VALUATION OF PUT OPTIONS ON LEVERAGED EQUITY

Dr Marco Realdon
Department of Economics and Related Studies
Helsington
York
YO10 5DD
UK
mr15@york.ac.uk

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Abstract

This paper presents new closed form solutions for the valuation of European put options and of "down-an-in" barrier options written on leveraged equity. Unlike in past literature (Toft and Prucyk, 1997) and in keeping with empirical evidence, the model allows equity to retain value even after the firm's default and reorganisation. This stylised fact can significantly alter the valuation of equity put and "down-and-in" options as bankruptcy costs, bargaining power of equity holders, debt maturity and other firm parameters change. The value of "in-the-money" puts often decreases in the firm's assets volatility. The model can produce a variety of realistic implied equity volatility "skews".

Keywords: equity put options, leveraged equity, default and reorganisation, barrier options, "down-and-in" options.

JEL classification: G13; G33.

1 INTRODUCTION

This paper studies the valuation of put options and "down-and-in" barrier options written on leveraged equity, whereby equity is itself a claim on the firm's assets and is subject to default risk. New closed form solutions for valuing put options are provided on the assumption that the underlying equity may retain value even after the firm's default and reorganisation. Such assumption (hereafter the "recovery assumption") differentiates this paper from previous contributions, notably Toft and Prucyk's (1997). The focus on the valuation of put options and "down-and-in" options is due to the fact that the "recovery assumption" affects these options more than others.

The "recovery assumption" is supported by substantial empirical evidence as reported in Gilson, Long and Lang (1990), Weiss (1990), Franks and Torous (1994) and others. Such evidence tells of absolute priority rule violations during firm reorganisations following default, either during private debt renegotiations or during formal bankruptcy proceedings. So equity holders keep a valuable claim on the assets of the firm even after default and even if debtors have not been paid in full. Moreover, even after the firm has defaulted and its stock has been delisted, the stock can keep trading "over the counter".

This paper shows how these stylised facts can be reflected in the valuation of equity put options and "down-and-in" barrier options, for such options appreciate precisely when equity value decreases and the firm approaches distress. Hence closed form solutions for these options are provided on the assumptions that equity can be valuable even after default and that option holders close out their position at the time of default. The latter assumption is buttressed by the provisions of the ISDA 2002 master agreement, which pertains also to over-the-counter equity derivatives, and by the regulations of some option exchanges, in that option positions are "closed out" and liquidated when the underlying stock is delisted following the firm's default and bankruptcy. Then the "close out" value of the option depends on the firm's equity value after bankruptcy.

A central result is that the "recovery assumption", bankruptcy costs and the bargaining power of equity holders during reorganisation can significantly affect the valuation of put options and "down-and-in" barrier options. One consequence is that the value of an equity put can decrease in the firm's assets volatility as the put is deep in the money and the firm approaches default. Moreover, the value of a put decreases in the firm's debt average maturity. These results are more material for the valuation of long term put options, such as "Leaps" or over the counter options, since the longer is the time to expiry, the more likely is the default of the underlying equity and the more critical is the "recovery assumption".

The "recovery assumption" as well as debt maturity seem capable to explain a variety of patterns for the volatility "skews" that are implied by observed equity option prices.

The paper is organised as follows. After a reference to the most relevant literature, the equity put valuation model is presented and comparative statics are performed. Then the analysis moves to the implied equity volatility "skews" that the put valuation model can explain and to the valuation of "down-and-in" options. The conclusions follow.

1.1 Literature

This paper views and values equity put options and "down-and-in" options as compound claims, i.e. claims written on equity, whereby equity is itself a claim on the firm's assets. The valuation of equity options as compound claims started with Geske (1979) and has recently developed through the work of Toft (1994), Toft and Prucyk (1997) and Ericsson (2002).

Geske and Ericsson view the equity underlying an option as a claim of finite maturity on the firm's assets. Instead in Toft and Prucyk and in this paper equity is viewed as a time independent claim of indefinite maturity, which simplifies the option valuation model and seems realistic.

Geske, Ericsson, Toft and Prucyk concentrate on the valuation of "European" equity call options, whereas this paper considers the valuation of equity put options and of "down-and-in" barrier options, for which the accurate modelling of financial distress is much more critical than for the valuation of call options. For example, Toft and Prucyk assumed that equity becomes worthless and that the equity call option is lost upon the firm's default, which is reasonable because default implies that the call will almost certainly not be exercised. But an equity put option is not expected to be lost upon default and it seems inaccurate to assume that equity is worthless after default, since the stock and the put option can keep trading after default even if the stock is delisted.

The literature on "strategic" structural models of credit risk, e.g. Anderson, Sundaresan and Tychon (1996), Mella-Barral and Perraudin (1997), Fan and Sundaresan (2000), has already recognised that in financial distress equity retains value when debt can be renegotiated. So this paper brings the results of such literature to bear for the valuation of equity options.

