Revisiting Static Portfolio Theory for HARA Investors*

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Abstract

The implications of the two-fund separation theorem have been carefully examined in the literature for the case of mean-variance preferences. However, even though the two-fund theorem applies to the whole class of HARA utility functions, its implications for the efficiency sets spanned by these preferences are much less known. Without dealing with general equilibrium issues, the goal of this paper is to show how most of the well-known constructions which arise in connection with the former subclass, extend in a relatively natural way to the whole latter set of preferences. Furthermore, graphical illustrations of the HARA portfolio problem that parallel mean-variance geometry are also provided. Along the same lines, It is also shown how the general problem can be seen as a choice between two parameters, one measuring reward and the other one measuring risk.

1 Introduction

The youth years of financial economics are clearly dominated by the early results of portfolio choice theory. Among other things, the contributions of Markowitz (1957) together with the derivation by Tobin (1958) of the two-fund theorem, always in the context of mean-variance preferences, play a crucial role in developing the first general equilibrium model of asset pricing, i.e. the CAPM. These two fields of finance marched closely together until the seminal work of Lucas (1978) which implies a radical change of perspective. From this point onwards, it becomes clear that it is not necessary to solve a portfolio choice problem in order to model the behavior of asset prices. The consumption-based approach allows asset pricing to (partially) abandon its early companion and walk its way mostly alone. The GMM technology of Hansen (1982) pushes further in this direction and it helps creating a new approach where the stochastic discount factor (SDF) is to be the main focus.

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The portfolio problem unexpectedly pops up again in asset pricing a few years later with Hansen and Jagannathan (1991) and their diagnostic tool based on the well-known volatility bounds on the stochastic discount factor. These authors show a duality relationship between the mean-variance problem and the minimum variance SDF. This connection relies heavily on the orthogonal decomposition of the mean-variance frontier previously derived by Hansen and Richard (1987). The link has recently been further emphasized by Cochrane (2001) in a series of results labeled equivalence theorems.

However, it is somehow surprising that all these constructs and links have mainly been presented as properties confined to the mean-variance world when they can in fact be seen as implications of the two-fund theorem. For example, in the work of Snow (1991), Stutzer (1995) and Bansal and Lehnman (1997) where new bounds on the SDF are established, the underlying presence of the portfolio problem is once again relatively hidden. As it turns out, the two-fund theorem applies to a much more general set of preferences (Cass and Stiglitz (1970)). As a consequence, the previous results obtained in the mean-variance framework, extend in a very natural way to whole family of HARA utility functions.

The first goal of our paper is to prove this point. In particular, we present a generalization of Hansen and Richard's decomposition and its links with the derivation of bounds on the SDF and the portfolio problem. Also, we formulate Cochrane's equivalence theorems in the context of the efficiency sets spanned by the above general class of preferences. Finally, we show how Roll's tracking error minimization (1992) can be posed in this more general setup. Some of our results in this direction try to formalize some connections already pointed out by Cerny (2003).

Our second goal is more illustrative and it is aimed at providing new insights. Most practitioners have in most cases turned their backs to the use of preferences that satisfy sensible economic assumptions. This may be partially explained by the fact that, as opposed to the other elements of the HARA family, mean-variance analysis provides an appealing two-parameter framework together with a whole battery of geometric interpretations which after all are a strong selling point. With this observation in mind, we show that similar graphical illustrations for the general HARA portfolio problem can be produced. Furthermore, we argue that the investor's decision in this context can also be seen as a choice between two parameters, one being a measure of reward and the other one a measure of risk. Once again, the two-fund theorem is at the root of these derivations.

The paper is organized as follows. The second section deals with notational issues. Section 3 presents some key results whose implications for portfolio theory are presented in Section 4. The link between the SDF and the portfolio problem is further discussed in Section 5. Finally, Section 6 concludes.

2 Preliminary and notational issues

Consider a two-period frictionless economy where a set of N basis assets are currently available at a known price and their payoffs are continuous random variables whose support is an interval of the real line and at least one of them is nondegenerated. Without loss of generality, our basis assets are assumed to have a strictly positive price and hence, their vector of returns denoted by $\mathbf{R} = (R_1, \ldots, R_N)$ can be defined. The space of attainable payoffs is thus given by[1]

$$\underline{X} \equiv \left\{ x : \exists \alpha \in \Re^N \text{ s.t. } \alpha^T \mathbf{R} = x \right\}.$$

Arbitrage opportunities are absent, this guarantees the existence of strictly positive random variables, m, satisfying

$$E(m\mathbf{R}) = \mathbf{1}$$

where $\mathbf{1}$ is a N-vector of ones. Denote by $\underline{M}_+$ the set of such random variables, also known as SDF's, and define the sets

$$\underline{R} \equiv \left\{ R \in \underline{X} : \exists \alpha \in \Re^N \text{ s.t. } \alpha^T \mathbf{R} = R \text{ and } \alpha^T \mathbf{1} = 1 \right\}$$

and

$$\underline{R}^e \equiv \left\{ R^e \in \underline{X} : \exists \alpha \in \Re^N \text{ s.t. } \alpha^T \mathbf{R} = R^e \text{ and } \alpha^T \mathbf{1} = 0 \right\},$$

that is, the set of returns and zero-price payoffs, respectively.

Consider the static portfolio problem for HARA utility functions given by

$$\max_{R \in \underline{R}} E\left[u_{\gamma,\delta}(R)\right] \tag{2.1}$$

where

$$u_{\gamma,\delta}(R) \equiv \begin{cases} \frac{1}{\gamma+1}(R-\delta)^{\gamma+1} & \text{if } \gamma < 0,\ \gamma \neq -1 \\ \ln(R-\delta) & \text{if } \gamma = -1 \\ -\frac{1}{\gamma+1}(\delta - R)^{\gamma+1} & \text{if } \gamma > 0 \\ -\exp(-\delta R) & \text{if } \gamma = \infty \end{cases} \tag{2.2}$$

and $\delta \in \Re$. For $\gamma < 0$ ($\gamma > 0$), δ can be interpreted as a subsistence (satiation) level, while in the case of negative exponential utility ($\gamma = \infty$) it stands for the coefficient of absolute risk aversion. Each set of utility functions above with common parameter γ will be referred to as an HARA utility class.

