Risk Factor Beta Conditional Value-at-Risk*

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Abstract

This paper proposes a new approach to calculate the portfolio VaR. Based on the assumption that the same macro factors affect returns of all assets, this methodology use a multifactor pricing model to generate the sequence of hypothetical future equilibrium portfolio returns given the values of the underlying macro factors and the assets betas with respect to these macro factors. The standard HS approach is then applied to these simulated hypothetical future equilibrium portfolio returns to calculate the VaR. We refer the obtained VaR to as the market factor beta conditional VaR.

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Keywords: historical simulation approach, Monte Carlo simulation, multifactor asset pricing model, Value at Risk, variance-covariance approach.

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1 Introduction

Value-at-Risk (VaR) is a common risk measure in the banking and financial business. It is widely used by security houses, investment banks, pension funds, and other financial institutions to measure the market risk of their asset portfolios. VaR is defined as an estimate of the maximum loss to be expected of a portfolio over a given time horizon that will not be exceeded with a given probability. In practice, the probability is typically chosen to be 5 % or 1%. The standard time horizon for calculating VaR is 1 trading day for most financial institutions with active trading portfolios. However, for some problems, other horizons may be appropriate. Portfolio VaR calculations for a 10-day holding period, for example, is usually required in calculating a bank's capital for market risk.

There are three basic approaches to the estimation of VaR. They are the variance-covariance approach, Monte Carlo simulation, and the historical simulation approach. Dowd (1998) and Jorion (2000) provide excellent surveys of these approaches. The variance-covariance model is based on the assumption that the individual securities are jointly normally distributed. This implies that the portfolio return is itself normal and therefore the variance of the portfolio return is a linear function of the variances and covariances of the individual assets. The normality assumption makes it easy to calculate VaR as a multiple of the portfolio standard deviation. However, the assumption of normality of individual returns does not correspond well with reality. Returns on stocks, for example, are often negatively skewed and have a heavier loss tail than is implied by the normal distribution.

To build up a probability distribution of the portfolio return, the Monte Carlo simulation approach assumes particular theoretical return processes. Making a specific distributional assumption about the random variables (normality is not required), this approach simulates large numbers of random paths the asset returns could follow. If the number of simulations is sufficiently large, the simulated distribution of the portfolio return will be close to the unknown true distribution and we can then infer the VaR as the appropriate percentile of the simulated probability distribution. The drawback of the Monte Carlo simulation approach is that individual returns are assumed to be coming from some arbitrary probability distribution.

The historical simulation methodology consists in using historical data to generate the hypothetical distribution of the portfolio return. In contrast to the variance-covariance and Monte-Carlo simulation approaches, the historical simulation technique does not rely on any particular parametric model and imposes no structure on the distribution of returns. The drawback of this approach is that the number of alternative scenarios is limited by historical data and we hence need a sufficiently long data to calculate VaR with any degree of precision. This may be a problem when in the portfolio there are assets with little history. If this is the case, the number of available observations on asset returns may be too small to draw reliable inferences about the tail of the distribution of profits and losses.

This paper proposes a new approach to calculate the portfolio VaR. Based on the assumption that the same macroeconomic factors affect returns of all assets in a portfolio, this methodology allows to generate the sequence of hypothetical future equilibrium portfolio returns given the historical values of the underlying macroeconomic factors and the asset betas with respect to these factors. VaR can then be found as an appropriate percentile of the generated hypothetical distribution of the portfolio return. Since this value of VaR is conditional on the estimated values of the risk factor betas, we refer it to as *the risk factor beta conditional VaR*.

Given that the number of the historical observations on the risk factors is usually much larger than that on the returns on individual assets, the number of the hypothetical portfolio returns we can generate within this approach may be much larger than the number of the portfolio returns calculated using historical data on individual asset returns, what may be especially important when in the portfolio there are assets with little history. This allows to estimate VaR more accurately compared to the standard historical simulation methodology.

The rest of the paper is as follows. Different approaches to the estimation of VaR are discussed in Section 2. Section 3 introduces the multifactor model based method for calculating VaR and discuss how this methodology can improve the historical simulation approach. In Section 4, we test this new approach empirically. Conclusions are presented in Section 5.

2 Approaches to the Estimation of VaR

Suppose there are N assets in our portfolio. The portfolio return $R_{p,t}$ over period t is

$$R_{p,t} = \mathbf{w}_t' \mathbf{R}_t, \tag{1}$$

where $\mathbf{R}_t$ is an $N \times 1$ vector of returns for N assets (or portfolios of assets) with $\mathbf{R}_t = [R_{1,t}\ R_{2,t}\ ...\ R_{N,t}]'$ and $\mathbf{w}_t$ is an $N \times 1$ vector of the relative weights of the assets in the portfolio with $\mathbf{w}_t = [w_{1,t}\ w_{2,t}\ ...\ w_{N,t}]'$.

The portfolio variance can be written as a linear function of the individual asset variances and covariances:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}, \tag{2}$$

where σ_i^2 is the variance on asset i and σ_{ij} is the covariance between asset i and j returns.