2 PUT OPTIONS VALUATION WHEN EQUITY IS STILL VALUABLE AFTER DEFAULT

This section presents a valuation model for put options written on leveraged equity, in the spirit of Toft and Prucyk (1997). The firm whose equity underlies the put option can default and can be reorganised. But unlike in Toft and Prucyk, here the underlying equity retains value even after default and reorganisation, which is a stylised fact featuring also in various recent structural models of credit risk and confirmed by the empirical literature.

In particular, the put valuation model follows Fan and Sundaresan (2000) in assuming that the firm defaults and is reorganised as soon as the firm's assets value V drops to a lower barrier V_s. Firm reorganisation averts bankruptcy and the associated bankruptcy costs. As in Fan and Sundaresan, reorganisation takes the form of a debt-equity swap, whereby debt holders exchange their debt claim for an equity claim on the firm's assets and previous equity holders retain a diluted equity claim. This formulation of reorganisation does not seem restrictive since a debt-equity swap may also proxy the payoffs to debt and equity holders that are associated with other forms of debt renegotiations, such as strategic debt service or negotiations in a formal bankruptcy proceeding.

The usual assumptions of structural models of credit risk underlie also this model. In particular perfect markets, absence of arbitrage and dynamic market completeness are assumed. The risk neutral process of the value of the firm's assets follows a geometric Brownian motion such as

$$dV = V \cdot (r - b) \cdot dt + V \cdot s \cdot dz \tag{1}$$

where b is the assets pay-out rate, r is the default free short interest rate that is assumed constant over time, s is the assets volatility and dz is the differential of a Wiener process.

The model for equity value $E(V)$ treats debt of finite average maturity as per Leland (1998). So at any time the firm generated net cash flows for equity holders are equal to

$$cf = bV - C(1 - t_x) + m(D(V) - F) \tag{2}$$

where bV is the assets generated cash flow, C are the coupon payments to debt holders, m is the yearly debt "roll-over" rate as in Leland (1998), $D(V)$ is total debt value, F is total debt face value, t_x is the corporation tax rate. In any short interval dt, a fraction $m \cdot dt$ of debt is retired at face value $m \cdot dt \cdot F$ and substituted by newly issued debt worth $m \cdot dt \cdot D(V)$, so that the nominal amount of outstanding debt is constant over time. Coupons payments generate a tax shield equal to Ct_x.

Using standard valuation arguments, we can deduce that the value of the firm's equity $E(V)$ must satisfy the following equation and boundary conditions:

$$\frac{1}{2}\frac{d^2E(V)}{dV^2}s^2V^2 + \frac{dE(V)}{dV}(r - b)V - rE(V) + cf = 0 \tag{3a}$$

$$E(V \to \infty) \to V + \frac{Ct_x}{r} - \frac{C + mF}{m + r} \tag{3b}$$

$$E(V_s) = yaV_s \tag{3c}$$

$$\left[\frac{dE(V)}{dV}\right]_{V=V_s} = ya \tag{3d}$$

where a denotes the fraction of assets value V that would be lost in case of assets liquidation and where y is a coefficient comprised between 0 and 1 that captures the bargaining power of equity holders in the reorganisation process that follows default. Condition 3b is a no-bubbles condition. Condition 3c states the payoff to equity holders upon reorganisation at V_s. Condition 3d is a "smooth pasting" condition, which implies that equity holders choose the reorganisation barrier V_s endogenously so as to maximise equity value $E(V)$.

In this model firm reorganisation always prevents liquidation and liquidation costs are never incurred. The y parameter is key. When $t_x = 0$, y can be thought of as the fraction of the bankruptcy costs saved through reorganisation that is attributed to equity holders.

The solutions to equation 3a for equity and the endogenous reorganisation barrier V_s are

$$E(V) = V - \frac{C+mF}{m+r} - \left(-\frac{C+mF}{m+r} + V_s(1-ya)\right)\left(\frac{V}{V_s}\right)^{q_m} + T_x(V) \quad (4)$$

$$T_x(V) = \frac{Ct_x}{r}\left(1 - \left(\frac{V}{V_s}\right)^q\right) \quad (5)$$

$$V_s = \frac{\frac{C+mF}{m+r}q_m - \frac{Ct_x}{r}q}{(1-ya)(q_m-1)} \quad (6)$$

with $q_m = \frac{-\left(r-b-\frac{1}{2}s^2\right)-\sqrt{\left(r-b-\frac{1}{2}s^2\right)^2+2(r+m)s^2}}{s^2}$ and $q = \frac{-\left(r-b-\frac{1}{2}s^2\right)-\sqrt{\left(r-b-\frac{1}{2}s^2\right)^2+2rs^2}}{s^2}$.

$T_x(V)$ is the value of the tax shield associated with coupon payments. The balance sheet identity is $E(V) + D(V) = V + T_x(V)$, where

$$D(V) = \frac{C+mF}{m+r} + \left(-\frac{C+mF}{m+r} + V_s(1-ya)\right)\left(\frac{V}{V_s}\right)^{q_m}. \quad (7)$$

Some remarks about these results are fitting. The parameter y can capture either the effect of concessions from debt holders to equity holders when debt is privately renegotiated or the effect of violations to the absolute priority rule during a formal bankruptcy proceeding. So the above pricing model seems suitable even when reorganisation tales place within a bankruptcy proceeding. The model just requires an estimate of ya, whereas y and a need not be known individually.