For a given γ, let $\underline{\delta}_\gamma$ be the set of values of δ for which a unique solution to (2.1) exists and let $R_{\gamma,\delta}$ denote the corresponding optimal return. Also, define

$$\underline{R}_\gamma \equiv \left\{ R_{\gamma,\delta} : \delta \in \underline{\delta}_\gamma \right\}.$$

It is well-known that quadratic utility ($\gamma = 1$), is equivalent to mean-variance preferences and therefore, the set $\underline{R}_1$ contains all returns that lie on the mean-variance frontier. The properties of this particular set, together with its links

[1] We will omit the label "almost sure" in all relationships involving random variables.

with asset pricing objects like SDF's, have been largely examined in the literature and they are mostly rooted in the two-fund separation theorem. It is also well-known that this theorem applies to $\underline{R}_\gamma$ for all γ. However, up to our knowledge, no serious effort has been done in order to establish, with enough generality, how the above properties and links translate for all the remainder efficiency sets ($\gamma \neq 1$) spanned by HARA utility functions. As already advanced, this will be our goal. It should be noted that even though our results apply, after minor modifications, to all γ, in order to favor their readability and avoid a further complication of our notation, we will focus most of our exposition on the *frontiers* associated to values of $\gamma < -1$.

In the absence of a risk-free asset, we will consider the augmented set $\underline{X}^v$ resulting from adding an *artificial* constant unit payoff with price v to $\underline{X}$. In such case, the objects $\underline{R}^v$, $\underline{M}_+^v$, and $\underline{R}_\gamma^v$ will be defined as the counterparts of $\underline{R}^v$, $\underline{M}_+$, and $\underline{R}_\gamma$ in $\underline{X}^v$, respectively. Obviously, the set $\underline{M}_+^v$ is not empty as long as there is no arbitrage in $\underline{X}^v$, or equivalently, as long as v belongs to the interval $(\underline{v}, \overline{v})$ where[2]

$$\underline{v} \equiv \sup \left\{ \alpha^T \mathbf{1} : \alpha \in \Re^N \text{ and } \alpha^T \mathbf{R} \leq 1 \right\}$$

and

$$\overline{v} \equiv \inf \left\{ \alpha^T \mathbf{1} : \alpha \in \Re^N \text{ and } \alpha^T \mathbf{R} \geq 1 \right\}.$$

Finally, whenever a riskless payoff is assumed to exist, its return will be denoted by R^f.

3 The index SDF and its return

Hansen and Richard (1987) and Hansen and Jagannathan (1991) study the properties of the unique SDF which is also a payoff and its associated return. These constructs have a clear parallel within each HARA utility class which we know turn to present.

Lemma 3.1 *For any strictly monotonic function f, there exists at most one payoff $x^* \in \underline{X}$ such that $f(x^*) \in \underline{M}_+$.*

Proof. Reasoning by contradiction, suppose there is x_f^* and x_f^{**} satisfying

$$E\left[f\left(x^{**}\right) x\right] = E\left[f\left(x^*\right) x\right], \qquad \forall x \in \underline{X}$$

where $x^{**}, x^* \in \underline{X}$. Then

$$E\left[f\left(x^*\right) x^*\right] = E\left[f\left(x^{**}\right) x^*\right]$$

and

$$E\left[f\left(x^*\right) x^{**}\right] = E\left[f\left(x^{**}\right) x^{**}\right].$$

[2] We assume there exists $x \in \underline{X}$ such that $x \geq 1$.

Hence,

$$E\{[f(x^*) - f(x^{**})]\, x^*\} = E\{[f(x^*) - f(x^{**})]\, x^{**}\}$$

or equivalently,

$$E\{[f(x^*) - f(x^{**})]\,(x^* - x^{**})\} = 0. \tag{3.1}$$

Now, without loss of generality assume that the function is strictly increasing. From (3.1) whenever $x^* > x^{**}$, it must be the case that $f(x^*) > f(x^{**})$ and if $x^* < x^{**}$, then $f(x^*) < f(x^{**})$. This means that the value inside the expectation is always strictly positive in those states of nature where $x^* \neq x^{**}$ and zero otherwise. Thus, in order for (3.1) to hold it must be the case that $x^* = x^{**}$. Similar arguments will give that if the function is strictly decreasing, the value inside the expectation is always strictly negative whenever $x^* \neq x^{**}$ and zero otherwise, which implies also again that $x^* = x^{**}$.

□

Proposition 3.2 *For a given $\gamma < -1$, there exists at most one payoff $x_\gamma \in \underline{X}$ such that $x_\gamma^\gamma \in \underline{M}_+$.*

This proposition merely guarantees that there is only one payoff (given its existence) which raised to the γ power is also an strictly positive SDF. Its proof is a straightforward application of an auxiliary Lemma stated in the Appendix. In the sequel, the random variable x_γ will be referred to as the γ index SDF.

For $\gamma = 1$ (mean-variance), Hansen and Richard (1987) show that the law of one price is a sufficient condition for the existence of a unique payoff (not necessarily positive) which is also an SDF. For more general cases, it is not easy to specify these conditions, although a characterization in terms of the existence of a solution to the portfolio problem is possible as we shall see below.

Also, the above authors prove that the return of x_1 has the minimum second moment in $\underline{R}$. This finding has a very natural extension.

Proposition 3.3 *For a given $\gamma < -1$, assume that x_γ exists and let R_γ be its return. Then*

$$E\left(R_\gamma^{\gamma+1}\right) = \min_{R \in \underline{R}} E\left(R^{\gamma+1}\right). \tag{3.2}$$

Proof. By definition, x_γ has price equal to $E\left(x_\gamma^{\gamma+1}\right)$ and hence,

$$R_\gamma = \frac{x_\gamma}{E\left(x_\gamma^{\gamma+1}\right)}. \tag{3.3}$$

Also, by definition, x_γ satisfies

$$E\left(x_\gamma^\gamma R\right) = 1, \qquad \forall R \in \underline{R}$$

which, from (3.3), can be equivalently stated as

$$E\left(R_\gamma^\gamma R\right) = \left[E\left(x_\gamma^{\gamma+1}\right)\right]^{-\gamma}, \qquad \forall R \in \underline{R}, \tag{3.4}$$

giving that R_γ satisfies the first-order conditions of problem (3.2). Since for any $\gamma \in \underline{\gamma}$, the function $f(x) = x^{\gamma+1}$ is strictly concave in $\Re_{++}$, from Proposition 3.2 it follows that the above conditions are sufficient and necessary conditions for a (unique) global minimum.

