

Our analysis is concerned with the limitations of a bond rating system which relies only on assessments either of default probabilities or of expected losses due to default. It is straightforward to show that a system which relies only on default probabilities is easy to game – by selling securities with lower recovery rates than the securities on which the ratings are based. Only slightly more subtly, a system which relies on expected default losses is also easy to game. This is because a simple measure of expected default loss takes no account of the states of the world in which the losses occur. The investment banker may profit then by selling securities whose default losses are allocated to states with the highest state prices per unit of probability.¹⁴ Rating agencies, by providing information about probabilities of default or expected default losses, are providing information about the *total risk* of the securities that they rate. Although it has been well known for over 40 years that equilibrium values must depend on measures of systematic rather than total risk, this insight has not so far affected the practices of the credit rating agencies. The failure of the credit rating agencies to recognise the distinction between total and systematic risk creates an opportunity for

by the structurers of the securities they were purchasing. These investors likely to rely on their repeated relationships with bankers and on ratings.'

⁹ Shareholder Report on UBS's Write-Downs, UBS, 18 April, 2008, page 39.

¹⁰ Ross (1989) has previously drawn attention to the marketing role of the investment banker for an institution that wishes to sell off some of its low grade assets. However, he does not consider the role of the credit rating agencies.

¹¹ Brennan and Kraus (1987), De Marzo and Duffie (1999), DeMarzo (2005).

¹² See also Plantin (2004) and Riddiough (1997).

¹³ See Gaur *et al.* (2005).

¹⁴ Coval *et al.* (2009) make a similar point.

investment banks to exploit by designing collateral and security characteristics to raise the systematic risk of the securities they issue above that of corporate securities with similar (total) default risk on which the credit ratings are based. We emphasise that our analysis does not rest on any assumption of bias or inaccuracy in the default probability and loss assessments which underly the ratings assigned by the agencies.

We assume that the underlying collateral against which the structured debt claims are written is properly valued. We also assume that bond ratings are calibrated with respect to single debt claims issued against a 'reference firm', by which we mean a firm with pre-specified risk characteristics. We then show under fairly general assumptions that under a rating system that is based either on default probabilities (e.g. Standard & Poor's and Fitch) or on expected default losses (e.g. Moody's) the optimal strategy for the issuer is to maximise the number of differently rated debt tranches. If the risk characteristics of the collateral can be chosen, then they will be chosen to have the maximum beta and the minimum idiosyncratic risk. A rating system that is based on expected losses (e.g. Moody's) reduces, but does not eliminate all of, the pricing anomalies and the issuer's marketing gains.

Our analysis is most closely related to that of Coval *et al.* (2009) who show that it is possible to exploit investors who rely on default probability based ratings for pricing securities, by selling bonds whose default losses occur in high marginal utility states. However, their theory has no explicit role for debt tranching as ours does. They use a structural bond pricing model to predict yield spreads on CDX index tranches and conclude that there is severe market mispricing: the market spreads are much too low for the risk of the tranches, and this is particularly true for the highly rated tranches. In contrast, our model suggests that highly rated tranches will be subject to the least mispricing, and that the highest marketing gains will come primarily from the junior tranches. This is consistent with the fact that UBS is reported to have retained the 'Super Senior' AAA-rated tranches of the CDO's it originated, while selling the junior tranches to third-party investors.¹⁵

Other important contributions include Longstaff and Rajan (2008) who estimate a multinomial Poisson process for defaults under the risk neutral density from the prices of CDO tranches, and Cuchra (2005) who provides empirical evidence on the determinants of initial offering spreads on structured bonds.

An important implication of the fact that tranching securities are typically written against diversified portfolios of securities is that defaults of tranching securities of a specified rating will tend to be much more highly correlated than defaults of securities of the same rating issued by a typical undiversified firm – in the limit the defaults of the tranching securities will be perfectly correlated. This, together with the systematic event of a decline in underwriting standards and a bubble in house prices, accounts for the fact that we see almost all highly rated securities issued against portfolios of subprime mortgages made in 2006 and 2007 experiencing ratings deterioration at the same time. This has profound implications for regulatory systems for bank capital that depend on bond ratings.¹⁶ A portfolio of n A rated CDO tranches will in general be much more risky than a portfolio of n A-rated bonds issued by corporations. However, an analysis

¹⁵ UBS. *ibid.*

¹⁶ Under Basel I the regulatory capital requirement was independent of the creditworthiness of the borrower. Under Basel II capital requirements depend either on external ratings, as discussed here, or on an approved internal rating system, which takes default probabilities and expected losses in case of default into account. Global regulators are re-examining the

of the regulatory implications of credit rating systems is beyond the scope of this paper. But our analysis does have implications for the emerging debate as to whether structured products should be rated on a different scale from other credit instruments.¹⁷

Section 2 provides an introduction to the market for structured bonds and Section 3 discusses credit ratings and the market for CDOs. Section 4 presents a general analysis of the investment banker's problem of security design and characterises his marketing profit. Section 5 illustrates the potential marketing gains to issuers of structured products in the context of a simple analytical model based on the CAPM and the Merton model of debt pricing. In section 6 the marketing gains from tranching corporate debt issues are analysed. In Section 7 the model is extended to the securitisation of corporate bonds.

2. Structured Bonds

In 1970 the US Government National Mortgage Association (GNMA) sold the first securities backed by a portfolio of mortgage loans. In subsequent years GNMA developed further these securitisation structures through which portfolios of commercial or residential mortgages are sold to outside investors. Beginning in the mid 1980s the concept was transferred to other asset classes such as auto loans, corporate loans, corporate bonds, credit card receivables, etc. Since then the market for the so-called asset backed securities (ABS) has seen tremendous growth. According to the Bank of England (2007) the global investment volume in the ABS market was USD 10.7 trillion by the end of 2006.

In a securitisation transaction a new legal entity, a Special Purpose Vehicle (SPV), is created to hold a designated portfolio of assets. The SPV is financed by a combination of debt and equity securities. A key feature is the division of the liabilities into tranches of different seniorities: payments are made first to the *senior* tranches, then to the *mezzanine* tranches, and finally to the *junior* tranches. This prioritisation scheme causes the tranches to exhibit different default probabilities and different expected losses. The junior tranches bear most of the default risk, and the super-senior tranche should have almost no default risk.¹⁸

Typically the SPV issues two to five rated debt tranches and one non-rated equity or first loss piece (FLP).¹⁹ In an empirical study of European securitisation transactions, Cuchra and Jenkinson (2005) found that a rather high percentage of the total portfolio volume is sold in tranches with a rating of A or better (on average 77%). AAA tranches on average accounted for 51% of the transaction but with a high variation across transactions types (between 30% and 89%). As shown by Franke *et al.* (2007) the size of the FLP varies significantly across transactions – from 2% to 20% in their sample of European CDOs.

degree to which regulatory frameworks have become dependent of credit ratings. *Financial Times*, 12 June 2008.

¹⁷ A managing director at Moody's said: 'we did go out and ask the community whether they wanted a different category of rating (for structured products) because this idea was floated by regulators but the strong response was please don't change anything'. *Financial Times*, 11 June 2008.

¹⁸ This is only a very brief and simplified description of these transactions. For a more detailed discussion on securitisation structures see Hein (2007).

¹⁹ Ashcraft and Schuerman (2008) describe a vehicle whose liabilities were divided into 16 tranches with 12 different credit ratings.

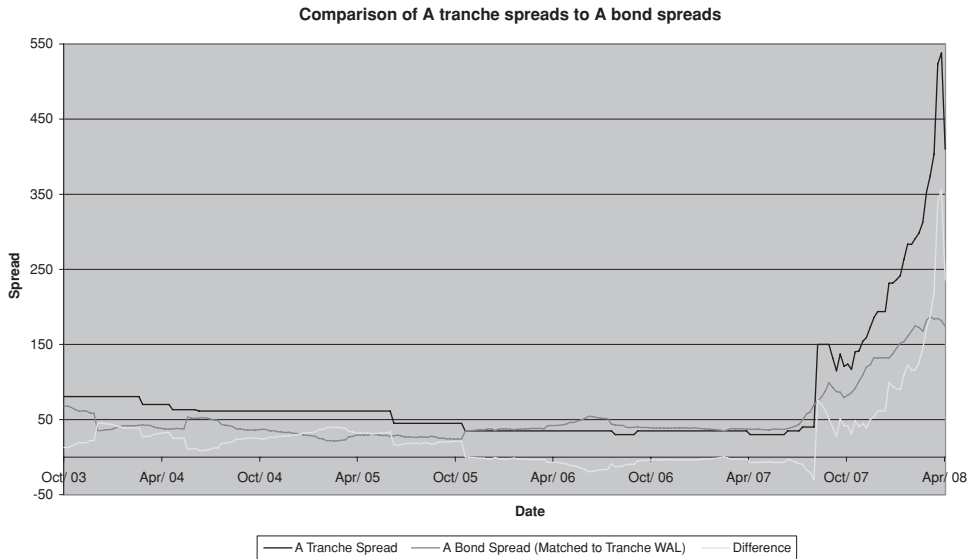


Fig. 1. CDO issuance and the spread differential between CDO and corporate bond yields

Data on CDO issuance is taken from the website of the Securities Industry and Financial Markets Association. The spread differential for each rating class is defined as the difference between average tranche spreads on European CLOs as reported by HSBC Global ABS Research and corporate bond spreads of the same rating class. The tranche spreads are quoted over EURIBOR/LIBOR since the tranches are floating rate notes. The corporate bond spreads are derived by comparing yields on the corresponding iBoxx Corporate (AAA/A/BBB) index to the iBoxx Sovereign index with the same maturity.

The originator of the CDO specifies in advance the number of tranches and their desired ratings. Due to information asymmetries between the originator and the investors concerning the quality of the underlying portfolio, the tranches need to be rated by an external rating agency. After analysing the transaction using cash flow simulations and stress testing,²⁰ two or three of the leading rating agencies assign ratings to the tranches. These ratings reflect assessments of the tranches' default probability (Standard & Poor's and Fitch) or expected default losses (Moody's), and are used by investors as an indicator of the tranche's quality.

Figure 1 displays the quarterly issuance volumes of balance sheet and arbitrage CDOs from 2004 to the first quarter of 2008 as reported by SIFMA. Total issuance of CDO's exploded in the years leading up to the sub-prime crisis with total quarterly issuance rising from \$25.0 billion in the first quarter of 2004 to \$178 billion in the fourth quarter of 2006. Even more significant is the fact that most of the growth in CDO offerings came from 'arbitrage' CDOs which SIFMA describes as motivated by mismatches between yields on the collateral and the average yields on the liability tranches sold against the collateral. It is the role of credit ratings in creating this mismatch that is the focus of our analysis.

²⁰ Beside this quantitative analysis, which plays a major role in the rating process, rating agencies also take into account qualitative aspects such as the servicer's, asset manager's and trustee's skills and reputation as well as legal aspects.

Figure 1 also shows the spread differential between CDO tranches and equivalently rated corporate bonds.²¹ Alongside the enormous growth in CDO issuance volumes, we see a sharp decline in the spread differential for different rating classes, especially for the BBB grade. From the first quarter of 2005 up to the third quarter of 2007, just before the subprime crisis, the spread differential was negligible with tranche spreads being even slightly below those of equivalently rated corporate bonds. During that period, the spread on AAA (A) rated tranches was, on average, 4.75 (8.19) basis points smaller than the corresponding bond spreads. From the beginning of the subprime crisis in mid 2007 issuance volumes dried out and the spread differentials increased sharply.