The variance-covariance approach exploits the assumption that the individual asset returns are multivariate normal. This implies that the portfolio return is itself normally distributed. The normality assumption makes it easy to calculate VaR from the z-score corresponding to the chosen confidence level.[1] The variance-covariance approach is intuitive and easy to implement. It is excellent when dealing with portfolios that have normal returns.

The accuracy of this approach depends on the assumption that normality is an adequate empirical description of the portfolio return. There is a large amount of empirical evidence that many individual returns are not normally distributed. Asset returns are often negatively skewed. Besides, many return distributions have fat tails, what implies that the approach based on the assumption of normality of asset returns can considerably underestimate "true" VaR.

However, the assumption of normality of the portfolio return is not too unreasonable, even if the individual asset returns are not normally distributed. According to the central limit theorem, we may assume that the portfolio return is normal if the portfolio is fairly well diversified and the returns on individual assets in the portfolio are independent of each other, even if we conclude that individual asset returns are non-normal.

A less restrictive approach is the portfolio-normal approach (see, for example, Wilson (1994) and Frain and Meegan (1996)). It is based on the assumption that the portfolio return is normal without requiring that individual asset returns are normally distributed or the portfolio is well diversified and individual asset returns are independent. When applying this approach, we must however always investigate whether normality is an adequate empirical description of the particular portfolio at hand.

Another method to estimate VaR is Monte Carlo simulation. According to this approach, we first select mathematical models for the prices of the assets in the portfolio and estimate their parameters. Then, we use a random number generator to construct fictitious paths for the random variables involved in these models and simulate the random precesses governing the prices of the assets in the portfolio using the generated values of the random variables. Each simulation gives us a hypothetical value for the portfolio as a whole. Repeating these simulations many times, we generate the probability distribution of portfolio values. VaR is calculated as the appropriate percentile of this simulated probability distribution. A serious downside of Monte-Carlo simulation, and one which this approach shares with the variance-covariance approach, is that the changes in the asset prices are usually assumed to be normally distributed.

The idea behind the historical simulation approach is that the forward distribution of the return on asset i, $R_{i,t+1}$, may be well approximated by the empirical distribution of the most recent S observations, $\{R_{i,t+1-s}\}_{s=1}^{S}$. This involves using past data as a guide to what might be the distribution of $R_{i,t+1}$ without imposing any further assumptions and provides us with S different scenarios for what might happen tomorrow. The first scenario is where the return on

[1] Under normality, the portfolio VaR is equal to $(\mu_p - \alpha\sigma_p)V_p$, where α reflects the level of confidence on which VaR is predicated (for example, for the 95% confidence level α equals -1.65), μ_p is the mean portfolio return, σ_p is the standard deviation of the portfolio return, and V_p is the initial portfolio value. When the portfolio return is normally distributed, to estimate VaR we need only to estimate μ_p and σ_p and to plug these estimates into the VaR formula.

asset i in period t is assumed to occur. The value of the asset return in the second scenario is the same as in period $t-1$, and so on. Finally, the Sth scenario is where the asset return is the same as it was in period $t+1-S$. The hypothetical portfolio returns can be calculated as

$$R_{p,t+1-s} = \sum_{i=1}^{N} w_{i,t} R_{i,t+1-s}, \ s = 1, 2, ..., S. \tag{3}$$

We then translate from a sample distribution of hypothetical portfolio returns $\{R_{p,t+1-s}\}_{s=1}^{S}$ to the distribution of portfolio profits and losses.[2] The expected portfolio VaR can be calculated as the relevant percentile of this simulated distribution. Each time period, the historical simulation estimate of VaR is updated using the most recent S periods of data.

There is no general agreement on which of the three described above approaches for calculating VaR is better. One advantage of the historical simulation approach is that this method is easy to implement and conceptually simple, what makes it easier to report results to senior management. The historical simulation approach implicitly presumes that the market events, as captured in the historical data set, will be reproduced in the future and hence is free from arbitrary statistical and distributional assumptions. This is, perhaps, the main advantage of historical simulation. A practical limitation of historical simulation is that when applying this approach, we need the longest possible run of historical data to maximize the accuracy of our inferences. A long estimation period would however lead our VaR estimate to be insufficiently sensitive to the newer, more useful information contained in more recent observations. If S is chosen to be too large, the most recent observations, which may have more information about tomorrow's distribution, will be assigned very small probabilities and the dynamic pattern of the obtained estimates of VaR might be too smooth.

To make allowance for the fact that the most recent values may be the most relevant for next period's distribution, $\{R_{i,t+1-s}\}_{s=1}^{S}$ may be assigned probability weights declining exponentially through the past:

$$\pi_{s,t+1} = \pi_{t+1-s} = a^{s-1}(1-a)/(1-a^n), \ s = 1, ..., S. \tag{4}$$

This is the weighted historical simulation approach. There is no guidance on how to choose $a > 0$. Typically, a is chosen to be between 0.95 and 1.[3] The declining weights make events further back in time less relevant than recent events. Although weighted historical simulation improves the standard historical simulation approach in what the former accounts for the fact that more recent states have more probability to occur tomorrow, the both these estimation

[2] One usually assumes that different scenarios have equal probabilities of occurring, $\pi_{s,t+1} = \pi_{t+1-s} = 1/S > 0$, $s = 1, ..., S$.

