

# Expected Life-Time Utility and Hedging Demands in a Partially Observable Economy

**Frederik Lundtofte**  
February, 2005

Dept. of Economics, Lund University, P. O. Box 7082, S-220 07 Lund, Sweden.  
Phone: +46 46 222 49 74. Fax: +46 46 222 46 13. E-mail: frederik.lundtofte@nek.lu.se

## **Abstract**

This paper analyzes the expected life-time utility and the hedging demands in a Lucas (1978) economy, in which the dividend drift term is unknown and mean-reverting. An expression for the individual investor's expected life-time utility in equilibrium is derived, and his hedging demand is analyzed. The hedging demand consists of two components, which could work in opposite directions so that a conservative investor may end up having a positive hedging demand. Interestingly, this differs from the theoretical findings in Brennan (1998), who analyzes the portfolio choice problem of an agent who learns about a constant expected stock return.

**Keywords:** learning, incomplete information, equilibrium, hedging demands

**JEL Classification Codes:** C13, G11, G12

I thank David Feldman, Michael Brennan, Yihong Xia, Carsten Sørensen and Björn Hansson for helpful discussions, and conference participants at the meetings of The Asian Finance Association in Taipei 2004 and The Econometric Society in Madrid 2004. Financial support from *Bankforskningsinstitutet* is gratefully acknowledged.

# 1 Introduction

A recent strand of literature studies the effects of incomplete information on various aspects of the economy. While traditional models assume that investors already know all the parameters, some more recently developed models incorporate the fact that investors have to learn the true dynamics of the economy - they do not know the truth with certainty. Instead, they observe the realizations of different variables in the economy, and from this they make an assessment of the true dynamics of the economy. Indeed, this point of departure is appealing because it corresponds more closely to the actual behavior of real investors.

Early contributions within the field of incomplete information are Williams (1977), Detemple (1986), Dothan and Feldman (1986) and Feldman (1989). Williams (1977) studies the CAPM under heterogeneous Gaussian distributions of investor beliefs consistent with available information. Detemple (1986) analyzes incomplete information regarding technology-relevant state variables, while Dothan and Feldman (1986) and Feldman (1989) analyze equilibrium interest rates and multiperiod bonds under partial information.

More recent contributions include Brennan (1998), Yan (2000), Ziegler (2000), Riedel (2000), Feldman (2002, 2003) and Cvitanić et al. (2004). Brennan (1998) studies the portfolio optimization problem when the agent learns about a constant drift in the stock return. In Brennan (1998), the dynamics of the stock price are exogenously given. Yan (2000) solves for the equilibrium asset prices and analyzes the volatility of stock index options. In contrast to Brennan (1998), Yan (2000) develops an equilibrium model, in which agents learn about a mean-reverting drift term in the dividend growth rate. Goldstein and Zapatero (1996) analyze asset prices and stock index options in the equivalent full information economy where there is a perfect positive correlation between the dividend growth rate and the true dividend drift term. Ziegler (2000) studies the portfolio choice problem of an agent whose beliefs about the dividend growth rate differ from the market beliefs. Riedel (2000) shows that in a Lucas (1978) economy in which there is an unobservable constant dividend growth rate, the term structure of interest rates decreases to negative infinity. The result in Riedel (2000) is in contrast to the result in Feldman (1989)

who shows that in a similar economy, where the unobservable productivity factors are stochastic, equilibrium term structures are bounded. Feldman (2003) resolves the apparent contradiction between Feldman (1989) and Riedel (2000). Feldman (2002) is providing a theoretical framework for empirical tests of asset-pricing models with unobservable productivity factors. Cvitanić et al. (2004) use the incomplete information framework in order to assess the economic value of analysts' recommendations.

The purpose of this paper is to analyze the expected life-time utility and the hedging demands in a pure Lucas (1978) economy, in which agents learn about an unknown mean-reverting drift term in the dividend growth rate process. The economy under consideration is thus equivalent to that in Yan (2000) but, instead of focusing on asset prices and option volatility, our paper focuses on the expected life-time utility and the hedging demands. As such, the main contribution of this paper is the analysis of the expected life-time utility and the hedging demand in a partially observable economy with a stochastic mean-reverting dividend drift. We derive an expression for the expected life-time utility and the hedging demand in equilibrium.

In our model as well as in Yan's (2000), the fact that the true drift term is stochastic and mean-reverting causes the filtering error to remain in the long run, unless a specific technical condition applies and there is a perfect correlation between the dividend growth rate and the true dividend drift. With a constant drift as in Brennan (1998), the filtering error will eventually vanish.

Agents are assumed to have a constant relative risk aversion ( $\gamma$ ) and maximize expected life-time utility of consumption. Empirical evidence suggests that the coefficient of relative risk aversion should be greater than unity. It is shown that a conservative ( $\gamma > 1$ ) investor dislikes both variability in his estimate and covariation between his estimate and the dividends. These results are in line with Ziegler (2003, Chapter 2), who primarily analyzes the case of a constant drift term.<sup>1</sup>

We also analyze the individual investor's hedging demand derived in this paper. The hedging demand is shown to consist of two components. The first is a hedging

---

<sup>1</sup>In addition, Ziegler (2003, Chapter 2) briefly discusses the effect of a stochastic drift term, which is uncorrelated with the dividend growth rate, and has zero drift.

component that arises because the true dividend drift is stochastic and there is a correlation between the true dividend drift and the dividend growth. The second is a hedging component that arises because the agent has to consider the fact that there is a difference between his estimate and the true drift term, i.e. he has to take into account the presence of an estimation error. In the case of a negative correlation between the true drift term and the dividend growth rate, the two hedging components work in opposite directions, and, assuming a positive equity premium, the investor could end up having a non-negative hedging demand. Interestingly, this differs from the theoretical findings in Brennan (1998), who analyzes optimal portfolio choice when agents learn about a constant drift term in the stock return. He shows that, assuming a positive equity premium, a conservative agent will always have a negative hedging demand. This paper makes it clear that this is not true in an equilibrium model where agents learn about a mean-reverting drift in the dividend growth rate.

A negative equity premium can occur naturally in this economy, and it is easy to provide a natural, straightforward explanation for a negative equity premium. It turns out that, with a negative equity premium, the stock returns and innovations in aggregate consumption are perfectly negatively correlated. The representative agent accepts a negative equity premium, since the stock acts as a hedge against a low future consumption. Since a negative equity premium can occur naturally in this economy, we analyze the implications of a negative equity premium for hedging demands.

Further, it is shown that the hedging components are related to the risk premium. Assuming a positive equity premium and conservative agents, the normalized risk premium is shown to be lower (higher) when agents have positive (negative) hedging demands compared to a situation where the agents' hedging demands are zero. If the agents' hedging demands are zero, then the normalized risk premium will equal the relative risk aversion of the agents.

The organization of this paper is as follows. In section 2, the nature of the economy under consideration is described. The theoretical results are derived in section 3: first, the filtering problem of the partially informed agents is examined in section 3.1, then the equilibrium expected life-time utility under partial information

is analyzed in section 3.2, and its relation to the dynamics of stock prices is examined. The individual investors' hedging demands are analyzed in section 3.3. Finally, section 4 concludes the paper.

## 2 The Economy

The economy considered here is a pure exchange Lucas (1978) economy. We assume that there is one riskbearing asset (stock) in the economy, which pays a continuous stream of dividends ( $D$ ). The consumption good is assumed to be perishable, so that in each period, all dividends are consumed. There is a complete probability space  $(\Omega, F, P)$ . The mean growth in dividends is assumed to be stochastic and to follow a mean-reverting Ornstein-Uhlenbeck process,

$$\frac{dD_t}{D_t} = \mu_t dt + \sigma_D dB_t \quad (1)$$

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + \sigma_\mu dZ_t \quad (2)$$

with  $\sigma_D$ ,  $\kappa$  and  $\sigma_\mu$  being positive constants, and where  $B$  and  $Z$  are Brownian motions defined over the complete probability space  $(\Omega, F, P)$ .  $B$  and  $Z$  are assumed to have instantaneous correlation  $\rho$ , where  $-1 \leq \rho \leq 1$ .