Note that if $y = 0$, $m = 0$ and $t_x = 0$, then $V_s = \frac{\frac{C+mF}{m+r}q_m - \frac{Ct_x}{r}q}{(1-ya)(q_m-1)}$ reduces to $V_s = \frac{Cq}{r(q-1)}$, which is special case of the endogenous default barrier in Leland (1994a). If $a = 0$ and $t_x = 0$, then $V_s = V_s = \frac{\frac{C+mF}{m+r}q_m - \frac{Ct_x}{r}q}{(1-ya)(q_m-1)}$ reduces to $V_s = \frac{\frac{C+mF}{m+r}q_m}{(q_m-1)}$, which is a special case of Leland (1994b).

Having presented the model for the claims on the underlying firm, we can now value a put written on the firm's equity.

2.1 The put option

Standard valuation arguments imply that the value of a "European" put option $P(V,t)$ on the firm's equity $E(V)$ must satisfy the following equation and conditions:

$$\frac{dP(V,t)}{dt} + \frac{1}{2}\frac{d^2P(V,t)}{dV^2}s^2V^2 + \frac{dP(V,t)}{dV}(r-b)V - r\cdot P(V,t) = 0 \quad (8)$$

$$P(V,T) = \max(X - E(V), 0) \quad (9)$$

$$P(V \to \infty, t) \to 0 \quad (10)$$

$$P(V_s, t) = P_{di}(yaV, t, V_s) \quad (11)$$

where $P_{di}(yaV, t, V_s)$ is the value of a "down-and-in" put option on yaV, with strike X, time to expiry T and with "down-and-in" barrier set equal to V_s. Without loss of generality, today's date is $t = 0$ so that T is the expiry date and also measures the residual life of the option.

Condition 9 is the put option payoff at maturity T. Condition 10 states that the put is approximately worthless as the firm's assets V and hence the firm's equity $E(V)$ become very valuable. Condition 11 states that, as $V = V_s$ for some $t \leq T$, the firm is reorganised, equity holders receive yaV_s and the value of the equity put becomes equal to the value of a "down-and-in" put on yaV_s that is "knocked-in" precisely as and when $V = V_s$. In other words, condition 11 states that, as the firm is reorganised, the nature of the equity claim on the firm's assets irreversibly changes from $E(V)$ to yaV and that after reorganisation the put on equity no longer is a claim on $E(V)$ but a claim on yaV.

The meaning of and the solution to equation 8 and respective conditions are clearer if we write the equity put value as

$$P(V, t) = P_{do}(E(V), t, V_s) + P_{di}(yaV, t, V_s) \tag{12}$$

where $P_{do}(E(V), t, V_s)$ is the value of a "down-and-out" put option on $E(V)$, with strike X, time to expiry T and with "down-and-out" barrier equal to V_s. So, when the underlying firm can default and be reorganised, the equity put $P(V, t)$ can be viewed and valued as the sum of a "down-and-out" put option on equity value before default $E(V)$ plus a "down-and-in" put option on equity value after default yaV. The "down-and-in" and "down-and-out" barriers are the same and are equal to the default and reorganisation barrier V_s.

This put valuation model acknowledges that the put and the equity claim are not lost when the firm defaults. After default equity is still valuable and the put is still "alive". More precisely, since upon default and reorganisation equity value is $E(V_s) = yaV_s$, after default equity value is yaV. And if, as in Toft and Prucyk (1997), we assume that equity is worthless after default or equivalently that $y = 0$, we implicitly assume that $P_{di}(0, t, V_s) = 0$. When valuing call equity options, it may be safe to assume that defaulted equity is worthless, but not so when valuing put options or "down and in" options, whose value heavily depends on the "down side" of equity. The comparative statics below confirm this point.

From standard references (e.g. Wilmott page 202, 1998) we know that the value of a "down-an-in" European put option on yaV with time to expiry T, strike X and "in" barrier V_s is

$$P_{di}\left(yaV,t,V_s\right)=Xe^{-rT}\cdot \tag{13}$$
$$\cdot\left(1-N\left(d\left(\frac{V}{V_s},1\right)\right)+\left(\frac{V}{V_s}\right)^{1-2\frac{(r-b)}{s^2}}\left(N\left(d_1\right)-N\left(d_2\right)\right)\right) \tag{14}$$
$$-yaV\left(1-N\left(d\left(\frac{V}{V_s},1\right)-s\sqrt{T}\right)+\left(\frac{V}{V_s}\right)^{-1-2\frac{(r-b)}{s^2}}\left(N\left(d_3\right)-N\left(d_4\right)\right)\right)$$

where

$$d_1 = \frac{\ln\frac{VX}{(V_s)^2ya}-\left(r-\frac{1}{2}s^2\right)T}{s\sqrt{T}} \tag{15}$$
$$d_2 = \frac{\ln\frac{V}{V_s}-\left(r+\frac{1}{2}s^2\right)T}{s\sqrt{T}} \tag{16}$$
$$d_3 = \frac{\ln\frac{VX}{(V_s)^2ya}-\left(r+\frac{1}{2}s^2\right)T}{s\sqrt{T}} \tag{17}$$
$$d_4 = \frac{\ln\frac{V}{V_s}-\left(r-\frac{1}{2}s^2\right)T}{s\sqrt{T}}. \tag{18}$$