[3] As $a \to 1$, $\pi_{s,t+1} \to 1/S$ and thus weighted historical simulation gets equivalent to standard historical simulation, which assumes that different scenarios have equal probability of occurring, $\pi_{s,t+1} = \pi_{t+1-s} = 1/S$, $s = 1, ..., S$.

techniques require a large number of historical observations to get a reliable estimate of VaR. Since the number of simulated portfolio returns is limited to the longest time-series of individual asset returns, getting a large number of historical portfolio returns might be a problem when in the portfolio there are assets with little or no history.

As a plausible solution to this problem, in this paper we propose the following approach to calculate VaR. Our approach is based on the assumption that there are systematic common factors that affect the returns of all securities. Under this assumption, we can estimate the sensitivities of individual returns to these common risk factors using the n most recent observations. Assuming that the forward joint distribution of the risk factors may be well proxied by their historical joint distribution, we can use the obtained estimates of the risk factor betas to simulate the hypothetical equilibrium individual asset returns we would have had if we had held the assets over the observation period and the sensitivities of the asset returns to the risk factors had been the same as those observed over the last n periods.

Following the historical simulation approach, we can then assume that the forward distributions of individual returns may be well proxied by their simulated historical distributions. The number of possible scenarios for each asset return will therefore be the same and equal to the number of historical observations on the macroeconomic factors. Since long time series of such factors are usually available, this technique allows us to generate desirably long time series of asset returns. Given these simulated values of the returns on individual assets, we can calculate the portfolio return corresponding to each scenario. When this approach is used, the number of the generated expected portfolio returns equals the number of the observations on the risk factors and hence may be large even if in the portfolio there are individual assets with little history. VaR can be calculated as a percentile point of the corresponding distribution of portfolio profits and losses. Since the obtained estimate of VaR is conditional on the estimates of the risk factor betas, we call it the risk factor beta conditional VaR. The next section describes this approach in details.

3 Multifactor Specification of Risk and VaR

Let us denote the excess return on asset i over the risk-free rate as $Z_{i,t} \equiv R_{i,t} - R_{f,t}$. The Arbitrage Pricing Theory postulates that the excess return on each security is given by

$$Z_{i,t} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_{K,t} + \varepsilon_{i,t}, \tag{5}$$

where a_i is the intercept of the factor model, $\mathbf{f}_{K,t}$ is a $K \times 1$ vector of the systematic common factor risk premia, $\boldsymbol{\beta}_i$ is a $K \times 1$ vector of factor sensitivities for asset i, and the disturbance term $\varepsilon_{i,t}$ denotes the firm i specific effect, $E\left[\varepsilon_{i,t} \middle| \mathbf{f}_{K,t}\right] = 0$, $cov(\varepsilon_{i,t}, \varepsilon_{j,t}) = 0$.

For the system of N assets,

$$\mathbf{Z}_t = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_{K,t} + \boldsymbol{\varepsilon}_t, \tag{6}$$

$E[\boldsymbol{\varepsilon}_t] = \mathbf{0}$, $E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'] = \boldsymbol{\Sigma}$, $E[\mathbf{f}_{K,t}] = \boldsymbol{\mu}_K$, $E\left[(\mathbf{f}_{K,t} - \boldsymbol{\mu}_K)(\mathbf{f}_{K,t} - \boldsymbol{\mu}_K)'\right] = \boldsymbol{\Omega}_K$, $cov(\mathbf{f}_{K,t}, \boldsymbol{\varepsilon}_t') = \mathbf{O}$, where $\mathbf{Z}_t$ is an $N \times 1$ vector of excess returns for N assets (or portfolios of assets) with $\mathbf{Z}_t = [R_{1,t} - R_{f,t}\ R_{2,t} - R_{f,t}\ ...\ R_{N,t} - R_{f,t}]'$, α is an $N \times 1$ vector of asset return intercepts with $\boldsymbol{\alpha} = [\alpha_1\ \alpha_2\ ...\ \alpha_N]'$, $\mathbf{B}$ is the $N \times K$ matrix of factor sensitivities with $\mathbf{B} = [\boldsymbol{\beta}_1\ \boldsymbol{\beta}_2\ ...\ \boldsymbol{\beta}_N]'$, $\boldsymbol{\varepsilon}_t$ is an $N \times 1$ vector of asset return disturbances with $\boldsymbol{\varepsilon}_t = [\varepsilon_{1,t}\ \varepsilon_{2,t}\ ...\ \varepsilon_{N,t}]'$, $\boldsymbol{\Sigma}$ is the variance-covariance matrix of the disturbances, $\boldsymbol{\Omega}_K$ is the variance-covariance matrix of the factors, and $\mathbf{O}$ is a $K \times N$ matrix of zeroes.