Agents are assumed not to know the true value of the drift term ( $\mu_t$ ). Instead, they have to estimate it from their observations of the realized values of the dividend process. However, they are assumed to know the long-run value of the growth in dividends ( $\bar{\mu}$ ), standard deviations ( $\sigma_D$  and  $\sigma_\mu$ ), the correlation ( $\rho$ ), and the value of the reversion parameter ( $\kappa$ ). Formally, agents are said to have the filtration  $\mathbf{G} = \{G_t\}$  where  $G_t = \sigma(D_s; s \leq t)$ .

All agents are assumed to maximize expected life-time utility of intermediate consumption through a CRRA utility function, subject to a wealth constraint. The instantaneous utility of intermediate consumption is assumed to be of the form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where  $\gamma > 0$  and  $\gamma \neq 1$ . The case of logarithmic utility is a special case of CRRA utility, where the coefficient of relative risk aversion equals one, and it deserves to

be treated separately.<sup>2</sup> However, the case of logarithmic utility can be analyzed in a similar manner, and the details of this case are left to the interested reader. As shown by Merton (1971), logarithmic preferences induce myopic behavior. All agents are assumed to have identical preferences, information and prior beliefs. Thus, the aggregation results from Rubinstein (1974) will hold, and we can use a representative agent framework, where this agent has constant relative risk aversion and maximizes expected utility of consumption conditional on his information set at time  $t$ ,  $G_t$ ,

$$U(\{c_s\}_t^T) = E \left[ \int_{s=t}^T e^{-\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \middle| G_t \right]. \quad (3)$$

Since this is a pure exchange Lucas (1978) economy with a perishable consumption good, the aggregate consumption will equal the aggregate dividend in each period, i.e.  $c_t = D_t$  for all  $t$ .

### 3 Theoretical Results

In this section, we will first analyze the filtering problem of the agents. Thereafter, we will examine the equilibrium properties of the interest rate, the stock price and the expected life-time utility. Finally, we will analyze the portfolio choice problem. The filtering problem is analyzed in section 3.1 below. In section 3.2, we examine the equilibrium properties of the interest rate, the stock price, and the expected life-time utility. The portfolio choice problem is analyzed in section 3.3.

#### 3.1 The Filtering Problem

As mentioned earlier, agents will have to estimate the unobserved drift term in the dividend growth equation, basing their estimates on their observations of the realized values of dividends. As noted by Feldman (2004), we need to know the conditional distribution of the unknown drift in order to re-represent the agent's original optimization problem as a Markovian one. Assuming a Gaussian prior, finding the posterior distribution of the drift becomes a standard filtering problem,

---

<sup>2</sup>Note that, although the function  $c^{1-\gamma}/(1-\gamma)$  explodes as  $\gamma \rightarrow 1$ ,  $(c^{1-\gamma} - 1)/(1-\gamma)$  goes to  $\ln c$ .

which fits into the Kalman-Bucy framework. First, we will derive the evolution of the conditional mean, and thereafter, examine its properties in steady state.

### 3.1.1 The conditional mean of the unknown drift

Here we will derive the evolution of the conditional mean of the unknown drift  $\mu_t$ . Applying Theorem 12.1 in Liptser and Shiryaev (2001), we can find the SDEs of the conditional mean  $m_t = E[\mu_t | G_t]$  and the conditional variance  $v_t = E[(\mu_t - m_t)^2 | G_t]$  of  $\mu_t$ .  $v_t$  is sometimes called the "filtering error," since it measures the conditional mean squared error. The conditional mean, i.e. the expected value of the unknown drift conditional on all available information, can be interpreted as the agents' estimate of the drift term.

**Proposition 1** *If agents' prior distribution over  $\mu_0$  is Gaussian with mean  $m_0$  and variance  $v_0$ , then the conditional mean  $m_t = E[\mu_t | G_t]$  satisfies*

$$dm_t = \kappa(\bar{\mu} - m_t)dt + \left( \frac{\rho\sigma_\mu\sigma_D + v_t}{\sigma_D^2} \right) \left( \frac{dD_t}{Dt} - m_t dt \right) \quad (4)$$

where the conditional variance  $v_t = E[(\mu_t - m_t)^2 | G_t]$  of  $\mu_t$  satisfies the Riccati equation

$$\frac{dv_t}{dt} = -2\kappa v_t + \sigma_\mu^2 - \left( \frac{\rho\sigma_\mu\sigma_D + v_t}{\sigma_D} \right)^2. \quad (5)$$

Furthermore, the posterior distribution of  $\mu_t$  is also Gaussian, with  $\mu_t | G_t \sim N(m_t, v_t)$ .

**Proof.** See Theorem 12.1 in Liptser and Shiryaev (2001). ■

Note that we can write

$$\frac{dD_t}{Dt} = m_t dt + \sigma_D d\bar{B}_t \quad (6)$$

where

$$d\bar{B}_t \equiv \frac{1}{\sigma_D} \left( \frac{dD_t}{Dt} - m_t dt \right) = dB_t + \left( \frac{\mu_t - m_t}{\sigma_D} \right) dt.$$

Note that  $d\bar{B}_t$  is the normalized unanticipated innovation in dividend growth, since  $d\bar{B}_t = \frac{1}{\sigma_D} \left( \frac{dD_t}{Dt} - E \left[ \frac{dD_t}{Dt} | G_t \right] \right)$ . Moreover, according to standard filtering theory,  $\bar{B}_t$  is a Brownian motion with respect to the agents' filtration  $G_t$  (see Liptser and Shiryaev (2001)). Inserting equation (6) into equation (4), we have

$$dm_t = \kappa(\bar{\mu} - m_t)dt + \left( \frac{\rho\sigma_\mu\sigma_D + v_t}{\sigma_D} \right) d\bar{B}_t. \quad (7)$$

This can be rewritten as

$$dm_t = \kappa(\bar{\mu} - m_t)dt + \sigma_m(v_t)d\bar{B}_t, \quad (8)$$

where the diffusion coefficient is

$$\sigma_m(v_t) = \frac{\rho\sigma_\mu\sigma_D + v_t}{\sigma_D}. \quad (9)$$

Note that this diffusion coefficient might be negative, since we can have  $\rho < 0$ , with  $\rho\sigma_\mu\sigma_D$  "beating"  $v_t$ .

### 3.1.2 Steady state

We will now examine the steady-state properties of the agents' estimate. First, we need to define what is meant by a steady state in this context. The steady-state value of the filtering error is defined as the constant solution to equation (5). In the following proposition, we determine the stable steady-state value of the filtering error ( $v_t$ ).

**Proposition 2** *The stable steady-state value of the filtering error ( $v_t$ ) is given by*

$$v^* = -(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) + \sqrt{(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D)^2 + (1 - \rho^2)\sigma_\mu^2\sigma_D^2} = \quad (10)$$

$$-\kappa\sigma_D^2 - \rho\sigma_\mu\sigma_D + \sigma_D\sqrt{\kappa^2\sigma_D^2 + 2\kappa\sigma_D\rho\sigma_\mu + \sigma_\mu^2} \quad (11)$$

*The solution  $v^*$  is always non-negative.*

**Proof.** See Feldman (1989). ■

Feldman (1989) shows that in general there are two steady states, of which one is unstable and the other one is stable. It follows from his analysis that the steady state  $v^*$  considered above is stable. In our model, the unstable steady state is always non-positive. Further, if  $v_0 > v^*$ , then the filtering error ( $v_t$ ) will decline monotonically towards its stable steady state ( $v^*$ ), but never reach it. If  $v_0 < v^*$ , then  $v_t$  will increase monotonically towards  $v^*$ , but never reach it. If the variance of the prior equals the stable steady-state value ( $v_0 = v^*$ ), then  $v_t$  will always be equal to the stable steady-state value ( $v_t = v^*$  for all  $t > 0$ ). Hence, in the limit, the filtering error ( $v_t$ ) will converge towards its stable steady state  $v^*$ , i.e.

$$\lim_{t \rightarrow \infty} v_t = v^*.$$

When the agent learns about a constant drift, as in Brennan (1998), the filtering error will eventually disappear as  $t$  goes to infinity. In contrast, when the agent learns about a stochastic, mean-reverting dividend drift, the filtering error will remain, even as  $t$  goes to infinity, unless there is a perfect correlation between the dividend drift and the dividend growth and a specific technical condition is satisfied.