$\square$

Remark 3.4 *From the above proof, it is easy to see that*

$$x_\gamma = \frac{R_\gamma}{\left[E\left(R_\gamma^\gamma\right) R^f\right]^{1/\gamma}}. \tag{3.5}$$

We divide our next set of results in two subsections covering the existence and absence of a riskless asset.

4 Portfolio theory results for HARA utility

$\underline{X}$ has a riskless payoff

Assume there is a constant payoff in $\underline{X}$ with return R^f. As it turns out, problem (2.1) has a solution a solution for any $\delta < R^f$ given the existence of x_γ. Actually, these two facts can be proved to be equivalent thereby giving the following characterization.

Proposition 4.5 *For a given $\gamma < -1$, x_γ exists if and only if $\underline{\delta}_\gamma = \left(-\infty, R^f\right)$.*

Proof. (necessity) For a given $\delta \in \Re$, the first-order conditions of portfolio problem (2.1) give that if a return $R_{\gamma,\delta}$ is optimal, it must satisfy

$$E\left[\left(R_{\gamma,\delta} - \delta\right)^\gamma R\right] = \alpha, \qquad \forall R \in \underline{R} \tag{4.6}$$

for some constant $\alpha \in \Re_{++}$. Given that by assumption our basis assets have a continuous probability distribution whose support is an interval of $\Re$, it is also clear that

$$R_{\gamma,\delta} - \delta > 0 \tag{4.7}$$

Since x_γ exists, it is clear from (3.4) that

$$R_{\gamma,\delta} = R_\gamma + \delta\left(1 - \frac{R_\gamma}{R^f}\right)$$

satisfies (4.6) and (4.7) as long as $\delta \in \left(-\infty, R^f\right)$. Furthermore, from identical arguments as the one in the proof of Proposition 3.3, it is clear that (4.6) and (4.7) are necessary and sufficient conditions for a unique global maximum of the portfolio choice problem. It remains to check that no solution exists for $\delta \notin \left(-\infty, R^f\right)$ but this is trivial since the absence of arbitrage guarantees that there is no $R \in \underline{R}$ such that

$$R - R^f > 0$$

which from (4.7) gives that no solution exists for $\delta \geq R^f$.

(Sufficiency) Assume that $\underline{\delta}_\gamma = \left(-\infty, R^f\right)$. Then, there is a solution to the portfolio problem for $\delta = 0$, or equivalently, there exists a return $R^* \in \underline{R}$ satisfying

$$E\left[(R^*)^\gamma R\right] = \alpha, \qquad \forall R \in \underline{R}$$

for some constant $\alpha \in \Re_{++}$. Hence,

$$\frac{R^{*\gamma}}{\alpha} \in \underline{M}_+$$

which together with

$$\frac{R^*}{\alpha^{1/\gamma}} \in \underline{X}$$

gives the desired result.

$\square$

Hansen and Richard (1987) introduce an orthogonal decomposition of the mean-variance frontier. In particular, they show that a return R is mean-variance if and only if it can be expressed as

$$R = R_1 + wR_1^e$$

where w is any real number and R_1^e is the zero-price payoff that solves

$$\min_{R^e \in \underline{R}^e} E\left[\left(1 - R^e\right)^2\right]. \tag{4.8}$$

Furthermore, R_1 and R_1^e satisfy $E\left(R_1 R_1^e\right) = 0$. This construction holds even in the absence of a risk-free asset; however, when a constant payoff exists, then

$$R_1^e = 1 - \frac{R_1}{R^f}.$$

Again, this decomposition turns out to be a particular case of a more general result. Define the excess return

$$R_\gamma^e \equiv 1 - \frac{1}{R^f} R_\gamma.$$

Proposition 4.6 *For a given $\gamma < -1$, assume that x_γ exists. $R \in \underline{R}_\gamma$ if and only if there is a constant $w < R^f$ such that*

$$R = R_\gamma + wR_\gamma^e.$$

Furthermore,

$$R_{\gamma,\delta} = R_\gamma + \delta R_\gamma^e, \qquad \forall \delta \in \underline{\delta}_\gamma. \tag{4.9}$$

We will not show this statement since its proof follows very similar arguments to the ones used in showing Proposition 4.5. In fact, the former can also be seen as a corollary of the latter. The same line of reasoning gives that under the same conditions, R_{γ}^{e} solves

$$\min_{R^e \in \underline{R}^e} E\left[(1-R^e)^{\gamma+1}\right] \tag{4.10}$$

and it satisfies

$$E\left[(R_\gamma)^\gamma R_\gamma^e\right] = 0. \tag{4.11}$$

As we see, the parallel of all these constructions is quite clear. A mean-variance return can always be written as the sum of the minimum second moment return, R_1, and a constant times the excess return that solves (4.8), R_1^e. Similarly, a return in $\underline{R}_\gamma$ can always be written as the return that minimizes $E\left(R^{\gamma+1}\right)$, R_γ, and a constant time the excess return that solves (4.10), R_γ^e. The orthogonality between the two objects in the mean-variance case has its counterpart in their satisfying (4.11).

Obviously, just as the two-fund separation theorem fails to hold in the absence of a risk-free asset when $\gamma \neq 1$, so does the above decomposition is such scenario.