When the factors are traded portfolios, the K-factor model (6) is:

$$\mathbf{Z}_t = \boldsymbol{\alpha} + \mathbf{B}\mathbf{Z}_{K,t} + \boldsymbol{\varepsilon}_t, \tag{7}$$

$E[\boldsymbol{\varepsilon}_t] = \mathbf{0}$, $E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'] = \boldsymbol{\Sigma}$, $E[\mathbf{Z}_{K,t}] = \boldsymbol{\mu}_K$, $E\left[(\mathbf{Z}_{K,t} - \boldsymbol{\mu}_K)(\mathbf{Z}_{K,t} - \boldsymbol{\mu}_K)'\right] = \boldsymbol{\Omega}_K$, $cov(\mathbf{Z}_{K,t}, \boldsymbol{\varepsilon}_t') = \mathbf{O}$, where $\mathbf{Z}_{K,t}$ is the $K \times 1$ vector of factor portfolio excess returns.

We can write the excess return on the portfolio of assets as

$$R_{p,t} - R_{f,t} = Z_{p,t} = \mathbf{w}_t'\mathbf{Z}_t = \mathbf{w}_t'\boldsymbol{\alpha} + \mathbf{w}_t'\mathbf{B}\mathbf{Z}_{K,t} + \mathbf{w}_t'\boldsymbol{\varepsilon}_t = \alpha_p + \boldsymbol{\beta}_p'\mathbf{Z}_{K,t} + \varepsilon_{p,t}, \tag{8}$$

where α_p is the portfolio return intercept, $\alpha_p = \mathbf{w}_t'\boldsymbol{\alpha}$, $\boldsymbol{\beta}_p$ is a $K \times 1$ vector of sensitivities of the portfolio return to the risk factors, $\boldsymbol{\beta}_p' = \mathbf{w}_t'\mathbf{B}$, and $\varepsilon_{p,t} = \mathbf{w}_t'\boldsymbol{\varepsilon}_t$.

Suppose that we observe the risk-free rate and the factor portfolio excess returns over a number of periods, $t = 1, 2, ..., T$. The beta coefficients in (7) may be estimated from regressions of asset excess returns on risk factor excess returns using the n most recent observations, $t = T - n + 1, ..., T$.

For the model in (7), the maximum likelihood estimators are just the OLS estimators:

$$\widehat{\boldsymbol{\alpha}} = \widehat{\boldsymbol{\mu}} - \widehat{\mathbf{B}}\widehat{\boldsymbol{\mu}}_K \tag{9}$$

$$\widehat{\mathbf{B}} = \left[\sum_{t=T-n+1}^{T} (\mathbf{Z}_t - \widehat{\boldsymbol{\mu}})(\mathbf{Z}_{K,t} - \widehat{\boldsymbol{\mu}}_K)'\right]\left[\sum_{t=T-n+1}^{T} (\mathbf{Z}_{K,t} - \widehat{\boldsymbol{\mu}}_K)(\mathbf{Z}_{K,t} - \widehat{\boldsymbol{\mu}}_K)'\right]^{-1}, \tag{10}$$

where $\widehat{\boldsymbol{\mu}} = n^{-1}\sum_{t=T-n+1}^{T} \mathbf{Z}_t$ and $\widehat{\boldsymbol{\mu}}_K = n^{-1}\sum_{t=T-n+1}^{T} \mathbf{Z}_{K,t}$.

We now can use the obtained estimates of the risk factor betas to simulate the hypothetical equilibrium returns we would have had if we had held our current portfolio over the observation period, $t = 1, 2, ..., T$, and the sensitivities of the excess returns on the assets in the portfolio to the risk factors had been the same as those observed over the last n periods, $t = T - n + 1, ..., T$.

For practical purposes, the nonsystematic risk of a well-diversified portfolio, which is given by $\sigma^2(\varepsilon_{p,t})$, is negligible (is zero). Because the expected value of $\varepsilon_{p,t}$ is zero, this follows that any realized value of $\varepsilon_{p,t}$ will be virtually zero. The absence of arbitrage in large economies implies that $\boldsymbol{\alpha}$ equals zero, that is $\alpha_p = \mathbf{w}_t'\boldsymbol{\alpha} = 0$. Hence, we conclude that for a well-diversified portfolio the predicted equilibrium return is

$$\widehat{R}_{p,t} = R_{f,t} + \widehat{\boldsymbol{\beta}}_p'\mathbf{Z}_{K,t},\ t = 1, 2, ..., T, \tag{11}$$

where $\widehat{\boldsymbol{\beta}}_p' = \mathbf{w}_T'\widehat{\mathbf{B}}$ with the estimate of the $N \times K$ matrix of factor sensitivities, $\widehat{\mathbf{B}}$, obtained using the n most recent observations on the asset excess returns and the factor portfolio excess returns.

