

A Dozen Consistent CAPM-Related Valuation Models

- So Why Use the Incorrect One?

by

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First draft: January 15, 2006

This version: May 29, 2006

*For presentation at the
European Financial Management Association
2006 Annual Meeting in Madrid (Spain)*

Abstract

This paper focuses on applications of the CAPM in capital budgeting and in valuation of "mispriced" financial assets. Most textbooks in finance do not warn against a common pitfall in discounting expected cash flows by risk adjusted discount rates that are conceptually inconsistent with the CAPM. Betas computed from returns based on investment cost rather than on market value, may give systematically inappropriate discount rates and numerically incorrect present values for non-zero NPVs and "mispriced" assets. The paper provides a self contained collection of a dozen consistent CAPM-related methods, that all give correct valuation results. The models include approaches based on certainty equivalents, equilibrium and disequilibrium required discount rates, simplified discounting rules based on absence of arbitrage for particular cash flow formulations, as well as required adaptations to make valuations from more advanced valuation methods consistent with correct CAPM procedures. Derivations of the valuation methods are shown in an appendix. A running base case numerical example illustrates the various procedures. Further illustrations are provided by a textbook example that also demonstrates how some simple procedures work for more complex cases than previously recognized.

JEL classifications: G31, G12, G11

Key words and phrases: CAPM present values, disequilibrium valuation, cost based betas, simple rules, Jensen's alpha, absence of arbitrage, CAPM adaptations

* The author is grateful to Jonas Andersson, Frøystein Gjesdal, Richard D. F. Harris, Hayne Leland, Steve Ross, and Knut K. Aase for helpful discussions and comments on previous versions. The usual disclaimer applies.

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1. Introduction

This paper takes a "back to basics" view on valuation of risky assets and projects, focusing on the conceptual foundations for applications of the Capital Asset Pricing Model¹ (CAPM) in computing consistent net present values (NPV) and theoretical market prices. Both the NPV and the CAPM are among the most important ideas and key concepts in finance², discussed at great length in introductory and intermediate finance courses³, and widely used in practice⁴. A basic CAPM property is that a quantifiable measure of the relevant risk of an individual asset may be derived from its covariance with the market return, often represented by beta. A practical risk adjusted discounting procedure ostensibly relies on the CAPM, but uses a beta concept that is inconsistent with the CAPM. This conceptual fallacy may result in a systematic bias in computed NPVs or in the apparent asset "mispricing", compared to benchmarks from the theoretical model.

The CAPM appears in many versions. This paper considers a "baker's dozen" simplified CAPM related approaches within an essentially single period context. All but one model are consistent in giving the exact same numerical valuation answer. Unfortunately, the one model giving an inconsistent theoretical value, may very well be the one selected by

¹ The CAPM was originally developed by Sharpe, Lintner, and Mossin. Consistency with the expected utility hypothesis requires restrictions on preferences and/or probability distributions.

² Brealey *et al.* (2006:957) list NPV and CAPM as the first two of "the seven most important ideas in finance".

³ Womack (2001) finds that in a typical core finance course in top MBA programs, roughly one half of the class time was spent on present value concepts, portfolio theory, CAPM and capital budgeting.

⁴ Graham and Harvey (2001) report that about 75% of US surveyed CFOs use NPV and a similar percentage use CAPM for determining the cost of capital. Brounen *et al.* (2004) report use by about one half of CFOs in their companion survey of European firms.

analysts, practitioners and other decision makers having had some exposure to finance as reflected in popular textbooks.

Consider the following overly simplified but transparent base-case example: A one-period investment project has an investment cost $I = 50$. Its end of period cash flow⁵ depends on the business cycle represented by the future, unknown state of economy, which may be either Good, So-so, or Bad. These three mutually exclusive states (or scenarios) are equally probable. The stochastic future cash flow \tilde{X} will be 160 in the Good state, 100 in the So-so state, but only 40 in the Bad state. The stochastic return \tilde{R}_M of the market portfolio is 40% in the Good state, 10% in the So-so state, and -20% in the Bad state. For simplicity, the risk free rate of interest R_F is zero.

The project's gross present value (PV) denoted by P , being the fair or equilibrium market value of the uncertain cash flow, is found by discounting the expected cash flow $E(\tilde{X})$ at a suitable risk adjusted discount rate (RADR) k . By subtracting the investment cost, the desired net present value is found to be $NPV = P - I$. According to the CAPM, the RADR may be computed as the sum of the risk free rate and an asset risk premium, where the risk premium in one formulation equals beta times the expected excess return over the risk free rate: $k = R_F + [E(\tilde{R}_M) - R_F] \cdot \beta$. In the example, the expected cash flow $E(\tilde{X})$ and the expected market return $E(\tilde{R}_M)$ are computed to be 100.00 and 10%, respectively. Thus, with the risk free rate $R_F = 0.00$, the discount rate (RADR) $k = 0.1\beta$, and gross PV

$$P = \frac{E(\tilde{X})}{1+k} = \frac{100}{1+0.1\beta}.$$

The one remaining parameter is beta. The analyst may recall beta being the covariance between the returns to the asset and to the market, divided by the variance of the market

⁵ The term cash flow actually refers to the end of period value for a longer lived asset, including cash flows occurring at the end of that period. For multi period applications, see e.g. Fama (1977) or Fama (1996).

return. The denominator $\text{Var}(\tilde{R}_M)$ is computed as 0.06. The crucial lacking information is then the covariance between the returns. The return to the asset is so far not defined. Based on the available information, knowing the investment cost but not the theoretical market price, the project analyst defines the cost based return or internal rate of return (IRR) by dividing the cash flow by the investment cost, and then subtracting one: $\tilde{r} \equiv \frac{\tilde{X}}{I} - 1$. Hence, the project's return in the Good state will be $r(\text{Good}) = \frac{160}{50} - 1 = 2.20$. Similarly, the return in the So-so state will be 1.00, and in the Bad state -0.20. Hence, with equally probable states, the expected return is 1.00 (i.e., 100%). Furthermore, the return covariance $\text{Cov}(\tilde{r}, \tilde{R}_M)$ is found to be 0.24. Thus, beta $\beta = \text{Cov}(\tilde{r}, \tilde{R}_M) / \text{Var}(\tilde{R}_M) = 0.24 / 0.06 = 4.00$. The discount rate

becomes $k = 0.1\beta = 0.1 \cdot 4.0 = 0.40$. The gross present value $P = \frac{E(\tilde{X})}{1+k} = \frac{100}{1+0.4} = \frac{500}{7} \approx 71.43$. The net present value $NPV = P - I = \frac{500}{7} - 50 = \frac{150}{7} \approx 21.43$.

Exhibit 1 illustrates the assumptions as well as the computations. But unfortunately, the stated gross and net present values are dead wrong! The whole CAPM inspired computational scheme in Exhibit 1 is numerically and technically correct, but it does not make much economic sense. The fundamental problem is the wrongful use of the cost based rate of return (or IRR) in computing the beta entering the discount factor, in conflict with the CAPM being an equilibrium model⁶.

The standard CAPM in its extensive form⁷

⁶ In Markowitz (1984) "the founding father of modern portfolio theory" warns about another "beta trap" caused by confusing properties of betas from the CAPM and from the related market model (MM) or single index model (SIM). These models are often used in conjunction with the CAPM, but the MM (or the SIM) and the CAPM do not require their companion model.

⁷ See e.g. Sharpe *et al.* (1999) Eq. (9.6), Danthine and Donaldson (2005) Eq. (7.2), or Elton *et al.* (2003:300).

$$E(\tilde{R}) = R_F + \frac{E(\tilde{R}_M) - R_F}{\text{Var}(\tilde{R}_M)} \text{Cov}(\tilde{R}, \tilde{R}_M) \quad (1)$$

applies to equilibrium market based returns

$$\tilde{R} \equiv \frac{\tilde{X}}{P} - 1, \quad (2)$$

i.e., with the price rather than the investment cost in the denominator. In fact, it can be shown that for the base case example the correct market value is $P = 80.00$, and hence that $NPV = 30.00$, when the CAPM is correctly applied. These values will be derived from twelve different CAPM related approaches in the subsequent sections.

A great number of valuation methods are available, from simple rules of thumb to highly sophisticated and complex theoretical models and proprietary software. This paper's focus on the CAPM should not be interpreted as a claim that the CAPM is a superior or recommended valuation approach⁸. Rather, *if* the CAPM is applied, its users should be aware whether the procedure is consistent with the conceptual foundations of the CAPM. The paper points out the direction of the systematic bias caused by inconsistent betas⁹, as well as providing lots of alternatives for CAPM consistent valuation of risky alternatives.

Proper valuation is essential when assessing a real or financial risky investment opportunity, whether the net present value of a real investment project within capital budgeting or the "fair" market value or return of a security or portfolio within financial investments. Valuation is particularly important when it comes to capital budgeting projects having non-zero net present values, and also when considering "mispriced" financial assets. In some disequilibrium cases the sign of the NPV or of the mispricing may suffice to make an

⁸ Jagannathan and Meier (2002) question whether the CAPM is needed for capital budgeting.

⁹ Another related but different pitfall is not distinguishing between the firm and the project discount rates caused by different risks that should be reflected in different betas, as pointed out by Rubinstein (1973:172) and in textbooks such as Ross *et al.* (2005:330).

accept/reject or buy/sell decision, whereas exact and correct numerical valuation measures may be required in more complex decision situations.

Admittedly, the methods reviewed are by themselves not original, but may be found scattered in the literature. The previous related literature on the properties of CAPM-related cost based (disequilibrium) risk versus market based (equilibrium) required rates of return is rather limited, but includes notable contributions by Rubinstein (1973), Fama (1977), Rendleman (1978), and Weston and Chen (1980), among others. The topic is mostly absent from most popular textbooks¹⁰, with Grinblatt and Titman (1998) as a significant exception.