**Proposition 3** *Assuming  $\sigma_D$ ,  $\sigma_\mu$  and  $\sigma_D$  are positive, we have*

$$v^* = 0 \Leftrightarrow (\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) \geq 0 \text{ and } \{\rho = +1 \text{ or } \rho = -1\}$$

**Proof.**  $\Rightarrow$ ) From equation (10), we see that for  $v^*$  to be zero, we must have  $(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) \geq 0$ . Then,

$$0 = -(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) + \sqrt{(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D)^2 + (1 - \rho^2)\sigma_\mu^2\sigma_D^2}$$

implies

$$(1 - \rho^2)\sigma_\mu^2\sigma_D^2 = 0.$$

Thus, it must be that  $\rho = +1$  or  $\rho = -1$ .

$\Leftarrow$ ) Insert  $\rho = +1$  or  $\rho = -1$  into equation (10). Assuming  $(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) \geq 0$ , the result follows. ■

The above proposition states that the filtering error will eventually vanish if and only if a technical condition holds and the true drift term is perfectly correlated with the dividends.

The stable steady-state value of  $v_t$  can be used to compute the corresponding stable steady-state value of the diffusion coefficient of the conditional mean. As  $v_t \rightarrow v^*$ ,  $\sigma_m(v_t) \rightarrow \sigma_m(v^*) \equiv \sigma_m^*$ , and thus the following proposition holds.

**Proposition 4** *In the stable steady state, the diffusion coefficient of the estimate ( $\sigma_m(v_t)$ ) is given by*

$$\sigma_m^* \equiv \sigma_m(v^*) = -\kappa\sigma_D + \sqrt{\kappa^2\sigma_D^2 + 2\kappa\sigma_D\rho\sigma_\mu + \sigma_\mu^2}. \quad (12)$$

**Proof.** Insert the relation (11) into equation (9), and the result follows. ■

In fact, it is possible to show that, in steady state, the variance of the estimate actually never exceeds the variance of the true drift. This result should not come

as a surprise, once it is realized that the process of the drift is unobserved and is estimated only by observing the realized dividend process. The key is to differentiate between the estimation process and the true drift process. Note that a natural choice for an ignorant investor, who does not use any of his observations, is to choose the long-run value as an estimate of the drift term ( $m_t = \bar{\mu}$ ), in which case the variance of the estimate is zero, whereas  $\sigma_\mu^2 > 0$ . In the light of this example, the proposition below should be a natural and intuitive result.

**Proposition 5** *In the stable steady state, the variance of the estimate is lower than or equal to the variance of the true drift, i.e.  $\sigma_m^{*2} \leq \sigma_\mu^2$ .*

**Proof.** As can be seen in equation (12), the sign of  $\sigma_m^*$  depends on whether  $\rho$  is less than or greater than  $-\frac{\sigma_\mu}{2\kappa\sigma_D}$ . We will prove the proposition by contradiction, but we need to divide the proof into two cases:  $\sigma_m^* \geq 0$ , and  $\sigma_m^* < 0$ .

Case i)  $\sigma_m^* \geq 0$

Suppose  $\sigma_m^* > \sigma_\mu$ . This means that  $\frac{\sigma_m^*}{\sigma_\mu} > 1$ , implying

$$-\frac{\kappa\sigma_D}{\sigma_\mu} + \sqrt{\frac{\kappa^2\sigma_D^2}{\sigma_\mu^2} + \frac{2\kappa\sigma_D\rho}{\sigma_\mu} + 1} > 1$$

$\Leftrightarrow$

$$-1 - \frac{\kappa\sigma_D}{\sigma_\mu} + \sqrt{\frac{\kappa^2\sigma_D^2}{\sigma_\mu^2} + \frac{2\kappa\sigma_D\rho}{\sigma_\mu} + 1} > 0.$$

Completing the square inside the square root, we have

$$-1 - \frac{\kappa\sigma_D}{\sigma_\mu} + \sqrt{\left(1 + \frac{\kappa\sigma_D}{\sigma_\mu}\right)^2 + \frac{2\kappa\sigma_D(\rho - 1)}{\sigma_\mu}} > 0.$$

For this to hold, we must have  $\rho > 1$ . Since  $-1 \leq \rho \leq 1$ , this is a contradiction and the supposition must be false.

Case ii)  $\sigma_m^* < 0$

Suppose  $-\sigma_m^* > \sigma_\mu$ . Then,  $-\frac{\sigma_m^*}{\sigma_\mu} > 1$ . This implies,

$$\frac{\kappa\sigma_D}{\sigma_\mu} - \sqrt{\frac{\kappa^2\sigma_D^2}{\sigma_\mu^2} + \frac{2\kappa\sigma_D\rho}{\sigma_\mu} + 1} > 1$$

$\Leftrightarrow$

$$\frac{\kappa\sigma_D}{\sigma_\mu} - 1 - \sqrt{\frac{\kappa^2\sigma_D^2}{\sigma_\mu^2} + \frac{2\kappa\sigma_D\rho}{\sigma_\mu} + 1} > 0.$$

Completing the square, we have

$$\frac{\kappa\sigma_D}{\sigma_\mu} - 1 - \sqrt{\left(\frac{\kappa\sigma_D}{\sigma_\mu} - 1\right)^2 + \frac{2\kappa\sigma_D(\rho + 1)}{\sigma_\mu}} > 0.$$

If  $\frac{\kappa\sigma_D}{\sigma_\mu} - 1 \leq 0$ , then the whole expression is non-positive, and we are done. If  $\frac{\kappa\sigma_D}{\sigma_\mu} - 1 > 0$ , it must be that  $\rho < -1$ . Since  $-1 \leq \rho \leq 1$ , this is a contradiction and the supposition must be false.

Hence, in total,  $|\sigma_m^*| \leq \sigma_\mu$ , implying  $\sigma_m^{*2} \leq \sigma_\mu^2$ . ■

### 3.2 The Partially Informed Equilibrium

Since, in each period, all dividends are consumed ( $c_t = D_t$ ), the stochastic discount factor is given by  $\Lambda_s = e^{-\beta(s-t)}D_s^{-\gamma}$ . Applying Ito's lemma, we can obtain the dynamics of the stochastic discount factor with respect to the agents' filtration  $G_t$ ,

$$\frac{d\Lambda_s}{\Lambda_s} = (-\beta - \gamma m_s + \frac{1}{2}\gamma(\gamma + 1)\sigma_D^2)ds - \gamma\sigma_D d\bar{B}_s. \quad (13)$$

By no arbitrage, the interest rate can be endogenously determined as

$$r_s = \beta + \gamma m_s - \frac{1}{2}\gamma(\gamma + 1)\sigma_D^2. \quad (14)$$

We see that the interest rate is an increasing function of the estimated growth in dividends, and a decreasing function of the variance in dividend growth. As the estimated growth in dividends increases, investors allocate more of their resources to the stock, thus decreasing the demand for bonds, so that the bond prices will decrease, or equivalently, the short interest rate will increase. Similarly, as the variance of the dividends increases, investors will invest less in the stock, and more in bonds, thus elevating the bond prices, or equivalently decreasing the short interest rate. Note that the interest rate is time-varying with the estimated growth in dividends. This means that the diffusion of the interest rate is given by

$$dr_s = \gamma dm_s = \gamma\kappa(\bar{\mu} - m_s)ds + \gamma\sigma_m(v_s)d\bar{B}_s. \quad (15)$$

Further, we see that the diffusion of the interest rate is of the Vasicek (1977) type,

$$dr_s = \kappa(\bar{r} - r_s)ds + \sigma_{r_s}d\bar{B}_s \quad (16)$$

with a long-run interest rate of  $\bar{r} = \beta + \gamma\bar{\mu} - \frac{1}{2}\gamma(\gamma+1)\sigma_D^2$ , and a diffusion coefficient of  $\sigma_{rs} = \gamma\sigma_{ms}$ . Note that the variance of the short rate,  $\sigma_{rs}^2 = \gamma^2\sigma_{ms}^2$ , is unambiguously increasing in the coefficient of relative risk aversion ( $\gamma$ ). This follows from the fact that, with an increasing coefficient of relative risk aversion, the interest rate becomes more sensitive to changes in the estimate ( $m_s$ ), as seen in equation (14). In the full information case, the variance of the short rate is given by  $\gamma^2\sigma_\mu^2$ . Thus, by Proposition 5, the variance of the short rate under partial information in steady state is lower than or equal to the variance of the short rate under full information.