3. Credit Ratings

Seven rating agencies have received the Nationally Recognized Statistical Rating Organization (NRSRO) designation in the USA, and are overseen by the SEC: Standard & Poor's, Moody's, Fitch, A. M. Best, Japan Credit Rating Agency, Ltd., Ratings and Investment Information, Inc. and Dominion Bond Rating Service. The three major rating agencies, S&P, Moody's, and Fitch, dominate the market with approximately 90–95% of the world market share.²² Moody's ratings are based on estimates of the expected losses due to default, while S&P and Fitch base their ratings on estimates of the probability that the issuing entity will default.²³

S&P's ratings for structured products have broadly the same default probability implications as their ratings for corporate bonds.²⁴ Before 2005 the implied default probabilities for corporate and structured product ratings were the same. In 2005 corporate ratings were 'delinked from CDO rating quantiles' in order to 'avoid potential instability in high investment-grade scenario loss rates'. Now, as a result, 'CDO rating quantiles are higher than the corporate credit curves at investment grade rating levels, and converge to the corporate credit curves at low, speculative-grade rating levels'.²⁵ Thus, in 2005 S&P liberalised the ratings for structured bonds. Table 1 shows cumulative default frequencies for corporate bonds by rating and maturity as reported by Standard and Poor's (2005), and Table 2 shows the cumulative default frequency for CDO tranches. For example, the five-year cumulative default probability implied by a B rating for a CDO tranche is now 26.09% as compared with 24.46% for a corporate bond. If the

²¹ The spread differential is defined as the difference between average tranche spreads on European CLOs as reported by HSBC Global ABS Research and corporate bond spreads of the same rating class. The tranche spreads are quoted over EURIBOR/LIBOR since the tranches are floating rate notes. The corporate bond spreads are derived by comparing yields on the corresponding iBoxx Corporate (AAA/A/BBB) index to the iBoxx Sovereign index with the same maturity.

²² Cantor and Packer (1996) discusses sovereign ratings, and Kou and Varotto (2008) show that for Eurobonds, credit ratings lag market information.

²³ S&P explicitly state that 'Our rating speaks to the likelihood of default, but not the amount that may be recovered in a post-default scenario.' Standard and Poor's (2007).

²⁴ For Standard and Poor's at least, the rating assigned to a particular tranche does not depend upon the size of the tranche, but only on the total face value of the tranche and tranches that are senior to it: 'Tranche thickness generally does not affect our ratings, nor their volatility, since our ratings are concerned with whether or not a security defaults, *not how much loss it incurs in the event of default.*' Standard and Poor's (2007).

²⁵ See Standard & Poor's (2005).

Table 1

Cumulative default frequencies for corporate issues

The table reports historical cumulative default frequencies (in percent) for the period 1981 to 2003 for 9,740 companies of which 1,386 defaulted. Source: Standard & Poor's (2005).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.02	0.03	0.06	0.10	0.14
AA	0.01	0.04	0.09	0.14	0.22	0.31	0.42
A	0.02	0.08	0.17	0.30	0.46	0.66	0.89
BBB	0.29	0.68	1.16	1.71	2.32	2.98	3.67
BB	2.30	4.51	6.60	8.57	10.42	12.18	13.83
B	5.30	10.83	15.94	20.48	24.46	27.95	31.00

Table 2

Cumulative default frequencies for CDO tranches

The table reports cumulative default frequencies (in percent) based on "quantitative and qualitative considerations" (Standard & Poor's, 2005, p. 10).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.03	0.07	0.12	0.19	0.29
AA	0.01	0.06	0.14	0.23	0.36	0.51	0.70
A	0.03	0.12	0.26	0.46	0.71	1.01	1.37
BBB	0.35	0.83	1.41	2.07	2.81	3.61	4.44
BB	2.53	4.95	7.23	9.38	11.40	13.31	15.11
B	5.82	11.75	17.15	21.92	26.09	29.73	32.90

Table 3

Cumulative 'idealized loss rates' according to Moody's (2005)

	1	2	3	4	5	6	7
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Aa	0.00	0.00	0.01	0.03	0.04	0.05	0.06
A	0.01	0.04	0.12	0.19	0.26	0.32	0.39
Baa	0.09	0.26	0.46	0.66	0.87	1.08	1.33
Ba	0.86	1.91	2.85	3.74	4.63	5.37	5.89
B	3.94	6.42	8.55	9.97	11.39	12.46	13.21

investors are aware of the different default rates implied by the same rating for corporate bonds and CDO tranches, then we should expect the tranches to sell at higher yields for this reason alone.

Moody's ratings for both corporate and structured bonds are based on the cumulative 'Idealized Loss Rates' which are shown in Table 3. According to Moody's, 'the idealized loss rate tables were derived based on a rough approximation of the historical experience as observed and understood as of 1989. In addition we assumed extra conservative (low)

loss rates at the highest rating levels . . . we use the idealized loss rates to model the ratings.²⁶ Although it would seem more reasonable to base credit ratings on expected default losses rather than simply on default probability, Cuchra (2005, p 16) reports that in European markets for structured finance ‘S&P ratings explain the largest share of the total variation in (new issue) spreads, followed by Moody’s and Fitch.’

4. Theoretical Framework for Rating Based Pricing and Tranching

Among the primary roles of the investment banker are the marketing of new issues of securities and the provision of advice on the appropriate mix of securities to finance a given bundle of assets. The mix of securities sold may be important for valuation on account of control, incentive, tax, liquidity, information and bankruptcy cost considerations, and advice on these issues provides a legitimate role for the investment banker. In our model the marketing gains from the choice of the financing mix arise from the difficulty in evaluating the different cash flow claims in the capital structure of a bond issuer. This forces many investors to rely on credit ratings as the sole basis of their evaluation and, as mentioned above, these ratings do not reflect the *systematic risk* characteristics of the securities being rated.²⁷ Thus our fundamental assumption is that investment bankers are able to sell new issues of structured bonds at yields to maturity that are the same as the yields on equivalently rated bonds issued by a reference firm.²⁸ The main difference between these two types of security is that the reference bond is assumed to be secured by the assets of a single firm and represents a senior claim with respect to equity, whereas the structured bond is either a subordinated bond within a tranching debt structure of a single firm or a tranche in the liabilities of an SPV that is holding the collateral.

Throughout this section we shall use an asterisk to denote variables that correspond to the rating agency’s reference bond or its issuer, and use the same variables without the asterisk to denote the corresponding variable for the structured bond or its issuer. Thus, let W_k^* and W_k denote the values of pure discount debt securities with face values B_k^* and B_k , rating k , and maturity τ ²⁹ when issued by the reference firm with asset value, V^* , and an arbitrary corporate bond issuer or an SPV holding collateral with asset value V .

Let y_k^* denote the yield to maturity, and $\phi_k^* \equiv W_k^*/B_k^* \equiv e^{-y_k^*\tau}$ the ratio of the market value of a k rated pure discount corporate bond to its face value when issued by the reference firm. Let S_k denote the *sales price* of a pure discount structured debt security with face value B_k and rating k issued by an arbitrary bond issuer or an SPV. Our

²⁶ Private communication from Moody’s.

²⁷ Cuchra (2005) shows that ‘the relation between price and credit rating for each tranche is very close indeed and consistent across all types of securitizations . . . this relationship seems considerably stronger than in the case of corporate bonds’. He also remarks that ‘the tranche-specific, composite credit rating . . . is the primary determinant of (launch) spreads.’

²⁸ This assumption also seems to be consistent with the expectations of the rating agencies. For example, ‘Do ratings have the same meaning across sectors and asset classes? The simple answer is “yes”. Across corporates, sovereigns and structured finance, we seek to ensure to the greatest extent possible that the default risk commensurate with any rating category is broadly similar’ Standard and Poor’s (2007). Similarly, the ‘idealized loss rates’ to which Moody’s structured product ratings are calibrated are taken from corporate bond experience.

²⁹ For simplicity we will drop the maturity subscript τ in the following.

assumption is that the *sales price*, S_k , at which a new debt security can be sold, bears the same relation to its face value as does the value of an equivalently rated debt security with the same maturity issued by the reference firm:

Pricing Assumption:

$$S_k = \phi_k^* B_k = e^{-y_k^* \tau} B_k.$$

The *Pricing Assumption* essentially states that, for a given maturity, yields depend only on ratings.

Let P^* denote the *physical* probability distribution of the asset value of the reference firm at the maturity of the bond, and let P denote the corresponding probability distribution for the corporate bond issuer or of the collateral held by the SPV. The price of any contingent claim written on the value of the reference firm, V^* , or on the value of the structured bond collateral, V , can be expressed as the discounted value of the contingent claim payoff under the equivalent martingale measures Q^* and Q . The link between the physical and risk neutral measures is given by the conditional pricing kernels for contingent claims on the underlying assets, $m^*(v)$ and $m(v)$, with $f_{Q^*}(v) = m^*(v)f_{P^*}(v)$ and $f_Q(v) = m(v)f_P(v)$, where $f(v)$ is the density function of the terminal underlying asset value v under the corresponding measure.

We consider two different rating systems:

(i) *Default probability based rating*

The bond rating, k , is a monotone decreasing function of the probability of default, $\Pi: \mathcal{R}_P(\Pi), \mathcal{R}'_P(\Pi) < 0$.

(ii) *Expected default loss based rating*

The bond rating, k , is a monotone decreasing function of the expected default loss, $\Lambda: \mathcal{R}_L(\Lambda), \mathcal{R}'_L(\Lambda) < 0$.

We assume for simplicity that all defaults take place at maturity, and denote the default loss rate for a bond with rating k and maturity τ by Λ_k , and denote the probability of default by Π_k . The probabilities of default and the expected default loss rates are determined by the *physical* probability distributions, P and P^* , while the market values of the instruments are determined by the face value and the risk neutral probability distributions, Q and Q^* , as illustrated below:

Agency Rated Reference Bond:

$$\Lambda_k, \Pi_k \xleftarrow{P_k^*} B_k^* \xrightarrow{Q_k^*} W_k^* \equiv \phi_k^* B_k^* \equiv e^{-y_k^* \tau} B_k^*$$

Agency Rated Structured Bond:

$$\Lambda_k, \Pi_k \xleftarrow{P_k} B_k \xrightarrow{Q_k} W_k \equiv \phi_k B_k \equiv e^{-y_k \tau} B_k$$

Thus the fair *market value* of the structured bond is:

$$W_k = \phi_k B_k = e^{-y_k \tau} B_k$$

which usually differs from the ratings based *sales price* as defined above. In effect, we assume that the investment banker is able to sell the security at a price that reflects the risk neutral probability distribution, Q_k^* , that is appropriate for a typical corporate issuer of a bond with the same probability of default or expected loss.

First we consider the gains from rating based pricing and tranching within a general model of valuation. In our subsequent analysis, we present two parametric

models to quantify the marketing gains that the issuer can reap under ratings based pricing.