The remainder of the paper is organized as follows. Section 2 formalizes discounting expected cash flows using cost based return betas, resulting in an incorrect NPV. Sections 3 through 7 discuss a dozen CAPM consistent procedures giving correct present values. Three certainty equivalent formulations are presented in Section 3. Two risk adjusted discount factor formulations derived from market based returns are shown in Section 4. Section 5 uses relations between three different betas to express present values in two different ways. For particular cash flow patterns, Section 6 shows two simple discounting rules based on absence of arbitrage and using conditional expected cash flow in one single state or scenario. Section 7 provides recipes for adapting three more general and advanced models to be consistent with the CAPM. Section 8 takes a closer look at the betas of disequilibrium versus equilibrium assets. A Security Market Line (SML) illustration is included in Section 9, discussing a possible ambiguity as to the interpretation of Jensen's alpha mispricing measure and the transition to an equilibrium. Section 10 concludes the main paper. Derivations of the valuation results are collected in Appendix 1. Appendix 2 contains some numerical calculations for the

¹⁰ Bodie *et al.* (2005:291) have a terse, four line paragraph stating that the CAPM is useful in capital budgeting decisions, by providing the required rate of return that the project needs to yield, based on its beta. It does not explain how the beta is found. Also, this CAPM required rate is suggested being used as an IRR hurdle rate, rather than for computing NPV.

base example used throughout the paper. Appendix 3 applies the various CAPM related methods to a more complex example introduced in the Grinblatt and Titman (1998) textbook.

2. Discounting factor for expected cash flows using cost based return betas

It may perhaps seem natural to define returns based on the ratio of cash flows to investment costs. After all, generally the theoretical market price may not be known at the outset, but is rather to be found by a suitable method. This cost based rate of return

$$\tilde{r} \equiv \frac{\tilde{X}}{I} - 1 \quad (3)$$

is also the internal rate of return (IRR) in a one period model, as a slight rearrangement of Eq.

3 shows that $I = \frac{\tilde{X}}{1 + \tilde{r}}$. The expected cost based return $E(\tilde{r})$ is also the IRR of the expected

cash flow in a single period model, as taking expectations of $I(1 + \tilde{r}) = \tilde{X}$ implies

$$I = \frac{E(\tilde{X})}{1 + E(\tilde{r})}. \text{ If the theoretical equilibrium market price } P \text{ according to the CAPM differs}$$

from the investment cost I , then the asset has a non-zero net present value, or is alternatively "mispriced" if it were traded separately. In the example, the expected disequilibrium return

$$E(\tilde{r}) = \frac{E(\tilde{X})}{I} - 1 = \frac{100}{50} - 1, \text{ such that } E(\tilde{r}) = 1.00 \text{ or } 100\%.$$

The cost (or IRR) based disequilibrium beta is the corresponding cost based return covariance term divided by the market return variance,

$$\beta(\tilde{r}) \equiv \frac{\text{Cov}(\tilde{r}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} \quad (4)$$

From analogy with the CAPM, it yields a cost (or IRR) disequilibrium risk-adjusted discount factor (RADR)

$$k(\tilde{r}) = R_F + \left[E(\tilde{R}_M) - R_F \right] \cdot \beta(\tilde{r}) \quad (5)$$

This RADR cannot generally be used for discounting expected cash flows, whenever the exact numerical values of gross or net present value are of interest¹¹:

$$P \neq \frac{E(\tilde{X})}{1 + k(\tilde{r})} \quad \text{for } P \neq I \quad (6)$$

However, the sign of the net present value using Eq. (6) will be the same as the sign of the correctly computed CAPM equilibrium net present value. Also, the difference $E(\tilde{r}) - k(\tilde{r})$ between the expected disequilibrium return and the cost based RADR will have the same sign as the correctly computed equilibrium net present value. In the example, $E(\tilde{r}) - k(\tilde{r}) = 1.00 - 0.40 = 0.60 > 0$. For ranking different investment projects, the cost based beta and RADR may thus be used. But it should be avoided for discounting expected cash flows, whenever correct numerical values are required, say, in case of selling or buying non-zero NPV projects or mispriced assets. Nevertheless, it will shortly be shown that the disequilibrium RADR from Eq. (5) may still be useful for computing net present values directly, without first computing gross present values.

3. CAPM certainty equivalent approaches

The CAPM may be written in certainty equivalent (CE) form, as¹²

$$P = \frac{E[\tilde{X}] - \lambda \text{Cov}(\tilde{X}, \tilde{R}_M)}{1 + R_F} \quad (7)$$

¹¹ Grinblatt and Titman (1998) apply this method for computing gross present value in their Example 10.5, but commendably comment that these betas are not really correct and thus the PV is also wrong. In contrast, Bossaerts and Ødegaard (2001:60) explicitly recommend finding present values by discounting expected future cash flow by a discount rate using the cost based beta and illustrate it by a numerical example. In a forthcoming revision, Bossaerts and Ødegaard (2006:60) state that actually the astute reader will have noticed that the above procedure is not correct, despite its widespread usage. Afterwards, Bossaerts and Ødegaard (2001:69-70; 2006:60) also show the correct market price based beta for discounting expected risky cash flows.

¹² See e.g. Copeland *et al.* (2005) Eq. (6.20) or Brealey *et al.* (2006:227).

The CE in the numerator adjusts the expected cash flow by deducting a risk correction.

Defining the "market price of risk" lambda as $\lambda \equiv \frac{E(\tilde{R}_M) - R_F}{\text{Var}(\tilde{R}_M)}$, the CE risk correction is the

product of lambda and a covariance term involving the project's (absolute) cash flow \tilde{X} rather than its (relative) return \tilde{R} .

In the example, the market price of risk is $\lambda = \frac{0.10 - 0.00}{0.06} = \frac{5}{3} \approx 1.67$. The cash flow

covariance with the market return is $\text{Cov}(\tilde{X}, \tilde{R}_M) = 12$. Hence, the equilibrium market price

$$P = \frac{100 - \frac{5}{3} \cdot 12}{1 + 0.00} = \frac{100 - 20}{1} = 80.00. \text{ The net present value } NPV = 80.00 - 50.00 = 30.00.$$

For a slight variation of this CE method, define the cash flow beta $\beta(\tilde{X})$ as the ratio of the covariance between the cash flow and the market portfolio return, divided by the variance of the market portfolio return:

$$\beta(\tilde{X}) \equiv \frac{\text{Cov}(\tilde{X}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} \quad (8)$$

Then the risk correction term in the CE is the product of the cash flow beta and the expected excess market return above the risk free rate. The gross present value becomes¹³

$$P = \frac{E[\tilde{X}] - [E(\tilde{R}_M) - R_F] \cdot \beta(\tilde{X})}{1 + R_F} \quad (9)$$

For the example, the cash flow beta $\beta(\tilde{X}) = \frac{\text{Cov}(\tilde{X}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} = \frac{12}{0.06} = 200$. Hence, the

$$\text{theoretical market price } P = \frac{100 - [0.10 - 0.00] \cdot 200}{1 + 0.00} = 80.00.$$

¹³ See e.g. Grinblatt and Titman (1998), Result 10.7.

Computations of the disequilibrium expected return $E(\tilde{r})$ and the disequilibrium RADR $k(\tilde{r}) = R_F + [E(\tilde{R}_M) - R_F] \cdot \beta(\tilde{r})$ of Eq. (5) are not necessarily wasted effort. In the certainty single period case, the net present value may be computed as

$$NPV = \frac{r - k}{1 + k} \cdot I, \quad (10)$$

where r is the internal rate of return and k the discount rate (presumably the risk free rate). The fraction may be interpreted as the "quality" of the project reflected in the discounted excess of the IRR over the required rate, and the investment cost as the "scale" of the project. In the uncertainty case, Eq. (10) carries over in two different versions, depending on a consistent choice of k in numerator and denominator.

Using the cost based RADR, the net present value is¹⁴

$$NPV = \frac{E(\tilde{r}) - k(\tilde{r})}{1 + R_F} \cdot I \quad (11)$$

With both expected return and RADR being disequilibrium ones, the difference is discounted at the risk free rate. The numerator $[E(\tilde{r}) - k(\tilde{r})]$ is similar to "Jensen's alpha" used in performance analysis, indicating the vertical distance to the security market line (SML).

Multiplying by the investment cost, the amount $[E(\tilde{r}) - k(\tilde{r})] \cdot I$ may be interpreted as the *net* future value certainty equivalent. For the base example, substitution into Eq. (11) yields

$$NPV = \frac{1.00 - 0.40}{1 + 0.00} \cdot 50 = 0.60 \cdot 50 = 30.00.$$

So far, all three certainty equivalent formulations, Eqs. (7), (9) and (11), have given the same net present value $NPV = 30.00$.

¹⁴ See Rubinstein (1973:174) and Weston and Chen (1980) Eq. (2a).

4. CAPM market based return discount factor approaches

The standard CAPM is an equilibrium single period model. All returns are based on market prices, as in Eq. (2). In equilibrium, all assets satisfy the fundamental relation given by Eq. (1) in the extensive form of the standard version of the CAPM. The asset's market based beta is the return covariance term $\text{Cov}(\tilde{R}, \tilde{R}_M)$ divided by the market return variance

$$\beta(\tilde{R}) \equiv \frac{\text{Cov}(\tilde{R}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} \quad (12)$$

Combining Eqs. (1) and (12), the equilibrium expected market based asset return $E(\tilde{R})$ translates into the equilibrium risk-adjusted discount factor (RADR)

$$k(\tilde{R}) = R_F + [E(\tilde{R}_M) - R_F] \cdot \beta(\tilde{R}) \quad (13)$$

This RADR should generally be used for discounting expected cash flows under the assumptions of the CAPM, whenever the exact numerical values of gross or net present value are of interest:

$$P(\tilde{R}) = \frac{E(\tilde{X})}{1 + k(\tilde{R})} \quad (14)$$

In applications, the theoretical price P and hence the asset's market based return itself may be unknown initially. One approach may be to "guesstimate" a market price P , compute the asset's stochastic market based return \tilde{R} by Eq. (2), and then proceed to Eqs. (12)-(14) to compute its beta, RADR and market price, all conditional on the initial "guesstimated" market price¹⁵. If the computed market price does not coincide with its guesstimate, then start over again with a better initial value. By a suitable iterative procedure (or plain trial and error), the

¹⁵ Grinblatt and Titman (1998:387) comment that if the analyst made a lucky guess and selected the correct PV number, then the returns would have a beta and an associated discount rate that would generate the original PV as the discounted expected future cash flow. However, Eq. (15) below provides the equilibrium beta.

