Recovering Risk-Neutral Densities of Spot and Option Markets under Stochastic Volatility and Price Jumps

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ABSTRACT

This paper examines the relative importance of stochastic volatility and price jumps in option pricing by exploiting the differences in option-based and index return-based risk-neutral densities. As evinced by the persistent smirked volatility smile, this study extracts the risk-neutral densities by exploiting the information embedded in the implied volatilities that are smoothed by a kernel regression. On the other hand, a canonical valuation approach is adopted to identify the risk-neutral density from the observed index returns. Statistical tests and implied risk aversion are constructed to investigate the differences between two risk-neutral densities. The 30-day S&P 500 options under stochastic volatility are found efficiently priced. By using a longer horizon of underlying returns, however, we are able to partly reconcile the differences between the index and option-implied risk-neutral densities after adding a jump component to the index dynamics. Option investors are found more risk averse than stock traders except for the 30-day jump-diffusion with stochastic volatility. The finding may help illustrate the puzzle that implicit volatilities are greater than subsequent realized volatilities.

Key Words**:** stochastic volatility, price jumps, risk-neutral distributions, canonical valuation, risk aversion

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I. Introduction

It has been widely documented that stock returns exhibit both stochastic volatility and jumps. The importance of such risk factors arises not only from time-series studies of stock prices, but also from cross-sectional studies of stock options (Bakshi et al., 1997; Bates, 2000). An arbitrage-free option pricing model can be reduced to the specification of a density function (the state-price density; SPD) assigning probabilities to the various possible values of the underlying asset price at the option's expiration (Ross, 1976; Banz and Miller, 1978; Breeden and Litzenberger, 1978; and Harrison and Kreps, 1979).[1] This brings us to ask an important question of whether option markets after introducing stochastic volatility and jumps correctly prices the probabilities of the dynamics of underlying returns. It is certainly tempting to answer this question simply by comparing features of the SPD implied by S&P 500 option prices to features of the observable time series of the underlying asset price. A number of econometric methods are now available to infer SPDs from option prices, either by relaxing the Black-Scholes (1973) and Merton (1973) log-normal assumption in specific directions (for example, Cox and Ross, 1976; Merton, 1976; Jarrow and Rudd, 1982; Bates, 1991; Goldberger, 1991; Madan and Milne, 1994; Melick and Thomas, 1997; and Bakshi et al., 1997) or by explicitly incorporating the deviations from the Black-Scholes model when estimating the option-implied SPD and pricing other derivative securities (for example, Shimko, 1993; Derman and Kani, 1994; Rubinstein, 1994; Stutzer, 1996; Jackwerth and Rubinstein, 1996; Campa et al., 1998; Aït-Sahalia and Lo, 1998; and Dumas et al., 1998). This paper uses a semiparametric option pricing theory that is more in line with the combination of a nonparametric implied volatility surface with the parametric option pricing formula. Observed implied volatilities are obtained based on the three parametric option pricing models of Black-Scholes' (1973) constant volatility (BS), Heston's (1993) stochastic volatility (SV), and Bakshi et al.'s (1997) jump-diffusion with stochastic volatility (SVJ). A nonparametric kernel regression technique (Aït-Sahalia and Lo, 1998) is adopted to reconcile both option pricing and observed implied volatility surfaces, and to obtain the corresponding SPD from option prices.

Previous studies in the literature have always compared the observed option data to the observed underlying returns data, i.e., a risk-neutral density to an actual density (Derman et al., 1997; Rosenberg and Engle, 1997; Aït-Sahalia and Lo, 2000;

[1] This approach relies on the assumption of market completeness to tie down the option prices to a SPD.

Jackwerth, 2000). However, very little is known about aggregate investors' preferences, and there is wide disagreement in the literature regarding what constitutes a reasonable value of the coefficient of relative risk aversion (RRA) (Mehra and Prescott, 1985; Cochrane and Hausen, 1992). Therefore, there is no reason a priori to compare the dynamics of the underlying asset that are implied by the option data to the dynamics implied by the actual time series, unless for some reason one holds a strong prior view on preferences. Other related studies are Chernov and Ghysels (2000) and Pan (2002) who provide an illustration of how affine models can exploit multiple information sources in the estimation method in an internally consistent fashion. However, econometric techniques that pool cross-sectional option prices and time-series information should be treated with caution. "Measurement error" can obscure a fundamental incompatibility between the time series properties and cross-sectional derivatives prices. Following the work of Aït-Sahalia et al. (2001), this paper instead compares two risk-neutral distributions implied by options prices and time-series underlying returns. The chief difference is that our paper identifies the risk-neutral density from the observed unadjusted index returns based on a canonical approach rather than an empirical Girsanov's change of measure adopted by Aït-Sahalia et al. (2001). In the existing literature, the difficulty at this point is that we do not observe the data that would be needed to infer directly the time-series SPD, since we only observe the actual realized values of the S&P 500, not its risk-neutral values. The main novelty in this paper is to show that it is nevertheless possible to use the observed asset prices to infer 'directly' the time-series SPD that should equal the option-implied cross-sectional SPD. This paper relies on a canonical valuation approach using the relative entropy principle to derive the SPD from underlying returns. It is well known in information theory that the relative entropy principle can be justified automatically and is consistent with Bayesian method of statistical inference. For option pricing, the notion of relative entropy is previously utilized in Buchen and Kelly (1996), Stutzer (1996) and Duan (2002).