Then, we can apply the historical simulation approach and assume that the simulated distribution of the equilibrium portfolio returns $\widehat{R}_{p,t},\ t = 1, 2, ..., T$, well approximates the time $T+1$ distribution of the portfolio return. This provides us with the $S = T$ alternative scenarios for what could be the portfolio return at $T+1$, $\left\{\widehat{R}_{p,s}\right\}_{s=1}^{S=T}$. The time $T+1$ VaR can then be calculated as the relevant percentile of this simulated distribution times the value of the portfolio at time T.

To calculate VaR at time $T+2$, we estimate the individual asset betas using information from $t = T-n+2$ to $T+1$ and calculate $\widehat{\boldsymbol{\beta}}_p'$ as $\widehat{\boldsymbol{\beta}}_p' = \mathbf{w}_{T+1}'\widehat{\mathbf{B}}$. Then, we simulate

$$\widehat{R}_{p,t} = R_{f,t} + \widehat{\boldsymbol{\beta}}_p'\mathbf{Z}_{K,t},\ t = 2, 3, ..., T+1 \tag{12}$$

and find the time $T+2$ VaR as the relevant percentile of the simulated distribution $\left\{\widehat{R}_{p,s}\right\}_{s=2}^{S=T+1}$ times the value of the portfolio at time $T+1$. VaR at $T+3$ and so on can be calculated analogously.

Within this approach, the number of points in the simulated forward distribution of the portfolio return is limited to the number of observations on the risk-free rate $R_{f,t}$ and the factor portfolio excess returns $\mathbf{Z}_{K,t}$. Since the number of the historical observations on the risk-free rate and the risk factors is usually much larger than that on the returns on individual assets, the number of the hypothetical portfolio returns we can generate within our approach can be much larger than the number of the portfolio returns calculated using historical data on individual asset returns. This may be especially important when in the portfolio there are assets with little history. The larger number of points in the simulated distribution allows to estimate VaR more accurately compared to the standard historical simulation methodology.

Another attractive feature of our approach is that it allows to generate the expected distribution of the portfolio return in equilibrium. Since we can not predict whether the portfolio in question will be fairly priced tomorrow, this seems to be more appropriate than just to use the historical realized returns which might be different from the equilibrium returns.

If historical data on some assets in the portfolio are not available, but the values of the risk factor betas for these assets are known, we can calculate the values of the portfolio betas using the property that the portfolio beta is equal to the weighted sum of betas of individual assets with the weights which are the same as those of assets in the portfolio and then simulate the forward distribution of the portfolio return using equation (11) with $\widehat{\boldsymbol{\beta}}_p$ calculated as the weighted sum of the individual asset betas. This allows to generate the hypothetical forward distribution of the portfolio return even when there are assets with no history, what is particularly important for emerging markets.

Since within our approach VaR is conditional on the risk factor betas estimated over a time period, which may be much shorter than T, the obtained estimate of VaR is more sensitive to changes in the market conditions and therefore more volatile compared to the conventional historical simulation estimate of VaR.

4 Empirical Results

In this section, we use the approach described in the previous section to calculate 1-day and 10-day 1% and 5% risk factor beta conditional VaR for the six Fama/French benchmark portfolios and the S&P500 index. The Fama/French benchmark portfolios are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint determines the buy range for the Small and Big portfolios and is given by the median NYSE market equity. The BE/ME breakpoints determine the buy range for the Growth, Neutral, and Value portfolios and are the 30th and 70th NYSE percentiles.

The Fama/French benchmark portfolios returns are from Kenneth French's webpage. S&P500 daily return data are obtained from the Center for Research in Security Prices (CRSP) of the University of Chicago. Table I reports sample moments of the distribution of the Fama/French benchmark portfolios and the S&P500 index for July 1, 1963 to December 31, 2004. Panel A reports statistics for daily returns. The daily returns have high sample excess kurtosis ranging from 5.66 to 25.07, a clear sign of fat tails. The skewness estimates are negative for all the portfolios, ranging from -1.10 to -0.47. Panel B reports sample moments for 10-day portfolio returns. These are considerably less leptokurtic than daily returns. Sample estimates of excess kurtosis tend to be close to zero (between -0.86 and 0.54) for the first three Fama/French benchmark portfolios but large and positive (from 3.61 to 8.31) for the fourth to sixth Fama/French benchmark portfolios and the S&P500 index, indicating that these portfolio returns have more mass in the tails of the distribution than predicted by a normal distribution. Sample estimates of skewness for 10-day portfolio returns are negative and range from -1.00 to -0.63.