CAPM consistent equilibrium market price P should be found. This theoretical price may differ from the investment cost I , yielding a non-zero net present value.

Suppose the gross present value 80.00 from the CAPM certainty equivalent approach is selected as the initial guesstimate of P . The resulting market based return \tilde{R} is then

$R(\text{Good}) = 1.00$, $R(\text{So-so}) = 0.25$, and $R(\text{Bad}) = -0.50$, with a mean of $E(\tilde{R}) = 0.25$. The return covariance $\text{Cov}(\tilde{R}, \tilde{R}_M) = 0.15$, and hence the market based beta

$\beta(\tilde{R}) = \text{Cov}(\tilde{R}, \tilde{R}_M) / \text{Var}(\tilde{R}_M) = 0.15 / 0.06 = 2.50$. Thus, the equilibrium RADR

$k(\tilde{R}) = 0.00 + (0.10 - 0.00) \cdot 0.25 = 0.25$. Discounting the expected cash flow $E(\tilde{X}) = 100$ at 25% yields the gross present value of 80.00, which was the starting point, confirming that $P = 80.00$ is correct. Hence, using the CAPM consistent RADR of 25% provides the correct gross market value and the correct net present value.

In fact, there is no need for using an initial guesstimate of the theoretical market price for finding beta and RADR. Using the cash flow beta $\beta(\tilde{X})$ from Eq. (8), the equilibrium return beta is given by¹⁶

$$\beta(\tilde{R}) = \frac{\beta(\tilde{X}) \cdot (1 + R_F)}{E(\tilde{X}) - \beta(\tilde{X}) \cdot [E(\tilde{R}_M) - R_F]} \quad (15)$$

Substituting it into Eq. (13) for the equilibrium risk-adjusted discount factor, provides the closed form cash flow beta RADR

$$k(\tilde{R}) = \frac{R_F E(\tilde{X}) + [E(\tilde{R}_M) - R_F] \cdot \beta(\tilde{X})}{E(\tilde{X}) - [E(\tilde{R}_M) - R_F] \cdot \beta(\tilde{X})} \quad (16)$$

or alternatively using the market price of risk lambda¹⁷,

¹⁶ Equivalent formulations have been derived by Ehrhardt and Daves (2000) Eq. (4) and by Lund (2002) Eq. (4). They cannot be used for projects having both non-zero expected cash flows and zero gross PVs, as there is then no finite RADR that would yield a PV of zero.

$$k(\tilde{R}) = \frac{R_F E(\tilde{X}) + \lambda \text{Cov}(\tilde{X}, \tilde{R}_M)}{E(\tilde{X}) - \lambda \text{Cov}(\tilde{X}, \tilde{R}_M)} \quad (17)$$

Either equilibrium RADR may then be used for computing a consistent PV.

$$\text{Plugging into Eq. (15), } \beta(\tilde{R}) = \frac{200 \cdot (1 + 0.00)}{100 - 200 \cdot [0.10 - 0.00]} = \frac{200}{100 - 20} = 2.50, \text{ as asserted.}$$

The RADR of 0.25 is verified by $k(\tilde{R}) = \frac{0.00 \cdot 100 + [0.10 - 0.00] \cdot 200}{100 - [0.10 - 0.00] \cdot 200} = \frac{20}{80}$ and by

$$k(\tilde{R}) = \frac{0.00 \cdot 100 + \frac{5}{3} \cdot 12}{100 - \frac{5}{3} \cdot 12} = \frac{20}{80}.$$

A further interesting use of the equilibrium RADR, is the following adaptation of Eq. (10) for computing the NPV in the case of uncertainty¹⁸:

$$NPV = \frac{E(\tilde{r}) - k(\tilde{R})}{1 + k(\tilde{R})} \cdot I \quad (18)$$

Compare Eqs. (11) and (18). In the latter formulation, the market based RADR has replaced both the cost based RADR in the numerator and the risk free rate in the denominator. The NPV is still related to a "Jensen's alpha" measure, but now interpreted as the excess of the expected cost based return (or expected IRR) over the equilibrium RADR. Using previously

computed values, $NPV = \frac{1.00 - 0.25}{1 + 0.25} \cdot 50 = \frac{0.75}{1.25} \cdot 50 = 30.00$.

5. Multi beta present value computations

So far, three different betas have been computed: The cost based return beta $\beta(\tilde{r}) = 4.00$ defined in Eq. (4), the equilibrium beta $\beta(\tilde{R}) = 2.50$ defined in Eq. (12), and the cash

¹⁷ In unpublished course materials Hayne Leland has independently derived an equivalent lambda form equilibrium RADR, corresponding to Eq. (17) after dividing through by the expected cash flow.

¹⁸ See Weston and Chen (1980) Eq.(1).

flow beta $\beta(\tilde{X})=200$ defined in Eq. (8). All have the same denominator, viz. the variance $\text{Var}(\tilde{R}_M)$ of the return to the market portfolio. In their numerators, all three betas have a covariance of the market portfolio return to a different function of the asset cash flow, when recognizing Eqs. (2) and (3), respectively, for the definitions of market based and cost based asset returns \tilde{R} and \tilde{r} .

Dividing Eq. (4) by Eq. (12) shows that the two return betas are related by

$\beta(\tilde{R}) = \frac{I}{P} \cdot \beta(\tilde{r})$. Thus, for assets having a positive NPV, investment cost is less than market price, market based beta is less than the cost based beta, the equilibrium RADR is less than the disequilibrium RADR, and the computed disequilibrium present value is necessarily too low. For the example, $\beta(\tilde{R}) = \frac{50}{80} \cdot 4.00 = 2.50$, as shown previously.

If both return betas are somehow available, the theoretical gross present value equals the investment cost multiplied by the ratio of the cost based beta to the market based beta:

$$P = \frac{\beta(\tilde{r})}{\beta(\tilde{R})} \cdot I \quad (19)$$

As a check, $P = \frac{4.00}{2.50} \cdot 50 = 80.00$.

From dividing Eq. (8) by Eq. (12), the ratio of the cash flow beta to the equilibrium return beta is simply the gross present value¹⁹:

$$P = \frac{\beta(\tilde{X})}{\beta(\tilde{R})} \quad (20)$$

Recall that the correct equilibrium beta is given by Eq. (15). The CAPM consistent theoretical price $P = 80$ is verified from the beta ratio $200/2.50=80.00$.

¹⁹ A similar result appears in Grinblatt and Titman (1998:391). Of course, the equilibrium beta cannot be zero, to avoid division by zero.

6. Conditional expected cash flow discounting by absence of arbitrage

Using some particular cash flow formulations, risk adjustments may be simplified. Without any loss of generality, let the asset's cash flow be a linear function of the market portfolio return:

$$\tilde{X} = a + b\tilde{R}_M + \tilde{\varepsilon} \quad (21)$$

where a and b are constants, such that by construction $\tilde{\varepsilon}$ is a mean zero residual which is uncorrelated with the market return \tilde{R}_M . This cash flow generating process is similar to the market model (MM) and the single index model (SIM) or single factor model, which are often used in conjunction with the CAPM, but with individual asset return rather than asset cash flow being determined²⁰. It may be noted that the constant b equals the cash flow beta $\beta(\tilde{X})$, as $\text{Cov}(\tilde{X}, \tilde{R}_M) = b \cdot \text{Var}(\tilde{R}_M)$ for uncorrelated residuals. Taking unconditional expectations, the constant a is $a = E(\tilde{X}) - bE(\tilde{R}_M)$ for a mean zero residual. For further

interpretations of the constants, rewrite the cash flow as $\tilde{X} = \frac{a-b}{1+R_F}(1+R_F) + b(1+\tilde{R}_M) + \tilde{\varepsilon}$.

Here the first two terms form a portfolio tracking the asset's cash flow, with the residual $\tilde{\varepsilon} = \tilde{X} - a - b\tilde{R}_M$ being a tracking error. The tracking portfolio is composed by investing the amount b in the market portfolio combined with a risk free lending of $[(a-b)/(1+R_F)]$.

By value additivity, and letting $V(\bullet)$ be a general valuation operator, the asset cash flow's value $V(\tilde{X}) = V(a) + bV(\tilde{R}_M) + V(\tilde{\varepsilon})$. As a is a constant, $V(a) = \frac{a}{1+R_F}$. By absence of arbitrage, $V(1+\tilde{R}_M) = V(1+R_F) = 1$, and hence $V(\tilde{R}_M) = V(R_F) = 1 - \frac{1}{1+R_F} = \frac{R_F}{1+R_F}$. The

²⁰ The nontrivial feature of these models is that the residuals are assumed uncorrelated across assets, which is not an issue here.

difficult part is $V(\tilde{\varepsilon})$, i.e., valuing the residual or tracking error. In case of perfect tracking, $\tilde{\varepsilon} = 0$, with an obvious zero market value $V(\tilde{\varepsilon}) = 0$. Otherwise, some asset pricing model is needed.