Our results, using options traded on the S&P 500 index between January 1993 and December 1995, show that the majority of option-implied SPDs exhibits systematic excessive skewness and kurtosis with respect to the index-implied SPDs and reject the null hypothesis that the S&P 500 options can be efficiently priced within the limited context of a specification under either stochastic volatility or stochastic volatility with price jumps. However, for short-term investment horizon the stochastic volatility provides superior performance, while adding a jump component to the index dynamics partly reconciles the differences between the index and option-implied risk-neutral densities for long days to expiration. We propose a peso-problem interpretation of this evidence: cross-sectional option prices capture a premium as compensation for the risk of a market crash, but actual realizations of that jump are too infrequent to be consistently observed and reflected in estimates drawn from the time series of asset returns. The prediction of this model with jumps is that time series SPD estimates should be insufficiently skewed and leptokurtic relative to their cross-sectional counterparts, which is exactly what we find empirically. The finding is not just restating that the option data exhibit an implied volatility smile, which is well-known by now. Instead, this demonstrates that the implied volatility smile is maturity-compatible with the dynamics of the S&P 500 returns as captured by a stochastic volatility dynamic with/without jumps. We then design implied risk aversion to exploit the SPD differences as well as statistical tests to show how they capture discrepancy across maturities, due to the irrationality of option prices (at least during the period under consideration).

The plan of the paper is as follows. Data and their screening procedures are discussed first. Following that, the models for obtaining the risk-neutral densities either from the time-series index returns or from the cross-sectional option prices are presented. In the following section, empirical results for distribution similarity comparison and implied risk aversion are analyzed. The final section summarizes the main findings.

II. DATA

The raw data on the S&P 500 index (SPX) options and contemporaneous index levels come from the quotation price history provided by the Chicago Board Options Exchange. The SPX option contract is the world's leading traded European-style index option. The period for this paper covers three years, 1993−1995. Trading volume for SPX options is significantly higher during the 1990s than during the 1980s. This sample period should therefore remove any relative disadvantages of using SPX options. Option price calculation requires the risk-free interest rate and the expected dividend stream for the S&P 500 index. The Treasury bill rate, taken from Datastream, provides the risk-free interest rate. Following standard practice, the actual dividends on the index, from S&P Corporation, is used for the expected dividend stream. For options expiring in a month or less, the delay between announcement and payment of dividends likely eliminates the difference between expected and actual dividends. The discrepancy may remain for longer maturities.

This study uses the last quotation before 15.00 for each option contract. Quotations are taken from Wednesday to minimize the effect of holidays, and from the third Wednesday of each month to infer risk-neutral distributions for 30, 100 and 300 days to expiry. SPX options actually expire on the Saturday immediately following the third Friday of the expiration month before or on 24 August 1992. After 24 August 1992, SPX contracts cease the trading in the morning of the third Friday of maturity month. Therefore, the precise horizon from the third Wednesday of each month to the expiration date of the following month is around thirty days. Hence, there are in total 36 "one-month" subperiods from January 1993 to December 1995, resulting 324(=36×3×3) risk-neutral distributions of 30, 100 and 300 days to expiration across three models. Although the choice of estimation horizon involves an element of arbitrariness, one month is a standard and natural choice of investment horizon. Besides, one-month horizon is chosen to eliminate overlapping observations by maintaining a constant estimation window. An occasional, slight overlap arises due to the nature of the options expiration calendar, but this is of the order of one or two days and should not significantly affect the results. All options prices are the midpoints of the bid-ask quotes. Finally the integrity of the data is carefully checked to discard clearly invalid observations and observations where options quotations violate standard upper and lower boundary arbitrage conditions and convexity relations. When estimating the risk-neutral distributions, to avoid problems of density instability caused by market microstructure effects, options priced at or below 25¢ and options with five or less days to maturity are excluded.[2] Table 1 presents characteristics of the whole data sample across maturity and moneyness, where moneyenss is defined as $(S_t - D_t)/X$ and D_t is the present value of all cash dividends paid over the life of the option. Average option quotes range from 0.65 dollar for deep OTM short-term calls to 74.77 dollar for deep ITM long-term calls. Bid-ask spreads range from 0.16 dollar for short-term deep OTM calls to 1.05 dollar for long-term deep ITM calls. The time-series data from Datastream consist of the S&P 500 index and one-month, three-month and one-year Treasury bill rates, as the risk-free rates for 30, 100 and 300 days to expiration, on a daily basis from January 1990 to December 1995. The risk-neutral distributions are the "canonical" densities of the 30-, 100- and 300-day overlapping returns over the 3 years prior to January 1993–December 1995, respectively.