To test whether the portfolio returns are normally distributed, we use the Jarque-Bera statistic.[4] Because the value of the Jarque-Bera statistic is greater than the 5% critical value

[4] The Jarque-Bera statistic is given by $JB = \frac{T}{6}\left(S^2 + \frac{K^2}{4}\right)$, where S is skewness and K is excess kurtosis. When the portfolio returns are normally distributed, this statistic is chi-squared distributed with 2 degrees of freedom, $JB \sim \chi^2(2)$.

for any portfolio under consideration, there is sufficient evidence to conclude that the normal distribution assumption is unreasonable and therefore the portfolio-normal approach (Wilson (1994) and Frain and Meegan (1996)) may not be applied to find the portfolio VaR.[5]

Since the approach we propose in this paper is completely free from any arbitrary statistical and distributional assumptions, we use this approach to calculate 1-day and 10-day 1% and 5%VaR for each portfolio. When applying this approach, we must first specify macroeconomic factors with considerable ability to explain the returns of the portfolios in question and then estimate the portfolio betas with respect to these factors. The portfolio betas can be estimated from equation

$$R_{p,t} - R_{f,t} = \alpha + \boldsymbol{\beta}'\mathbf{Z}_{K,t} + \epsilon_t. \tag{13}$$

Fama and French (1993) find that only three factor portfolios (mimicking portfolios) are necessary to explain the cross section of expected stock returns. These factors are the return on the value-weighted stock index from CRSP, the return on a portfolio of low market value of equity firms minus a portfolio of high market value of equity firms, and the return on a portfolio of high book-to-market value firms minus a portfolio of low book-to-market value firms.[6]

Given this result, we use the Fama/French three factor model to explain the excess portfolio returns:

$$R_{p,t} - R_{f,t} = \alpha + \beta_1(R_{m,t} - R_{f,t}) + \beta_2 smb_t + \beta_3 hml_t + \epsilon_t, \tag{14}$$

in which the benchmark factors are the excess return on the market portfolio (the value-weighted return on all stocks listed on the NYSE, AMEX, and NASDAQ (obtained from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates)), $R_{m,t} - R_{f,t}$, the average return on the three small value-weighted size/book-to-market portfolios minus the average return on the three big portfolios, smb_t, and the average return on the two value portfolios minus the average return on the two growth portfolios, hml_t. The benchmark factors are from Kenneth French's webpage.

We use the 50 most recent returns (1-day returns for a 1-day VaR and 10-day returns for a 10-day VaR) to calculate the regression parameters in (14). To estimate the factor betas, we use standard regression techniques. Then, we assume that in the next holding period the values of the risk factor betas will be the same as those estimated for the 50 most recent periods and generate the next period distribution of the portfolio return using the obtained estimates of the factor betas and the historical values of the risk factors observed over the period from July 1, 1963 to December 31, 2004.

For well-diversified portfolios ϵ_t is virtually zero. Since equilibrium market prices are rational in that they rule out risk-free arbitrage opportunities, we assume $\alpha = 0$ and use the following

[5] The 5% critical value from $\chi^2(2)$ is 5.99.

[6] Fama and French (1993) also find that when bond portfolios are included then the returns on a term structure factor and a default risk factor must be added as factors.

pricing relationship that guaranties the absence of risk-free arbitrage opportunities to predict the value of $R_{p,t}$:

$$\widehat{R}_{p,t} = R_{f,t} + \widehat{\beta}_1(R_{m,t} - R_{f,t}) + \widehat{\beta}_2 smb_t + \widehat{\beta}_3 hml_t. \tag{15}$$

Equation (15) gives us the multidimensional security characteristic line with three factors. The number of the predicted equilibrium values of $R_{p,t}$ equals the number of the past values of the risk factors. These predicted values define the expected next holding period distribution of $R_{p,t}$ conditional that in the next period the risk factor betas are the same as they were over the 50 most recent periods. After generating the expected distribution of the portfolio return, we assume, as in the historical simulation approach, that either all the possible states are equally-weighted (equally-weighted scenarios) or that the most recent values have more probability to occur tomorrow (geometrically-weighted scenarios) and calculate the appropriate percentile of the next period distribution of the portfolio return. The VaR of the portfolio can be calculated as the decrease in portfolio value corresponding to this percentile point.

To test how well our estimation approach would have performed in the past, we carry out a back test. When backtesting our approach, we calculate a 1-day VaR for the 3000 most recent days and a 10-days VaR for the 500 most recent 10-day holding periods. Backtesting involves estimating how often the profit and the loss in a holding period exceeded VaR calculated for that period. We first assume that the simulated returns have equal probabilities of occurring in the next holding period. Then, we calculate VaR under the assumption that the most recent returns may be the most relevant for next period's distribution and when the simulated historical returns are assigned probability weights declining exponentially through the past. The results are reported in Panel B of Table II (for a 1-day 1% VaR), Table III (for a 1-day 5% VaR), Table IV (for a 10-day 1% VaR), and Table V (for a 10-day 5% VaR).

Empirical evidence is that the estimates of 1% and 5% 1-day VaR are not accurate when scenarios are assumed to have equal probabilities of occurring. However, if probability weights are assumed to decline exponentially through the past, the results are reasonably accurate when $a = 0.99$. The results are much more accurate when VaR is calculated for a 10-day holding period. For a 10-day horizon, the percentage of downside and upside excessions over time is close to that corresponding to the chosen confidence level of VaR when a is equal or close to 1. This result is robust to the confidence level of VaR. This suggests that our approach yields an accurate estimate of VaR when the tails of the empirical distribution of the portfolio return are not very heavy (as in the case of 10-day returns). We observe that when the tails are very fat, the approach we propose in this paper may significantly underestimate the true VaR.