It is important to note that our result also identifies the constant in the decomposition as the subsistence level δ of the corresponding utility function. For $\gamma = 1$, this allows to establish the link between the utility-based maximization and the corresponding mean-variance problem. Specifically, from Proposition 4.6 it is easy to see that, $R \in \underline{R}$ solves[3]

$$\max_{R \in \underline{R}} -\frac{1}{2}(\delta - R)^2 \tag{4.12}$$

if and only if it solves

$$\min_{R \in \underline{R}} var(R)$$
$$\text{s.t. } E(R) = \delta E\left(R_1^e\right) + E(R_1).$$

For the efficient part of the mean-variance frontier, the above equivalence can be stated in an alternative way. Namely, for any $\delta \geq R^f$, $R \in \underline{R}$ solves (4.12) if and only if it solves

$$\max_{R \in \underline{R}} E(R) \tag{4.13}$$
$$\text{s.t. } var(R) = \left(1 - \frac{\delta}{R^f}\right)^2 var(R_1).$$

Also, it is well-known that the efficient part of the mean-variance frontier gives a straight line when plotted in mean-standard deviation space. In particular (see Hansen and Jagannathan (1991) or Cochrane (2001)), we have that

$$E(R) - R^f = \frac{\sigma(R_1)}{E(R_1)}\sigma(R) \qquad \forall R \in \underline{R}_1 \tag{4.14}$$

[3] The statement of Proposition 4.6 holds for any $\gamma \neq 0$, except for positive even integers.

or equivalently,

$$E(R) - R^f = \frac{R^f - E(R_1)}{\sigma(R_1)} \sigma(R) \qquad \forall R \in \underline{R}_1 \tag{4.15}$$

where $\left[R^f - E(R_1)\right] / \sigma(R_1)$ gives the highest Sharpe ratio available in $\underline{X}$. This equation relates the two relevant parameters of the mean-variance problem.

Clearly, a link between δ and the mean and variance of the optimal return can be easily derived for $\gamma \in \underline{\gamma}$, since under the conditions of Proposition 4.6, we also have that

$$E(R_{\gamma,\delta}) = \delta E\left(R_\gamma^e\right) + E(R_\gamma) \tag{4.16}$$

and

$$var(R_{\gamma,\delta}) = \left(1 - \frac{\delta}{R^f}\right)^2 var(R_\gamma), \tag{4.17}$$

which by factoring out δ in (4.17) and plugging it into (4.16) gives a relationship that replicates (4.15), that is,

$$E(R) - R^f = \frac{E(R_\gamma) - R^f}{\sigma(R_\gamma)} \sigma(R) \qquad \forall R \in \underline{R}_\gamma. \tag{4.18}$$

However, this equation has less interest when $\gamma \neq 1$ since in that case the mean and standard deviation are not proper parameters of the portfolio choice problem (note that the slope of the line given in (4.18) is not necessarily the highest Sharpe ratio attainable in $\underline{X}$). In other words, when $\gamma \neq 1$ these two moments are no longer the relevant measures of reward and risk, respectively (nor the Sharpe ratio the appropriate reward-to-risk measure). At this point a natural question may be raised: is it possible to give economic intuition to the elements of R_γ in a way that parallels the arguments above when $\gamma \neq 1$? The following results suggest a potentially positive answer to this question for the case.[4]

Proposition 4.7 *For a given $\gamma \in \underline{\gamma}$, assume that x_γ exists. Then, $R_{\gamma,\delta}$ and δ solve*

$$\max_{\lambda \in \Re, R \in \underline{R}} \lambda \tag{4.19}$$

$$s.t.\ E\left[(R - \lambda)^{\gamma+1}\right] = \left(1 - \frac{\delta}{R^f}\right)^{\gamma+1} E\left[(R_\gamma)^{\gamma+1}\right] \tag{4.20}$$

for any $\delta \in \underline{\delta}_\gamma$.

By a close inspection of (2.1) together with Proposition 4.6, the result above can almost be regarded as tautological and hence, we skip its proof. Nevertheless, we believe it helps motivating our reasoning below. For any constant λ and

[4] The mathematical results we will present also apply to the case $\gamma > 0$. However, their economic interpretation is much less appealing.

a given $\gamma \in \underline{\gamma}$, consider the values

$$\|R-\lambda\|_{\gamma+1} \equiv \left\{E\left[(R-\lambda)^{\gamma+1}\right]\right\}^{1/(\gamma+1)} \tag{4.21}$$

and

$$\|R-\lambda\|_{\gamma+1}^{\gamma+1} = E\left[(R-\lambda)^{\gamma+1}\right]. \tag{4.22}$$

From (2.1), it is easy to see that $R_{\gamma,\lambda}$ minimizes (4.22), or equivalently, maximizes (4.21). Also, the former is infinity, or equivalently, the latter is zero if $R-\lambda$ can take zero values (or negative for that matter). The variance gives the second moment of a return in excess of is mean, whereas (4.22) gives the "$\gamma+1$ moment" of the return in excess of λ. The standard deviation expresses the variance in the same units as $R-E(R)$ just as (4.21) transforms (4.22) into the same units as $R-\lambda$. Thus, (4.21) is a natural replacement of the standard deviation. Last but not least, as opposed to the variance, (4.21) has a meaning which is in consonance with standard assumptions of economic theory. Namely, it gives, for some $\gamma \in \underline{\gamma}$, the CRRA-certainty equivalent of the return R in excess of λ.

On the other hand, λ is a guaranteed level of return (as long as risk is not infinite/zero). The elements of $\underline{R}_\gamma$ happen to maximize this floor value for a given level of risk just as the elements of $\underline{R}_1$ maximize the mean for a given value of the standard deviation. It can thus be associated with a measure of reward.

Furthermore, a straight line links the optimal values of these two proxies of reward and risk, just as the optimal pairs of mean and variance relate through (4.15). Specifically, from Proposition 4.7 we have that

$$R^f - \delta = \frac{R^f}{\|R_\gamma\|_{\gamma+1}} \|R_{\gamma,\delta} - \delta\|_{\gamma+1} \tag{4.23}$$

Also, the slope of this line is the highest attainable reward-to-risk ratio in $\underline{X}$, a measure that defines a *generalized* Sharpe ratio.[5] We will refer to the set of the optimal pairs of reward and risk as the γ *frontier* which we illustrate in Figure 1. The vertical axis represents a guaranteed value λ while the horizontal axis gives the CRRA-certainty equivalent of the corresponding *excess* return. For any $R \in R$, the pairs λ and $\|R-\lambda\|_{\gamma+1}$ must give a point below the straight line. All optimal values lie on the line.

Also, if we denote by α_γ the portfolio weight of the risk-free asset in R_γ, it is easy to see that the return

$$R_{\gamma,\delta^{ta}} = R_\gamma + \delta^{ta} R_\gamma^e$$

[5] Cerny (2002) introduces an extension of the Sharpe ratio to the entire CRRA family of utility functions. For $\gamma \in \underline{\gamma}$ and $R \in \underline{R}$, his generalized Sharpe ratio, $h_\gamma(R)$, can be implicitly defined (in the presence of a risk-free asset) as

$$1 + h_\gamma^2(R) \equiv \left(\frac{R^f}{\|R\|_{\gamma+1}}\right)^{2\gamma}.$$

Our ratio is thus equivalent to Cerny's since it gives an identical ranking of investments.