Then appendix A shows that the solution for the "down-and-out" put on $E\left(V\right)$ is

$$P_{do}\left(E\left(V\right),t,V_s\right)=O\left(X,t,V_s\right)-O\left(E\left(V\right),t,V_s\right) \tag{19}$$

where

$$O\left(X,t,V_s\right)=e^{-rT}X\cdot \tag{20}$$
$$\cdot\left(N\left(d\left(\frac{V}{V_s},1\right)-s\sqrt{T}\right)-N\left(d\left(\frac{V}{V_x},1\right)-s\sqrt{T}\right)\right)+$$
$$-e^{-rT}X\left(\frac{V}{V_s}\right)^{1-2\frac{(r-b)}{s^2}}\left(N\left(d\left(\frac{V_s}{V},1\right)-s\sqrt{T}\right)-N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},1\right)-s\sqrt{T}\right)\right)$$
$$O\left(E\left(V\right),t,V_s\right)=O\left(V,t\right)+O\left(T_x\left(V\right),t\right)-O\left(D\left(V\right),t\right) \tag{21}$$

$$O\left(V,t,V_s\right)=e^{-bT}VN\left(d\left(\frac{V}{V_s},1\right)\right)-N\left(d\left(\frac{V}{V_x},1\right)\right)+ \tag{22}$$
$$-\left(\frac{V}{V_s}\right)^{-2\frac{(r-b)}{s^2}}e^{-bT}V_s\left(N\left(d\left(\frac{V_s}{V},1\right)\right)-N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},1\right)\right)\right)$$

$$O\left(D\left(V\right),t,V_s\right)=e^{\left(-r+n\left(q_m\right)\right)T}\cdot\left(-\frac{C+mF}{m+r}+V_s\left(1-ya\right)\right)\cdot \tag{23}$$

$$\cdot\left(\frac{1}{V_s}\right)^{q_m}\cdot\left(V^{q_m}\cdot\Omega_1\left(V,T\right)-\left(\frac{V}{V_s}\right)^{-2\frac{n(q_m)}{q_m\cdot s^2}}\left(V_s\right)^{q_m}\cdot\Omega_2\left(V,T\right)\right)+$$

$$+e^{-rT}\frac{C+mF}{m+r}\left(N\left(d\left(\frac{V}{V_s},1\right)-s\sqrt{T}\right)-\left(\frac{V}{V_s}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{V_s}{V},1\right)-s\sqrt{T}\right)\right)$$

$$-e^{-rT}\frac{C+mF}{m+r}\left(N\left(d\left(\frac{V}{V_x},1\right)-s\sqrt{T}\right)-\left(\frac{V}{V_s}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},1\right)-s\sqrt{T}\right)\right)$$

$$O\left(T_x\left(V\right),t,V_s\right)=-e^{\left(-r+n\left(q\right)\right)T}\cdot\frac{Ct_x}{r}\cdot\left(\frac{1}{V_s}\right)^{q}\cdot \tag{24}$$

$$\cdot\left(V^{q}\cdot\Omega_3\left(V,T\right)-\left(\frac{V}{V_s}\right)^{-2\frac{n(q)}{q\cdot s^2}}\left(V_s\right)^{q}\cdot\Omega_4\left(V,T\right)\right)+$$

$$+e^{-rT}\frac{Ct_x}{r}\left(N\left(d\left(\frac{V}{V_s},1\right)-s\sqrt{T}\right)-\left(\frac{V}{V_s}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{V_s}{V},1\right)-s\sqrt{T}\right)\right)$$

$$-e^{-rT}\frac{Ct_x}{r}\left(N\left(d\left(\frac{V}{V_x},1\right)-s\sqrt{T}\right)-\left(\frac{V}{V_s}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},1\right)-s\sqrt{T}\right)\right)$$

with $\Omega_1\left(V,T\right)=N\left(d\left(\frac{V}{V_s},q_m\right)\right)-N\left(d\left(\frac{V}{V_x},q_m\right)\right)$, $\Omega_2\left(V,T\right)=N\left(d\left(\frac{V_s}{V},q_m\right)\right)-N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},q_m\right)\right)$, $\Omega_3\left(V,T\right)=N\left(d\left(\frac{V}{V_s},q\right)\right)-N\left(d\left(\frac{V}{V_x},q\right)\right)$, $\Omega_4\left(V,T\right)=N\left(d\left(\frac{V_s}{V},q\right)\right)-N\left(d\left(\frac{(V_s)^2}{V\cdot V_x},q\right)\right)$, $d\left(z,w\right)=\frac{w\ln(z)+\left(n(w)+\frac{1}{2}s^2w^2\right)T}{w\cdot s\sqrt{T}}$ and $n\left(w\right)=(r-b)w+\frac{1}{2}w\left(w-1\right)s^2$. V_x needs to be found numerically and is such that $E\left(V_x\right)=X$.