[2] These exclusion criteria should reduce problems caused by price discreteness and illiquidity, though they are less severe than similar criteria imposed in similar studies.

Table 1 Sample Properties of S&P 500 Index Options

This study includes Wednesday options between 14:45 and 15:00 over the period, January 1993–December 1995. The option price is calculated as the midpoint of bid and ask quotes. *Moneyness* is defined as $(S_t – D_t)/X$ and D_t is the present value of all cash dividends paid over the life of the option.

		Days to Expiration								
		All Options			***Calls***			***Puts***		
Moneyness		***<60***	***60–180***	***>180***	***<60***	***60–180***	***>180***	***<60***	***60–180***	***>180***
< 0.94	Option price	46.07	35.07	27.36	0.65	2.38	7.42	47.92	44.64	42.93
	Bid-ask spread	0.96	0.70	0.75	0.16	0.25	0.46	0.99	0.83	0.97
	Observations	280	826	561	11	187	246	269	639	315
0.94–0.97	Option price	12.62	14.13	21.25	1.21	5.04	19.47	20.08	21.21	22.95
	Bid-ask spread	0.54	0.56	0.78	0.17	0.36	0.73	0.78	0.72	0.83
	Observations	698	811	297	276	355	145	422	456	152
0.97–1.00	Option price	6.56	12.37	23.81	4.05	12.29	29.34	9.06	12.45	17.79
	Bid-ask spread	0.35	0.55	0.80	0.26	0.56	0.89	0.44	0.54	0.71
	Observations	1394	930	393	695	448	205	699	482	188
1.00–1.03	Option price	7.83	15.91	26.41	12.48	22.33	38.37	3.41	7.81	13.22
	Bid-ask spread	0.38	0.61	0.79	0.55	0.78	0.95	0.23	0.40	0.62
	Observations	1425	975	450	694	544	236	731	431	214
1.03–1.06	Option price	14.40	21.65	29.87	24.59	32.34	48.71	1.60	4.80	9.49
	Bid-ask spread	0.56	0.65	0.75	0.86	0.86	0.99	0.17	0.32	0.49
	Observations	1150	868	329	640	531	171	510	337	158
> 1.06	Option price	33.62	34.98	45.91	46.98	52.67	74.77	0.90	2.67	6.13
	Bid-ask spread	0.74	0.69	0.79	0.97	0.92	1.05	0.16	0.27	0.44
	Observations	1277	1026	352	907	663	204	370	363	148

III. SPD Inferred from the Time Series of Returns

Our time-series estimator of the SPD g^* is based on inferring the density from the specific evolution of the asset price, using the canonical valuation approach. Here, we assume the dynamic feature contains stochastic volatility with/without jumps and occurs only in the one-period conditional mean μ_t and variance σ_t of continuously compounded returns { R_t , t=1, 2,…,n}. The standardized return { $x_t = (R_t - \mu_t)/\sigma_t$; t =1, 2,…,n} forms an *i.i.d.* sequence. Let $G(\cdot)$ be the empirical density function of the standardized returns, i.e., $x_t \sim G(0,1)$. Given the normalized return, $z_t = \Phi^{-1}[G(x_t)]$, with $\Phi^{-1}(\cdot)$ being the cumulative density function inversion of a standard normal *r.v.*, the z_t becomes a normal-distributed *r.v.*, i.e., $z_t \sim \mathrm{N}(0,1)$. A key feature of the risk-neutral density is its expected return equal to the risk-free rate. Other than the expected return condition, there is no a prior reason for the risk-neutral distribution to deviate from the objective distribution. Thus, one can call upon the information theory to find the risk-neutral density function that minimizes the relative entropy subject to its expected value condition. Using the relative entropy principle, the risk-neutral density $g^*(z_t)$ for the normalized return z_t is the solution to the Eq.(1). For some value c_t,

$$\begin{aligned} &\min_{g^*(z_t)} \int_{-\infty}^{\infty} g^*(z_t)\ \ln\left(\frac{g^*(z_t)}{\phi(z_t)}\right)\ dz_t \\ &\text{subject to} \\ &\int_{-\infty}^{\infty} g^*(z_t)\ dz_t = 1 \\ &\int_{-\infty}^{\infty} z_t\ g^*(z_t)\ dz_t = c_t \end{aligned} \tag{1}$$

where $\phi(z_t)$ is the standard normal probability density function. It is well known in the information theory that Eq.(1) has the solution as shown in Eq (2).