4.1 Issuing a single bond

As a starting point, we characterise the marketing gain from ratings-based pricing when issuing a single bond against a given portfolio of assets. When ratings are based on *default probability*, the face value of the bond with rating k and default probability Π_k issued by the reference firm, B_k^* , and the face value of the single debt issue with the same rating and default probability when issued by an arbitrary firm or SPV, B_k , are defined by

$$\int_0^{B_k^*} f_{P^*}(v) dv = F_{P^*}(B_k^*) = \Pi_k = F_P(B_k) = \int_0^{B_k} f_P(v) dv \tag{1}$$

where $F_{P^*}(F_P)$ denotes the *cdf* with respect to the physical probability measure $P^*(P)$ and $f_{P^*}(f_P)$ are the corresponding density functions.

When ratings are based on *expected default loss*, the face values are defined by

$$\Lambda_k = \frac{\mathcal{L}^*}{B_k^*} = \frac{\mathcal{L}}{B_k} \tag{2}$$

with

$$\mathcal{L}^* = \int_0^{B_k^*} (B_k^* - v) f_{P^*}(v) dv, \quad \mathcal{L} = \int_0^{B_k} (B_k - v) f_P(v) dv$$

The marketing gain, Ω , from issuing the security is equal to the difference between the sales price, S_k , and the market value W_k :

$$\Omega = S_k - W_k = [\phi_k^* - \phi_k] B_k \tag{3}$$

Setting the interest rate equal to zero for simplicity, the value of the new security is given by:

$$W_k = \int_0^{B_k} v f_Q(v) dv + B_k \int_{B_k}^{\infty} f_Q(v) dv \equiv \phi_k B_k \tag{4}$$

Similarly, ϕ_k^* is defined implicitly by the valuation of the reference corporate liability:

$$W_k^* = \int_0^{B_k^*} v f_{Q^*}(v) dv + B_k^* \int_{B_k^*}^{\infty} f_{Q^*}(v) dv \equiv \phi_k^* B_k^* \tag{5}$$

Combining (4) and (5) with (3), the marketing gain may be written as:

$$\Omega = B_k \left\{ \frac{1}{B_k^*} \int_0^{B_k^*} v f_{Q^*}(v) dv + \int_{B_k^*}^{\infty} f_{Q^*}(v) dv \right\} - B_k \left\{ \frac{1}{B_k} \int_0^{B_k} v f_Q(v) dv + \int_{B_k}^{\infty} f_Q(v) dv \right\} \tag{6}$$

where B_k^* and B_k are given by equation (1) under a default probability rating system, and by equation (2) under a default probability rating system. Sufficient conditions for the marketing gain to be positive or negative are given in the following lemmas:

Lemma 1. Default probability rating system

- (a) The marketing gain, Ω , will be positive if P first order stochastically dominates P^* ($P >^{FSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the marketing gain will be negative if $P^* >^{FSD} P$ and $Q \geq^{SSD} Q^*$.
- (b) Moreover if two issuers have the same risk-neutral distribution Q and their physical distributions, P_1 and P_2 , are such that $P_2 >^{FSD} P_1 \geq^{FSD} P^*$, and $Q^* \geq^{SSD} Q$, then the marketing gain from issuing a structured bond with a given rating k will be greater for the second issuer (SPV_2) than for the first issuer (SPV_1).

Proof: See Appendix.

Lemma 2. Expected default loss rating system

- (a) The marketing gain, Ω , will be positive if P second order stochastically dominates P^* ($P >^{SSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the marketing gain will be negative if $P^* >^{SSD} P$ and $Q \geq^{SSD} Q^*$.
- (b) Moreover if two issuers have the same risk-neutral distribution Q and their physical distributions, P_1 and P_2 , are such that $P_2 >^{SSD} P_1 \geq^{SSD} P^*$ and $Q^* \geq^{SSD} Q$, then the marketing gain from issuing a structured bond with a given rating k will be greater for the second issuer (SPV_2) than for the first issuer (SPV_1).

Proof: See Appendix.

As a direct application of part (a) of Lemmas 1 and 2, consider the situation in which returns are normally distributed, the CAPM holds, and V and V^* have the same total risk. The risk neutral measures will then be identical: $Q \equiv Q^*$. P will first and second order stochastically dominate P^* whenever the assets of the bond issuer have a beta coefficient higher than that of the reference firm, because this will imply a higher mean return for the bond issuer. Part (b) of the lemmas implies that, for a given total risk and bond rating, the marketing gain will be monotonically increasing in the beta of the issuer's collateral.

4.2 Issuing multiple tranches

Lemmas 1 and 2 characterise conditions under which the marketing gain from a single debt issue is positive under our pricing assumption. However, some corporations issue several subordinated debt tranches, and most asset securitisations involve multiple tranches.³⁰ In this section we consider when the marketing gain can be increased by issuing additional tranches. To analyse the gains from introducing multiple tranchised securities, consider the gain from replacing a single debt issue with face value B_k and rating k with two tranches. Denote the face value of the senior tranche by B_{1,k_1} and its rating by k_1 , and denote the face value of the junior tranche by $B_{2,k_2} \equiv B_k - B_{1,k_1}$ and its rating by k_2 .³¹

³⁰ Cuchra and Jenkinson (2005) report that in 2003 the average number of tranches in European securitisations was 3.93 and in US securitisations 5.58.

³¹ Note that in our notation, B_{j,k_j} , j denotes the seniority of the tranche issued and k_j denotes its rating. Note that neither the payoff nor the rating of a given tranche depend on the existence or characteristics of more junior tranches.

Under a *default probability* rating system, the default probability of the single tranche, Π_k , is equal to the default probability of the junior tranche of the dual tranche structure, since in both cases the SPV defaults when its terminal value, V , is less than $B_k = B_{1,k_1} + B_{2,k_2}$. Hence, under rating based pricing the junior tranche sells at the same (corporate bond) yield as the single tranche: $\phi_{k_2}^* = \phi_k^*$. On the other hand, the senior tranche defaults in only a subset of the states in which the single tranche issue defaults so that it sells at a lower yield and $\phi_{k_1}^* > \phi_k^*$: the extra gain from switching from a single-tranche to a two-tranche structure is $(\phi_{k_1}^* - \phi_k^*)B_{1,k_1}$. It is straightforward to extend this argument to additional tranches as stated in the following lemma:

Lemma 3. Default probability rating system

Under a default probability rating system it is optimal to subdivide a given tranche into a junior and a senior tranche with different ratings.

The Lemma implies that it is optimal to have as many tranches as there are different rating classes. A similar result holds for the expected default loss rating system:

Lemma 4. Expected default loss rating system

Under an expected default loss rating system, if a given tranche is profitable, then it is optimal to subdivide the tranche into a junior and a senior tranche with different ratings, whenever the pricing kernel for the reference issuer, $m^(v)$, is a decreasing function of the underlying asset value.*

Proof: See Appendix.

Lemmas 3 and 4 are consistent with the findings of Cuchra and Jenkinson (2005) that the number of tranches in European securitisations has displayed a secular tendency to increase, and that securitisations characterised by greater information asymmetry tend to have more tranches with different ratings.

5. Parametric Model of Ratings Yields

In order to quantify the gains from tranching and securitisation when bond issues are made at yields that reflect only their ratings it is necessary to have a model of yields as a function of ratings. We assume that bond ratings are based on the risk characteristics of a reference firm, the value of whose assets (V^*) follows a geometric Brownian motion:

$$dV^* = \mu^* V^* dt + \sigma^* V^* dz^* \tag{7}$$

where $\mu^* = r_f + \beta^* (r_m - r_f)$, r_f denotes the risk-free rate, $(r_m - r_f)$ the excess market return, and β^* the CAPM beta coefficient.³² The total risk σ^* can be decomposed into a systematic and a residual risk component: $\sigma^* = \sqrt{(\beta^* \sigma_m)^2 + \sigma_\epsilon^{*2}}$, where σ_m denotes the market volatility and σ_ϵ^* denotes the residual risk.

When ratings are based on *default probabilities*, the face value of the reference bond with rating k , B_k^* , depends on its default probability Π_k , i.e. the probability that the

³² While our analysis is based on the CAPM it is straightforward to recast it in terms of a more general pricing kernel formulation.

assets of the reference firm are less than B_k^* at maturity:³³

$$\Pi_k = \mathcal{N}\left(-\frac{\ln(V^*/B_k^*) + (\mu^* - 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}}\right) \tag{8}$$

where \mathcal{N} denotes the cumulative standard normal distribution. Inverting equation (8), the face value of the reference bond per unit of total asset value B_k^*/V^* , may be expressed as a function of Π_k :

$$\frac{B_k^*}{V^*} \equiv \frac{1}{\exp\{-\mathcal{N}^{-1}[\Pi_k]\sigma^*\sqrt{\tau} - (\mu^* - 0.5\sigma^{*2})\tau\}} \tag{9}$$

When ratings are based on *expected default losses*, the face value of a reference bond with rating k , B_k^* , depends on its loss rate Λ_k :

$$B_k^* = \frac{\mathcal{L}_k^*}{\Lambda_k} \tag{10}$$

where the expected default loss, \mathcal{L}_k^* , is given by

$$\mathcal{L}_k^* = B_k^*\mathcal{N}(-d_2^{P*}) - V^*e^{\mu^*\tau}\mathcal{N}(-d_1^{P*}) \tag{11}$$

with

$$d_1^{P*} = \frac{\ln(V^*/B_k^*) + (\mu^* + 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}} \tag{12}$$

$$d_2^{P*} = d_1^{P*} - \sigma^*\sqrt{\tau} = \frac{\ln(V^*/B_k^*) + (\mu^* - 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}}. \tag{13}$$

The market value of the rating k reference bond, W_k^* , is given by the Merton (1974) formula:

$$W_k^* = B_k^*e^{-r_f\tau}\mathcal{N}(d_2^{Q*}) + V^*\mathcal{N}(-d_1^{Q*}) \tag{14}$$

where d_1^{Q*} and d_2^{Q*} are defined as in equations (12) and (13) substituting r_f for μ^* .

Given the market value and the face value of the reference bond, we get the bond yield for rating class k as

$$\frac{W_k^*}{B_k^*} = \phi_k^* = e^{-y_k^*\tau} \tag{15}$$

Given the probability of default or expected loss corresponding to a particular bond rating, different values of $\mu^*(\beta^*)$ and σ^* for the reference firm will imply different values of W_k^* and B_k^* , and hence of ϕ_k^* and y_k^* . Therefore in what follows we will also explore the implications of the risk characteristics of the reference firm for the marketing gains from securitisation and tranching.

6. Marketing Gains from Rating Based Pricing of Corporate Debt

In this section we quantify the potential gains from ratings-based pricing when the asset value of the *issuer* (V) also follows a geometric Brownian motion with parameters

³³ For convenience we again drop the maturity subscript τ , although both Π_k and B_k^* depend on the time to maturity.

(μ, σ) , where $\mu = r_f + \beta(r_m - r_f)$.³⁴ This assumption allows us to obtain quasi-analytic solutions and also to quantify the marketing gains resulting from differences in the risk characteristics of issuer and reference firm and from tranching. In this case it is natural to think of the issuer as another firm whose asset risk and capital structure differ from those of the reference firm. In the following section we will consider the gains from securitising a portfolio of corporate bonds and tranching the securities sold against the corporate bond collateral: in that case the distributions of returns on the collateral portfolio and the reference firm do not belong to the same family, precluding a direct analysis of the effects of differences in the risk characteristics of the issuing firm and the reference firm.