Equations 20, 21, 22, 23, 24 provide the values of claims that pay respectively X, $E\left(V\right)$, V, $D\left(V\right)$ and $T_x\left(V\right)$ at maturity T if at T equity value is $yaV_s<E\left(V\right)<X$, and only if assets value V has not reached the "down-and-out" barrier V_s before T. Having presented the put valuation model, the comparative statics of such model follow.

2.2 Comparative statics

We consider a base case scenario with the following base case parameters: $X=50$, $b=3\%$, $s=20\%$, $r=4\%$, $F=50$, $m=0$, $C=0.05\cdot F$, $a=20\%$, $y=1$, $t_x=0$, $T=1$. Figure 1 shows how put value in the base case when y is either 1 or 0, whereby y is a proxy for the bargaining power of equity holders during reorganisation. Figure 1 shows that the range of variation of put values can be wide as the bargaining power of equity holders varies. Put value decreases in y, since equity value $E(V)$ increases in y. Put values exhibit a kink when V equals V_s (notice that V_s decreases as y decreases). The kink is due to default and reorganisation of the equity claim. After reorganisation the equity value function $E(V)$ becomes less steep in V. The kink suggests that $E(V)$ is concave in the firm's assets value V as V approaches V_s from above. It follows that soon before default and reorganisation put value decreases if the firm's assets volatility s increases. This feature is absent in past compound option models.

Figure 1 shows that as V increases and the put gets more "in-the-money", a rise is y produces a greater absolute decrease in put value. The reason is that a rise in y produces a greater absolute increase in equity value $E(V)$ when V is low and that the the absolute value of the put "delta" is greater when V is low. Instead, a rise in y causes a greater percentage drop in put value when V is low and the put is less "out–of-the-money".

Since in the put valuation formulas the bargaining power parameter y is everywhere multiplied by the bankruptcy cost parameter a and vice versa, the effect on put value of a given percentage change in y is the same as the effect on put value of that same percentage change in a. So all that has been said about the effect of y on put value $P(V,t)$ is valid also for the effect of a on $P(V,t)$. Thus the assumption about the recovery value of equity after default, the bargaining power of equity holders and potential bankruptcy costs can significantly affect equity put option value. This is even more the case as T increases.

Figure 2 shows how put value $P(V,T)$ changes around the base scenario value as, ceteris paribus, assets volatility s, assets payout rate b and debt average maturity m change in turn.

Put value decreases in assets payout b, since $E(V)$ increases in b. Put value may decrease as well as increase in the firm's assets volatility s. Figure 2 shows that, as usual, put value increases in assets volatility when the put is not "deep-in-the-money", but decreases in volatility when the firm approaches default and the put becomes "deep-in-the-money". In fact, especially when V approaches V_s, equity value $E(V)$ increases in volatility. Moreover, as V approaches V_s, Figure 1 shows how put value is a kinked concave function of the firm's assets value V.

Unlike in Toft and Prucyk (1997), this put valuation model encompasses the case in which the firm's debt has finite average maturity. Figure 2 shows how put value generally increases in m and debt average maturity $\frac{1}{m}$ decreases. The reason is that, ceteris paribus, shorter debt maturity decreases equity value $E(V)$ when V approaches the default barrier V_s and the option is "in-the-money". Moreover, increasing m increases V_s. When V is very high, shorter maturity increases equity value, but the option is "out-of-the money". As the put gets very "far from the money", changes in m produce significant proportional changes in put value.

Put value decreases in r because equity value rises in r and because the present value of the final payoff decreases in r. As the firm's leverage rises, i.e. as F and C rise, the default barrier V_s rises, so that reorganisation becomes a more likely prospect and bankruptcy costs increase too since such costs are equal to aV_s. As a result put value becomes more sensitive to the prospect of firm reorganisation and to y and a.

Put value usually increases as time to expiry T gets longer, but not so when the put is "deep-in-the-money" and the firm approaches default, since put value after default decreases in T. Finally, unreported simulations showed that also in the present model early exercise of the "American" put can be optimal even in the absence of dividends.

We can conclude that the differences between the above comparative statics and those of ordinary puts confirm the significant impact that firm reorganisation can have on put value.