$$g^*(z_t;\lambda_t) = \frac{e^{\lambda_t z_t}\ \phi(z_t)}{\int_{-\infty}^{\infty} e^{\lambda_t z_t}\ \phi(z_t)\ dz_t} = \phi(z_t - \lambda_t) \tag{2}$$

where λ_t corresponds to a given value of c_t. Note that the value of λ_t is determined by the fact that the risk-neutral density must give rise to an expected asset return equal to a continuously compounded risk-free rate r minus the dividend yield d (continuously compounded). That is, $E^*_{t-\Delta}(S_t) = S_{t-\Delta}e^{(r-d)\Delta}$ where d is the annualised continuously compounded dividend rate, defined by $d\Delta = \ln(1 - Div_t/S_{t-\Delta})$ with Div_t the cumulative value of all cash dividends paid over the period between t–Δ and t. Consequently, λ_t^* solves the one-period expected gross return of the index in Eq.(3),

$$\int_{-\infty}^{\infty} e^{\mu_t+\sigma_t G^{-1}[\Phi(z_t)]}\,\phi\left(z_t-\lambda_t^*\right)dz_t = e^{(r-d)\Delta} \tag{3}$$

Using the transformation, the sequence of continuously compounded index returns is expressed as $\left\{\ R_t;\ t=1,2,...,n\ \right\}=\left\{\ \mu_t+\sigma_t G^{-1}[\Phi(z_t)]\ ;\ t=1,2,...,n\ \right\}$ where z_t is the normalized return and has the objective density function of $\phi(z_t)$ and the risk-neutral density function of $\phi\left(z_t-\lambda_t^*\right)$. In other words, under the objective probability measure $z_t=\Phi^{-1}\left[G\left((R_t-\mu_t)/\sigma_t\right)\right]\sim \mathrm{N}(0,1)$, whereby under the risk-neutral probability measure $z_t^*=\Phi^{*-1}\left[G\left((R_t-\mu_t)/\sigma_t\right)\right]\sim \mathrm{N}\left(\lambda_t^*,1\right)$. Therefore, the risk-neutral asset dynamics become Eq.(4),

$$S_t^* = S_{t-\Delta} e^{\mu_t+\sigma_t G^{-1}\left[\Phi^*\left(z_t^*\right)\right]} \tag{4}$$

where $z_t^*=\Phi^{*-1}\left[G\left((R_t-\mu_t)/\sigma_t\right)\right]\sim \mathrm{N}\left(\lambda_t^*,1\right)$. The path of risk-neutral index price, S_t^*, is produced and its risk-neutral density $g^*(S_\tau)$ is obtained via a kernel density function as shown in Eq.(5),

$$g^*(S_\tau)=\frac{1}{nh_S}\sum_{t=1}^{n} K_S\left(\frac{S_i-S_t^*}{h_S}\right) \tag{5}$$

where τ corresponds to the 30-, 100- or 300-day period return; S_i is the interpolated index price; n is the sample size of observed data; h_S is the bandwidth of index price; and $K_S(\cdot)$ is a second-order Gaussian kernel function, defined by $K_{(2)}(u)=\exp\left(-u^2/2\right)/\sqrt{2\pi}$.

While implementing the canonical valuation method, we consider a simple procedure of identifying the empirical distribution $G(\cdot)$ from a sample of one-period continuously compounded asset returns $\{R_t,\ t=1,\ 2,\ldots,n\}$. We first estimate the sample mean μ_t and standard deviation σ_t. The empirical distribution function for a sample, $R=\{(R_t-\mu_t)/\sigma_t;\ t=1,2,...,n\}$, is formally defined as $\hat{G}(x;R)\equiv\frac{1}{n}\sum_{t=1}^{n} 1_{\{(R_t-\mu_t)/\sigma_t\le x\}}$ where $1_{\{\cdot\}}$ is an indicator function giving a value of 1 if the condition is true and 0 otherwise. Note that the canonical valuation method is silent on the period-by-period risk-neutral price dynamic and may limit the ability to reflect the market condition. In a spirit similar to the canonical valuation method of Stutzer (1996), this study formalizes the risk-neutralization process as a GARCH type with/without jumps so that one can infer directly from the price dynamic of the underlying asset to establish the risk-neutral pricing dynamic. Different from Duan's (2002) work, this study proposes the discrete-time approximations, as the dynamics of index returns, of Heston's (1993) stochastic volatility and Bakshi et al.'s (1997) diffusion-jump with stochastic volatility. The resultant risk-neutral densities, inferred from the index returns, are thus directly comparable to corresponding counterparts implicit in option prices incorporating either volatility or jump risk. For comparison purposes, an underlying dynamic of constant volatility is also considered.