6.1 *Single debt issue*

Consider first the case in which a single debt security with credit rating, k , is issued. When ratings are based on *default probabilities [expected default losses]*, the face value of the bond, B_k , is derived by substituting (V, μ, σ) for the corresponding variables in equation (9) [(11)] as given in the previous section.

Under the ratings-based *Pricing Assumption*, the bond is sold at the yield determined by its rating. Hence, the sales price is based on the bond yield as derived in (15): $S_k = \phi_k^* B_k$, and the marketing gain is $\Omega = S_k - W_k$. The marketing gain will depend on the relation between (μ, σ) and (μ^*, σ^*) as discussed in Lemmas 1 and 2. If the parameters of the reference firm and the corporate issuer are the same, i.e. $\mu = \mu^*$ and $\sigma = \sigma^*$, then the marketing gain will be zero.

6.2 *Multiple debt tranches*

In considering subordinated issues it is convenient to define B_{k_i} , the cumulative face value, as the sum of the face values of all tranches senior to the tranche with rating k_i , including the k_i rated tranche itself, so that B_{i,k_i} , the face value of tranche i with rating k_i is given by $B_{i,k_i} = B_{k_i} - B_{k_{i-1}}$, where k_{i-1} denotes the rating of the immediate senior tranche. The face value of the most senior tranche, B_{1,k_1} , is equal to B_{k_1} .

Under a *default probability* rating system, B_{k_i} is derived as before by substituting the appropriate parameters in equation (9).

The calculation of the cumulative face value of subordinated debt is less direct under the *expected default loss* rating system. In this case the expected loss, \mathcal{L}_{i,k_i} , on the i th tranche with face value B_{i,k_i} , is $\mathcal{L}_{i,k_i} = \mathcal{L}_{k_i} - \mathcal{L}_{k_{i-1}}$ with \mathcal{L}_{k_i} and $\mathcal{L}_{k_{i-1}}$ as defined in (11). Hence the expected loss rate on the i th tranche is:

$$\Lambda_{k_i} = \frac{\mathcal{L}_{i,k_i}}{B_{i,k_i}} = \frac{\mathcal{L}_{k_i} - \mathcal{L}_{k_{i-1}}}{B_{k_i} - B_{k_{i-1}}} \quad \text{for } i > 1 \tag{16}$$

and for the most senior tranche

$$\Lambda_{k_1} = \frac{\mathcal{L}_{k_1}}{B_{1,k_1}} = \frac{\mathcal{L}_{k_1}}{B_{k_1}} \tag{17}$$

which corresponds to equation (10). From $\Lambda_{k_1}, \dots, \Lambda_{k_i}$ the implicit equations for B_{i,k_i} , (16) and (17), may be solved recursively starting with the most senior tranche.

³⁴ In contrast to the previous section, the parameter values here do not have an asterisk * which is only used for the reference bond.

The market value of the i th tranche with face value B_{i,k_i} is equal to the difference between market values of adjacent cumulative tranches: $W_{i,k_i} = W_{k_i} - W_{k_{i-1}}$ with W_{k_i} and $W_{k_{i-1}}$ as determined in the single tranche case.

Using the ratings-based *Pricing Assumption*, the sales price of the i th tranche, S_{i,k_i} , is given by

$$S_{i,k_i} = \phi_{k_i}^* B_{i,k_i} = e^{-y_{k_i}^* \tau} B_{i,k_i} = \frac{W_{k_i}^*}{B_{k_i}^*} B_{i,k_i} \quad (18)$$

where $y_{k_i}^*$ is derived from the reference bond as described in section 5. Note that $y_{k_i}^* \neq y_{i,k_i}^*$, that is the reference bond yield is calculated based on a single debt issue and applied to equivalently rated subordinated bond within a tranching structure. The marketing gain on the i th tranche is $\Omega_i = S_{i,k_i} - W_{i,k_i}$, and the total marketing gain is $\Omega = \sum_i \Omega_i$.

6.3 Numerical examples

In this section we present estimates of the gains to ratings-based pricing and tranching for a corporate issuer as described in sections 4.1 and 4.2, assuming a risk-free interest rate of 3.5%, a market risk premium of 7%, and a market volatility of 14%.³⁵ Panels A of Tables 4 and 5 report the rating-implied 5-year corporate bond yields, $y_{k_i}^*$, for each rating class under the assumptions that the asset beta of the reference corporate issuer is 0.80, and its residual risk, σ_ε , is 25% p.a. For each rating class the reference corporate issuer is assumed to issue a *single* bond with face value, $B_{k_i}^*$, chosen to yield the appropriate default probability, Π_{k_i} , or expected default loss, Λ_{k_i} . Table 4 relates to a *Default Probability* (S&P) rating system, and Table 5 to an *Expected Default Loss* (Moody's) system.

Figure 2 plots ratings-implied yields from the model for the two rating systems along with 'actual' corporate yields which are constructed by adding 3.5% to the CDS yield spreads reported in Table 1 of Coval *et al.* (2009).³⁶ Although the parameters of the reference corporate issuer were chosen somewhat arbitrarily, the model yields fit the actual yields surprisingly well: for the first five rating classes the difference between the actual yield and the average of the Moody's and S&P ratings-implied yields is less than 20 *basis points* and averages only 4 *basis points*. For the B-rated bonds the average of the two ratings-implied yields overpredicts the actual yield by 109 *basis points*.

Panel B of Tables 4 and 5 shows the valuation and pricing of 5-year maturity tranches under the *default probability* and the *expected default loss* rating systems respectively, when the asset betas of both the arbitrary corporate issuer and the reference firm is 0.8 and the residual risk, σ_ε , is 25% p.a., so that the marketing gains reported are attributable entirely to tranching. For each tranche the sales price is calculated by multiplying the nominal value of the tranche by the multiplier, ϕ_k^* , that is calculated for that rating using

³⁵ From 1927 to 2007 the US equity market risk premium has averaged about 8.2% and the risk-free rate has averaged about 3.8%. (see Kenneth R. French Data Library: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. Welch (2000) reports that the arithmetic long-term equity premium consensus forecast is about 7%. The marketing gains are increasing in the assumed value of the market risk premium so we are adopting a conservative position. The annualised monthly standard deviation of the Fama-French market factor from January 1946 to March 2008 is 14.5%.)

³⁶ The spreads are the average 5-year bond-implied CDS spreads provided by Lehman Brothers for the period 2004.9–2006.9.

Table 4
Corporate bond valuation and pricing under the *Default Probability* rating system (S&P)

In Panel A ratings based yields are calculated by setting the face value of the bond issued by the reference firm, $B_{k_i}^*$ to yield the default probability, Π_{k_i} appropriate for its rating class. The bond is valued using the Merton model and its yield, $y_{k_i}^*$, is the rating-based yield. In Panel B, the tranche face values are calibrated to yield the default probability, Λ_{k_i} appropriate for its rating class, and are *valued* using the Merton model. The *Sales Price* of each tranche is calculated by multiplying the face value by the rating appropriate multiplier, $\phi_{k_i}^*$, calculated in Panel A. Parameter assumptions: $V^*(0) = V(0) = 100$, $\tau = 5$, $r = 3.5\%$, $r_m = r_f = 7\%$, $\sigma_m = 0.14$, $(\beta^*; \sigma_\varepsilon^*) = (0.8; 0.25)$.

Panel A: Ratings based yields

Rating <i>i</i>	S&P rating k_i	Probability of default Π_{k_i}	Face value $B_{k_i}^*$	Equilibrium value $W_{k_i}^*$	Rating-based bond yield $y_{k_i}^*$	$\phi_{k_i}^* \equiv W_{k_i}^*/B_{k_i}^*$
1	AAA	0.061%	18.02	15.12	3.51%	0.839
2	AA	0.219%	22.81	19.12	3.53%	0.838
3	A	0.459%	26.49	22.17	3.56%	0.837
4	BBB	2.323%	38.59	31.96	3.77%	0.828
5	BB	10.424%	60.47	47.90	4.66%	0.792
6	B	24.460%	85.54	62.41	6.31%	0.730

Panel B: Valuation and pricing of *tranchéd debt* issued by a single corporate entity

Tranche <i>i</i>	S&P rating k_i	Probability of default Λ_{k_i}	Face value tranche B_{i,k_i}	Equil. value tranche W_{i,k_i}	Equilibrium yield to maturity	Sales price $S_{i,k_i} = \phi_{k_i}^* B_{i,k_i}$	Gain $S_{i,k_i} - W_{i,k_i}$
1	AAA	0.061%	18.02	15.12	3.51%	15.12	0.00
2	AA	0.219%	4.79	4.00	3.60%	4.01	0.01
3	A	0.459%	3.68	3.05	3.74%	3.08	0.03
4	BBB	2.323%	12.10	9.79	4.24%	10.02	0.23
5	BB	10.424%	21.88	15.94	6.34%	17.33	1.39
6	B	24.460%	25.97	14.51	10.93%	18.29	3.78
Total:	Equity			100.00		105.45	5.45

Table 5
Corporate bond valuation and pricing under the *Expected Default Loss* rating system (Moody's)

In Panel A ratings based yields are calculated by setting the face value of the bond issued by the reference firm, $B_{k_i}^*$, to yield the expected loss, Λ_{k_i} , appropriate for its rating class. The bond is valued using the Merton model and its yield, $y_{k_i}^*$, is the rating-based yield. In Panel B, the tranche face values are calibrated to yield the expected loss, Λ_{k_i} , appropriate for its rating class, and are *valued* using the Merton model. The *Sales Price* of each tranche is calculated by multiplying the face value by the rating appropriate multiplier, ϕ_k^* , calculated in Panel A. Parameter assumptions: $V^*(0) = 100$, $\tau = 5$, $\eta = 3.5\%$, $r_m - \eta = 7\%$, $\sigma_m = 0.14$, $(\beta^*, \sigma_\varepsilon^*) = (0.8; 0.25)$.