2.3 When markets are incomplete

The model above assumes dynamic market completeness, which allows us to regard the value of the firm's assets V as the price of a traded asset and to assume that its risk neutral process is as in equation 1. But even if the market is incomplete the proposed model is still valid if only $(r - b)$ is substituted with $(n - \lambda s)$, where n is the real drift of V and λ is the market price of V-risk. The reason for this adjustment is that market incompleteness causes the risk neutral process for V to be no longer as per equation 1 but

$$dV = V\,(n - \lambda s)\,dt + sV dz. \tag{25}$$

Market incompleteness seems more appropriate an assumption when the firm's stock is not traded in the stock market or when the stock has been delisted after default. In this regard, Ericsson (1998) argues that, as long as the firm's stock is traded, the value process of the firm's assets V can be replicated by trading in the stock. But replication is more unlikely after delisting because the stock would not be trading in the stock market any more. So we may want to assume market incompleteness just after default. In such case $(n - \lambda s)$ should substitute $(r - b)$ just in equations 13, 15, 16, 17 and 18.

2.4 Calibrating the model and the implied volatility skew

The above presented put valuation model depends on more and different parameters than the Black and Scholes put model. Some parameters like m, F and C can be estimated from balance sheet data, some like r and t_x from the financial environment and some like b, s, a and y can be inferred from (the time series of) equity prices and put prices. The parameters b, s, a and y can be simply "calibrated" to present equity and put prices or estimated through a maximum likelyhood method from times series of prices as in Ericsson and Reneby (2001). Such parameters offer more "flexibility" in calibrating the put valuation model than the Black and Scholes model does. If the above model is used in substitution to the Black and Scholes model, need to employ implied volatility skews to explain the observed prices of put options with different strikes and expiry dates can be eliminated. Moreover, the calibration of the above model provides parameter estimates that can be used to value also the firm's debt. In other words the proposed model established a link between equity put options and spreads on the firm's debt.

A virtue of the proposed put valuation model is that it can predict various patterns of implied equity volatility "skews" as y, a, m, F and b change as shown in Table 3. Overall the model seems quite capable of explaining the types of volatility skews that can be estimated from observed option prices.

The first two sections of table 3 show that, even when the firm is far from default ($V = 200$ and $F = 50$) implied volatility decreases as bankruptcy costs and the bargaining power of equity holders (i.e. as a and y) increase. The third section shows how implied volatility rises as debt average maturity ($\frac{1}{m}$) decreases and the fourth shows how volatility increases as leverage increases. A volatility skew can be detected in all the sections of the table, whereby implied equity volatility is high when the strike price of the put is lower, in keep with empirical evidence and with the fact that leveraged equity is more volatile when the firm approaches default.

After analysing how the firm's default and reorganisation affect the valuation of plain put options, the following section turns to the valuation of barrier options.

3 VALUATION OF BARRIER OPTIONS

Like put options, "down-and-in" barrier options on leveraged equity are here of interest because their value is more sensitive than the value of other options to the "recovery assumption" for defaulted equity. Like put options, "down-and-in" options are amenable to closed form solutions. Given the above framework and the "recovery assumption" for equity value after default, the formula for a "down-and-in" put with "in" barrier at $E(V_u) = U$ such that $V_x > V_u > V_s$ is:

$$P_{di}(V, t, U) = P(V, t) - P_{do}(V, t, U) \tag{26}$$

where $P(V, t)$ is as before given by equations 12, 19, 20, 21, 22, 23, 24 and where $P_{do}(V, t, U)$ is the value of a "down-an-out" put on equity $E(V)$ with "out" barrier at $E(V_u) = U$, with strike X and residual life T. It follows that

$$P_{do}(V, t, U) = O(X, t, U) - O(E(V), t, U) \tag{27}$$

where, if only V_s is substituted with V_u, the formula for $O(X, t, U)$ is the same as the right hand side of equation 20 and the formula for $O(E(V), t, U)$ is the same as the right hand side of equation 21.

Notice that, since $V_u > V_s$ the value of the "down and out" put option $P_{do}(V, t, U)$ is not affected by default and the "recovery assumption", since the put is "knocked out" before default. Instead the value of the plain put $P(V, t)$ is clearly depends on default and the "recovery assumption" as apparent from equations 12, 19, 20, 21, 22, 23 and 24. So from equation 26 it follows that the value of the "down-and-in" put on leveraged equity $P_{di}(V, t, U)$ also depends on default and the "recovery assumption". Moreover, any percentage change in $P(V, t)$ produced by a change in the recovery parameters y or a will correspond to an even greater percentage change in $P_{di}(V, t, U)$, precisely because y or a affect $P(V, t)$ but do not affect $P_{do}(V, t, U)$.

Similar formulas and arguments are valid also for "down-and-in" call options as shown in appendix B. These arguments highlight that the precise modelling of equity during financial distress is particularly important for both "down-and-in" put and call options, but not for "down-and-out" options.

4 CONCLUSIONS

This paper has presented new closed form solutions for the valuation of equity put options and "down-and-in" barrier options on leveraged equity. The basic assumption ("recovery assumption") is that such options are not lost and that equity retains value even after the firm defaults, in keeping with empirical evidence on the recovery value of the firm's equity after default. The "recovery assumption" distinguishes this paper from past literature, in particular from Toft and Prucyk (1997).