According to the CAPM CE Eq. (7), $V(\tilde{\varepsilon}) = 0$ under the zero mean and zero correlation assumptions. Hence, with the assumed linear cash flow pattern given in Eq. (21) and with the assumed tracking error properties, the gross present value is simply

$$P = \frac{a + bR_F}{1 + R_F} \quad (22)$$

This is another certainty equivalent formulation, which does not require any difficult computations. The CE follows from Eq. (21), by simply replacing the stochastic market portfolio return by the risk free rate, and ignoring the stochastic residual term. The numerator is thus the cash flow from the tracking portfolio's cash flow with perfect tracking, *if* the market return should equal the risk free rate. It is also analogous to a point lying on the usual OLS linear regression line.

It may be somewhat surprising that seemingly different approaches give the same CEs, but the reconciliation is straightforward. Substituting the values for the constants a and b implied by mean zero uncorrelated residuals, and reorganizing, the cash flow CE becomes

$$a + bR_F = E(\tilde{X}) - \frac{\text{Cov}(\tilde{X}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)} [E(\tilde{R}_M) - R_F].$$

The expression on the right hand side may be

recognized as the cash flow beta CE of Eq. (9), as well as the lambda CE of Eq. (7).

A related approach focuses on the tracking portfolio. Its gross present value is the sum of the investment in the market portfolio plus the amount lent: $P = b + \frac{a - b}{1 + R_F}$, which can

easily be reorganized as Eq. (22). With perfect tracking, the asset market value is also given

by Eq. (22), without reference to any particular asset pricing model²¹ but assuming no arbitrage.

The challenge is to extend this result to the case of imperfect tracking. As observed above, with a mean zero residual $\tilde{\varepsilon}$ which is uncorrelated with the market return \tilde{R}_M , imposing the CAPM will do the trick. Further properties of the tracking error have then no additional effect on valuation. It does not matter whether the tracking error variance may be considered "quite large". It is also irrelevant whether the conditional expected cash flow is nonlinear in the market return²². The residuals are not required to be independent of the market return, such that the conditional expected residual may be nonzero for some market returns²³. The difference between the conditional expected cash flow and the conditional expected tracking error, $a + bR_M = E(\tilde{X} | R_M) - E(\tilde{\varepsilon} | R_M)$, appears as the CE in the numerator of Eq. (22) when conditioning on the risk free rate. It is immaterial whether there is in fact any such state or scenario, where the market portfolio rate actually equals the risk free interest rate.

For the base case, the cash flow pattern satisfies Eq. (21), with the constants $a = 80$, $b = 200 (= \beta(\tilde{X}))$, and $\tilde{\varepsilon} = 0$, i.e., a noiseless generating cash process which allows perfect

²¹ This is the "simple discounting rule" of Black (1988), who assumes perfect tracking. For an extension, let the asset cash flow be a linear function of the returns on one or more arbitrary but fairly priced portfolio or security returns, but retain the perfect tracking assumption. Eq. (22) then still holds, with the market return sensitivity constant being replaced by the sum of the individual sensitivity constants of the individual risky return components. See Black (1988), who notes that this is a special case of more general results obtained by Ross (1978). Rephrased, the cash flow condition is that the investment's cash flow is spanned by portfolios (or securities) that are being priced according to their competitive equilibrium values. This spanning argument is similar to the one underlying the "unanimity approach" to valuation in incomplete markets, see e.g. Ekern and Wilson (1974).

²² Grinblatt and Titman (1998:394-396) incorrectly claim that the risk free discounting only works if the conditional expected cash flow is linear in the return, i.e., with the conditional expected residual always being zero. In contrast, the crucial requirement under the CAPM is that the unconditional expected residual is zero and that the residual is uncorrelated with the market return. Appendix 3 illustrates this increased applicability, with the Adonis Travel Agency example from Grinblatt and Titman (1998:385-392). Using a dataset provided by Hayne Leland, it is also unproblematic to value a highly nonlinear contingent claim involving the squared market return by the simple discounting rule, as it is just another way of providing a CAPM consistent CE.

²³ Recall that independent random variables have no correlation, whereas uncorrelated variables may be stochastically dependent. In general, the conditional expectations $E(\tilde{\varepsilon} | R_M) = 0$ of a residual (or tracking error) is a sufficient but not necessary condition for the unconditional expectation $E(\tilde{\varepsilon}) = 0$.

tracking. The conditional expected cash flow $E(\tilde{X} | R_M) = a + bR_M$ is trivially equal to both the asset's cash flow itself and to the tracking portfolio's cash flow. The arbitrage free market

value $P = \frac{a + bR_F}{1 + R_F} = \frac{80 + 200 \cdot 0.00}{1 + 0.00} = 80.00$, as for the other CAPM consistent methods. It

does not matter that there is no state s such that $R_M(s) = R_F$.

The risk free discounting in Eq. (22) and the traditional CAPM risk adjusted discounting as in Eq. (14) may be combined and generalized to a conditional risk adjusted discounting approach. Rather than using the equilibrium RADR as given in Eq. (13), the conditional risk adjusted discount rate method uses some arbitrary market return value R_M instead of the unconditional expected market return $E(\tilde{R}_M)$:

$$k(R_M) \equiv R_F + (R_M - R_F) \cdot \beta(\tilde{R}) \quad (23)$$

Then use the conditional cash flow $a + bR_M = E(\tilde{X} | R_M) - E(\tilde{\varepsilon} | R_M)$, conditioning on the same but arbitrary market portfolio return. The equilibrium present value is then found by discounting the tracking portfolio's conditional cash flow, or equivalently the asset's conditional expected cash flow in excess of the conditional expected tracking error, at this conditional RADR²⁴:

$$P = \frac{a + bR_M}{1 + k(R_M)} \quad (24)$$

In the case of conditional expected tracking error always being zero, the numerator is simply the conditional expected cash flow. By itself this approach requires neither the unconditional expectation nor the variance of the market portfolio return, but the correct market based beta $\beta(\tilde{R})$ is needed for computing the conditional discount rate $k(R_M)$. The practical

²⁴ Black (1988) contains a verbal but imprecise description of a somewhat similar procedure.

applicability of Eq. (24) may thus be somewhat limited²⁵, as Eq. (22) using risk free discounting is even simpler.

For the base example²⁶, consider the Good state, with a market return of $R_M(\text{Good}) = 0.40$. The conditional discount rate $k(R_M) = 0.00 + (0.40 - 0.00) \cdot 2.50 = 1.00$. The conditional cash flow $E(\tilde{X} | \text{Good}) = 80 + 200 \cdot 0.40 = 160$, consistent with $X(\text{Good}) = 160$ for this noiseless asset cash flow. Conditional discounting yields $P = \frac{160}{1+1.00} = 80.00$. The conditional discount rates become 0.25 and -0.50 for the So-so and Bad states, respectively. Eq. (24) then gives the same gross present value of 80.00, independent of the state.

7. CAPM adaptations of more general valuation models

Relying on its mean-variance foundations, the CAPM is a rather special valuation model, strictly holding for only particular preferences or return distributions. Finance provides a plethora of more general models, including the state preference model, the martingale risk-adjusted probability model, and the stochastic discount factor model. By the Fundamental Theorem of Asset Pricing, all three latter models are equivalent to absence of arbitrage and also to optimal portfolio choice by some economic agent preferring more to less²⁷. Consistency with the CAPM imposes additional particular restrictions on the pricing factors.

²⁵ If unconditional expectations are available, then the market based beta may be computed from Eq. (15), with the cash flow beta replaced by the constant b .

²⁶ Appendix 3 also illustrates the conditional discount rate method for the more complex Adonis Travel Agency example from Grinblatt and Titman (1998:385-392), where market portfolio return and tracking error are uncorrelated but not independent, and thus the asset's conditional expected cash flows may be different from the tracking portfolio's cash flow.

²⁷ Ross (2005) provides a concise account of modern neoclassical asset pricing theory.

The State-preference model²⁸ (SP) is a positive linear pricing rule pricing assets based on their payoffs, or cash flows in the current setting. With a finite set of states s , the theoretical asset price is given as

$$P = \sum_s \varphi(s) X(s) \quad (25)$$

Here $X(s)$ is the assumed known cash flow if state s occurs, whereas $\varphi(s)$ is the state price for an elementary state-contingent claim paying one monetary unit if and only if state s is obtained²⁹. Basically, the state prices reflect state-contingent marginal utility for some optimally adapted economic agent, state probability, and time preference. The state prices may possibly be derived from market prices in a (dynamically) complete market. But if one makes the heroic assumption that both the state preference model and the CAPM hold simultaneously and yield the same asset value, then the state prices must be given as³⁰

$$\varphi(s) \equiv \frac{f(s)}{1+R_F} \left\{ 1 - \lambda \left[R_M(s) - E(\tilde{R}_M) \right] \right\} \quad (26)$$

where lambda $\lambda \equiv \frac{E(\tilde{R}_M) - R_F}{\text{Var}(\tilde{R}_M)}$ as in the CAPM CE formulation in Eq. (7). It will be seen

that the state prices sum to the risk free discount factor $1/(1+R_F)$. A state price will exceed its "discounted probability" $f(s)/(1+R_F)$ if and only if the market portfolio return $R_M(s)$ is less than the expected market return $E(\tilde{R}_M)$, if CAPM holds.

Recalling equally probable states, zero risk free rate, the "Good" state market portfolio return $R_M(\text{Good}) = 0.40$, expected market return $E(\tilde{R}_M) = 0.10$, and market price of risk

²⁸ The state-preference approach to asset pricing was pioneered by Arrow and Debreu. For current textbook expositions, see Danthine and Donaldson (2005) Ch. 8 or Copeland *et al.* (2005) Ch. 4.

²⁹ The elementary state-contingent claims are also referred to as primitive securities or Arrow-Debreu certificates. Another term for state prices is Arrow-Debreu prices.