In the economy under consideration, there is a close relationship between asset prices and expected life-time utility. Note that the price of a stock is the discounted flow of dividends, i.e.

$$S(t) = E \left[ \int_t^T \frac{\Lambda_s}{\Lambda_t} D_s ds \middle| G_t \right] = E \left[ \int_t^T e^{-\beta(s-t)} \left( \frac{D_s}{D_t} \right)^{-\gamma} D_s ds \middle| G_t \right], \quad (17)$$

while the expected life-time utility is given by

$$J(D_t, m_t, t) = E \left[ \int_t^T e^{-\beta(s-t)} \frac{D_s^{1-\gamma}}{1-\gamma} ds \middle| G_t \right]. \quad (18)$$

Hence, the relation between the stock and the expected utility is

$$J(D_t, m_t, t) = \frac{D_t^{-\gamma}}{(1-\gamma)} S(t). \quad (19)$$

Using Proposition 3 in Yan (2000) and the relation between the stock price and the expected life-time utility, we can establish the following proposition.

**Proposition 6** *The expected life-time utility of the representative agent is given by*

$$J(D_t, m_t, t) = \frac{D_t^{1-\gamma}}{1-\gamma} \int_t^T \exp(\Psi(t, s, m_t)) ds,$$

where

$$\begin{aligned} \Psi(t, s, m_t) &= \left[ -\beta + (1-\gamma) \left( \bar{\mu} - \frac{1}{2}\sigma_D^2 \right) \right] (s-t) + \\ &+ \frac{(1-\gamma)^2}{2} \int_t^s \left( \sigma_D + \left( \rho\sigma_\mu + \frac{v_\tau}{\sigma_D} \right) \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) \right)^2 d\tau + \\ &+ (1-\gamma)(m_t - \bar{\mu}) \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right). \end{aligned}$$

**Proof.** See Proposition 3 in Yan (2000). Use the relation in equation (19). ■

From the above proposition, it follows directly that the expected life-time utility is increasing in the estimated dividend growth,

$$\frac{\partial J}{\partial m_t} = D_t^{1-\gamma} \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) ds > 0.$$

This is simply a result of non-satiation. As the estimated expected growth rate increases, the expected future consumption rises.

The second derivative with respect to the estimated expected growth rate is

$$\frac{\partial^2 J}{\partial m_t^2} = (1 - \gamma) D_t^{1-\gamma} \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right)^2 \exp(\Psi(t, s, m_t)) ds.$$

This second derivative reveals that the investor's expected life-time utility is strictly concave in  $m_t$  if and only if the investor is conservative ( $\gamma > 1$ ). Also, it is strictly convex in  $m_t$  if and only if the investor is aggressive ( $\gamma < 1$ ). This means that a conservative investor dislikes variability in his estimate, whereas an aggressive investor likes variability in his estimate.

The same results apply to the covariation between the estimate and the dividend, as revealed by the following cross-derivative,

$$\frac{\partial^2 J}{\partial m_t \partial D_t} = (1 - \gamma) D_t^{-\gamma} \int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) ds.$$

Consequently, a conservative investor dislikes both variability in the estimate of the expected growth rate and covariation between the estimate of the expected growth rate and the dividend. An aggressive investor likes both variability in the estimate of the expected growth rate and covariation between the estimate of the expected growth rate and the dividend.

Another interesting aspect is the response in utility to changes in the long-run growth in dividends ( $\bar{\mu}$ ). We can study it directly through the partial derivative,

$$\frac{\partial J}{\partial \bar{\mu}} = D_t^{1-\gamma} \int_t^T \left( (s-t) - \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \right) \exp(\Psi(t, s, m_t)) ds.$$

This means that the sign of the response to changes in the long-run growth in dividends is independent of relative risk aversion. A quick investigation of the

function  $c(x) = x - \left(\frac{1-e^{-\kappa x}}{\kappa}\right)$  reveals that it is strictly convex (since  $\kappa > 0$ ), and reaches its minimum at  $x = 0$ . Since  $c(0) = 0$ , the function  $c$  is non-negative. Hence, the sign of the partial derivative  $\frac{\partial J}{\partial \bar{\mu}}$  is positive. That is, the effect on expected life-time utility of a rising long-run growth in dividends ( $\bar{\mu}$ ) is positive for both aggressive and conservative agents.

The close relationship between the stock price and expected life-time utility, as expressed in equation (19), allows us to shed new light on the factors driving the expected rate of return and the volatility of stocks.

As shown by Yan (2000), the stock price is separable,

$$S_t = D_t f(m_t, t).$$

Applying Ito's lemma, we obtain

$$dS_t = f dD_t + D_t f_m dm_t + \frac{1}{2} D_t f_{mm} (dm_t)^2 + D_t f_t dt,$$

or

$$\frac{dS_t}{S_t} = \left( m_t + \frac{f_m}{f} \kappa (\bar{\mu} - m_t) + \frac{1}{2} \frac{f_{mm}}{f} \sigma_m^2 + \frac{f_t}{f} \right) dt + \left( \sigma_D + \sigma_m \frac{f_m}{f} \right) d\bar{B}_t.$$

The relation between the stock price and expected life-time utility together with the separability of the stock price implies

$$f(m_t, t) = \frac{1 - \gamma}{D_t^{1-\gamma}} J.$$

Thus, the dynamics of the return process can be written as

$$\begin{aligned} \frac{dG_t}{S_t} &= \frac{dS_t}{S_t} + \frac{D_t}{S_t} dt = \\ &\left( m_t + \frac{J_m}{J} \kappa (\bar{\mu} - m_t) + \frac{1}{2} \frac{J_{mm}}{J} \sigma_m^2 + \frac{J_t}{J} + \left( \frac{D_t^{1-\gamma}}{1-\gamma} \right) \frac{1}{J} \right) dt + \left( \sigma_D + \sigma_m \frac{J_m}{J} \right) d\bar{B}_t. \end{aligned}$$

The cum-dividend expected return on the stock is therefore given by

$$\mu_t^S = m_t + \frac{J_m}{J} \kappa (\bar{\mu} - m_t) + \frac{1}{2} \frac{J_{mm}}{J} \sigma_m^2 + \frac{J_t}{J} + \left( \frac{D_t^{1-\gamma}}{1-\gamma} \right) \frac{1}{J}.$$

Let us focus on the case of a conservative ( $\gamma > 1$ ) representative investor. As can be seen in the previous analysis of the expected life-time utility,  $\frac{J_m}{J} < 0$ , and  $\frac{J_{mm}}{J} > 0$  for a conservative representative investor.

The first term in the above expression for the cum-dividend expected return is just the estimate of the expected dividend growth rate. The second term states that the cum-dividend expected return also depends on the difference between the estimated drift ( $m_t$ ) and the long-run drift ( $\bar{\mu}$ ). The second term is positive if the estimated drift exceeds the long-run drift, and negative if the estimated drift is lower than the long-run drift. The third term is related to the curvature of the expected life-time utility and the variance of the estimate. This term is always positive. The fourth term is related to the value of time, and hence indirectly to the interest rate. The fifth term is simply the dividend yield, and it is always positive.

Yan (2000) solves explicitly for the coefficients of the return process, and we adopt the following proposition directly from him.