The analysis has shown that the equity put option is very sensitive to the "recovery assumption", to bankruptcy costs, to the bargaining power of equity holders during reorganisation and to debt average maturity. Unlike is the Black and Scholes model, equity put value decreases in assets volatility when the put is "deep-in-the-money" and the firm approaches distress. These results are more material when the life of the option is longer."Down-and-in put" options are much more sensitive then plain puts to the modelling of distressed equity and to the "recovery assumption".

The "recovery assumption" enables the put valuation model to predict a richer set of shapes of implied volatility skews of the type observed in the equity options markets.

These results also suggest that equity call option prices rather than equity put option prices should be used to imply equity volatility, since the former, unlike the latter, are relatively insensitive to the "recovery assumption" of distressed equity.

Finally future research can employ observed equity prices, put option prices and the closed form solutions here presented to estimate the firm's assets value, drift, volatility, bankruptcy costs and bargaining power parameters to be used in pricing the firm's debt.

Appendix A. Derivation of the Put option formula

This appendix derives equations 19, 20, 21, 22, 23 and 24.

We can write $D(V) = \frac{C+mP}{r+m} + AV^{q_m}$, with $A = \left(-\frac{C+mP}{r+m} + (1-ay) \cdot V_s\right) \cdot (V_s)^{-q_m}$. Then by applying Ito's lemma it follows that

$$d(D(V)) = d(AV^{q_m}) = n(q_m) \cdot AV^{q_m} \cdot dt + AV^{q_m} \cdot qs \cdot dz$$

where

$$n(w) = (r-b)w + \frac{1}{2}w(w-1)s^2. \tag{A-1}$$

So the term (AV^{q_m}) follows a geometric Brownian motion.

Then the present value of a claim $Q\left(D(V), V_x, T\right)$ that pays $D\left(V\right) = \frac{C+mP}{r+m} + AV^{q_m}$ at time T if $V_T \geq V_x$ and that pays nothing otherwise is:

$$Q\left(D(V), V_x, T\right) = Q\left(AV^{q_m}, V_x, T\right) + Q\left(\frac{C+mP}{r+m}, V_x, T\right) \qquad \text{(A-2)}$$

where

$$Q\left(AV^{q_m}, V_x, T\right) = AV^{q_m} \cdot e^{(-r+n(q_m))T} \cdot N\left(d\left(\frac{V}{V_x}, q_m\right)\right) \qquad \text{(A-3)}$$

$$Q\left(\frac{C+mP}{r+m}, V_x, T\right) = e^{-rT} \cdot \frac{C+mP}{r+m} \cdot N\left(d\left(\frac{V}{V_x}, 1\right) - s\sqrt{T}\right) \qquad \text{(A-4)}$$

and where $N\left(u\right)$ is the cumulative of the standard normal density with u as the upper limit of integration and with

$$d\left(z, w\right) = \frac{w \ln\left(z\right) + \left(n\left(w\right) + \frac{1}{2}s^2 w^2\right) T}{ws\sqrt{T}}. \qquad \text{(A-5)}$$

Using results for valuing "down-and-out" barrier options (see e.g. Wilmott (1998) at page 192), the value of a claim $Q\left(D\left(V\right), V_s, V_x, T\right)$ that pays V at time T only if $V_T \geq V_x$ *and only if* $V_t > V_s$ for any time $t < T$, and that pays nothing otherwise is:

$$Q\left(D\left(V\right), V_s, V_x, T\right) = Q\left(AV^{q_m}, V_s, V_x, T\right) + Q\left(\frac{C+mP}{r+m}, V_s, V_x, T\right) \qquad \text{(A-6)}$$

where

$$Q\left(AV^{q_m}, V_s, V_x, T\right) = A \cdot e^{(-r+n(q_m))T} \cdot \qquad (7)$$
$$\cdot \left(V^{q_m} N\left(d\left(\frac{V}{V_x}, q_m\right)\right) - \left(\frac{V}{V_s}\right)^{-2\frac{n(q_m)}{q_m \cdot s^2}} \left(V_s\right)^{q_m} \cdot N\left(d\left(\frac{\left(V_s\right)^2}{V \cdot V_x}, q_m\right)\right)\right)$$

and

$$Q\left(\frac{C+mP}{r+m}, V_s, V_x, T\right) = \frac{C+mP}{r+m} e^{-rT} \cdot \qquad (8)$$
$$\cdot \left(N\left(d\left(\frac{V}{V_x}, 1\right) - s\sqrt{T}\right) - \left(\frac{V}{V_s}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{\left(V_s\right)^2}{V \cdot V_x}, 1\right) - s\sqrt{T}\right)\right).$$

Then we can write

$$O(D(V),t) = Q(D(V),V_s,V_s,T) - Q(D(V),V_s,V_x,T) \quad \text{(A-9)}$$

which gives equation 23. Similarly we can derive

$$O(V,t) = Q(V,V_s,V_s,T) - Q(V,V_s,V_x,T) \quad \text{(A-10)}$$
$$O(T_x(V),t) = Q(T_x(V),V_s,V_s,T) - Q(T_x(V),V_s,V_x,T) \quad \text{(A-11)}$$
$$O(X,t) = Q(X,V_s,V_s,T) - Q(X,V_s,V_x,T). \quad \text{(A-12)}$$