Panel A: Ratings based yields

Rating i	Moody's rating k_i	Expected loss Λ_{k_i}	Face value $B_{k_i}^*$	Equilibrium value $W_{k_i}^*$	Rating-based bond yield $y_{k_i}^*$	$\phi_k^* \equiv W_{k_i}^*/B_{k_i}^*$
1	Aaa	0.002%	13.72	11.52	3.50%	0.839
2	Aa	0.037%	23.24	19.48	3.53%	0.838
3	A	0.257%	34.17	28.44	3.67%	0.832
4	Baa	0.869%	45.56	37.33	3.98%	0.820
5	Ba	4.626%	74.56	56.56	5.52%	0.759
6	B	11.390%	106.15	71.42	7.93%	0.673

Panel B: Valuation and pricing of tranchet debt issued by a single corporate entity

Tranche i	Moody's rating k_i	Exp. Loss Λ_{k_i}	Face value tranche B_{i,k_i}	Equilibrium tranche W_{i,k_i}	Equilibrium value	Equilibrium yield to maturity	Sales price $S_{i,k_i} = \phi_k^* B_{i,k_i}$	Gain $S_{i,k_i} - W_{i,k_i}$
1	Aaa	0.002%	13.72	11.52	11.52	3.50%	11.52	0.00
2	Aa	0.037%	5.09	4.27	4.27	3.53%	4.27	0.00
3	A	0.257%	8.52	7.08	7.08	3.69%	7.09	0.01
4	Baa	0.869%	5.91	4.83	4.83	4.05%	4.85	0.02
5	Ba	4.626%	24.78	18.56	18.56	5.78%	18.80	0.24
6	B	11.390%	8.67	5.63	5.63	8.64%	5.83	0.20
-	Equity			48.11	48.11		48.11	0.00
Total:				100.00	100.00		100.47	0.47

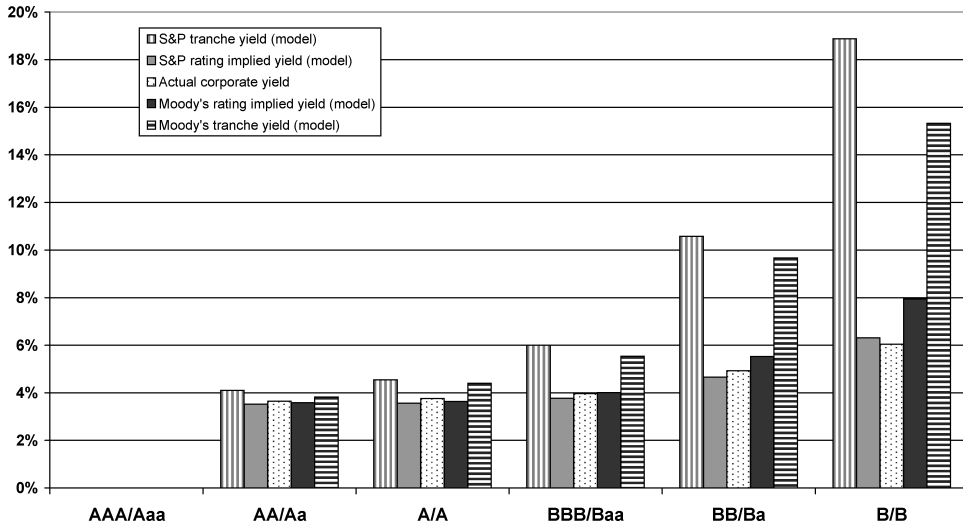


Fig. 2. Model rating-implied yields, model tranche yields, and actual corporate yields 2004.9–2006.9

The ‘actual’ corporate yields are constructed by adding 3.5% to the CDS yield spreads reported by Coval *et al.* (2009). The Moody’s and S&P rating-implied yields are constructed from the ratings agencies’ default data as applied to a standard reference firm as shown in Tables 4 and 5. Moody’s and S&P tranche yields are taken from the examples presented in Table 7.

the reference firm in Panel A, while the equilibrium value is determined by the Merton model. Despite the fact that the risk characteristics of the issuer and the reference firm are identical, the gain to tranching the debt is 5.45% under the default probability rating system and 0.47% under the expected default loss rating system. Except for the AAA-rated tranche, the marketing gain is positive for all tranches and the profit is greatest for the most subordinated tranche which has the lowest rating. No profit is assigned to the unrated equity or first loss piece which is assumed to be retained by the issuer.

While the pure gains from tranching reported in Tables 4 and 5 are positive, they are small. Therefore in Table 6 we explore the effect on the marketing gains of varying the risk characteristics of both issuer and reference firm in the presence of tranching. We also explore the effect of changing the number of tranches. Figures shown in bold correspond to the basic examples presented in Tables 4 and 5.

Ω_B^M , which is reported in the second to last column of the table, is the (percentage) marketing gain from issuing six debt tranches whose lowest rated tranche has a B rating. As we vary β and σ_ϵ for both the issuer of the tranching debt and the reference firm from which the ratings yields are calculated, the marketing gain ranges from 2.41% to 11.19% under the *default probability* system and from -0.36% to 3.26% under the *expected default loss* system. The right hand column, labelled Ω_B^S in Panel A and Ω_B^S in Panel B shows the marketing gain from issuing the same total amount of debt (Total Debt) in a single issue.³⁷ Consistent with Lemma 3, the marketing gain from replacing the single

³⁷ Under default probability rating the single bond will be rated B. But under expected default loss rating the rating of the single bond will not correspond to any of the Moody’s classifications; we price the single bond by assuming that its yield is the same as that of a bond issued by the reference firm with the same expected default loss.

debt security with multiple debt tranches is always positive, i.e. $\Omega_{\bullet}^M > \Omega_{\bullet}^S$, and the gain from issuing six tranches always exceeds that from issuing five tranches ($\Omega_B^M > \Omega_{BB}^M$ and $\Omega_B^M > \Omega_{Ba}^M$) if tranching is profitable. The gain from multiple tranching is increasing in the systematic risk of the issuer, β , and decreasing in the residual risk, σ_{ϵ} .

In summary, while there are theoretical marketing gains available under ratings-based pricing, these appear to be modest for corporate issuers in most scenarios under *expected default loss* rating, although they can be much larger under *default probability* rating, which performs particularly poorly for subordinated debt issues. In the following section we analyze the magnitude of the marketing gains available from ratings-based pricing for a securitizer of corporate bonds.

Table 6
Marketing gains from tranching corporate debt

The table shows the marketing gains for a corporation from from creating five or six subordinated debt tranches with S&P ratings AAA, AA, A, BBB, BB (and B) in Panel A and Moody's ratings Aaa, Aa, A, Baa, Ba, B in Panel B. $r_f = 3.5\%$, $r_m - r_f = 7\%$ and $\sigma_m = 0.14$. First, the characteristics of the reference firm $(\beta^*, \sigma_{\epsilon}^*) = (0.8, 0.25)$ are fixed and the systematic and idiosyncratic risk parameters $(\beta, \sigma_{\epsilon})$ of the arbitrary corporate issuer are varied. The last four lines in each Panel show the reverse case, holding $(\beta, \sigma_{\epsilon}) = (0.8, 0.25)$ fixed. Lemmas 1(a) and 2(a) provide sufficient conditions for a gain (\checkmark) or a loss (\times) from a issuing single debt. Total Debt is the sum of the equilibrium market values of the tranches. Ω_{BB}^M (Ω_B^M) is the marketing gain from a five (six) tranche securitisation expressed as percent of the underlying collateral value. Ω_{\bullet}^S is the marketing gain from a *single* debt issue with the same total amount of debt as the corresponding multi-tranche issue. Note that under the expected default loss rating system the rating of the single debt issue is no longer Ba (B), and the corresponding single debt issues are denoted ba and b. The numbers presented in bold fonts correspond to the basic examples presented in Tables 4 and 5.

Panel A: Under a Default Probability Rating System (S&P)

Issuer			Five tranches			Six tranches		
β	σ_{ϵ}	Lemma 1 (a)	Total debt	Ω_{BB}^M	Ω_{BB}^S	Total debt	Ω_B^M	Ω_B^S
0.5	0.15	x	67.1	1.58	-1.24	78.3	4.56	-3.47
	0.25		46.5	0.90	-0.74	60.5	3.31	-1.94
	0.35		30.3	0.54	-0.36	44.2	2.41	-0.84
0.8	0.15		67.4	2.96	0.18	79.2	7.84	-0.27
	0.25		47.9	1.67	0.00	62.4	5.45	0.00
	0.35		31.7	0.96	0.03	46.1	3.79	0.36
1.1	0.15		65.8	4.34	1.71	78.4	11.19	3.23
	0.25		48.1	2.53	0.88	63.1	7.82	2.31
	0.35		\checkmark	32.3	1.45	0.51	47.3	5.33
Reference firm								
β^*	σ_{ϵ}^*							
1.1	0.25		47.9	1.29	-0.86	62.4	4.43	-2.20
0.5	0.25		47.9	2.00	0.77	62.4	6.39	2.07
0.8	0.15		47.9	1.60	-0.13	62.4	5.44	0.22
0.8	0.35		47.9	1.66	-0.05	62.4	5.30	-0.48

Table 6
Continued.

Panel B: Under an Expected Default Loss Rating System (Moody's)

Issuer			Five tranches			Six tranches		
β	σ_ϵ	Lemma 2 (a)	Total debt	Ω_{Ba}^M	Ω_{ba}^S	Total debt	Ω_B^M	Ω_b^S
0.5	0.15		66.5	0.32	-0.08	69.8	0.46	-0.12
	0.25	x	45.0	-0.31	-0.57	50.3	-0.41	-0.86
	0.35	x	28.5	-0.43	-0.57	34.2	-0.68	-0.97
0.8	0.15	✓	66.5	1.44	1.04	70.4	1.97	1.35
	0.25		46.3	0.26	0.00	51.9	0.47	0.00
	0.35		29.7	-0.15	-0.30	35.8	-0.19	-0.50
1.1	0.15	✓	65.7	2.71	2.30	68.2	3.26	2.72
	0.25	✓	46.2	0.91	0.65	52.3	1.48	1.00
	0.35		30.2	0.19	0.04	36.6	0.41	0.09
Reference firm								
β^*	σ_ϵ^*							
1.1	0.25	x	46.3	-0.22	-0.63	51.9	-0.24	-0.94
0.5	0.25	✓	46.3	0.74	0.60	51.9	1.18	0.93
0.8	0.15		46.3	-0.34	-0.82	51.9	-0.36	-1.18
0.8	0.35	✓	46.3	0.58	0.41	51.9	0.93	0.62

7. Marketing Gains from Corporate Bond Securitization

In the previous section we considered a *corporate* issuer of tranching debt. In this section we analyse a corporate bond securitisation through an SPV. We proceed by simulating under both the physical and risk neutral distributions, P and Q, the payoff on a portfolio of J bonds issued by J identical firms each with underlying asset value process:

$$dV = \mu V dt + \sigma V dz \quad \text{with } V(0) = 100 \tag{19}$$

where $\mu = r_f + \beta(r_m - r_f)$. The correlation between the returns on any two firms is $\rho \equiv \beta^2 \sigma_m^2 / (\beta^2 \sigma_m^2 + \sigma_\epsilon^2)$. Details of the simulation procedure are described in the Appendix. In addition to using the Merton model which assumes that the payoff on the bond is $\min[V, B]$, we also allow for a fixed recovery rate in the event of default which is triggered when $V < B$.

Table 7 reports the results for six-tranche securitisations of a portfolio of 125 B-rated underlying bonds under the *default probability* and the *expected default loss* rating systems. The risk parameters of the issuers of the underlying bonds, (β, σ_ϵ) , are the same as those for Tables 4 and 5.³⁸ Note first that the equilibrium yields for the tranching SPV debt shown in Table 7 are considerably above the equilibrium yields for the ratings-based yields reported in Tables 4 and 5, particularly for the more junior tranches. Under

³⁸ The valuation of the reference bonds is shown in Tables 4 and 5. The implied correlations between the returns on the issuers of the bonds in the portfolio is 0.17.