**Proposition 7** *The dynamics of the return process is given by*

$$\frac{dG_t}{S_t} = \mu_t^S dt + \sigma_{St} d\bar{B}_t,$$

where

$$\mu_t^S = r_t + \gamma\sigma_D^2 + (1 - \gamma)\gamma\sigma_D\sigma_{mt}\lambda(m_t, t)$$

$$\sigma_{St} = \frac{\mu_t^S - r_t}{\gamma\sigma_D} = \sigma_D + (1 - \gamma)\sigma_{mt}\lambda(m_t, t)$$

and

$$\lambda(m_t, t) = \frac{\int_t^T \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \exp(\Psi(t, s, m_t)) ds}{\int_t^T \exp(\Psi(t, s, m_t)) ds}.$$

**Proof.** See Yan (2000). ■

Note that  $0 < \lambda(m_t, t) < 1/\kappa$ . Although the equity premium and the stock volatility are stochastic in this economy, the market price of risk,  $(\mu_t^S - r_t)/\sigma_{St}$ , is in fact constant ( $= \gamma\sigma_D$ ). Further, the equity premium can be negative. Since  $\mu_t^S - r_t = \gamma\sigma_D\sigma_{St}$ , the signs of the equity premium and the stock diffusion coefficient are equal,

$$\text{sign}(\mu_t^S - r_t) = \text{sign}(\sigma_{St}).$$

Let us investigate what is needed for the equity premium to be positive, i.e. let us investigate the sign of  $\sigma_{St}$ . We will focus on the most interesting case when  $\gamma > 1$ . If  $\sigma_{mt}$  is negative, the sign of  $\sigma_{St}$  is positive, and we are done. If  $\sigma_{mt}$  is positive, then, since  $0 < \lambda(m_t, t) < 1/\kappa$

$$\sigma_{St} = \sigma_D + (1 - \gamma)\sigma_{mt}\lambda(m_t, t) > \sigma_D + \frac{(1 - \gamma)\sigma_{mt}}{\kappa}$$

and  $\sigma_D + \frac{(1 - \gamma)\sigma_{mt}}{\kappa} > 0$  guarantees a positive equity premium. This condition implies that

$$\sigma_{mt} < \frac{\kappa\sigma_D}{\gamma - 1}$$

or

$$v_t < \frac{\kappa\sigma_D^2}{\gamma - 1} - \rho\sigma_\mu\sigma_D.$$

That is, the filtering error must be sufficiently small. If  $\frac{\kappa\sigma_D^2}{\gamma - 1} - \rho\sigma_\mu\sigma_D < 0$ , this condition cannot be satisfied. Otherwise, one way of ensuring that the filtering error is always below  $\frac{\kappa\sigma_D^2}{\gamma - 1} - \rho\sigma_\mu\sigma_D$  is to have  $v^* \leq \frac{\kappa\sigma_D^2}{\gamma - 1} - \rho\sigma_\mu\sigma_D$  and  $v_0 < v^*$ . This corresponds to an overconfident investor, who believes very strongly in his prior. As shown by Feldman (1989), if  $v_0 < v^*$ , then  $v_t$  will increase monotonically through time towards the steady-state value  $v^*$ , but never reach it.

It should be stressed that a negative equity premium is by no means unreasonable in this model. It is easy to provide a natural, straightforward explanation for a negative equity premium, which occurs if e.g.  $\sigma_{mt}$  is positive and the coefficient of relative risk aversion is large enough. With a negative equity premium, the diffusion coefficient of the stock ( $\sigma_{St}$ ) is negative, and hence stock returns (see Proposition 7) and innovations in aggregate consumption (as represented by the dividend process in equation (6)) are perfectly negatively correlated. The representative agent accepts a negative equity premium, since the stock acts as a hedge against a low future consumption. On the one hand, the low future consumption originating from the stock makes the representative agent want to hold less of the stock. On the other hand, the future consumption is perfectly negatively correlated with the stock return, i.e. a low future consumption is correlated with a gain in the value of the stock. Thus, the representative agent still wants to hold the entire supply of shares in the stock (which can be normalized to one share).

### 3.3 The Individual Agents' Hedging Demands in Equilibrium

In this section, we will analyze the individual investors' hedging demands in equilibrium. First, we will assume a positive equity premium. The hedging demands are then shown to be related to the normalized risk premium. Finally, since a negative equity premium can occur naturally in this model, we will analyze the individual investors' hedging demands in the case of a negative equity premium.

The partially informed agent's problem (P) is

$$\begin{aligned} J(W_t, m_t, t) &= \max_{\{c_s, \alpha_s\}} E \left[ \int_{s=t}^T e^{-\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \middle| G_t \right] \\ \text{s.t. } dW_t &= W_t [r_t + \alpha_t(\mu_t^S - r_t)] dt - c_t dt + \alpha_t W_t \sigma_{St} d\bar{B}_t \\ dm_t &= \kappa(\bar{\mu} - m_t) dt + \sigma_{mt} d\bar{B}_t. \end{aligned}$$

The corresponding Hamilton-Jacobi-Bellman equation is given by

$$\begin{aligned} \max_{c_t, \alpha_t} \left\{ J_t - \beta J + u + J_W (W_t [r_t + \alpha_t(\mu_t^S - r_t)] - c_t) + J_m \kappa(\bar{\mu} - m_t) + \right. \\ \left. J_{Wm} \alpha_t W_t \sigma_{St} \sigma_{mt} + \frac{1}{2} J_{WW} \alpha_t^2 W_t^2 \sigma_{St}^2 + \frac{1}{2} J_{mm} \sigma_{mt}^2 \right\} = 0. \quad (20) \end{aligned}$$

The first order conditions for optimal consumption and optimal portfolio weights are

$$c_t : \quad u_C - J_W = 0 \quad (21)$$

$$\alpha_t : \quad J_W W_t (\mu_t^S - r_t) + J_{Wm} W_t \sigma_{St} \sigma_{mt} + J_{WW} \alpha_t W_t^2 \sigma_{St}^2 = 0 \quad (22)$$

This means that the optimal portfolio weight is given by<sup>3</sup>

$$\alpha_t = \frac{J_W}{-W_t J_{WW}} \frac{\mu_t^S - r_t}{\sigma_{St}^2} + \frac{J_{Wm}}{-W_t J_{WW}} \frac{\sigma_{mt}}{\sigma_{St}} \quad (23)$$

where the myopic part of the demand is given by

$$\alpha_{mt} = \frac{J_W}{-W_t J_{WW}} \frac{\mu_t^S - r_t}{\sigma_{St}^2}, \quad (24)$$

---

<sup>3</sup>It follows from the analysis in Merton (1971) that, in the case of logarithmic preferences,  $J_W / (-W_t J_{WW}) = 1$  and  $J_{Wm} = 0$ . Hence, logarithmic preferences induce myopic behavior. See Feldman (1992) for an extensive analysis of logarithmic preferences in the presence of incomplete information.

and the hedging demand is given by

$$\alpha_{ht} = \frac{J_{Wm} \sigma_{mt}}{-W_t J_{WW} \sigma_{St}}. \quad (25)$$

In the Appendix, we have derived an expression for the expected life-time utility in equilibrium. This expression is given in the proposition below.

**Proposition 8** *In equilibrium, the partially informed agent's value function is given by*

$$J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \left( \int_t^T \exp(\tilde{\Psi}(t, s, m_t)) ds \right)^\gamma, \quad (26)$$

where

$$\begin{aligned} \tilde{\Psi}(t, s, m_t) = & \left( -\beta - \frac{1}{2} \gamma (\gamma - 1) \sigma_D^2 \right) (s - t) + (1 - \gamma) \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t + \\ & (1 - \gamma) \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) (\kappa \bar{\mu} + (1 - \gamma) \sigma_D \sigma_{m\tau}) d\tau + \\ & + \frac{(1 - \gamma)^2}{2} \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right)^2 \sigma_{m\tau}^2 d\tau. \end{aligned}$$

**Proof.** See Appendix. ■

This means that we can write the value function as  $J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} g(m_t, t)$ , where

$$g(m_t, t) \equiv \left( \int_t^T \exp(\tilde{\Psi}(t, s, m_t)) ds \right)^\gamma.$$

Hence, the hedging demand in equation (25) can be written as

$$\alpha_{ht} = \frac{g_m \sigma_{mt}}{g \sigma_{St}}. \quad (27)$$

By equation (9), we know that  $\sigma_{mt} = \rho \sigma_\mu + v_t / \sigma_D$ , so the hedging demand can be split into two components,

$$\alpha_{ht} = \frac{g_m \rho \sigma_\mu}{g \sigma_{St}} + \frac{g_m v_t}{g \sigma_{St} \sigma_D}. \quad (28)$$

By non-satiation, it must be that  $J_W > 0$ , and further, the expected life-time utility must increase as the investment opportunities improve, so we also have  $J_m > 0$ . These results can be confirmed by investigating the expected life-time utility in

equation (26).  $J_W > 0$  implies that  $g(m_t, t) > 0$ . Further, since  $g(m_t, t) = (1 - \gamma)J/W_t^{1-\gamma}$ ,  $J_m > 0$  means that  $g_m < 0$  for a conservative ( $\gamma > 1$ ) investor. Hence, in the case of a conservative ( $\gamma > 1$ ) investor,  $g_m/g$  is negative.