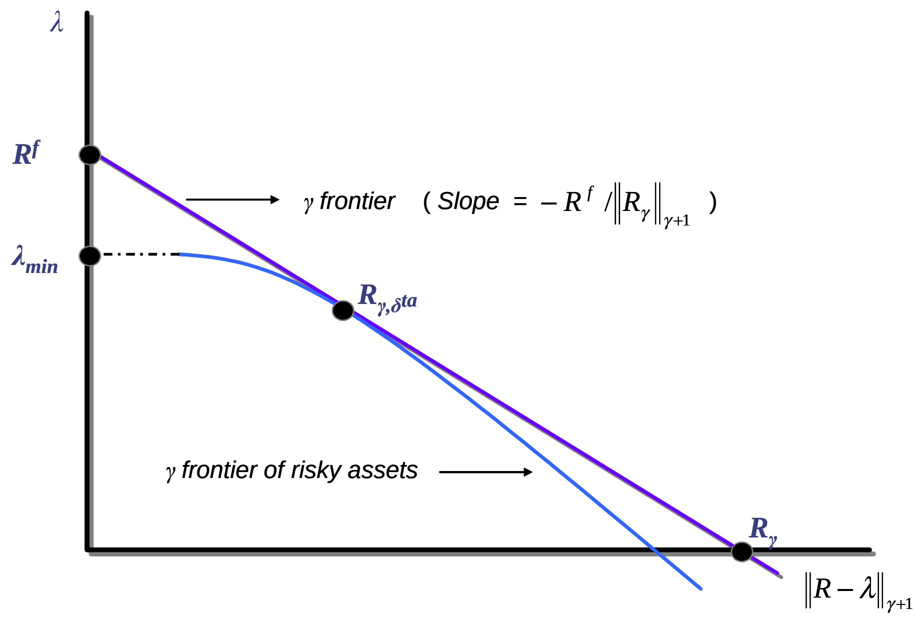

Figure 1: **Figure 1**: The γ frontier with and without a constant payoff.

where $\delta^{ta} \equiv -\alpha_\gamma R^f / (1-\alpha_\gamma)$, is optimal even if we remove the risk-free asset from $\underline{X}$. Thus, the above return is the tangency portfolio in Figure 1. Note that if α_γ is negative (the case depicted), $\delta^{ta} > 0$ and the tangency portfolio lies above the horizontal axis, that is,

$$\left\|R_{\gamma,\delta^{ta}} - \delta\right\|_{\gamma+1} > \|R_\gamma\|_{\gamma+1} .$$

Otherwise, $\delta^{ta} < 0$ and the tangency point is placed in the figure to the right of $\|R_\gamma\|_{\gamma+1}$ below the horizontal axis.

No risk-free asset

Suppose now that there is no risk-free asset and let $\underline{X}^v$ and $\underline{R}^v$ be the augmented set of payoffs and returns, respectively, resulting from adding a constant unit payoff with price v to $\underline{X}$. Define $\underline{M}^v_+$, x^v_γ, R^v_γ, R^{ev}_γ and $\underline{R}^v_\gamma$ as the counterparts of $\underline{M}_+$, x_γ, R_γ, R^e_γ and $\underline{R}_\gamma$ in $\underline{X}^v$, respectively. Obviously, the set $\underline{M}^v_+$ is not empty as long as there is no arbitrage in $\underline{X}^v$, or equivalently, as long as v belongs to the interval $(\underline{v}, \overline{v})$ where[6]

$$\underline{v} \equiv \sup\left\{\alpha^T \mathbf{1} : \alpha \in \Re^N \text{ and } \alpha^T \mathbf{R} \leq 1\right\}$$

[6] We assume there exists $x \in \underline{X}$ such that $x \geq 1$.

and

$$\overline{v} \equiv \inf \left\{ \alpha^T \mathbf{1} : \alpha \in \Re^N \text{ and } \alpha^T \mathbf{R} \geq 1 \right\}.$$

The random variable x_γ^v is now the unique element of $\underline{X}^v$ such that $\left(x_\gamma^v\right)^\gamma \in \underline{M}_+^v$. Furthermore,

$$R_\gamma^v = \frac{x_\gamma^v}{E\left[\left(x_\gamma^v\right)^{\gamma+1}\right]}$$

and

$$R_\gamma^{ev} = 1 - vR_\gamma^v.$$

Also, for a given $\gamma \in \underline{\gamma}$, from our arguments above, it is straightforward to see that the objects x_γ^v, R_γ^v and R_γ^{ev} exist as long as there is a solution to the portfolio problem in $\underline{R}^v$ for $\delta = 0$. Furthermore, it is also possible to give a characterization of this existence by using the original set of returns $\underline{R}$. Let α_γ^v the weight that R_γ^v assigns to $1/v$ and define

$$\delta_v \equiv \frac{-\alpha_\gamma^v}{v\left(1 - \alpha_\gamma^v\right)}.$$

Proposition 4.8 *For given $\gamma \in \underline{\gamma}$ and $v \in (\underline{v}, \overline{v})$, x_γ^v exists if and only if there is a solution in $\underline{R}$ to problem (2.1) for $\delta = \delta_v$.*

The proof is immediate by using Proposition 4.5 and noting that the return R_{γ,δ_v} belongs to both $\underline{R}_\gamma^v$ and $\underline{R}_\gamma$. In addition, since the decomposition in (4.9) applies in $\underline{R}_\gamma^v$, we have that

$$R_{\gamma,\delta_v} = R_\gamma^v + \delta_v R_\gamma^{ev}. \tag{4.24}$$

In Figure 1 we have also plotted the γ frontier in the absence of a constant payoff. It does not produce a straight line any longer and it obviously lies below the optimal pairs that the inclusion of R^f in $\underline{R}$ implied. Clearly, the ceiling on the values of δ for with a solution to (2.1) exists, $\lambda_{\min} \equiv \sup \underline{\delta}_\gamma$, must satisfy $\lambda_{\min} \leq \underline{v}$.