3.1 Constant Volatility

The BS option pricing model assumes that the asset price follows a geometric Brownian motion given below,

$$d\ln S_t = \left(\mu_S - \frac{1}{2}\sigma^2\right) dt + \sigma\, dw_{S,t} \tag{6}$$

where S_t is time-*t* index price; μ_S is the annualized expected percentage rate of return due to price risk; σ^2 is the annualized variance of index returns; and $w_{S,t}$ is the Wiener process. Following Nelson (1990), a discrete-time process becomes,

$$R_t = \mu_S\Delta - \frac{1}{2}\sigma^2\Delta + \sigma\sqrt{\Delta}\; x_{S,t} \tag{7}$$

where $x_{S,t} \sim \mathrm{N}(0,1)$ and Δ corresponds to 30, 100 or 300 days on an annualized basis. R_t is the one-period continuously compounded rate of return, defined by $R_t = (\ln S_t - \ln S_{t-\Delta})$, which follows a normal distribution with mean, $\mu_t = \mu_S\Delta - \sigma^2\Delta/2$, and standard deviation, $\sigma_t = \sigma\sqrt{\Delta}$. The likelihood function for an individual return conditional on the information set $I_{t-\Delta}$ is given by,

$$f(R_t | I_{t-\Delta}) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\Delta}} e^{-x_{S,t}^2/2} \tag{8}$$

The parameter space for this model is $\Xi = \{\mu_S, \sigma\}$. By maximizing the log-likelihood function of the returns $\{R_t, t = 1, 2, ..., n\}$, the estimated parameter vector, $\hat{\Phi} = \{\hat{\mu}_S, \hat{\sigma}\}$, is obtained. We then use the standardized returns, calculated as $\hat{x}_t = (R_t - \hat{\mu}_t)/\hat{\sigma}_t$ with $\hat{\mu}_t = \hat{\mu}_S\Delta - \hat{\sigma}^2\Delta/2$ and $\hat{\sigma}_t = \hat{\sigma}\sqrt{\Delta}$, to estimate the preference parameter $\lambda^*_{\text{smooth}}$ via Eq.(3).

Note that under the geometric Brownian motion assumption the one-period continuously compounded return R_t (or equivalently, its standardized return x_t) has a normal distribution, implying that $G(x_t) = \Phi(x_t)$ and $G^{-1}[\Phi(z_t)] = x_t$ where $z_t \sim \mathrm{N}(0,1)$. Thus, $\lambda^*_{\text{smooth}}$ satisfying Eq.(3) is $[(r-d)\Delta - \mu_S\Delta]/\sigma\sqrt{\Delta}$. Consequently, the risk-neutral index price dynamic becomes,

$$\begin{aligned} S_t^* &= S_{t-\Delta} e^{\hat{\mu}_t + \hat{\sigma}_t G^{-1}\left[\Phi^*(z_t^*)\,;\,\hat{\Xi}\right]} \\ &= S_{t-\Delta} e^{\hat{\mu}_t + \hat{\sigma}_t\, \hat{x}_t^*} = S_{t-\Delta} e^{\hat{\mu}_t + \hat{\sigma}_t\, (\hat{x}_t + \lambda^*_{\text{smooth}})} = S_{t-\Delta} e^{(r-d)\Delta - \hat{\sigma}_t^2/2 + \hat{\sigma}_t\, \hat{x}_t} \end{aligned} \tag{9}$$

where $z_t^* = \Phi^{*-1}[\Phi(x_t)] = x_t^* = x_t + \lambda_{\text{smooth}}^* \sim \mathrm{N}(\lambda_{\text{smooth}}^*, 1)$ and $x_t \sim \mathrm{N}(0,1)$; $R_t^* = R_t + \lambda_{\text{smooth}}^* \sigma\sqrt{\Delta}$. Its risk-neutral density $g^*(S_\tau)$ can be obtained via a kernel density function as shown in Eq.(5).