³⁰ The origin of Eq. (26) is not known to the author, but the expression has been around for decades. Also note that the market portfolio return must be bounded above, to avoid negative state prices for exceptionally high market returns. However, even with some negative state prices, the computed asset value would be consistent with the CAPM value.

$\lambda = 5/3 \approx 1.67$, the state price $\varphi(\text{Good}) = \frac{1/3}{1+0} \left\{ 1 - \frac{5}{3} [0.40 - 0.10] \right\} = \frac{1}{6}$. Corresponding

computations show that $\varphi(\text{So-so}) = \frac{1}{3}$ and $\varphi(\text{Bad}) = \frac{1}{2}$. From Eq. (25), the SP theoretical

market value of the asset is $P = \frac{1}{6} \cdot 160 + \frac{1}{3} \cdot 100 + \frac{1}{2} \cdot 40 = \frac{1}{6} (160 + 200 + 120) = \frac{480}{6}$, verifying

that $P = 80.00$ according to the state preference model as well.

The martingale risk-adjusted probabilities approach³¹ uses risk-adjusted probabilities $f^*(s)$ rather than the "true" probabilities $f(s)$ to compute the cash flow certainty equivalent as a risk-adjusted "expected" cash flow

$$E^*(\tilde{X}) \equiv \sum_s f^*(s)X(s) \quad (27)$$

This "expected" cash flow is then discounted at the risk free rate, giving the theoretical market value

$$P = \frac{E^*(\tilde{X})}{1 + R_F} \quad (28)$$

This procedure is particularly popular in option pricing³², but it has a wider applicability. It is also often referred to as risk-neutral pricing, as the "expected" cash flow is discounted at the risk free rate.

For consistency with the state preference model, the risk adjusted state probabilities $f^*(s) = (1 + R_F)\varphi(s)$, ensuring that they sum to unity. Using state prices $\varphi(s)$ from Eq. (26), the risk adjusted probability of state s occurring is

$$f^*(s) \equiv f(s) \left\{ 1 - \lambda [R_M(s) - E(\tilde{R}_M)] \right\} \quad (29)$$

³¹ It was pioneered by Cox and Ross (1976a, 1976b) and by Harrison and Kreps (1979).

³² Binomial option pricing is a convenient approach, consistent with Eq. (28).

With the zero risk free rate of the example, the risk adjusted state probabilities $f^*(s)$ have the same numerical value as the corresponding state price $\varphi(s)$. Hence, the certainty equivalent $E^*(\tilde{X}) = 80.00$ discounted at a zero rate results in the gross present value $P = 80.00$ by the risk adjusted martingale probabilities method as well.

Under certainty, discounting may be expressed as multiplying cash flows with their corresponding discount factors, and then summing to get present values. Under uncertainty, the stochastic discount factor (SDF) approach values a one period project as the (true) expectation of the product of the stochastic discount factors and cash flows,

$$P = E(\tilde{m}\tilde{X}) = \sum_s f(s)m(s)X(s) \quad (30)$$

Here \tilde{m} is the stochastic discount factor³³, with state-contingent value $m(s)$. For consistency with the two previous models, the SDF in state s is $m(s) = \varphi(s)/f(s)$, i.e., the state price normalized by its state probability. The CAPM then requires that

$$m(s) \equiv \frac{1}{1+R_F} \left\{ 1 - \lambda \left[R_M(s) - E(\tilde{R}_M) \right] \right\} \quad (31)$$

The three states in the example are equally probable. Hence, the SDFs $m(s)$ are three times the corresponding state prices $\varphi(s)$. Thus, $m(\text{Good}) = \frac{1}{2}$, $m(\text{So-so}) = 1$, and

$m(\text{Bad}) = \frac{3}{2}$. Computing the expected product of SDF and cash flow,

$$P = \frac{1}{3} \left[\frac{1}{2} \cdot 160 + 1 \cdot 100 + \frac{3}{2} \cdot 40 \right] = [80 + 100 + 60]/3 = 240/3, \text{ once again confirming } P = 80.00.$$

³³ Also referred to as the pricing kernel, the state price deflator, or the state price density.

8. A closer look at the betas

Discounting the expected cash flows $E(\tilde{X})=100$ by the discount rate $k(\tilde{r}) = 0.40$ based on the cost based beta $\beta(\tilde{r}) = 4.00$, resulted in the incorrect gross present value $PV = 500/7 \approx 71.43$. With the market return based beta $\beta(\tilde{R}) = 2.50$, the resulting RADR of $k(\tilde{R}) = 0.25$ yielded the correct theoretical market value of $P = 80.00$, as did the eleven other CAPM related methods discussed afterwards. The correct NPV is therefore $NPV = 30.00$.

To reconcile the different betas, recall the well known fact that the beta of a portfolio equals the weighted betas of its components, with market value proportions as weights. Consider decomposing the asset into a risky zero NPV component with stochastic cash flow \tilde{X}_0 and cost I , and another non-zero NPV component with cash flow \tilde{X}_{NPV} and a zero cost:

$$\tilde{X} = \tilde{X}_0 + \tilde{X}_{NPV} . \quad (32)$$

The asset or "portfolio" has a market value P , the zero NPV component has a market value $P_0 = I$, and the non-zero NPV component has a market value $P_{NPV} = NPV$. Thus, the correct beta of the asset is the weighted beta $\beta(\tilde{R}) = \frac{I}{P}\beta(\tilde{R}_0) + \frac{NPV}{P}\beta(\tilde{R}_{NPV})$. The remaining problem is then to decide on the decomposition and to compute the betas.

First, suppose that the non-zero NPV component is non-risky, with certain cash flow $X_{NPV} = NPV \cdot (1 + R_F)$, implying the zero NPV component cash flow $\tilde{X}_0 = \tilde{X} - NPV \cdot (1 + R_F)$. Deducing a constant from a stochastic variable has no impact on a covariance, such that $\text{Cov}(\tilde{X}_0, \tilde{R}_M) = \text{Cov}(\tilde{X}, \tilde{R}_M)$. The cost based betas for the asset and for its zero NPV component are therefore equal. Furthermore, for the zero NPV component, its cost based and market based betas are equal as the market price equals the investment cost. Thus, $\beta(\tilde{R}_0) = \beta(\tilde{r}_0) = \beta(\tilde{r})$. The non-risky non-zero NPV component obviously has a zero

market beta $\beta(R_{NPV}) = 0.00$, whereas its cost based beta $\beta(r_{NPV})$ would be undefined due to division by its zero cost. With $\beta(\tilde{R}_{NPV}) = 0.00$, the weighted beta expression simplifies to

$\beta(\tilde{R}) = \frac{I}{P} \beta(\tilde{R}_0)$. Furthermore, using $\beta(\tilde{R}_0) = \beta(\tilde{r})$, the desired relationship between

market and cost based betas is $\beta(\tilde{R}) = \frac{I}{P} \beta(\tilde{r})$. A corresponding expression was shown

directly in Section 5, simply by division of the two equations defining the two return betas.

Here it was demonstrated using decomposition and a portfolio approach. In the example,

$$\beta(\tilde{R}) = \frac{50}{80} \cdot 4.00 = 2.50, \text{ as asserted.}$$

Next, consider a proportional risky decomposition, where the risky zero NPV

component $\tilde{X}_0 = \frac{I}{P} \tilde{X}$ has the gross PV = I . Its cost and market based return betas both equal

the asset's beta, $\beta(\tilde{r}_0) = \beta(\tilde{R}_0) = \beta(\tilde{R})$. The risky non-zero NPV component $\tilde{X}_{NPV} = \frac{P-I}{P} \tilde{X}$

has a zero cost and a NPV = $P - I$. Its cost based return and cost based beta are not well

defined. Its market based beta equal the asset's beta, $\beta(\tilde{R}_{NPV}) = \beta(\tilde{R})$. With the asset

considered as a portfolio, the asset beta as a PV weighted combination of the components'

equal market based betas, is trivially the same beta. Thus, for this decomposition, the market

betas of both components equal the asset's beta: $\beta(\tilde{R}_0) = \beta(\tilde{R}_{NPV}) = \beta(\tilde{R}) = 2.50$.

9. A Security Market Line (SML) illustration

Loosely speaking, the SML relates the expected return of any asset to its beta according to

$$E(\tilde{R}) = R_F + [E(\tilde{R}_M) - R_F] \cdot \beta \quad (33)$$

Most often, expected returns are plotted along the vertical axis, and betas along the horizontal axis. Unfortunately, the exact definitions of the returns and particularly of the betas are often missing. However, whenever the CAPM holds exactly, all assets plot exactly on the SML, using market based returns and market based beta.

In disequilibrium, assets may plot off the SML, indicating mispricing or non-zero NPVs. Assets plotting above the SML are considered underpriced with a positive NPV. Assets plotting below the SML are considered overpriced with a negative NPV. A disequilibrium is generally considered a transient situation for traded assets. The transition to equilibrium is left unexplained, beyond statements like that according to the CAPM, asset prices will somehow adjust until equilibrium is established, but not how and to what. For non-traded assets, the costs may remain different from theoretical PVs.

Exhibit 2 illustrates the base case example. The SML has a zero intercept because of the risk free rate. Its slope of 0.10 is the expected excess market portfolio return above the risk free rate. In equilibrium, any asset would plot exactly along this SML. The equilibrium expected return would be $E(\tilde{R}) = k(\tilde{R}) = 0.25$ for a beta equal to the market based $\beta(\tilde{R}) = 2.50$, as indicated by the lower circle. For a beta equal to the cost based $\beta(\tilde{r}) = 4.00$, the equilibrium expected return would be $E(\tilde{R}) = k(\tilde{r}) = 0.40$, as indicated by the lower square.