5 Conclusions

In this paper, we proposed a new approach to the estimation of the portfolio VaR. This approach is based on the use of the risk factor betas of individual assets to simulate the hypothetical portfolio returns. The portfolio VaR is then calculated as an appropriate percentile point of the corresponding distribution of portfolio profit and losses. The obtained estimate of the portfolio VaR is conditional on the values of betas and is therefore called the risk factor beta conditional VaR.

To evaluate the quality of our approach, we compare the actual portfolio returns to those generated with the risk factor conditional VaR model for the six Fama/French benchmark portfolios and the S&P500 index. Evidence suggests that this approach works reasonably well in practice and we can feel reasonably comfortable with our methodology for calculating VaR when the tails of the distribution of the portfolio return are not extremely fat.

References

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Table I

Sample Moments of Portfolio Returns

This table reports summary statistics for 1-day and 10-day returns (in percent) on the six Fama/French benchmark portfolios and the S&P500 index for July 1, 1963 to December 31, 2004. JB is the Jarque-Bera statistic.

Statistic	Port#1	Port#2	Port#3	Port#4	Port#5	Port#6	S&P500
			A. 1-Day Returns				
Mean	0.0381	0.0592	0.0670	0.0426	0.0467	0.0541	0.0319
Variance	1.1158	0.5785	0.5422	1.0223	0.7190	0.7295	0.8945
Skewness	-0.6745	-0.8635	-0.8824	-0.4706	-1.1013	-0.8877	-0.9410
Excess Kurtosis	5.6626	7.7862	8.7582	11.2577	25.0697	18.2173	22.6843
JB	14751.1	27690.0	34748.5	55558.2	275714.8	145846.4	225554.2
			B. 10-Day Returns				
Mean	0.4192	0.6158	0.6970	0.4347	0.4747	0.5523	0.3231
Variance	18.6715	10.3136	10.2708	11.6655	8.6107	9.1105	9.6265
Skewness	-0.6318	-0.7243	-0.7135	-0.7300	-0.9123	-0.8314	-0.9996
Excess Kurtosis	-0.8568	0.0304	0.5403	4.4581	6.9243	3.6072	8.3105
JB	101.4	91.3	101.3	957.3	2230.5	686.3	3178.2

Table II

Backtesting Results for a 1-day 1% Risk Factor Beta Conditional VaR

This table reports the percentage of the days on which the profit and the loss in a day exceeded the 1-day 1% VaR calculated for that day. The 1-day 1% VaR for the six Fama/French benchmark portfolios and the S&P500 index are calculated for 3000 days. The risk factors are the excess return on the market portfolio, the average return on the three small value-weighted size/book-to-market portfolios minus the average return on the three big portfolios, and the average return on the two value portfolios minus the average return on the two growth portfolios. The factor betas are estimated using the 50 most recent daily observations. Daily returns from July 1, 1963 to December 31, 2004 are used. When the probability weights $\pi_{s,t+1}$ are assumed to decline exponentially through the past, $\pi_{s,t+1} = a^{s-1}(1-a)/(1-a^n)$, $s = 1, ..., S$, we set a equal to 0.95, 0.98, and 0.99.

a	Port#1	Port#2	Port#3	Port#4	Port#5	Port#6	S&P500
				A. Downside Excessions			
				A.1 Equally-Weighted Scenarios			
	1.97	1.40	1.40	2.10	1.90	2.27	2.00
				A.2 Geometrically-Weighted Scenarios			
0.95	2.87	2.47	2.53	2.63	2.67	2.83	2.43
0.98	1.90	1.60	1.50	1.53	1.77	1.57	1.50
0.99	1.33	1.37	1.17	1.33	1.37	1.47	1.37
				B. Upside Excessions			
				B.1. Equally-Weighted Scenarios			
	3.40	2.57	2.00	2.03	1.77	1.50	1.83
				B.2. Geometrically-Weighted Scenarios			
0.95	2.87	2.70	2.93	2.57	2.80	2.73	2.50
0.98	1.77	1.73	1.77	1.37	1.70	1.83	1.57
0.99	1.40	1.53	1.47	1.30	1.37	1.57	1.23

Table III

Backtesting Results for a 1-day 5% Risk Factor Beta Conditional VaR

This table reports the percentage of the days on which the profit and the loss in a day exceeded the 1-day 5% VaR calculated for that day. The 1-day 5% VaR for the six Fama/French benchmark portfolios and the S&P500 index are calculated for 3000 days. The risk factors are the excess return on the market portfolio, the average return on the three small value-weighted size/book-to-market portfolios minus the average return on the three big portfolios, and the average return on the two value portfolios minus the average return on the two growth portfolios. The factor betas are estimated using the 50 most recent daily observations. Daily returns from July 1, 1963 to December 31, 2004 are used. When the probability weights $\pi_{s,t+1}$ are assumed to decline exponentially through the past, $\pi_{s,t+1} = a^{s-1}(1-a)/(1-a^n)$, $s = 1, ..., S$, we set a equal to 0.95, 0.98, and 0.99.