3.2 Conditional Heteroskedasticity (GARCH-SV)

Heston (1993) assumes that asset return variance follows a mean-reverting square root process shown as follows,

$$
\begin{aligned}
&d\ln S_t = \left(\mu_S - \frac{1}{2}v_t\right) dt + \sqrt{v_t}\ dw_{S,t} \\
&dv_t = \kappa_v(\theta_v - v_t)\ dt + \sigma_v\sqrt{v_t}\ dw_{v,t} \\
&\mathrm{corr}(dw_{S,t}, dw_{v,t}) = \rho\ dt
\end{aligned}
\tag{10}
$$

where $w_{S,t}$ and $w_{v,t}$ are standard Wiener processes governing the dynamics of index returns and its instantaneous variance v_t. κ_v presents the mean-reverting speed of the variance process towards its long-term mean θ_v with variation coefficient σ_v. By the Euler method the stochastic difference process of Eq.(10) becomes,

$$
\begin{aligned}
&R_t = \mu_S\Delta - \frac{1}{2}v_t\Delta + \sqrt{v_t\Delta}\ x_{S,t} \\
&v_{t+\Delta}\Delta = v_t\Delta + \kappa_v\Delta\cdot\theta_v\Delta - \kappa_v\Delta\cdot v_t\Delta + \sigma_v\Delta\cdot\sqrt{v_t\Delta}\ x_{v,t} \\
&\mathrm{corr}(x_{S,t}, x_{v,t}) = \rho\ ,\ x_{S,t} \sim \mathrm{N}(0,1)\ ,\ x_{v,t} \sim \mathrm{N}(0,1)
\end{aligned}
\tag{11}
$$

Following Nelson (1990, 1991), Taylor and Xu (1993), Corradi (2000) and Heston and Nandi (2000), an appropriate GARCH-type model for continuously compounded returns R_t, which approximates Eq.(8) as time interval approaches zero, is proposed here. Define $h_t \equiv v_t\Delta$, $B \equiv 1 - \kappa_v\Delta$, $\mu_h \equiv \theta_v\Delta$, $C \equiv \rho\sigma_v\Delta$, $D \equiv \sigma_v\Delta\sqrt{(1-\rho^2)/\mathrm{var}|x_{S,t}|}$, and $x_{v,t} \equiv \rho\, x_{S,t} + \sqrt{(1-\rho^2)/\mathrm{var}|x_{S,t}|}\left(|x_{S,t}| - E|x_{S,t}|\right)$ in which the proposed $x_{v,t}$ has mean 0, variance 1 and is correlated with $x_{S,t}$ by ρ. The discrete-time version of Heston's stochastic volatility (henceforth, GARCH-SV) can be represented by,

$$
\begin{aligned}
&R_t = \mu_S\Delta - \frac{1}{2}h_t + \sqrt{h_t}\ x_{S,t} \\
&h_{t+\Delta} = \mu_h + B\ (h_t - \mu_h) + C\sqrt{h_t}\ x_{S,t} + D\sqrt{h_t}\ \left[\ |x_{S,t}| - E|x_{S,t}|\ \right] \\
&x_{S,t} \sim \mathrm{N}(0,1)
\end{aligned}
\tag{12}
$$

where $\mu_S > 0, B < 1, 0 < \mu_h < \Delta, \mathrm{sign}(C) = \mathrm{sign}(\rho), D > 0, 0 < h_t < \Delta$. The continuously compounded rate of returns R_t has mean $\mu_t = \mu_S \Delta - h_t/2$ and standard deviation $\sigma_t = \sqrt{h_t}$. The likelihood function for an individual return conditional on the information set $I_{t-\Delta}$ is shown as follows,

$$f(R_t | I_{t-\Delta}) = \frac{1}{\sqrt{h_t}} f(x_{S,t} | I_{t-\Delta}) \tag{13}$$

where $f(x_{S,t})$ is the probability density function of a standard normal-distributed random variable $x_{S,t}$ given by,

$$\begin{aligned} & f(x_{S,t} | I_{t-\Delta}) = \frac{1}{\sqrt{2\pi}} e^{-x_{S,t}^2/2} \\ & \text{with } x_{S,t} = \frac{R_t - \mu_S \Delta + h_t/2}{\sqrt{h_t}} \\ & \qquad E|x_{S,t}| = \sqrt{2/\pi}, \quad \mathrm{var}|x_{S,t}| = 1 - 2/\pi \end{aligned} \tag{14}$$

The parameter vector of interest, $\Xi = \{\mu_S, \mu_h, B, C, D, \delta_0\}$ where δ_0 is the parameter determining the initial value of variance, is estimated by maximizing the log-likelihood function of the daily returns $\{R_t, t = 1, 2, ..., n\}$.