In equilibrium, all agents will allocate all their wealth to the risky asset. To see this, note that if we (without loss of generality) normalize the supply of the risky asset to one share, then, if there are  $n$  agents in the economy, we must have

$$\sum_{i=1}^n \frac{\alpha_{it} W_{it}}{S(t)} = 1. \quad (29)$$

If all agents have identical information, priors and preferences, then since they have constant relative risk aversion, their share of wealth allocated to the stock must be the same, i.e.  $\alpha_{it} = \alpha_t$  for  $i = 1, 2, \dots, n$ . Inserting this into equation (29), we have

$$\alpha_t \sum_{i=1}^n W_{it} = S(t). \quad (30)$$

Individual wealth  $W_{it}$  is either allocated to the stock or to the bond, i.e.  $W_{it} = W_{it}^S + W_{it}^B$ , where  $W_{it}^S$  is the wealth allocated to the stock and  $W_{it}^B$  is the wealth allocated to the bond. Then, equation (30) can be rewritten as

$$\alpha_t \left( \sum_{i=1}^n W_{it}^S + \sum_{i=1}^n W_{it}^B \right) = S(t).$$

Since the bond is in zero net supply,  $\sum_{i=1}^n W_{it}^B = 0$ . Further, since  $\sum_{i=1}^n W_{it}^S = \sum_{i=1}^n \alpha_{it} W_{it} = S(t)$ , where the last equality follows from equation (29), it must be that  $\alpha_t = 1$ .

We will analyze the hedging demand given in equation (28), first assuming a positive equity premium, and then assuming a negative equity premium.

### 3.3.1 The case of a positive equity premium

In this subsection, we will analyze the individual investor's hedging demand assuming a positive equity premium, and, unless otherwise stated, we will assume a coefficient of relative risk aversion greater than unity ( $\gamma > 1$ ). We will also investigate how the hedging demand is related to the risk premium.

The first component of the hedging demand in equation (28) is a hedging component that arises because the true dividend drift is stochastic, and there is a correlation  $\rho$  between the true dividend drift and the dividend growth. The second

component is a hedging component that arises because the agent has to take into consideration that there is a difference between his estimate and the true drift term, i.e. he has to take into account that his estimate has an estimation error of  $v_t$ . If  $\gamma > 1$ , then the second component is negative, i.e. a conservative investor will hold less of the stock due to the estimation error. This is in line with the conclusions in Brennan (1998). However, if the correlation between the drift term ( $\mu_t$ ) and the dividend growth rate is negative ( $\rho < 0$ ), then the first component is positive, i.e. the investor will hold more of the stock due to the negative correlation. This is because, with a negative correlation, the stochastic component of the true dividend growth rate ( $\sigma_D dB_t$ ) works as a hedge against bad states (low  $\mu_t$ ). The lower the correlation ( $\rho$ ), the better the hedge against bad states, and the more of the stock held by a conservative investor.

In the case of a negative correlation between the dividend growth rate and the drift, the two components work in opposite directions. The first component makes the investor want to hold more of the stock, while the second component makes the investor want to hold less of the stock. It is possible that these components completely offset each other, or that the first component dominates the second component. Thus, a conservative ( $\gamma > 1$ ) investor can end up having a positive hedging demand. Interestingly, this contrasts with the conclusions in Brennan's (1998) model, where he shows that the hedging demand of a conservative agents is always negative. From the expression for the hedging demand, it can be seen that a conservative investor will have a positive hedging demand whenever  $-\rho\sigma_\mu > \frac{v_t}{\sigma_D}$ . If  $-\rho\sigma_\mu = \frac{v_t}{\sigma_D}$ , the investor will have a zero hedging demand, and if  $-\rho\sigma_\mu < \frac{v_t}{\sigma_D}$ , the investor will have a negative hedging demand.

It can be noted that there is an interesting relation between the hedging components and the normalized risk premium in the economy,

$$\frac{\mu_t^S - r_t}{\sigma_{St}^2} = \frac{\gamma\sigma_D}{\sigma_D + (1 - \gamma) \left( \rho\sigma_\mu + \frac{v_t}{\sigma_D} \right) \lambda(m_t, t)}.$$

Focusing on the most relevant case when agents are conservative ( $\gamma > 1$ ), we see that when  $-\rho\sigma_\mu > \frac{v_t}{\sigma_D}$ , so that the first component is positive and dominates the second component, the normalized risk premium is lower compared to a situation where the second component dominates the first component and the investor has a negative

hedging demand. Hence, the normalized risk premium is lower when the investors have a positive hedging demand compared to when they have a negative hedging demand. If the two hedging components completely offset each other, so that the investors' hedging demands are equal to zero, then the normalized risk premium is exactly equal to the coefficient of relative risk aversion ( $\gamma$ ). To conclude: if the investors have a positive hedging demand, then the normalized risk premium is lower than  $\gamma$ , and if the investors have a negative hedging demand, then the normalized risk premium is greater than  $\gamma$  (assuming that the conditions are met to ensure a positive equity premium). If the investors' hedging demands are equal to zero, then the normalized risk premium is exactly equal to  $\gamma$ . The findings regarding the relation between the hedging demand and the normalized risk premium are summarized in Table 1 below.

*Table 1* The relation between the hedging demand and the normalized risk premium (assuming  $\mu_t^S > r_t$  and  $\gamma > 1$ ).

case	$-\rho\sigma_\mu > \frac{v_t}{\sigma_D}$	$-\rho\sigma_\mu = \frac{v_t}{\sigma_D}$	$-\rho\sigma_\mu < \frac{v_t}{\sigma_D}$
<b>hedging demand</b>	$\alpha_{ht} > 0$	$\alpha_{ht} = 0$	$\alpha_{ht} < 0$
<b>normalized risk premium</b>	$< \gamma$	$= \gamma$	$> \gamma$

If the investor learns about a constant drift parameter as in Ziegler (2003, Chapter 2), the first component will be zero, and the second component will be close to zero in the long run, since, as  $t$  goes to infinity, the estimation error ( $v_t$ ) approaches its stable steady-state value ( $v^*$ ), which is zero in the case of a constant drift parameter. In contrast, when the investor learns about a stochastic mean-reverting drift, his hedging demand due to parameter uncertainty will generally not go to zero. Remember, by Proposition 3,  $v^* = 0$  if and only if  $(\kappa\sigma_D^2 + \rho\sigma_\mu\sigma_D) \geq 0$  and  $\{\rho = +1 \text{ or } \rho = -1\}$ . That is, a perfect correlation between the dividend growth rate and the true drift term is necessary for the estimation error (and hence its associated hedging component) to disappear in the long-run. In the long run, when learning is close to its stable steady state, the hedging demand  $\alpha_{ht}$  is close to its steady state value  $\alpha_{ht}^*$ , given by

$$\alpha_{ht}^* = \frac{g_m}{g} \frac{\left(-\kappa\sigma_D + \sqrt{\kappa^2\sigma_D^2 + 2\kappa\sigma_D\rho\sigma_\mu + \sigma_\mu^2}\right)}{\sigma_{St}}.$$

Therefore, the hedging demand in steady state is positive if  $\rho < -\frac{\sigma_\mu}{2\kappa\sigma_D}$ , and zero if  $\rho = -\frac{\sigma_\mu}{2\kappa\sigma_D}$ . If  $\rho > -\frac{\sigma_\mu}{2\kappa\sigma_D}$  however, the agent will have a negative hedging demand in steady state. As before, the hedging demands affect the normalized risk premium in steady state. Table 2 shows the relation between the hedging demand and the normalized risk premium in the stable steady state.

*Table 2.* The relation between the hedging demand and the normalized risk premium in the stable steady state (assuming  $\mu_t^S > r_t$  and  $\gamma > 1$ ).

case	$\rho < -\frac{\sigma_\mu}{2\kappa\sigma_D}$	$\rho = -\frac{\sigma_\mu}{2\kappa\sigma_D}$	$\rho > -\frac{\sigma_\mu}{2\kappa\sigma_D}$
hedging demand	$\alpha_{ht}^* > 0$	$\alpha_{ht}^* = 0$	$\alpha_{ht}^* < 0$
normalized risk premium	$< \gamma$	$= \gamma$	$> \gamma$

### 3.3.2 The case of a negative equity premium

In this subsection, we will analyze optimal portfolio choice under the assumption of a negative equity premium and a conservative investor ( $\gamma > 1$ ). From Proposition 7, it is easy to see that, under these assumptions,  $\sigma_{mt}$  needs to be positive.