Finally, boundary conditions 9, 10, 11 and the linearity of partial differential equation 8 allows us to write equation 12 and

$$P_{do}(E(V),T) = O(V,T) + O(T_x(V),T) - O(D(V),T). \quad \text{(A-13)}$$

Appendix B. The valuation of barrier call options on leveraged equity

For completeness the value of a "down-and-in" barrier option on leveraged equity is here reported. The value of a "down-and-in" call $C_{di}(V,t,U)$ on $E(V)$ with "in" barrier at $E(V_u) = U$ such that $V_x > V_u > V_s$, with maturity T and strike X is:

$$C_{di}(V,t,U) = C(V,t) - C_{do}(V,t,U)$$

where $C(V,t)$ is the value of a call option on leveraged equity $E(V)$ given that equity after default and reorganisation is worth Vay and where

$C_{do}(V,t,U)$ is a "down-and-out" call on $E(V)$ with "out" barrier $E(V_u) = U$. $C(V,t)$ and $C_{do}(V,t,U)$ also have strike X and maturity T.

We can derive a closed form solution for $C(V,t)$, since before default put-call parity allows us to write: $C(V,t)+Xe^{-rT} = P(V,t)+C_0(V,t)$, where $C_0(V,t)$ is the value of call similar to $C(V,t)$ but which has 0 strike price. $C_0(V,t)$ is equivalent to the right to receive equity $E(V)$ at time T.

Then, employing arguments similar to the ones in appendix A, we can write

$$C_{do}(V,t) = O_c(E(V),t) - O_c(X,t) \quad \text{(B-1)}$$

with

$$O_c(X,t) = e^{-rT}X\left(N\left(d\left(\frac{V}{V_x},1\right)\right) - \left(\frac{V}{V_u}\right)^{1-2\frac{(r-b)}{s^2}} N\left(d\left(\frac{(V_u)^2}{V \cdot V_x},1\right)\right)\right) \quad \text{(B-2)}$$

$$O_c\left(E\left(V\right),t\right) = O_c\left(V,t\right) + O_c\left(T\left(V\right),t\right) - O_c\left(D\left(V\right),t\right) \qquad \text{(B-3)}$$

$$O_c\left(V,t\right) = e^{-bT}\left(VN\left(d\left(\frac{V}{V_x},1\right)\right) - \left(\frac{V}{V_u}\right)^{-2\frac{(r-b)}{s^2}} V_s N\left(d\left(\frac{(V_u)^2}{V\cdot V_x},1\right)\right)\right) \qquad \text{(B-4)}$$

$$O_c\left(D\left(V\right),t\right) = \left(-\frac{C+mF}{m+r} + V_s\left(1-ya\right)\right)e^{-r+n(q)T}\cdot \qquad (5)$$
$$\cdot\left(V^{q_m}N\left(d\left(\frac{V}{V_x},q_m\right)\right) - \left(\frac{V}{V_u}\right)^{-2\frac{n(q_m)}{q_m\cdot s^2}}\left(V_u\right)^{q_m}N\left(d\left(\frac{\left(V_u\right)^2}{V\cdot V_x},q_m\right)\right)\right)$$
$$+\frac{C+mF}{m+r}e^{-rT}\left(N\left(d\left(\frac{V}{V_x},1\right)+s\sqrt{T}\right) - \left(\frac{V}{V_u}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_u)^2}{V\cdot V_x},1\right)+s\sqrt{T}\right)\right).$$

$$O_c\left(T_x\left(V\right),t\right) = -\frac{Ct_x}{r}e^{-r+n(q)T}\cdot \qquad (6)$$
$$\cdot\left(V^{q}N\left(d\left(\frac{V}{V_x},q\right)\right) - \left(\frac{V}{V_u}\right)^{-2\frac{n(q)}{q\cdot s^2}}\left(V_u\right)^{q}N\left(d\left(\frac{\left(V_u\right)^2}{V\cdot V_x},q\right)\right)\right)$$
$$+\frac{Ct_x}{r}e^{-rT}\left(N\left(d\left(\frac{V}{V_x},1\right)+s\sqrt{T}\right) - \left(\frac{V}{V_u}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_u)^2}{V\cdot V_x},1\right)+s\sqrt{T}\right)\right).$$

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Figure 1: Value of put $P(V,T)$ with base case parameters as y changes from 0 to 1.

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Figure 2: Differences in put value $P(V,T)$ from the base case scenario as, ceteris paribus, $s = 30\%$, $b = 6\%$ and $m = 40\%$.

Figure 3: The table shows the effect on implied volatility of departures from the base case scenario in which $V = 200$, $X = 50$, $b = 3\%$, $s = 20\%$, $r = 4\%$, $F = 50$, $m = 0$, $C = 0.05 \cdot F$, $a = 20\%$, $y = 1$, $t_x = 0$, $T = 1$.