Under these settings, $\lambda^*_{\text{smooth}}$ satisfying Eq.(3) is solved via a numerical procedure. Consequently, the risk-neutral index price dynamic becomes,

$$S_t^* = S_{t-\Delta} e^{\mu_t^* + \sqrt{h_t^*} G^{-1}\left[\Phi^*(z_t^*) ; \hat{\Xi}\right]} \tag{15}$$

$$h_{t+\Delta}^* = \mu_h + B\,(h_t^* - \mu_h) + C\sqrt{h_t^*}\; x_{S,t}^* + D\sqrt{h_t^*}\,\Big[\,|x_{S,t}^*| - E|x_{S,t}^*|\,\Big] \tag{16}$$

where $z_t^* \sim \mathrm{N}(\lambda^*_{\text{smooth}}, 1)$; $\mu_t^* = \mu_S \Delta - 0.5 h_t^*$; $x_{S,t}^* = x_{S,t} + \lambda^*_{\text{smooth}}$; $E|x_{S,t}^*| = E|x_{S,t}| + \lambda^*_{\text{smooth}}$; and $R_t^* = R_t - 0.5\,(h_t^* - h_t) + (\sqrt{h_t^*} - \sqrt{h_t})\, x_{S,t} + \lambda^*_{\text{smooth}} \sqrt{h_t^*}$. Its risk-neutral density $g^*(S_\tau)$ can be obtained via a kernel density function as shown in Eq.(5).

3.3 Conditional Heteroskedasticity and Jumps (GARCH-SVJ)

Given that the pure-diffusion model of Heston (1993) cannot produce enough excess skewness and kurtosis to reconcile observed tail-fatness of the stock return distribution (Andersen et al., 1998), nor can it easily match the "smirkiness" exhibited in the cross-sectional options data (Bakshi et al., 1997; Bates, 2003), the extension to include jumps is well motivated. Following the work of Bates (1996), Bakshi et al. (1997) and Pan (2002), we assume that the logarithmic index price follows a diffusion-jump with stochastic volatility process shown as follows,

$$
\begin{aligned}
& d\ln S_t = \left(\mu_S - \frac{1}{2}v_t\right)\, dt + \sqrt{v_t}\,dw_{S,t} + \ln(1+J_t)\, dq_t \\
& dv_t = \kappa_v(\theta_v - v_t)\, dt + \sigma_v\sqrt{v_t}\, dw_{v,t} \\
& \ln(1+J_t) \sim \mathrm{N}\left(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2\right) \\
& dq_t \sim \mathrm{Poisson}(\lambda_J dt) \\
& \mathrm{corr}(dw_{S,t}, dw_{v,t}) = \rho\, dt
\end{aligned}
\tag{17}
$$

where J_t is the percentage jump size of the index price following a lognormal with mean $\ln(1+\mu_J) - \sigma_J^2/2$ and variance σ_J^2. q_t is the jump frequency of the index price driven by a Poisson process with intensity λ_J. The Poisson process is assumed to be independent of two Wiener processes $w_{S,t}$ and $w_{v,t}$. The terms of μ_S and $\mu_J\lambda_J$ correspond to the expected percentage rates of return caused by the diffusion and jump factors, respectively, of the index price. As inspired by the literature, one of its discrete-time processes (henceforth, GARCH-SV-Jump) can be represented by,

$$
\begin{aligned}
& R_t = \mu_S\Delta - \frac{1}{2}h_t + \sqrt{h_t}\,x_{S,t} + \sum_{j=0}^{N_t}\ln Y_{j,t} \\
& h_{t+\Delta} = \mu_h + B(h_t - \mu_h) + C\sqrt{h_t}\; x_{S,t} + D\sqrt{h_t}\;\left[\,|x_{S,t}| - E|x_{S,t}|\,\right] \\
& \ln Y_{j,t} \sim \mathrm{N}\left(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2\right) \\
& N_t \sim \mathrm{Poisson}(\lambda_J\Delta)\;,\; x_{S,t} \sim \mathrm{N}(0,1)
\end{aligned}
\tag{18}
$$