As before, the total demand is the sum of a myopic demand and a hedging demand. The myopic demand is given by

$$\alpha_{mt} = \frac{\mu_t^S - r_t}{\gamma\sigma_{St}^2}. \quad (31)$$

Recall that the total demand is equal to one in equilibrium. Then, since a negative equity premium implies that the myopic demand is negative, the hedging demand must be greater than one,  $\alpha_{ht} > 1$ . Investigating equation (27), it can be confirmed that the hedging demand is positive, since  $\sigma_{mt}$  is positive, and  $\sigma_{St}$  is negative. The finding that the hedging component is greater than one corresponds well with the intuition regarding a negative equity premium in section 3.2.

## 4 Conclusions

We analyze a pure exchange economy in which agents learn about a mean-reverting drift in the dividend growth rate. Agents are assumed to have power utility and maximize expected utility of life-time consumption. First, we derive the expected

life-time utility of the representative agent and analyze its properties. Thereafter, we investigate the relation between the expected life-time utility of the representative agent and the return process of the risky asset. Finally, we derive an expression for the expected life-time utility of an individual investor in equilibrium, and analyze his hedging demand. The hedging demands of the individual agents are shown to be related to the normalized risk premium.

The expected life-time utility of the representative agent is found to be increasing in the estimated dividend growth. This is a result of non-satiation. As the expected growth rate increases, the expected future consumption rises. Further, in the most interesting case of a conservative representative agent ( $\gamma > 1$ ), which is consistent with most empirical studies, the expected life-time utility is concave in the estimated expected growth rate. This means that a conservative agent dislikes variability in his estimate. A conservative representative agent is also shown to dislike covariation between the estimate and the dividend.

The intimate relation between the stock price and expected life-time utility allows us to shed new light on the factors driving the moments of the return process of the risky asset. They are shown to be driven by the slope and curvature of the indirect utility function with respect to the estimate, and the slope with respect to time (the value of time).

It is shown that the hedging demand of an individual investor consists of two components. The first component is a hedging component that arises because of the correlation between the true drift and the dividend growth rate. The second component is a hedging component that arises because of the estimation error. In the case of a negative correlation between the drift term and the dividends, the two hedging components work in opposite directions, so that the conservative investor can have a non-negative hedging demand. Interestingly, this differs from the results in Brennan (1998). In a model where an agent learns about a constant drift in the stock return, Brennan (1998) finds that a conservative agent always has a negative hedging demand (provided that the equity premium is positive).

The hedging demands of the agents are related to the normalized risk premium. Assuming agents are conservative and that the equity premium is positive, it is shown that the normalized risk premium is lower (higher) when the agents' hedging

demands are positive (negative) compared to a situation where the agents' hedging demands are zero. If the agents' hedging demands are zero, then the normalized risk premium is equal to the agents' mutual coefficient of relative risk aversion ( $\gamma$ ).

## Appendix

This section derives an explicit expression for the partially informed agent's lifetime expected utility. It is shown that under incomplete information, the value function can be written as  $J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \theta(m_t, t)^\gamma$ , where  $\theta(m_t, t)$  is the solution to a linear PDE. Defining  $g(m_t, t) \equiv \theta(m_t, t)^\gamma$ , we can write  $J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} g(m_t, t)$ .

With  $J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \theta(m_t, t)^\gamma$ , the partial derivatives of the value function are given by

$$\begin{aligned} J_t &= \frac{W_t^{1-\gamma}}{1-\gamma} \gamma \theta^{\gamma-1} \theta_t = \gamma \frac{\theta_t}{\theta} J \\ J_W &= W_t^{-\gamma} \theta(m_t, t)^\gamma = (1-\gamma) \frac{J}{W_t} \\ J_{WW} &= -\gamma W_t^{-\gamma-1} \theta(m_t, t)^\gamma = -\gamma(1-\gamma) \frac{J}{W_t^2} \\ J_m &= \frac{W_t^{1-\gamma}}{1-\gamma} \gamma \theta^{\gamma-1} \theta_m = \gamma \frac{\theta_m}{\theta} J \\ J_{Wm} &= W_t^{-\gamma} \gamma \theta^{\gamma-1} \theta_m = \gamma(1-\gamma) \frac{\theta_m}{\theta} \frac{J}{W_t} \\ J_{mm} &= \frac{W_t^{1-\gamma}}{1-\gamma} \gamma (\theta^{\gamma-1} \theta_{mm} + (\gamma-1) \theta^{\gamma-2} \theta_m^2) = \left( \gamma \frac{\theta_{mm}}{\theta} - \gamma(1-\gamma) \left( \frac{\theta_m}{\theta} \right)^2 \right) J. \end{aligned}$$

The first-order condition for optimal consumption in equation (21) thus reads

$$c_t^{-\gamma} = W_t^{-\gamma} \theta^\gamma. \quad (32)$$

This means that the optimal consumption is given by

$$c_t = W_t \theta^{-1}. \quad (33)$$

By equation (23), the optimal portfolio weight is given by

$$\alpha_t = \frac{\mu_t^S - r_t}{\gamma \sigma_{St}^2} + \frac{\theta_m}{\theta} \frac{\sigma_{mt}}{\sigma_{St}}. \quad (34)$$

Inserting optimal consumption as given in equation (33) and the optimal portfolio weight given in equation (34) together with the partial derivatives into the HJB equation (20) yields

$$\begin{aligned}
& \gamma \frac{\theta_t}{\theta} J - \beta J + \frac{J}{\theta} + (1-\gamma) J \left( r_t + \frac{(\mu_t^S - r_t)^2}{\gamma \sigma_{St}^2} + \frac{\theta_m \sigma_{mt}}{\theta \sigma_{St}} (\mu_t^S - r_t) - \frac{1}{\theta} \right) + \gamma \frac{\theta_m}{\theta} J \kappa (\bar{\mu} - m_t) + \\
& + \gamma (1-\gamma) \frac{\theta_m}{\theta} J \left( \frac{\sigma_{mt}}{\gamma \sigma_{St}} (\mu_t^S - r_t) + \frac{\theta_m}{\theta} \sigma_{mt}^2 \right) - \frac{1}{2} \gamma (1-\gamma) J \left( \frac{\mu_t^S - r_t}{\gamma \sigma_{St}} + \frac{\theta_m}{\theta} \sigma_{mt} \right)^2 + \\
& + \frac{1}{2} J \left( \gamma \frac{\theta_{mm}}{\theta} - \gamma (1-\gamma) \left( \frac{\theta_m}{\theta} \right)^2 \right) \sigma_{mt}^2 \\
& = 0.
\end{aligned}$$

Canceling out the  $J$ s from the above expression and manipulating, we have

$$\gamma \frac{\theta_t}{\theta} - \beta + \frac{\gamma}{\theta} + (1-\gamma) r_t + \frac{1}{2} (1-\gamma) \frac{(\mu_t^S - r_t)^2}{\gamma \sigma_{St}^2} + \gamma \kappa (\bar{\mu} - m_t) \frac{\theta_m}{\theta} + (1-\gamma) \frac{\sigma_{mt}}{\sigma_{St}} (\mu_t^S - r_t) \frac{\theta_m}{\theta} + \frac{1}{2} \gamma \sigma_{mt}^2 \frac{\theta_{mm}}{\theta} = 0.$$

Multiplying by  $\theta$  and rearranging, this simplifies to

$$\begin{aligned}
& \gamma + \left( \frac{1}{2} (1-\gamma) \frac{(\mu_t^S - r_t)^2}{\gamma \sigma_{St}^2} + (1-\gamma) r_t - \beta \right) \theta + \gamma \theta_t + \\
& + \left( \gamma \kappa (\bar{\mu} - m_t) + (1-\gamma) \frac{\sigma_{mt}}{\sigma_{St}} (\mu_t^S - r_t) \right) \theta_m + \frac{1}{2} \gamma \sigma_{mt}^2 \theta_{mm} \\
& = 0
\end{aligned} \tag{35}$$

which is a linear PDE. From Proposition 7,  $\frac{\mu_t^S - r_t}{\sigma_{St}} = \gamma \sigma_D$ , and from equation (14),  $r_t = \beta + \gamma m_t - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2$ , so equation (35) further simplifies to

$$\begin{aligned}
& 1 + \left( (1-\gamma) m_t - \frac{1}{2} \gamma (1-\gamma) \sigma_D^2 - \beta \right) \theta + \theta_t + \\
& + (\kappa (\bar{\mu} - m_t) + (1-\gamma) \sigma_D \sigma_{mt}) \theta_m + \frac{1}{2} \gamma \sigma_{mt}^2 \theta_{mm} \\
& = 0
\end{aligned} \tag{36}$$

with terminal condition  $\theta(m_T, T) = 0$ . The terminal condition stems from the fact that  $J(W_T, m_T, T) = \max_{\{c_s, \alpha_s\}} E \left[ \int_{s=T}^T e^{\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \middle| G_T \right] = 0$  has to hold for all  $W_T$ .