a	Port#1	Port#2	Port#3	Port#4	Port#5	Port#6	S&P500
				A. Downside Excessions			
				A.1. Equally-Weighted Scenarios			
	10.17	9.00	8.43	8.27	7.67	7.57	7.90
				A.2. Geometrically-Weighted Scenarios			
0.95	7.10	6.67	6.40	6.17	6.73	6.70	5.93
0.98	5.87	5.80	5.37	5.27	5.83	5.93	5.33
0.99	5.70	5.03	5.37	5.10	5.43	5.80	5.40
				B. Upside Excessions			
				B.1. Equally-Weighted Scenarios			
	10.67	9.90	9.27	8.00	6.80	6.53	7.03
				B.2. Geometrically-Weighted Scenarios			
0.95	6.03	5.90	6.40	6.30	6.47	6.30	6.40
0.98	5.73	5.63	5.97	5.73	5.83	5.93	5.33
0.99	5.80	5.70	6.03	5.53	5.83	5.83	5.20

Table IV

Backtesting Results for a 10-day 1% Risk Factor Beta Conditional VaR

This Table reports the percentage of the 10-day periods in which the loss in 10 days exceeded the 10-day 1% VaR calculated for that day. The 10-day 1% VaR for the six Fama/French benchmark portfolios and the S&P500 index are calculated for 500 days. The risk factors are the excess return on the market portfolio, the average return on the three small value-weighted size/book-to-market portfolios minus the average return on the three big portfolios, and the average return on the two value portfolios minus the average return on the two growth portfolios. The factor betas are estimated using the 50 most recent 10-day returns. 10-day returns from July 12, 1963 to December 31, 2004 are used. When the probability weights $\pi_{s,t+1}$ are assumed to decline exponentially through the past, $\pi_{s,t+1} = a^{s-1}(1-a)/(1-a^n)$, $s = 1, ..., S$, we set a equal to 0.95, 0.98, and 0.99.

a	Port#1	Port#2	Port#3	Port#4	Port#5	Port#6	S&P500
				A. Downside Excessions			
				A.1. Equally-Weighted Scenarios			
	1.00	1.00	1.00	2.20	1.20	1.60	1.20
				A.2. Geometrically-Weighted Scenarios			
0.95	2.00	2.60	2.20	2.20	2.80	3.60	2.20
0.98	1.20	1.60	1.20	1.60	1.80	2.00	1.60
0.99	1.20	1.40	1.20	1.40	1.60	2.00	1.40
				B. Upside Excessions			
				B.1. Equally-Weighted Scenarios			
	2.00	0.80	0.80	1.00	1.20	1.00	1.00
				B.2. Geometrically-Weighted Scenarios			
0.95	2.40	2.20	3.00	3.00	3.00	3.00	3.20
0.98	1.00	1.20	1.40	1.20	2.20	2.40	2.00
0.99	1.40	1.00	1.40	1.20	1.40	1.80	1.20

Table V

Backtesting Results for a 10-day 5% Risk Factor Beta Conditional VaR

This Table reports the percentage of the 10-day periods in which the loss in 10 days exceeded the 10-day 5% VaR calculated for that day. The 10-day 5% VaR for the six Fama/French benchmark portfolios and the S&P500 index are calculated for 500 days. The risk factors are the excess return on the market portfolio, the average return on the three small value-weighted size/book-to-market portfolios minus the average return on the three big portfolios, and the average return on the two value portfolios minus the average return on the two growth portfolios. The factor betas are estimated using the 50 most recent 10-day returns. 10-day returns from July 12, 1963 to December 31, 2004 are used. When the probability weights $\pi_{s,t+1}$ are assumed to decline exponentially through the past, $\pi_{s,t+1} = a^{s-1}(1-a)/(1-a^n)$, $s = 1, ..., S$, we set a equal to 0.95, 0.98, and 0.99.

a	Port#1	Port#2	Port#3	Port#4	Port#5	Port#6	S&P500
				A. Downside Excessions			
				A.1. Equally-Weighted Scenarios			
	5.80	4.80	5.60	5.00	5.40	6.20	5.40
				A.2. Geometrically-Weighted Scenarios			
0.95	6.60	7.00	6.60	6.00	6.20	6.80	7.00
0.98	5.60	5.20	5.20	5.00	6.00	6.00	6.20
0.99	5.60	5.20	5.60	4.80	5.40	6.20	5.40
				B. Upside Excessions			
				B.1. Equally-Weighted Scenarios			
	5.40	5.20	6.00	6.00	5.40	5.00	5.00
				B.2. Geometrically-Weighted Scenarios			
0.95	5.80	6.80	6.80	7.60	7.40	7.20	5.80
0.98	5.60	5.40	5.80	6.60	6.00	7.00	5.40
0.99	5.60	5.00	5.60	5.80	5.80	6.40	5.60