Further intuition is given in Figure 4 which provides a graphical illustration of the result in the proposition above. It shows how the γ frontier can be swept out by means of the tangency portfolios associated with different values of v. For each one of them, one can construct the optimal set $\underline{R}_\gamma^v$ which when plotted in the space of the figure delivers a straight line with slope

$$-\frac{1}{v\left\|R_\gamma^v\right\|_{\gamma+1}}$$

whose tangency point with $\underline{R}_\gamma$ corresponds to the return (4.24). The plot depicts this formulation for two different prices of the *artificial* risk-free asset. The set of all these tangency points gives the γ frontier.

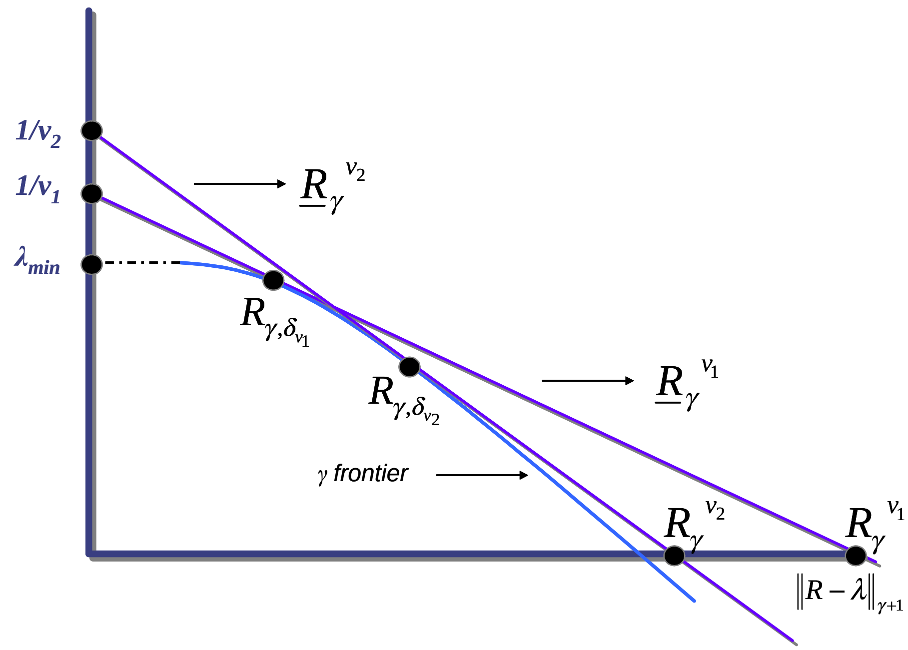

Figure 4: Sweeping out the γ frontier.

Performance evaluation

Given the incentives that many professional money managers face, their portfolio choice is often the solution of the following problem

$$\begin{aligned} &\min_{R \in \underline{R}} E\left(R - R^b\right)^2 \\ &\text{s.t. } E\left(R - R^b\right) = \mu \end{aligned} \tag{4.25}$$

where $R^b \in \underline{R}$ is a prespecified *benchmark* return and μ is a target expected value. The manager's goal is thus to minimize the variance of the return of its portfolio in excess of the benchmark's return (also known as tracking error) given a target expected performance relative to the benchmark. This problem is well-known and it has been studied in detail by Roll (1992). For each μ, the optimal portfolio can be easily shown to be equal to the sum of a particular excess return plus the benchmark itself. Furthermore, this excess return is identical, up to a constant, across any μ and any $R^b \in \underline{R}$. Also, the optimal return in this case is not mean-variance efficient, unless the benchmark is.

Obviously, the problem above can be posed in terms of quadratic utility as follows

$$\max_{R \in \underline{R}} -\frac{1}{2} E\left[\left(\delta - R + R^b\right)^2\right] \tag{4.26}$$

and in a similar way as in the previous section, for each target mean in (4.25) one can find a value of δ so that problems (4.25) and (4.26) are equivalent.

Furthermore, given the microeconomic flaws of the quadratic utility/mean-variance framework, it seems natural to use a HARA utility class which satisfies standard assumptions and thus, pose the tracking error problem along the same dimensions of reward and risk that we have already introduced. Hence, consider the extension of (4.26) resulting from replacing quadratic utility with any other HARA utility class. For given δ and γ, denote by $R^b_{\gamma,\delta}$ the optimal return of this general problem and let $\underline{R}^b_\gamma \in \underline{R}$ be the set of these solutions for any $\delta \in \Re$.[7]

Proposition 4.9 *For a given $\gamma < -1$, assume that x_γ exists. In the presence of a risk-free asset, $R \in \underline{R}^b_\gamma$ if and only if there is a constant $w < 0$ such that*

$$R = R^b + wR^e_\gamma.$$

Furthermore,

$$R^b_{\gamma,\delta} = R^b + \delta R^e_\gamma, \qquad \forall \delta < 0. \tag{4.27}$$

Proof. The result follows by plugging (4.27) into the corresponding HARA problem and using (4.10).

□

Remark 4.10 *For $\gamma = 1$ (mean-variance), the result above holds for any $\delta \in \Re$ although the efficient solutions correspond to $\delta > 0$. In addition, it is clear from the above result that $R \in \underline{R}$ solves (4.26) for a given δ if and only if it solves (4.25) for $\mu = \delta E(R^e_1)$.*

The *benchmark* γ-optimal set, $\underline{R}^b_\gamma$, contains all those returns that maximize the CRRA-certainty equivalent of the tracking error in excess of the guaranteed value δ. Obviously, δ must be negative since absence of arbitrage prevents any excess return from being positive. Note that Roll's results carry on to the general case. Specifically, the second term of the decomposition of the optimal return in (4.27) is always the same for a given δ, regardless of the particular benchmark. Also, this excess return is identical, up to a constant, for any $\delta < 0$.

5 The equivalence theorems and the bounds on the SDF

Next, we explore the connections between the set $\underline{R}_\gamma$ and its corresponding *index* SDF, x_γ. Cochrane (2001, Chapter 6) develops a whole set of results labeled *equivalence theorems*. Among other things, he shows that any return in the mean-variance frontier gives an SDF, after an affine transformation; and that an expected return-beta representation can be obtained by using R_1 as a single factor. Their natural counterparts read as follows.