The Feynman-Kac solution to equation (36) is

$$\theta(m_t, t) = E_t^{\tilde{Q}} \left[ \int_t^T \exp \left\{ \int_t^s \left( (1-\gamma) m_\tau - \frac{1}{2} \gamma (\gamma - 1) \sigma_D^2 - \beta \right) d\tau \right\} ds \right]. \tag{37}$$

In the probability measure  $\tilde{Q}$ ,  $m_\tau$  evolves according to

$$dm_\tau = (\kappa(\bar{\mu} - m_\tau) + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau + \sigma_{m\tau}dB_\tau^{\tilde{Q}} \quad (38)$$

where  $B_\tau^{\tilde{Q}}$  is a Brownian motion with respect to the probability measure  $\tilde{Q}$ .

Equation (38) implies that

$$m_s - m_t = \int_t^s (\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau - \kappa \int_t^s m_\tau d\tau + \int_t^s \sigma_{m\tau}dB_\tau^{\tilde{Q}}$$

or equivalently,

$$\int_t^s m_\tau d\tau = \frac{1}{\kappa} \int_t^s (\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau - \frac{m_s}{\kappa} + \frac{m_t}{\kappa} + \int_t^s \frac{\sigma_{m\tau}}{\kappa}dB_\tau^{\tilde{Q}}. \quad (39)$$

The solution to equation (38) is given by

$$m_s = e^{-\kappa(s-t)}m_t + \int_t^s e^{-\kappa(s-\tau)}(\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau + \int_t^s e^{-\kappa(s-\tau)}\sigma_{m\tau}dB_\tau^{\tilde{Q}}. \quad (40)$$

Inserting the solution (40) into equation (39), we have

$$\begin{aligned} \int_t^s m_\tau d\tau &= \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t + \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) (\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau + \\ &+ \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) \sigma_{m\tau}dB_\tau^{\tilde{Q}}. \end{aligned} \quad (41)$$

From Fubini's theorem and the normality of  $\int_t^s m_\tau d\tau$ , it follows that equation (37) can be written as

$$\begin{aligned} \theta(m_t, t) &= \int_t^T \exp \left\{ (1 - \gamma)E_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] + \frac{(1 - \gamma)^2}{2} \text{Var}_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] + \right. \\ &\left. + \left( -\frac{1}{2}\gamma(\gamma - 1)\sigma_D^2 - \beta \right) (s - t) \right\} ds. \end{aligned} \quad (42)$$

Given equation (41), it is now easy to calculate the moments;

$$\begin{aligned} E_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] &= \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) m_t + \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right) (\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau})d\tau \\ \text{Var}_t^{\tilde{Q}} \left[ \int_t^s m_\tau d\tau \right] &= \int_t^s \left( \frac{1 - e^{-\kappa(s-\tau)}}{\kappa} \right)^2 \sigma_{m\tau}^2 d\tau. \end{aligned}$$

Thus the solution  $\theta(m_t, t)$  is

$$\theta(m_t, t) = \int_t^T \exp\left(\tilde{\Psi}(t, s, m_t)\right) ds \quad (43)$$

where

$$\begin{aligned} \tilde{\Psi}(t, s, m_t) = & \left(-\beta - \frac{1}{2}\gamma(\gamma - 1)\sigma_D^2\right)(s - t) + (1 - \gamma) \left(\frac{1 - e^{-\kappa(s-t)}}{\kappa}\right) m_t + \\ & (1 - \gamma) \int_t^s \left(\frac{1 - e^{-\kappa(s-\tau)}}{\kappa}\right) (\kappa\bar{\mu} + (1 - \gamma)\sigma_D\sigma_{m\tau}) d\tau + \\ & + \frac{(1 - \gamma)^2}{2} \int_t^s \left(\frac{1 - e^{-\kappa(s-\tau)}}{\kappa}\right)^2 \sigma_{m\tau}^2 d\tau. \end{aligned}$$

Given the solution  $\theta(m_t, t)$  in equation (43), the solution to the value function can be written as  $J(W_t, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} g(m_t, t)$ , where  $g(m_t, t) = \theta(m_t, t)^\gamma$ .

## References

- Brennan, M. J. (1998): "The Role of Learning in Dynamic Portfolio Decisions," *European Finance Review*, Vol. 1, pp. 295-306.
- Cvitanić, J., A. Lazrak, L. Martellini, and F. Zapatero (2004): "Dynamic Portfolio Choice with Parameter Uncertainty and the Economic Value of Analysts' Recommendations," Working paper, University of Southern California.
- Detemple, J. B. (1986): "Asset Pricing in a Production Economy with Incomplete Information," *Journal of Finance*, Vol. 41, pp. 383-391.
- Dothan, M. U. and D. Feldman (1986): "Equilibrium Interest Rates and Multi-period Bonds in a Partially Observable Economy," *Journal of Finance*, Vol. 41, pp. 369-382.
- Feldman, D. (1989): "The Term Structure of Interest Rates in a Partially Observable Economy," *Journal of Finance*, Vol. 44, pp. 789-812.
- Feldman, D. (1992): "Logarithmic Preferences, Myopic Decisions, and Incomplete Information," *Journal of Financial and Quantitative Analysis*, Vol. 27, pp. 619-629.

- Feldman, D. (2002): "Production and the Real Rate of Interest: A Sample Path Equilibrium," *European Finance Review*, Vol. 6, pp. 247-275.
- Feldman, D. (2003): "The Term Structure of Interest Rates: Bounded or Falling?," *European Finance Review*, Vol. 7, pp. 103-113.
- Feldman, D. (2004): "Incomplete Information Equilibria: Separation Theorems and Other Myths," Working paper, The University of New South Wales. (Revised December 12, 2004.)
- Goldstein, R. and F. Zapatero (1996): "General Equilibrium with Constant Relative Risk Aversion and Vasicek Interest Rates," *Mathematical Finance*, Vol. 6, pp. 331-340.
- Liptser, R. S. and A. N. Shiryaev (2001): "Statistics of Random Processes I & II," Springer-Verlag. Berlin.
- Lucas, R. E. Jr. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, Vol. 46, pp. 1429-1445.
- Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, Vol. 3, pp. 373-413.
- Riedel, F. (2000): "Decreasing Yield Curves in a Model with an Unknown Constant Growth Rate," *European Finance Review*, Vol. 4, pp. 51-67.
- Rubinstein, M. (1974): "An Aggregation Theorem for Securities Markets," *Journal of Financial Economics*, Vol. 1, pp. 225-244.
- Vasicek, O. (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, Vol. 5, pp. 177-188.
- Williams, J. T. (1977): "Capital Asset Prices with Heterogeneous Beliefs," *Journal of Financial Economics*, Vol. 5, pp. 219-239.
- Yan, H. (2000): "Uncertain Growth Prospects, Estimation Risk, and Asset Prices," Working paper, University of Texas at Austin.

Ziegler, A. (2000): "Optimal Portfolio Choice under Heterogeneous Beliefs," *European Finance Review*, Vol. 4, pp. 1-19.

Ziegler, A. (2003): "Incomplete Information and Heterogeneous Beliefs in Continuous-Time Finance," Springer-Verlag. Berlin.