Table 7
Pricing the liabilities of a 5-year maturity SPV holding corporate bond collateral

In Panel A tranche face values, B_{i,k_i} , are set to yield the default probability, Π_{k_i} appropriate for its rating class. The tranche is valued using the Merton model and its sale price is determined using the rating-based multiplier, ϕ_k^* from Table 4. In Panel B, the tranche face values, B_{i,k_i} , are set to yield the expected loss, Λ_{k_i} appropriate for its rating class, and are valued using the Merton model. The *Sales Price* of each tranche is calculated by multiplying the face value by the rating appropriate multiplier, ϕ_k^* , from Table 5. Parameter assumptions: Collateral: 125 issues of B rated bonds; for issuers (β ; σ_e) = (0.8; 0.25), $\tau = 5$. The implied correlation between issuer returns is 0.17. The asset risk of the firm underlying the ratings is (β^* ; σ_k^*) = (0.8; 0.25) with $\tau_f = 3.5\%$, $\tau_m - \tau_f = 7\%$, $\sigma_m = 14\%$.

Panel A: Default probability rating system (S&P)

Tranche <i>i</i>	S&P rating k_i	Probability of default Λ_{k_i}	ϕ_k^*	Face value tranche B_{i,k_i}	Equilibrium value tranche W_{i,k_i}	Equilibrium yield to maturity	Sales price $S_{i,k_i} = \phi_k^* B_{i,k_i}$	Gain $S_{i,k_i} - W_{i,k_i}$
1	AAA	0.061%	0.839	93.46	78.37	3.52%	78.42	0.06
2	AA	0.219%	0.838	5.49	4.48	4.10%	4.61	0.13
3	A	0.459%	0.837	2.66	2.12	4.54%	2.23	0.11
4	BBB	2.223%	0.828	9.12	6.76	6.00%	7.56	0.80
5	BB	10.424%	0.792	8.15	4.81	10.57%	6.46	1.65
6	B	24.46%	0.730	5.55	2.16	18.88%	4.05	1.89
-	Equity Total				1.31		104.63	4.63

Panel B: Expected default loss rating system (Moody's)

Tranche <i>i</i>	Moody's rating k_i	Exp. loss Λ_{k_i}	ϕ_k^*	Face value tranche B_{i,k_i}	Equilibrium value tranche W_{i,k_i}	Equilibrium yield to maturity	Sales price $S_{i,k_i} = \phi_k^* B_{i,k_i}$	Gain $S_{i,k_i} - W_{i,k_i}$
1	Aaa	0.002%	0.839	80.82	67.81	3.51%	67.85	0.04
2	Aa	0.037%	0.836	9.49	7.84	3.82%	7.94	0.10
3	A	0.257%	0.834	5.86	4.70	4.40%	4.88	0.18
4	Baa	0.869%	0.818	7.66	5.81	5.54%	6.27	0.46
5	Ba	4.626%	0.759	14.09	8.69	9.67%	10.69	2.00
6	B	11.390%	0.672	1.75	0.82	15.33%	1.18	0.36
-	Equity Total				4.34		103.14	3.14

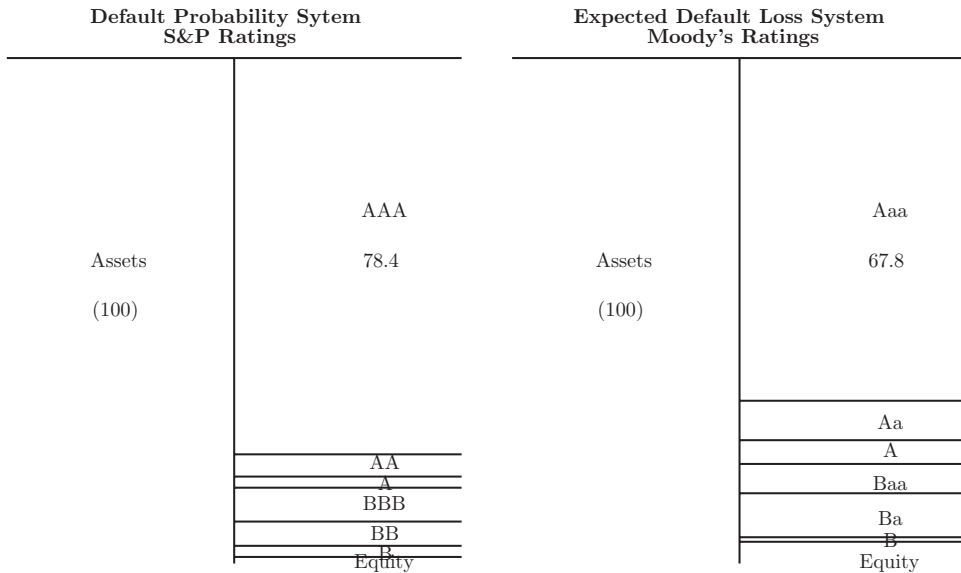


Fig. 3. Equilibrium market value capital structures of an SPV under two different rating systems (Parameter assumptions as in Table 7).

the *expected default loss (default probability)* rating system the ratings-implied yield on a Ba (BB) bond is 5.52% (4.66%) whereas the equilibrium yield on the correspondingly rated tranche is 9.67% (10.57%). Figure 2 shows the equilibrium tranche yields under both rating systems for our example along with the model ratings-based yields and the ‘actual’ corporate bond yields. While, as noted above, our calibration matches the *corporate* yield spreads quite well, it implies that the yields on equivalently rated tranches from bond securitisations should be much higher; for the BBB *default probability* rated tranche the implied spread between the yield on the SPV liability and the ratings-implied yield is $6.00 - 3.77 = 2.23\%$. Yet, as shown in Figure 1, the average spread between BBB CDO tranches and BBB corporate yields during the period 2005.4 to 2007.3 when Arbitrage CDO issuance was at its peak was only 3.7 *basis points*. It is this lack of spread between equivalently rated SPV and corporate liabilities which gives rise to the arbitrage opportunities from tranching and securitisation.

Comparing the tranche structure of the bond securitisation to the debt structure of the single corporate issuer of tranching debt, we see that the senior tranches of the securitisation are much larger as the result of both the seniority of the bond collateral relative to a pure equity claim and the effects of diversification: under the Moodys (S&P) rating system, the AAA tranche accounts for 67.8% (78.4%) of the liability value as compared with only 11.5% (15.1%) for the corporate issuer. Figure 3 shows to scale the equilibrium market value capital structures of the SPV for the examples in Table 7. Despite the conceptual differences between the Moody’s and S&P rating systems, the structures implied by the two systems are fairly similar and correspond to structures observed in the market.

In contrast to the case of a single firm issuing tranching debt, there is now a small positive marketing gain on the AAA-tranches, although the gains on the higher rated tranches are proportionally much smaller than those of lower rated tranches: the gain on the AAA tranche is only 5–8 *basis points*, while those on the B rated tranches are

46.3% and 43.9% of the tranche values. This result contrasts with the suggestion of Coval *et al.* (2009) who claim that ‘highly rated tranches should trade at significantly higher yield spreads than single name bonds with identical credit ratings’. Interestingly, this suggestion is contradicted by their finding that ‘triple-A rated tranches trade at comparable yields to triple-A rated bonds’, which is consistent with our results in Table 7. As derived above in Tables 4 and 5, the equilibrium yield on the AAA reference bond is 3.50–3.51%, while Panel A (B) of Table 7 shows that the equilibrium yield on the AAA tranche is 3.52% (3.51%) under the *default probability* (*expected default loss*) rating system.

There is a significant difference between the marketing gains for the corporate issuer and for the securitiser under the *expected default loss* rating system. The marketing gain is now 3.14% as compared with only 47 *basis points* for the single corporate issuer.³⁹ Moreover, the gain is proportionately much larger when the virtually riskless and presumably easy to sell AAA tranche is excluded: the proportionate gain on the non-AAA tranches is 21.1% for the *default probability* system and 9.75% for the *expected default loss* system.

Table 8 reports summary statistics on the marketing gains under a variety of different scenarios to assess the sensitivity of our findings to variation in the risk characteristics of the reference firm and of the bond collateral issuers, the rating of the bond collateral, the number of bonds in the collateral, the number of tranches, and the market risk premium and volatility. In this table the base examples analysed in Table 7 are repeated in bold font. We report two measures of the marketing gain: columns headed ‘A’ report the gain expressed as a proportion of the total value of the collateral, while the ‘B’ columns report the gain expressed as a proportion of the total value excluding the value of the AAA tranche. The marketing gains are most sensitive to the number of tranches and the rating of the bonds held as collateral. Excluding these two variables, the A measure gains calculated using the Merton model range from 3.79% to 6.38% for S&P ratings, and from 2.61% to 4.14% for the Moody’s ratings, while the B measure gains range from 16.55% to 22.26% under the S&P system, and from 7.12% to 13.95% under the Moody’s system. The A measure of gains is quite sensitive to the rating of the underlying bond collateral: under the Moody’s system the A measure gain drops from 3.14% to 1.95% as the bond rating increases from B to BB; however, the B measure gain changes only from 9.75% to 9.02%, since the higher quality collateral permits a much greater proportion of AAA debt to be issued. Both A and B measure gains are highly sensitive to the number of tranches and to the rating of the most junior tranche. Under the Moody’s system the A (B) measure drops from 3.14% (9.75%) to 1.76% (4.21%) as the number of tranches is reduced from six to two when the junior tranche of the two tranche issue is rated Baa, but the profit actually increases to 5.37% (16.68%) when the junior tranche has only a Ba rating.

As a further check on the robustness of our results, the Merton model for payoffs on both the reference bond and the bond held as collateral was replaced by a model with a fixed recovery rate with default being triggered at maturity by the condition that $V < B$. When the recovery rate is set at 40% the results are very similar to those obtained using the Merton model. Using the Moody’s rating system, the A measure gain falls from 3.96% to 2.15%, while the B measure rises from 8.94% to 10.75% as the assumed recovery rate is increased from 20% to 60%.

³⁹ Under the *default probability* system the gain actually decreases from 5.45% to 4.63%.

Table 8
Marketing gains from securitisation of corporate bonds

The table reports the marketing gains from securitising a portfolio of corporate bonds when tranches are sold at ratings-based yields according to S&P and Moody's ratings. Columns headed 'A' report the marketing gain expressed as a proportion of the total collateral value, while the 'B' columns report the gain expressed as a proportion of the total value excluding the value of the AAA tranche. The characteristics of the reference firm are set to $(\beta^*, \sigma_\varepsilon^*) = (0.8, 0.25)$; these parameters are varied in examples (ix) and (x). In addition, $r_f = 3.5$ and $r_m - r_f = 7.0, \sigma_m = 14.0$. For the base case, the SPV holds a portfolio of 125 B-rated bonds whose issuers with risk parameters $(\beta, \sigma_\varepsilon) = (0.8, 0.25)$. The SPV is assumed to issue 6 differently rated tranches corresponding to the ratings whose characteristics are described in Tables 1 and 3. In example (iv) the two tranches are first assumed to be rated AAA (Aaa) and BBB (Baa) and second AAA (Aaa) and B (Ba) by S&P (Moody's). For purpose of comparison the parameter and marketing gain of the base case are repeated in bold for each parameter perturbation. The last two columns show the results when assuming a fixed recovery rate of 40% for the underlying bonds. This assumption is varied in case (xi).