[7] Once again, we only treat the case $\gamma < -1$ even though the result applies to any γ after slight modifications.

Proposition 5.11 *For any $\gamma \in \underline{\gamma}$,*

a) *There exists $m \in \underline{M}_+$ such that*

$$m = (a + bR)^{\gamma} \tag{5.28}$$

for some $a, b \in \Re$ and $R \in \underline{R}$ if and only if $R \in \underline{R}_{\gamma}$.

b) *If x_{γ} exists, then*

$$E(R) = R^f + \beta_{R,R_{\gamma}} \left[E(R_{\gamma}) - R^f \right], \qquad \forall R \in \underline{R},$$

where $\beta_{R,R_{\gamma}} \equiv cov\left(R_{\gamma}^{\gamma}, R\right)/var\left(R_{\gamma}^{\gamma}\right)$.

Proof. a) Assume that there is m satisfying (5.28). Since $a + bR \in \underline{X}$, from Proposition 3.2 and (3.5), it follows that

$$a + bR = \frac{R_{\gamma}}{\left[E\left(R_{\gamma}^{\gamma}\right) R^f\right]^{1/\gamma}}. \tag{5.29}$$

By taking expectations on both sides, we get

$$a + bE(R) = \frac{E(R_{\gamma})}{\left[E\left(R_{\gamma}^{\gamma}\right) R^f\right]^{1/\gamma}}. \tag{5.30}$$

Also, by multiplying both sides of (5.29) by x_{γ}^{γ} and taking expectations, it follows that

$$a\frac{1}{R^f} + b = \frac{1}{\left[E\left(R_{\gamma}^{\gamma}\right) R^f\right]^{1/\gamma}}. \tag{5.31}$$

Now, (5.30) and (5.31) give a system of two equations whose solution for a and b is

$$\begin{aligned} a &= \frac{E(R) - E(R_{\gamma})}{E(R) - R^f} R^f \left[E\left(R_{\gamma}^{\gamma}\right) R^f\right]^{-1/\gamma} \\ b &= \frac{E(R_{\gamma}) - R^f}{E(R) - R^f} \left[E\left(R_{\gamma}^{\gamma}\right) R^f\right]^{-1/\gamma}. \end{aligned} \tag{5.32}$$

After plugging these expressions back into (5.29), we obtain that

$$R = R_{\gamma} - \frac{a}{b}\left(1 - \frac{R_{\gamma}}{R^f}\right) \tag{5.33}$$

From Proposition 4.6, it only remains to check that $-a/b < R^f$. Indeed, since $R_{\gamma} > 0$, it follows from (5.29) that

$$R > -\frac{a}{b}$$

which, in the absence of arbitrage, it can only hold if $-a/b < R^f$.

Conversely, assume that $R \in \underline{R}_\gamma$. From the first-order conditions of problem (2.1), it holds that

$$E\left[(R' - \delta)^\gamma R\right] = R^f E\left[(R' - \delta)^\gamma\right], \qquad \forall R \in \underline{R}$$

for some $\delta < R^f$ which together with the assumptions on the distribution of the basis payoffs gives that

$$(a + bR')^\gamma \in \underline{M}_+$$

for $a = -\delta b$ and $b = \left\{R^f E\left[(R' - \delta)^\gamma\right]\right\}^{-1/\gamma}$.

b) By definition x_γ satisfies

$$E\left(x_\gamma^\gamma R\right) = 1, \qquad \forall R \in \underline{R}$$

which gives

$$E(R) = R^f - R^f cov\left(x_\gamma^\gamma, R\right), \qquad \forall R \in \underline{R},$$

and from (3.5), the above can be written as

$$E(R) = R^f - \frac{R^f var\left(R_\gamma^\gamma\right)}{E\left(R_\gamma^\gamma\right) R^f} \beta_{R,R_\gamma}, \qquad \forall R \in \underline{R}.$$

Since this relationship also holds for R_γ and $\beta_{R_\gamma,R_\gamma} = 1$, it follows that

$$E(R_\gamma) - R^f = -\frac{R^f var\left(R_\gamma^\gamma\right)}{E\left(R_\gamma^\gamma\right) R^f}$$

which gives the desired result.

□

Hansen and Jagannathan (1991) derive a lower bound on the second moment of any $m \in \underline{M}$ based on the *index* SDF for mean-variance preferences, x_1. In particular they show that

$$E\left(m^2\right) \geq E\left(x_1^2\right) = \left[E\left(R_1^2\right)\right]^{-1}, \qquad \forall m \in \underline{M}.$$

This result has again a natural counterpart for any $\gamma \in \underline{\gamma}$. Note that R_γ satisfies

$$E\left(R_\gamma^\gamma R\right) = E\left(R_\gamma^\gamma\right) R^f \qquad \forall R \in \underline{R}. \tag{5.34}$$

Now, for any $m \in \underline{M}_+$, define

$$R_\gamma^m \equiv \frac{1}{E\left(m^{\frac{1}{\gamma}+1}\right)} m^{1/\gamma}$$

and let $\underline{R}_\gamma^m$ the augmented set that results from adding R_γ^m to $\underline{R}$. Note that since

$$E\left[\left(R_\gamma^m\right)^\gamma R\right] = E\left(m^{\frac{1}{\gamma}+1}\right)^{-\gamma} \qquad \forall R \in \underline{R}_\gamma^m,$$

we have that

$$E\left(m^{\frac{1}{\gamma}+1}\right)^{-\gamma}=E\left[\left(R_{\gamma}^{m}\right)^{\gamma+1}\right]=\min_{R\in\underline{R}_{\gamma}^{m}}E\left(R^{\gamma+1}\right)\leq\min_{R\in\underline{R}}E\left(R^{\gamma+1}\right)=E\left(R_{\gamma}^{\gamma+1}\right),$$

which gives,

$$E\left(m^{\frac{1}{\gamma}+1}\right)^{-\gamma}\leq E\left(R_{\gamma}^{\gamma+1}\right),$$

or equivalently,

$$E\left(m^{\frac{1}{\gamma}+1}\right)\leq\left[E\left(R_{\gamma}^{\gamma+1}\right)\right]^{-1/\gamma}=E\left(x_{\gamma}^{\gamma+1}\right). \tag{5.35}$$

The above arguments can be formalized as follows.