The asset in question has a non-equilibrium expected return of 1.00. But what beta should be used for plotting the asset in the diagram? The upper circle corresponds to using the market based $\beta(\tilde{R}) = 2.50$, whereas the upper square applies to the cost based $\beta(\tilde{r}) = 4.00$. The vertical distances between the two circles and between the two squares, respectively, correspond to two different versions of Jensen's alpha, being either 0.75 or 0.60. Both

alternative versions may be used for computing NPV, as demonstrated by Eqs. (18) and (11), respectively.

It appears that an unambiguous consensus as to the relevant beta for using Jensen's alpha has not yet been established. Focus on a benchmark predicted by the CAPM might favor the equilibrium market based beta, but the cost based beta or more generally other disequilibrium betas such as betas based on regression or factor models have also been suggested³⁴.

If the transition to equilibrium is supposed to take place through price changes, then the market based beta appears most relevant. For a traded asset, the price (and hence the cost) would converge to the CAPM theoretical price $P = 80.00$, plotting on the SML with an expected return of 0.25, corresponding to $\beta(\tilde{R}) = 2.50$.

However, the transition might also conceivably take place through revision of assumed cash flow properties. Consider the disequilibrium cash flow decomposition $\tilde{X} = \tilde{X}_0 + \tilde{X}_{NPV}$ of Eq. (32) in the previous section. Equilibrium would then be established when the second component \tilde{X}_{NPV} becomes zero, which may happen in different ways. From the first decomposition, a certain downward shift of $X_{NPV} = NPV \cdot (1 + R_f) = 30$ in assumed cash flow, would reduce the market price to the investment cost of 50. After such a (negative) additive shift in perceived cash flow to $\tilde{X}_0 = \tilde{X} - NPV(1 + R_f)$, the asset would then plot on the SML, with the equilibrium expected return of 0.40, and with the corresponding beta being $\beta(\tilde{r}) = 4.00$. However, with the second proportional risky cash flow decomposition, equilibrium occurs whenever the second component $\tilde{X}_{NPV} = \frac{P - I}{P} \tilde{X}$ reaches zero and the

³⁴ Levy and Post (2005:776) state that Jensen suggested regressing the asset excess return on the market excess return.

perceived cash flow distribution \tilde{X} undergoes a multiplicative shift to $\tilde{X}_0 = \frac{I}{P} \tilde{X}$. The asset would then also plot on the SML, but now with the equilibrium expected return of 0.25 and with the corresponding beta being $\beta(\tilde{R}) = 2.50$. Obviously, in general there are numerous combinations of additive and multiplicative shifts of the original cash flow distribution into some CAPM zero NPV distribution, leaving the resulting plot on the SML undetermined.

10. Conclusions

Judging from finance courses and finance textbooks as well as surveys of practitioners, the CAPM remains a central cornerstone in capital budgeting and security valuation, despite impressive advances in asset pricing theory. Suppose that for some unspecified reason, it is decided to use CAPM related valuation tools in a particular decision situation, say, for a capital budgeting project. If the analyst is not sufficient familiar with the conceptual CAPM foundations, she may apply a CAPM related procedure that is not conceptually sound and which causes a systematic numerical valuation bias compared to the one obtained from a correctly computed theoretical CAPM benchmark, possibly leading to an incorrect decision.

The CAPM is an equilibrium model, with returns based on equilibrium prices. In disequilibrium, the cost differs from market price, and cost (IRR) based returns are different from market based returns. Covariance terms for market based and cost based asset returns with the market portfolio return are different, causing the corresponding market and cost based betas to be different. Therefore, the expected returns used as required rates of returns in discount factors, are also different. If a cost (IRR) based beta is used for computing the risk adjusted discount rate in capital budgeting, the computed NPV will be systematically

underestimated compared to its theoretical CAPM counterpart, for projects having a positive NPV. Opposite bias effects occur for projects having negative NPVs.

For convenience, this paper has collected a dozen CAPM-related models, all yielding the same numerical values, and all being consistent with the conceptual foundations of the CAPM. The models include approaches based on certainty equivalents, equilibrium and disequilibrium required discount rates, simplified discounting based on absence of arbitrage for particular cash flow patterns, as well as required adaptations to make valuations from more advanced valuation methods consistent with correct CAPM procedures. It may also be handy to have the derivations of all twelve valuation expressions collected in one single appendix.

Considering the difficulties in obtaining adequate inputs to even a simple CAPM-related analysis in practice, the difference between cost based and market based returns, betas and RADRs may seem like a minor detail. Furthermore, it is by no means obvious that the CAPM should be used at all. However, if a CAPM-related method is used, it should be used correctly. Different valuation results may still be a major detail, at least from a conceptual point of view, and also for the effects on optimal decisions. A small step in the right direction may be to have more textbook discussions of how to apply CAPM-related valuation methods consistently. The conceptual inconsistency issue and its practical ramifications should be addressed at least in passing.

Summing up, with a dozen consistent CAPM-related models available, analysts should have wide opportunities to apply appropriate methods with which they are familiar. So why continue using an incorrect one of discounting expected cash flows by a RADR from a cost (IRR) based beta?

References

- Black, F. (1988), "A simple discounting rule", *Financial Management*, 17 (2, Summer), 7-16
- Bodie, Z., A. Kane and A. J. Marcus (2005), *Investments*, 6th Ed., McGraw-Hill
- Bossaerts, P. L. and B. A. Ødegaard (2001), *Lectures on corporate finance*, World Scientific
- Bossaerts, P. L. and B. A. Ødegaard (2006), *Lectures on corporate finance*, 2nd Ed., draft
- Brealey, R. A., S. C. Myers and F. Allen (2006), *Principles of corporate finance*, 8th Ed., McGraw-Hill
- Brounen, D., A. de Jong, and K. Koedijk (2004), "Corporate finance in Europe: Confronting theory with practice", *Financial Management*, 33 (4, Winter), 71-101
- Copeland, T. E., J. F. Weston and K. Shastri (2005), *Financial theory and corporate policy*, 4th Ed., Addison-Wesley
- Cox, J. C. and S. A. Ross (1976a), "The valuation of options for alternative stochastic processes", *Journal of Financial Economics*, 7, 229-263
- Cox, J. C. and S. A. Ross (1976a), "A survey of some new results in financial option pricing theory", *Journal of Finance*, 31 (1), 383-402.
- Danthine, J.-P. and J. Donaldson (2005), *Intermediate financial theory*, 2nd Ed., Elsevier Academic Press
- Ehrhardt, M. C. and P. R. Daves (2000), "Capital budgeting: The valuation of unusual, irregular, or extraordinary cash flows", *Financial Education and Practice*, 10 (2, Fall/Winter), 106-114
- Ekern, S. and R. Wilson (1974), "On the theory of the firm in an economy with incomplete markets", *Bell Journal of Economics and Management Science*, 5 (1, Spring), 171-180.
- Elton, E. J., M. J. Gruber, S. J. Brown, and W. N. Goetzmann (2003), *Modern portfolio theory and investment analysis*, 6th Ed., Wiley
- Fama, E. F. (1977), "Risk adjusted discount rates and capital budgeting under uncertainty", *Journal of Financial Economics*, 5 (August), 1-24
- Fama, E. F. (1996), "Discounting under uncertainty", *Journal of Business*, 69 (4, October), 415-428
- Graham, J. R. and C. R. Harvey (2001), "The theory and practice of corporate finance: Evidence from the field", *Journal of Financial Economics*, 61, 187-243
- Grinblatt, M. and S. Titman (1998), *Financial markets and corporate strategy*, McGraw-Hill

Harrison, M. and D. Kreps (1979), "Martingales and multi-period securities markets", *Journal of Economic Theory*, 20, 381-408

Jagannathan, R. and I. Meier (2002), "Do we need CAPM for capital budgeting?", *Financial Management*, 31 (4, Winter), 55-77

Levy, H. and T. Post (2005), *Investments*, Prentice Hall

Lund, D. (2002), "Taxation, uncertainty, and the cost of equity", *International Tax and Public Finance*, 9, 479-499

Markowitz, H. M. (1984), "The 'two beta' trap", *Journal of Portfolio Management*, 11 (1, Fall), 12-20

Rendleman, R. J., Jr (1978), "Ranking errors in CAPM capital budgeting applications", *Financial Management*, 8 (Winter), 40-44

Ross, S. A. (1978), "A simple approach to the valuation of risky streams", *Journal of Business*, 51 (3, July), 453-475

Ross, S. A. (2005), *Neoclassical finance*, Princeton University Press

Ross, S. A., R. W. Westerfield and J. F. Jaffe (2005), *Corporate finance*, 7th Ed., McGraw-Hill

Rubinstein, M. E. (1973), "A mean-variance synthesis of corporate financial theory", *Journal of Finance*, 28 (March), 167-181

Sharpe, W. F., G. J. Alexander and J. F. Bailey (1999), *Investments*, 6th Ed., Prentice Hall

Weston, J. F. and N-f. Chen (1980), "A note on capital budgeting and the three Rs", *Financial Management*, 9 (1), 12-14

Womack, K. L. (2001), "Core finance courses in the top MBA programs in 2001", Tuck School of Business at Dartmouth Working Paper No. 01-07, SSRN Electronic Paper Collection <http://papers.ssrn.com/abstract=291973>

Appendix 1: Derivations of valuation equations in the text

Lambda CE-form: From the CAPM in its extensive form Eq. (1), the definition of market return in Eq. (2), and the definition of the "market price of risk" lambda as

$$\lambda \equiv \frac{E(\tilde{R}_M) - R_F}{\text{Var}(\tilde{R}_M)}, \text{ the CAPM can be written as } E\left(\frac{\tilde{X}}{P}\right) - 1 = R_F + \lambda \text{Cov}\left(\frac{\tilde{X}}{P} - 1, \tilde{R}_M\right).$$