Variation	Merton model				Fixed recovery (40%)				
	S&P ratings		Moody's ratings		S&P ratings		Moody's ratings		
	A	B	A	B	A	B	A	B	
(i) Base Case	4.63%	21.10%	3.14%	9.75%	5.19%	18.40%	3.09%	9.53%	
(ii) β	1.0	6.38	22.20	4.14	10.10	6.58	19.46	4.05	10.66
	0.8	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	0.7	3.79	21.41	2.61	9.32	4.19	16.55	2.74	9.06
(iii) σ_ε	0.30	4.07	21.40	2.61	9.32	4.01	15.62	2.58	8.94
	0.25	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	0.20	5.36	21.30	3.73	10.20	6.58	19.47	4.05	10.65
(iv) No. of Tranches	2	0.97	16.78	1.76	4.21	0.48	1.58	1.47	5.08
	6	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	2	2.34	10.78	5.37	16.68	0.52	1.86	5.42	16.73
(v) No. of Bonds	62	4.52	19.36	3.12	7.48	4.75	15.63	3.03	10.47
	125	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	140	4.62	19.91	3.16	8.32	5.04	17.81	3.06	8.81
(vi) $r_m - r_f$	8	5.62	27.15	4.09	13.15	6.45	24.2	4.00	12.77
	7	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	6	3.74	16.55	2.34	7.05	4.05	13.78	2.33	6.96
(vii) σ_m	12	4.53	16.90	2.74	7.12	4.40	13.46	2.63	7.40
	14	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	16	4.72	28.95	3.60	13.95	5.82	24.25	3.77	13.23
(viii) Rating of Bonds	BB	2.40	19.05	1.95	9.02	3.47	15.95	2.13	10.24
	B	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
(ix) β^*	1.0	4.42	20.37	2.89	9.26	4.73	16.77	2.85	8.80
	0.8	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	0.6	4.83	22.26	3.39	10.86	5.66	20.21	3.37	10.40
(x) σ_ε^*	0.30	4.62	21.29	3.28	10.51	5.50	19.64	3.26	10.06
	0.25	4.63	21.10	3.14	9.75	5.19	18.40	3.09	9.53
	0.20	4.63	21.34	2.94	9.52	4.73	16.88	2.84	8.77
(xi) Recovery Rate	20	–	–	–	–	5.93	14.15	3.96	8.94
	40	–	–	–	–	5.19	18.53	3.09	9.54
	60	–	–	–	–	3.85	22.84	2.15	10.75

Overall, the results in Table 8 are consistent with the observations made in the previous section. Again, the marketing gains are higher under the S&P *default probability* rating system than under the Moody's *expected default* loss rating system. As shown in examples (ii) and (iii), the higher the systematic risk, β , and the smaller the residual risk, σ_{ε} , of the bond collateral issuers, the higher is the marketing gain from securitisation. Significantly, under *expected default loss* rating, the A measure of marketing gains from securitizing a portfolio of bonds and issuing tranching debt under ratings-based pricing is significantly larger than the gains from corporate tranching of debt reported in Table 6. The gains from securitisation are further magnified when they are expressed as a proportion of the value excluding the AAA tranche. Under *default probability* rating the A measure of gains is comparable across corporate issues and securitisations, but the B measure is much greater for securitisations because of the larger amount of AAA debt that can be supported by the SPV collateral.

8. Conclusion

In this paper we have analysed the gains from issuing tranching debt in a market in which bonds can be sold to investors at prices and yields that reflect only their credit rating. The rating can depend on default probabilities as in the case of Standard & Poor's or on expected default losses as in the case of Moody's. For both rating systems, we find general conditions under which tranching debt is overpriced. These conditions relate to the risk characteristics of the collateral relative to those of the reference firm from which ratings-based bond yields are derived.

We first quantify the marketing gains available to a corporate debt issuer under ratings-based pricing using the CAPM and the Merton (1974) structural debt model to value bonds. We find that the potential gains are greater under the S&P system than under the Moody's system and in most cases the marketing gain under the Moody's system is small, suggesting that this system is fairly robust for the purpose of pricing corporate liabilities.

However, the marketing gains are potentially much higher for Special Purpose Vehicles which hold corporate bonds as collateral. In particular, we show that the more junior tranches are likely to be significantly mispriced under ratings-based pricing, even when ratings depend on *expected default losses*. For example, in the example in Panel B of Table 7 the most junior, B-rated, tranche has an equilibrium yield to maturity of 15.33%, while the ratings-based yield shown in Panel A of Table 5 is only 7.93%. As a result, marketing gains of the order of 3–4% of the collateral value are easily attainable, and these are magnified to 9–11% if the easy to sell and properly priced AAA tranches are excluded from the calculation.

Thus, to the extent that investors relied on bond ratings in their evaluation of CDO tranches, the explosion in the issuance of 'arbitrage CDO's' during 2006 and 2007 can be explained by the mispricing that would be caused by ratings-based pricing. There is considerable anecdotal evidence of investor reliance on ratings, and Figure 1 shows that spreads on equivalently rated corporate bonds and CDO tranches were close to zero during this period. As shown in Figure 2, a simple calibration of the bond pricing model produces corporate yield spreads comparable to those observed in the period 2004.9–2006.9 and implies much higher equilibrium spreads on junior CDO tranches.

Our analysis implies that CDO liabilities with probabilities of default or expected default losses that are the same as those of corporate liabilities can be expected to trade

at significantly different yields, and this is particularly true for the most subordinated tranches. To the extent that investors and regulators rely on credit ratings as an indicator of risk and therefore of equilibrium yields, our analysis supports the case for introducing ratings modifiers for structured products as suggested by the SEC.

A. Proofs

A.1 Proof of Lemma 1

- (a) If $P \geq^{FSD} P^*$, the first order stochastic dominance ranking of the physical distributions implies that under a default probability rating system $B_k \geq B_k^*$. Then note that (6) can be written as:

$$\Omega = \frac{B_k}{B_k^*} E_{Q^*} \{ \min[B_k^*, V^*] \} - E_Q \{ \min[B_k, V] \} \tag{20}$$

$$= E_{Q^*} \left\{ \min \left[B_k, \frac{B_k}{B_k^*} V^* \right] \right\} - E_Q \{ \min[B_k, V] \}$$

$$\geq E_{Q^*} \{ \min[B_k, V^*] \} - E_Q \{ \min[B_k, V] \} \tag{21}$$

Ω is positive if $Q^* \geq^{SSD} Q$.

For the converse argument note that $P^* \geq^{FSD} P$ implies $B_k < B_k^*$.

- (b) Note that if $P_2 \geq^{FSD} P_1$ the face value of the k -rated bond issued by the second issuer, B_k^2 , is greater than the face value of bond issued by the first issuer, B_k^1 . This implies that Ω^2 is greater than Ω^1 since expression (20) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.2 Proof of Lemma 2

- (a) If $P \geq^{SSD} P^*$, the second order stochastic dominance ranking of the physical distributions implies that under an expected default loss rating system $B_k \geq B_k^*$. The rest of the proof follows from the proof of Lemma 1.
- (b) If $P_2 \geq^{SSD} P_1$ the face value of the k -rated bond issued by the second issuer, B_k^2 , is greater than the face value of bond issued by the first issuer, B_k^1 . This implies that Ω^2 is greater than Ω^1 since expression (20) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.3 Proof of Lemma 4

$$\Delta\Omega = \phi_{k_1}^* B_{k_1} + \phi_{k_2}^* B_{k_2} - \phi_k^* B_k \tag{22}$$

Now

$$\phi_{k_1}^* \equiv \frac{E_{Q^*} \min[B_{k_1}^*, V]}{B_{k_1}^*}, \quad \phi_{k_2}^* \equiv \frac{E_{Q^*} \min[B_{k_2}^*, V]}{B_{k_2}^*}, \quad \phi_k^* \equiv \frac{E_{Q^*} \min[B_k^*, V]}{B_k^*} \tag{23}$$

Therefore substituting from equations (23) in (22) and noting that $B_k = B_{1,k_1} + B_{2,k_2}$, we have:

$$\Delta\Omega = \frac{B_{1,k_1}}{B_{k_1}^*} E_{Q^*} \min[B_{k_1}^*, V] + \frac{B_{2,k_2}}{B_{k_2}^*} E_{Q^*} \min[B_{k_2}^*, V] - \frac{B_{1,k_1} + B_{2,k_2}}{B_k^*} E_{Q^*} \min[B_k^*, V] \tag{24}$$

Now, under an expected default loss rating system, the SPV bonds have the same expected payoff per unit of face value as do the correspondingly rated corporate bonds, so that:

- for the untranchched issue:

$$\frac{E_P \min[B_k, V]}{B_k} = \frac{E_{P^*} \min[B_k^*, V]}{B_k^*} \tag{25}$$

- for the senior tranche:

$$\frac{E_P \min[B_{1,k_1}, V]}{B_{1,k_1}} = \frac{E_{P^*} \min[B_{k_1}^*, V]}{B_{k_1}^*} \tag{26}$$

- for the junior tranche:

$$\frac{E_P \{ \min[B_k, V] - \min[B_{1,k_1}, V] \}}{B_{2,k_2}} = \frac{E_{P^*} \min[B_{k_2}^*, V]}{B_{k_2}^*} \tag{27}$$

Then substituting for B_k^* , $B_{k_1}^*$, and $B_{k_2}^*$ from equations (25)–(27) in (25):

$$\Delta\Omega = \left\{ \frac{E_{Q^*} \min[B_{k_1}^*, V]}{E_{P^*} \min[B_{k_1}^*, V]} - \frac{E_{Q^*} \min[B_{k_2}^*, V]}{E_{P^*} \min[B_{k_2}^*, V]} \right\} E_P \min[B_{1,k_1}, V] + \left\{ \frac{E_{Q^*} \min[B_{k_2}^*, V]}{E_{P^*} \min[B_{k_2}^*, V]} - \frac{E_{Q^*} \min[B_k^*, V]}{E_{P^*} \min[B_k^*, V]} \right\} E_P \min[B_k, V] \tag{28}$$

Define the bond payoffs, $\pi_1^*(v) = \min[B_{k_1}^*, v]$, $\pi_2^*(v) = \min[B_{k_2}^*, v]$, $\pi^*(v) = \min[B_k^*, v]$, $\pi_1(v) = \min[B_{1,k_1}, v]$, $\pi_2(v) = \min[B_{2,k_2}, v]$ and recall that $E_{Q^*}[v] = E_{P^*}[m^*(v)v]$. Then the incremental profit from the second tranche is

$$\Delta\Omega = \left\{ \frac{E_{P^*}[m^* \pi_1^*]}{E_{P^*}[\pi_1^*]} - \frac{E_{P^*}[m^* \pi_2^*]}{E_{P^*}[\pi_2^*]} \right\} E_P[\pi_1] + \left\{ \frac{E_{P^*}[m^* \pi_2^*]}{E_{P^*}[\pi_2^*]} - \frac{E_{P^*}[m^* \pi^*]}{E_{P^*}[\pi^*]} \right\} E_P[\pi_1 + \pi_2] = (E_P[\pi_1] + E_P[\pi_2]) E_{P^*}[m^*(v)w(v)] \tag{29}$$

where

$$w_x(v) = x \left(\frac{\pi_1^*(v)}{E_{P^*}[\pi_1^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]} \right) + (1 - x) \left(\frac{\pi_2^*(v)}{E_{P^*}[\pi_2^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]} \right) \tag{30}$$

and $x = E_P[\pi_1(v)] / (E_P[\pi_1(v)] + E_P[\pi_2(v)])$. A second tranche will be profitable if there exists an x such that $E_{P^*}[m^*(v)w_x(v)] > 0$. $w_x(v)$ is a piecewise linear function with slopes given by:

$$\frac{dw_x(v)}{dv} = \begin{cases} x \left[\frac{1}{E_{P^*}[\pi_1^*]} - \frac{1}{E_{P^*}[\pi_2^*]} \right] + \left[\frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]} \right] & \text{for } v < B_{k_1}^* \quad (i) \\ (1-x) \frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]} & \text{for } B_{k_1}^* < v < B_k^* \quad (ii) \\ (1-x) \frac{1}{E_{P^*}[\pi_2^*]} & \text{for } B_k^* < v < B_{k_2}^* \quad (iii) \\ 0 & \text{for } v > B_{k_2}^* \quad (iv) \end{cases}$$