Proposition 5.12 *For a given $\gamma\in\underline{\gamma}$, assume that x_{γ} exists. Then*

$$E\left(m^{\frac{1}{\gamma}+1}\right)\leq E\left(x_{\gamma}^{\gamma+1}\right)=\left[E\left(R_{\gamma}^{\gamma+1}\right)\right]^{-1/\gamma},\qquad\forall m\in\underline{M}_{+}.$$

Note that our derivation clearly shows the relationship between the bound and the portfolio problem as opposed to the more ad-hoc arguments in Hansen and Jagannathan (1991), Bansal and Lehmann (1997), Snow (1991) and Stutzer (1995). Even though Cerny (2003) also uses the portfolio problem, his derivation is more complex and relies on the corresponding dual maximization.

No risk-free asset

Similarly, these constructions can be exploited to obtain a region of bounds by replicating the derivation in (5.34)-(5.35).

Proposition 5.13 *For given $\gamma\in\underline{\gamma}$ and $v\in(\underline{v},\overline{v})$, assume that x_{γ}^{v} exists. Then*

$$E\left(m^{\frac{1}{\gamma}+1}\right)\leq E\left[\left(x_{\gamma}^{v}\right)^{\gamma+1}\right]=\left\{E\left[\left(R_{\gamma}^{v}\right)^{\gamma+1}\right]\right\}^{-1/\gamma},\qquad\forall m\in\underline{M}_{+}^{v}. \tag{5.36}$$

Note that the inequality above can be written as

$$\left[E\left(m^{\frac{1}{\gamma}+1}\right)\right]^{-\left(\frac{1}{\gamma}+1\right)}\geq E\left[\left(R_{\gamma}^{v}\right)^{\gamma+1}\right]^{1/(\gamma+1)},\qquad\forall m\in\underline{M}_{+}^{v} \tag{5.37}$$

or equivalently,

$$1/\left\|m\right\|_{\frac{1}{\gamma}+1}\geq\left\|R_{\gamma}^{v}\right\|_{\gamma+1},\qquad\forall m\in\underline{M}_{+}^{v}.$$

Hence, this construction can be graphically derived as follows in Figure 4. For a given $m\in\underline{M}_{+}$, draw a tangency line to the γ frontier whose interception with the vertical axis is its implied risk-free rate. Then, the lower bound (5.37) is given by the intersection of this tangent with the horizontal axis. The duality of the portfolio problem and the bounds on the SDF is further illustrated in Figure (5). As it can be seen, the choice of an implied risk-free rate in the vertical axis, for example $1/v_1$, ties the mean of the SDF and it is mapped in the horizontal axis at the point where the set $\underline{R}_{\gamma}^{v}$ crosses it.

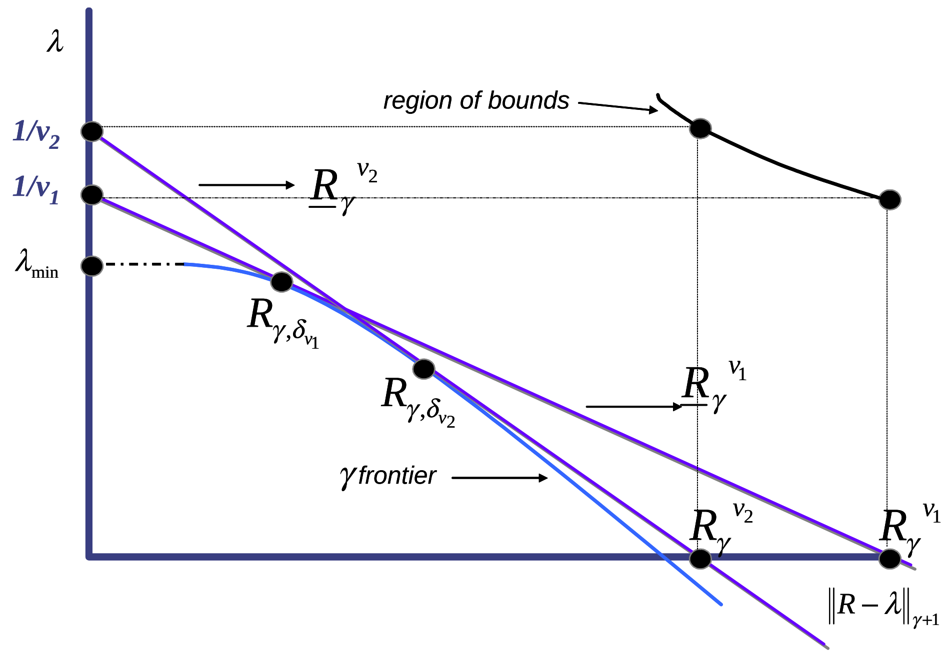

Figure 5: Optimal returns and bounds on the SDF

Finally, our extension of Cochrane's equivalence theorems reads as follows in the context of this subsection.

Proposition 5.14 *For any $\gamma \in \underline{\gamma}$,*

a) *There exists $v \in (\underline{v}, \overline{v})$ such that*

$$(a + bR)^{\gamma} \in \underline{M}_{+}^{v}$$

for some $a, b \in \Re$ and $R \in \underline{R}$ if and only if $R \in \underline{R}_{\gamma}$.

b) *If x_{γ}^{v} exists, then*

$$E\left(R\right) = \frac{1}{v} + \beta_{R,R_{\gamma}} \left[E\left(R_{\gamma}^{v}\right) - \frac{1}{v}\right], \qquad \forall R \in \underline{R},$$

where $\beta_{R,R_{\gamma}^{v}} \equiv cov\left[\left(R_{\gamma}^{v}\right)^{\gamma}, R\right] / var\left[\left(R_{\gamma}^{v}\right)^{\gamma}\right]$.

6 Conclusion

Practitioners have usually ignored the objections of economists to the use of mean-variance preferences. Although this may be partially explained by the difficulty that setting the value of the risk-aversion parameter involves, a contributing factor can also be found in the lack of intuitive relationships that financial academics have provided outside the mean-variance framework. This paper has tried to make an effort in this latter direction. In addition, we have emphasized some strong links that tie the static portfolio problem to well-known objects of asset pricing theory in the context of the whole family of HARA utility functions.

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