Deducting a constant has no effect on the covariance term. Multiplying through by P and rearranging give $(1 + R_F)P = E(\tilde{X}) - \lambda \text{Cov}(\tilde{X}, \tilde{R}_M)$. Solving for P yields Eq. (7). \square

Cash flow beta CE-form: Combine the definitions of cash flow beta from Eq. (8) and of lambda above, and substitute into Eq. (7). Eq. (9) follows immediately. For an alternative,

note that $\beta(\tilde{R}) = \frac{\beta(\tilde{X})}{P}$, substitute into $k(\tilde{R})$, discount $E(\tilde{X})$, multiply through by P on

RHS, cancel P in the equation, multiply through by denominator, and reorganize. \square

Cost based alpha CE-form: The cost based return definition Eq. (3) can be rearranged

into $I = \frac{\tilde{X}}{1 + \tilde{r}}$, showing that the cost based return is also the internal rate of return IRR. The

cash flow is therefore $\tilde{X} = (1 + \tilde{r})I$, with expected value $E(\tilde{X}) = [1 + E(\tilde{r})]I$ and covariance

$\text{Cov}(\tilde{X}, \tilde{R}_M) = \text{Cov}(\tilde{r}, \tilde{R}_M)I$. Substitution into Eq. (7), observing net present value

$NPV \equiv P - I$, and collecting terms, $NPV = \frac{I}{1 + R_F} \left\{ E(\tilde{r}) - [R_F + \lambda \text{Cov}(\tilde{r}, \tilde{R}_M)] \right\}$. As

$\lambda \text{Cov}(\tilde{r}, \tilde{R}_M) = [E(\tilde{R}_M) - R_F] \beta(\tilde{r})$, the term in square brackets is the cost based RADR

$k(\tilde{r})$ according to Eq. (5). Thus, Eq. (11) holds. \square

Market based RADR: Prior knowledge of the market price (gross PV) is not needed to apply Eq. (14). Combining Eqs. (2), (12) and (8) for market return, market based beta, and

cash flow beta, respectively, show that $\beta(\tilde{R}) = \beta(\tilde{X})/P$. Eq. (15) follows by substitution into the cash flow beta CE-formulation in Eq. (9). \square

Market beta based alpha: Substitution of $E(\tilde{X}) = [1 + E(\tilde{r})]I$ into the numerator of Eq. (14) for the market RADR based PV, and then deducting the investment cost, result in Eq. (18). \square

Cost based and market based betas: Using the definition of cost based return from Eq. (3), the cost based beta from Eq. (4) becomes $\beta(\tilde{r}) = \frac{1}{I} \frac{\text{Cov}(\tilde{X}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)}$. Similarly, the market based beta from Eq. (12) is $\beta(\tilde{R}) = \frac{1}{P} \frac{\text{Cov}(\tilde{X}, \tilde{R}_M)}{\text{Var}(\tilde{R}_M)}$ using Eq. (2). Division and cancellation of the common fraction provide Eq. (19) relating PV, investment, and cost and market based betas. \square

Cash flow and market return based betas: Eq. (20) follows immediately from division of the two betas and then cancellation. \square

Simple risk free discounting: Eq. (22) is derived in the text. \square

Simple conditional discounting: For any arbitrary cash flow satisfying the assumptions of Eq. (21), and using the conditional RADR defined in Eq. (23), consider the fraction

$$\frac{E(\tilde{X} | R_M) - E(\tilde{\varepsilon} | R_M)}{1 + k(R_M)} = \frac{a + bR_M}{1 + R_F + (R_M - R_F)\beta(\tilde{R})}. \text{ As the constant } b \text{ equals the cash flow}$$

beta $\beta(\tilde{X})$, the market based beta $\beta(\tilde{R}) = \frac{b}{P}$ by Eq. (20). Substitution and multiplication by

$$P \text{ convert the fraction into } P \frac{a + bR_M}{P(1 + R_F) + (R_M - R_F)b}. \text{ Substituting for } P \text{ from Eq. (22) in}$$

the denominator gives $P \frac{a + bR_M}{a + bR_F + bR_M - bR_F}$, which simplifies to P and verifies Eq. (24).

□

CAPM adaptation to state preference: If the CAPM should give correct pricing when applied to any risky asset, *then* it should also correctly price elementary state-contingent claims (or Arrow-Debreu certificates). The state price $\varphi(s)$ is the price of the elementary claim with cash flow \tilde{X}_s , which takes on the value 1 in state s and zero otherwise. From the

CAPM CE-formulation in Eq. (7), $\varphi(s) = \frac{E[\tilde{X}_s] - \lambda \text{Cov}(\tilde{X}_s, \tilde{R}_M)}{1 + R_F}$ if both the CAPM and

the SP hold. The particular structure of an Arrow-Debreu certificate imply that the expected cash flow equals the state probability, i.e., $E(\tilde{X}_s) = f(s)$. Furthermore, from the covariance

identity $\text{Cov}(\tilde{X}_s, \tilde{R}_M) = E(\tilde{X}_s \tilde{R}_M) - E(\tilde{X}_s)E(\tilde{R}_M)$, the covariance terms

$\text{Cov}(\tilde{X}_s, \tilde{R}_M) = f(s)R_M(s) - f(s)E(\tilde{R}_M)$, by the properties of \tilde{X}_s . Substituting, the state

price $\varphi(s) = \frac{f(s) - \lambda [f(s)R_M(s) - f(s)E(\tilde{R}_M)]}{1 + R_F}$, which can be rearranged as Eq. (26). □

CAPM adaptation to risk adjusted probabilities: Consistency of Eqs. (25), (27) and (28) requires that the risk adjusted state probabilities $f^*(s) = (1 + R_F)\varphi(s)$. Substitution from Eq. (26) then yields Eq. (29). □

CAPM adaptation to stochastic discount factors: For consistency between SP and SDF pricing, in Eqs. (26) and (30), respectively, the SDF in state s is $m(s) = \varphi(s)/f(s)$. Eq. (31) then follows immediately. □

Appendix 2: Some numerical calculations for base example

<i>Assumptions</i>					
Investment cost	I				50.00
Risk free rate of interest (tentative PV)	R_F				0.000 80.00
State of the economy	s	Good	So-so	Bad	Sum
Probability	$f(s)$	1/3	1/3	1/3	
Project's cash flow	$X(s)$	160	100	40	
Market portfolio return	$R_M(s)$	0.40	0.10	-0.20	
<i>Asset return</i>					
Cost based return	$r(s)$	2.200	1.000	-0.200	
Market based return (with $PV=80$)	$R(s)$	1.000	0.250	-0.500	
<i>Expected values</i>					
Cash flow	$E(\tilde{X})$	160/3	100/3	40/3	100.00
Market portfolio return	$E(\tilde{R}_M)$	2/15	1/30	- 1/15	0.100
Cost based return	$E(\tilde{r})$	11/15	1/3	- 1/15	1.000
Asset market return	$E(\tilde{R})$	1/3	1/12	- 1/6	0.250
<i>Variance</i>					
Market portfolio return	$\text{Var}(\tilde{R}_M)$	0.030	0.000	0.030	0.060
<i>Covariances with market portfolio return</i>					
Cash flow	$\text{Cov}(\tilde{X}, \tilde{R}_M)$	6.000	0.000	6.000	12.00
Cost based return	$\text{Cov}(\tilde{r}, \tilde{R}_M)$	0.120	0.000	0.120	0.240
Market based return	$\text{Cov}(\tilde{R}, \tilde{R}_M)$	0.075	0.000	0.075	0.150
<i>Beta with market portfolio return</i>					
Cash flow	$\beta(\tilde{X})$				200.00
Cost based return	$\beta(\tilde{r})$				4.000
Market based return	$\beta(\tilde{R})$				2.500
Lambda	λ				5/3

Appendix 3: The Grinblatt and Titman (1998) Adonis Travel Agency Example

Like the base example, the Grinblatt and Titman (1998:385-392) example has three scenarios (states), for which the investment project's cash flow and market portfolio return are specified. Unlike the base example, the states are not equally probable, making computations somewhat less transparent. Investment costs and risk free rates are also provided.

Grinblatt and Titman compute cost based betas as in Eq. (4) and discount the expected cash flow at the cost based RADR from Eq. (5). To their credit, they explicitly state that these betas are not really correct and they discuss the reasons why. They then move on to the certainty equivalent approach, using the asset beta to correct for risk as in Eq. (9), and computing a correct present value.