Note that the face value and therefore the expected payoff of a corporate bond is a decreasing function of its rating so that:

$$\frac{1}{E_{P^*}[\pi_1^*]} > \frac{1}{E_{P^*}[\pi^*]} > \frac{1}{E_{P^*}[\pi_2^*]}$$

Then for $0 \leq x \leq 1$ the slope dw_x/dv is negative in region (ii), positive in region (iii) and zero in region (iv). Note that $E_{P^*}[w_x(v)] = 0$. Consider $x = \hat{x}$ such that $w_{\hat{x}}(v) = 0$ in region (iv). Equation (30) implies that

$$\hat{x} = \frac{B_k^*/E_{P^*}[\pi^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}{B_{k_1}^*/E_{P^*}[\pi_1^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}$$

Since $E_{P^*}[w_x(v)] = 0$, the slope conditions in regions (ii) and (iii) imply that $w_{\hat{x}}(v) > 0$ in region (i), which is sufficient for $\Delta\Omega \propto E_{P^*}[m^*(v)w_x(v)] > 0$ if $m^*(v)$ is a decreasing function.

B. Simulating SPV Cash Flows

In the following we sketch our simulation procedure.

1. *Determination of debt face value*

Given the rating k and maturity τ of a bond issued by firm j we can determine the face value, \hat{B}_k , of each bond in the SPV portfolio. Under the default probability rating system \hat{B}_k is obtained from equation (9) using the historical default probability given by S&P.

Under the expected default loss rating system we have to solve equations (10) and (11) iteratively for \hat{B}_k until the expected loss rate, Λ_k , equal to that given by the Moody's rating.⁴⁰

2. *Simulation of SPV value*

For each firm associated with the bonds in the SPV portfolio we can simulate its asset value at τ under the physical measure by:

$$V_j(\tau) = V_j(0) \exp [(\mu - 0.5\sigma^2)\tau + \beta\sigma_m\sqrt{\tau}z_0 + \sigma_\varepsilon\sqrt{\tau}z_j] \quad (31)$$

$z_0, z_j \text{ iid } \mathcal{N}(0, 1) \quad j = 1, \dots, J$

⁴⁰ In case of using a fixed recovery rate of R , meaning that the bond pays off $R \cdot \hat{B}_k$ in any default state, equation (11) reduces to $\hat{L} = \hat{B}_k(1 - R)\mathcal{N}(-d_2^p)$.

Analogously the risk-neutral value, $V_j^Q(\tau)$, is given by the same formula with μ replaced by r_f . For each simulation run n , $V_j(\tau)$ is produced for all J firms, and the cashflow from bond j can then be determined as

$$CF_{j,n}(\tau) = \min[V_{j,n}(\tau), \hat{B}_k] \tag{32}$$

The bond defaults if $V_{j,n}(\tau) < \hat{B}_k$.⁴¹

The total portfolio cashflow under the physical measure is then given by

$$CF_{SPV,n}(\tau) = \sum_{j=1}^J CF_{j,n}(\tau) \tag{33}$$

and, analogously, under the risk-neutral measure

$$CF_{SPV,n}^Q(\tau) = \sum_{j=1}^J \min[V_{j,n}^Q(\tau), \hat{B}_k] \tag{34}$$

Performing N simulation runs, we get the distribution of the portfolio value in τ under both measures. The market value of the portfolio at $t = 0$ is then derived as:

$$W_{SPV} = e^{-r_f \tau} \frac{1}{N} \sum_{n=1}^N CF_{SPV,n}^Q(\tau) \tag{35}$$

3. Tranche valuation

We assume that the SPV issues I tranches with ratings $k_i (i = 1, \dots, I)$ against the portfolio of bonds. Under the default probability rating system, the aggregate face value B_{k_i} for the SPV portfolio is determined by taking the Π_{k_i} -quantile of the physical distribution of the SPV value obtained from step 2. Again, B_{k_i} has to be solved iteratively under the expected default loss rating system.

Given B_{k_i} , the total market value of the aggregate bond written on the SPV is then derived under the risk-neutral measure by

$$W_{k_i} = e^{-r_f T} \frac{1}{N} \sum_{n=1}^N \min[CF_{SPV,n}^Q, B_{k_i}] \tag{36}$$

The face and market values of each tranche are then calculated as the first differences of the aggregate values:

$$B_{i,k_i} = B_{k_i} - B_{k_{i-1}}, \tag{37}$$

$$W_{i,k_i} = W_{k_i} - W_{k_{i-1}}, \tag{38}$$

with the first tranche, $B_{1,k_1} = B_{k_1}$ and $W_{1,k_1} = W_{k_1}$. The market value of the equity piece can then be derived as

$$W_{equity} = W_{SPV} - \sum_{i=1}^I W_{i,k_i} \tag{39}$$

⁴¹ In case of using a fixed recovery rate, equation (32) is replaced by $CF_{j,n}(\tau) = \hat{B}_k$ for $V_{j,n}(\tau) \geq \hat{B}_k$ and $CF_{j,n}(\tau) = R \cdot \hat{B}_k$ for $V_{j,n}(\tau) < \hat{B}_k$.

4. Sales price and profit

First the yield on the reference bonds with ratings k_i is determined. Given the risk characteristics (β^*, σ^*) of the reference firm on which ratings are based, we can again determine the face value, $B_{k_i}^*$, of the reference bond and the corresponding market value, $W_{k_i}^*$ according to Merton's formula as given by equation (14).⁴² Then the yield is defined as

$$y_{k_i}^* = \frac{1}{T} \ln \frac{B_{k_i}^*}{W_{k_i}^*} \quad (40)$$

According to our pricing assumption, the sales price of tranche i is given by

$$S_{i,k_i} = e^{-y_{k_i}^* T} B_{i,k_i} \quad (41)$$

such that the profit on tranche i is derived as

$$\Omega_i = S_{i,k_i} - W_{i,k_i} \quad (42)$$

The total profit is given by $\Omega = \sum \Omega_i$ which equals a percentage profit of $\frac{\Omega}{W_{SPV}}$ on the portfolio's market value.

References

- Ashcraft, A. B. and Schuermann, T., 'Understanding the securitization of subprime mortgage credit', *Federal Reserve Bank of New York Staff Reports*, No. 318 (2008).
- Black, F. and Scholes, M., 'The pricing of options and corporate liabilities', *Journal of Political Economy*, Vol. 81, 1973, no. 3, pp. 637–54.
- Bank of England, *Financial Stability Report, October 2007*, Issue No. 22 (London, 2007).
- Boot, A. and Thakor, A., 'Security design', *Journal of Finance*, Vol. 48, 1993, pp. 1349–78.
- Brennan, M. and Kraus, A., 'Efficient financing under asymmetric information', *Journal of Finance*, Vol. 42, 1987, pp. 1225–43.
- Cantor, R. and Packer, F., 'Sovereign risk assessment and agency credit ratings', *European Financial Management*, Vol. 2, 1996, pp. 247–56.
- Coval, J. D., Jurek, J. W. and Stafford, E., 'Economic catastrophe bonds', *American Economic Review*, Vol. 99, 2009, pp. 628–666.
- Cuchra, F.-M., 'Explaining launch spreads on structured bonds', *Discussion Paper* (University of Oxford, 2005).
- Cuchra, F.-M. and Jenkinson, T., 'Why are securitisation issues tranching?' *Working Paper* (Department of Economics, Oxford University, 2005).
- DeMarzo, P. and Duffie, D., 'A liquidity-based model of security design', *Econometrica*, Vol. 67, 1999, pp. 65–99.
- DeMarzo, P., 'The pooling and tranching of securities: a model of informed intermediation', *The Review of Financial Studies*, Vol. 18, 2005, pp. 1–35.
- Franke, G., Weber, Th. and Herrmann, M., 'How does the market handle information asymmetries in securitisations?' *Discussion Paper* (University of Konstanz, 2007).
- Gaur, V., Seshadri, S. and Subrahmanyam, M., 'Intermediation and value creation in an incomplete market: implications for securitisation', *Working Paper* (Leonard N. Stern School of Business, New York University, 2005).
- Gorton, G., 'The subprime panic', *European Financial Management*, Vol. 15, 2009, pp. 10–46.

⁴² Using the assumption of a fixed recovery rate R for the reference bond the value of this bond is given by $W_{k_i}^* = B_k^* e^{-r_f \tau} \mathcal{N}(d_2^{Q^*}) + R \cdot B_k^* e^{-r_f \tau} \mathcal{N}(-d_2^{Q^*})$.

- Hein, J., 'Optimization of credit enhancements in collateralized loan obligations – the role of loss allocation and reserve account', *Discussion Paper* (University of Konstanz, 2007).
- Kou, J. and Varotto, S., 'Timeliness of spread implied ratings', *European Financial Management*, Vol. 14, 2008, pp. 503–27.
- Longstaff, F. A. and Rajan, A., 'An empirical analysis of the pricing of collateralized debt obligations', *Journal of Finance*, Vol. 63, 2008, pp. 529–563.
- Merton, R. C., 'On the pricing of corporate debt: the risk structure of interest rates', *Journal of Finance*, Vol. 29, 1974, pp. 449–70.
- Modigliani, F. and Miller, M. H., 'The cost of capital, corporation finance and the theory of investment', *American Economic Review*, Vol. 48, 1958, no. 3, pp. 261–97.
- Moody's Investors Service, *Special Comment: Default & Loss Rates of Structured Finance Securities: 1993-2004* (New York, 2005).
- Plantin, G., 'Tranching', *Working Paper* (<http://ssrn.com/abstract=650839>), (2004).
- Riddiough, T., 'Optimal design and governance of asset-backed securities', *Journal of Financial Intermediation*, Vol. 6, 1997, pp. 121–52.
- Ross, S., 'Institutional markets, financial marketing, and financial innovation', *Journal of Finance*, Vol. 44, 1989, no. 3, pp. 541–56.
- Securities Industry and Financial Markets Association, *Research*, <http://www.sifma.org/research/global-cdo.html> (2008).
- Standard & Poor's, *CDO Evaluator Version 3.0: Technical Document* (London, 2005).
- Standard & Poor's, *Structured Finance: Commentary* (2007).
- Subrahmanyam, A., 'A theory of trading in stock index futures', *Review of Financial Studies*, Vol. 4, 1991, pp. 17–51.
- Welch, I., 'Views of financial economists on the equity premium and on professional controversies', *Journal of Business*, Vol. 73, 2000, pp. 501–37.