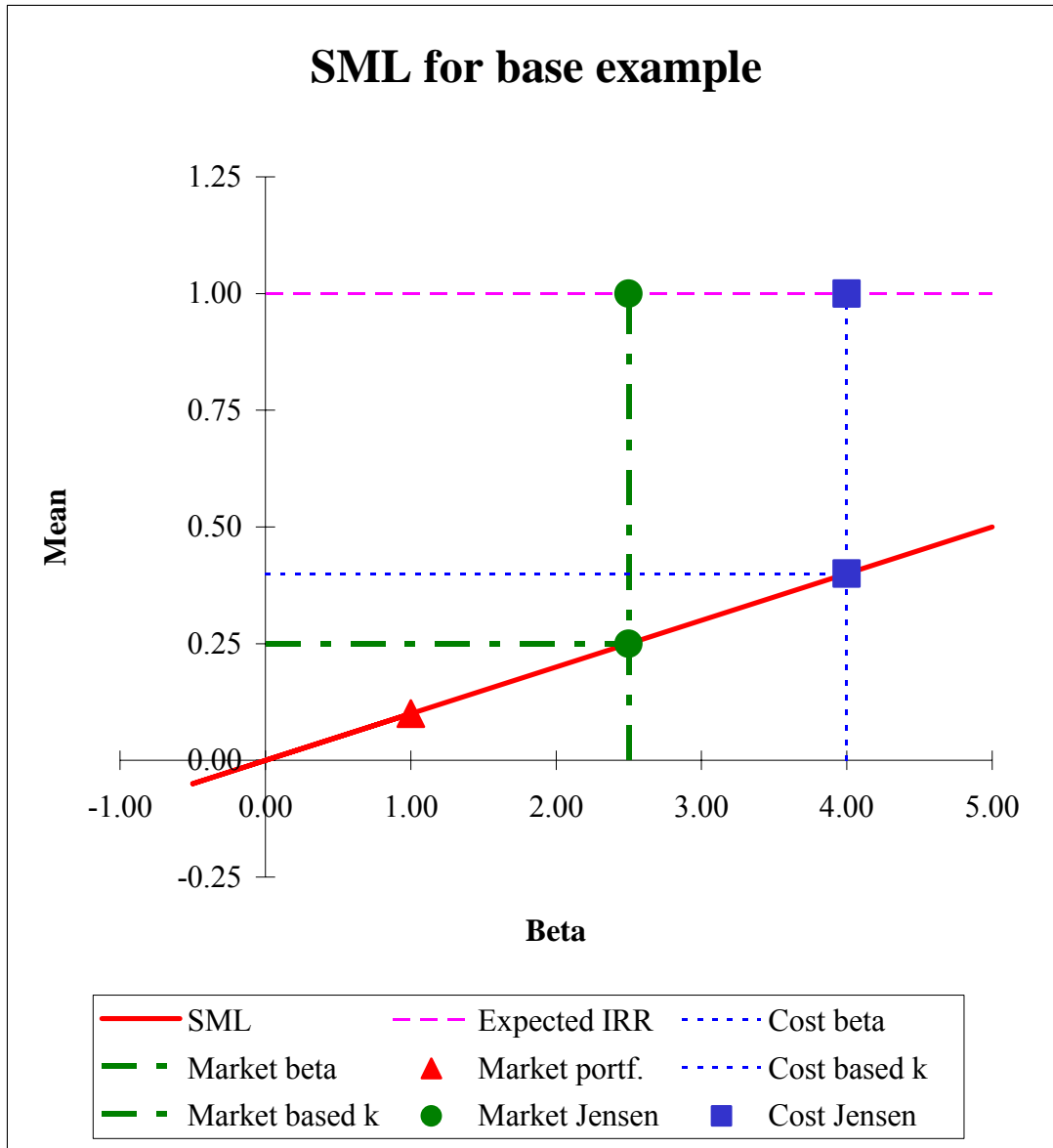
Importantly, their input specifications cause the conditional (expected) cash flows to be non-linear in the market return, implying non-zero conditional expected residuals. Apparently overlooking the possibility for still using the cash flow of the tracking portfolio in Eq. (22) or Eq. (24) for valuation, they do not use this example for illustrating the power of these simple discounting rules.

Exhibit 3 provides the assumptions and basic computations for this example. Exhibit 4 verifies the incorrect present value caused by cost based betas, as well as the correct present values as computed by the dozen CAPM consistent methods. In particular, Exhibit 4 shows how Eqs. (22) and (24) both work, even for this example with a non-zero conditional expected tracking error. Exhibit 5 illustrates a thought experiment with hypothetical outcomes consistent with the assumed properties, yielding the assumed slope and intercept and a zero mean residual uncorrelated with the market return, but with residuals systematically clustered (here in three single points) above or below the regression line for different market returns.

**Exhibit 1: Assumptions (in italics) and cost based computations
Single period base case example**

Investment cost	I		50.00		
Risk free rate of interest	R_F		0.000		
State of the economy	Proba- bility	Project's cash flow	Market return	Project return	
s	$f(s)$	$X(s)$	$R_M(s)$	$r(s)$	
Good	$1/3$	160	0.40	2.20	
So-so	$1/3$	100	0.10	1.00	
Bad	$1/3$	40	-0.20	-0.20	
Expected values	$E(\bullet)$	100	0.10	1.00	
Variance	$\text{Var}(\tilde{R}_M)$		0.06		
Covariance with market	$\text{Cov}(\tilde{r}, \tilde{R}_M)$			0.24	
Beta	$\beta(\tilde{r})$			4.00	
Required rate of return (RADR)	$k(\tilde{r})$			0.40	
Gross present value	PV			$500/7$	71.43
Net present value	NPV			$150/7$	21.43

Exhibit 2: Security market line (SML) for base example



**Exhibit 3 Assumptions and basic computations
Grinblatt and Titman (1998) Adonis Travel Agency Example**

Assumptions (in italics)					
	Investment cost	I			100.000
	Risk free rate of interest	R_F			8.625 %
	(tentative PV)				76.379
	State of the economy	s	Recovery	Recession	Depression
	Probability	$f(s)$	3/4	3/16	1/16
	Project's cash flow	$X(s)$	150	35	5
	Market portfolio return	$R_M(s)$	0.25	-0.01	-0.15
Asset return					
Eq.3	Cost based return	$r(s)$	0.500000	-0.650000	-0.950000
Eq.2	Market based return	$R(s)$	0.963901	-0.541756	-0.934537
	(with tentative PV)				
Expected values					
	Cash flow	$E(\tilde{X})$	112.500	6.563	0.313
	Market portfolio return	$E(\tilde{R}_M)$	0.188	-0.002	-0.009
	Cost based return	$E(\tilde{r})$	0.375	-0.122	-0.059
	Asset market return	$E(\tilde{R})$	0.723	-0.102	-0.058
Variance					
	Market portfolio return	$\text{Var}(\tilde{R}_M)$	0.004	0.007	0.007
Covariances with market portfolio return					
	Cash flow	$\text{Cov}(\tilde{X}, \tilde{R}_M)$	1.694	2.947	2.332
	Cost based return	$\text{Cov}(\tilde{r}, \tilde{R}_M)$	0.017	0.029	0.023
	Market based return	$\text{Cov}(\tilde{R}, \tilde{R}_M)$	0.022	0.039	0.031
Beta with market portfolio return					
Eq.8	Cash flow	$\beta(\tilde{X})$			404.542
Eq.4	Cost based return	$\beta(\tilde{r})$			4.04542
Eq.12	Market based return	$\beta(\tilde{R})$			5.29653
Eq.15	Market based return	$\beta(\tilde{R})$			5.29653
	Lambda	λ			5.22165
Risk adjusted discount rate					
Eq.5	Cost based beta	$k(\tilde{r})$			45.034 %
Eq.16	Market based beta	$k(\tilde{R})$			56.294 %
Linear cash flow generating process					
	Sensitivity parameter	b			404.542
	Constant	a			48.075
Eq.23	Conditional RADR	$k(R_M)$	0.95356	-0.42354	-1.16506
More general models					
Eq.26	State prices	$\varphi(s)$	0.42456	0.34048	0.15556
Eq.29	Adjusted state prob.	$f^*(s)$	0.46118	0.36985	0.16897
Eq.31	SDF	$m(s)$	0.56608	1.81591	2.48890

**Exhibit 4 Market/present value computations
Grinblatt and Titman (1998) Adonis Travel Agency Example**

Eq.		Either Or Or	Numerator Factor Recovery	Denominator Factor Recession Depression	<i>P</i> <i>NPV</i> <i>PV</i>
Market value/Gross present value					
Eq.6	Cost based RADR		119.37500	1.45034	82.308
Eq.14	Market based RADR		119.37500	1.56294	76.379
Eq.7	CE - lambda		82.96624	1.08625	76.379
Eq.9	CE - cash flow beta		82.96624	1.08625	76.379
Eq.19	Cost and market betas		0.76379	100.000	76.379
Eq.20	Cash flow and market betas		404.54175	5.29653	76.379
Net present values					
	CE definition		76.37859	100.000	-23.621
Eq.11	Jensen cost based		-0.23621	100.000	-23.621
Eq.18	Jensen market based		-0.23621	100.000	-23.621
Market value/Gross present value					
Eq.22	Risk free conditional CE		82.96624	1.08625	76.379
Eq.24	Conditional RADR	Recovery	149.20995	1.95356	76.379
		Recession	44.02910	0.57646	76.379
		Depression	-12.60674	-0.16506	76.379
	(arbitrary R_M)	0.2	128.98287	1.68873	76.379
Eq.25	State prices		63.68390	11.91691	0.77778
Eq.27	"Expected" cash flow		69.17664	12.94474	0.84486
Eq.28	Risk adjusted probabilities		82.96624	1.08625	76.379
Eq.30	SDF		63.68390	11.91691	0.77778

Exhibit 5 Thought experiment: Regression
Grinblatt and Titman (1998) Adonis Travel Agency Example

State	Observation	Cash flow	Market	Regression	Residual
Recovery	1	150.00	0.25	149.21	0.79
Recovery	2	150.00	0.25	149.21	0.79
Recovery	3	150.00	0.25	149.21	0.79
Recovery	4	150.00	0.25	149.21	0.79
Recovery	5	150.00	0.25	149.21	0.79
Recovery	6	150.00	0.25	149.21	0.79
Recovery	7	150.00	0.25	149.21	0.79
Recovery	8	150.00	0.25	149.21	0.79
Recovery	9	150.00	0.25	149.21	0.79
Recovery	10	150.00	0.25	149.21	0.79
Recovery	11	150.00	0.25	149.21	0.79
12 Recovery	12	150.00	0.25	149.21	0.79
Recession	13	35.00	-0.01	44.03	-9.03
Recession	14	35.00	-0.01	44.03	-9.03
3 Recession	15	35.00	-0.01	44.03	-9.03
1 Depression	16	5.00	-0.15	-12.61	17.61

Regressing on market return

Slope	404.542	0.000
Intercept	48.075	0.000
Correlation	0.994	0.000

Recall cash flow generating process

Sensitivity parameter	<i>b</i>	404.542
Constant	<i>a</i>	48.075