where $B < 1, 0 < \mu_h < \Delta, \mathrm{sign}(C) = \mathrm{sign}(\rho), D > 0, 0 < h_t < \Delta, \lambda_J > 0, \sigma_J > 0$. $\ln Y_{j,t}$ is the jump size of the logarithmic return given a Poisson-distributed j^{th} jump occurring at time *t*. The logarithmic return therefore follows a mixture distribution of normal and Poisson densities with mean $\mu_t = \mu_S\Delta - h_t/2 + [\ln(1+\mu_J) - \sigma_J^2/2]\lambda_J\Delta$ and variance $\sigma_t^2 = h_t + \left\{\,[\ln(1+\mu_J) - \sigma_J^2/2]^2 + \sigma_J^2\,\right\}\lambda_J\Delta$. The likelihood function for an individual return conditional on the information set $I_{t-\Delta}$ is shown as follows,

$$
f(R_t|I_{t-\Delta}) = \sum_{j=0}^{\infty}\frac{e^{-\lambda_J\Delta}(\lambda_J\Delta)^j}{j\,!}\frac{1}{\sqrt{h_t + j\sigma_J^2}}\, f\left(x_{\mathrm{SVJ},\,t}|N_t = j\right)
\tag{19}
$$

where $f\left(x_{\mathrm{SVJ},t}|N_t = j\right)$ is the probability density function of standard normal-distributed random variable $x_{\mathrm{SVJ},t}$ with mean 0 and variance 1 conditional on the occurring of *j* jumps, displayed as follows,

$$f\left(x_{\text{SVJ},t}\middle|N_t=j\right)=\frac{1}{\sqrt{2\pi}}e^{-x_{\text{SVJ},t}^2/2}$$
$$\text{with } x_{\text{SVJ},t}=\frac{R_t-\mu_S\Delta+h_t/2-j\left[\ln(1+\mu_J)-\sigma_J^2/2\right]}{\sqrt{h_t+j\sigma_J^2}} \tag{20}$$

The parameter space for GARCH-SV-Jump is $\Xi=\{\mu_S,\mu_h,B,C,D,\delta_0,\lambda_J,\mu_J,\sigma_J\}$ with δ_0 determining the initial value of variance, which can be estimated via a maximum likelihood method.

Given the parameter Ξ, the cumulative density function of the standardized return $x_t=(R_t-\mu_t)/\sigma_t$ is computed as

$$\begin{aligned}F\left(x_t\middle|I_{t-\Delta}\right)&=\sum_{j=0}^{\infty}\frac{e^{-\lambda_J\Delta}(\lambda_J\Delta)^j}{j\,!}\Phi\left(x_{\text{SVJ},t}^{\text{standardized}}\middle|N_t=j\right)\\&=\Phi\left(x_{\text{smooth},t}\right)+\sum_{j=0}^{\infty}\frac{e^{-\lambda_J\Delta}(\lambda_J\Delta)^j}{j\,!}\Phi\left(x_{\text{jump},t}\middle|N_t=j\right)\end{aligned} \tag{21}$$

where $\Phi\left(x_{\text{SVJ},t}^{\text{standardized}}\middle|N_t=j\right)$ is the cumulative density function of normal-distributed $x_{\text{SVJ},t}^{\text{standardized}}$ with mean $(j-\lambda_J\Delta)\;\theta_J/\sigma_t$ and variance $\left(h_t+j\sigma_J^2\right)/\sigma_t^2$, given j jumps occurring. Alternatively, this density function can be decomposed into two components: $\Phi\left(x_{\text{smooth},t}\right)$ is the cumulative density function of normal-distributed $x_{\text{smooth},t}$ with mean 0 and variance h_t/σ_t^2 representing the smooth innovations of the standardized returns, while $\Phi\left(x_{\text{jump},t}\right)$ is the cumulative density function of normal-distributed $x_{\text{jump},t}$ with mean $(j-\lambda_J\Delta)\left(\ln(1+\mu_J)-\sigma_J^2/2_J^2\right)/\sigma_t$ and variance $j\sigma_J^2/\sigma_t^2$ given the occurring of j jumps representing the jump innovations of the standardized returns. Define $z_t=\Phi^{-1}\left[F\left(x_t\middle|I_{t-\Delta}\right)\right]$ given as follows.

$$\begin{aligned}z_t&=\Phi^{-1}\left[F\left(x_t\middle|I_{t-\Delta}\right)\right]\\&=x_{\text{smooth},t}+\sum_{j=0}^{\infty}\frac{e^{-\lambda_J\Delta}(\lambda_J\Delta)^j}{j\,!}x_{\text{jump},t}\\&=\frac{\sqrt{h_t}}{\sigma_t}x_{S,t}+\sum_{j=0}^{\infty}\frac{e^{-\lambda_J\Delta}(\lambda_J\Delta)^j}{j\,!}x_{\text{jump},t}\end{aligned} \tag{22}$$

The new *r.v.*, z_t, is a compounded-jump distributed random variable with zero mean and unit variance having a probability density function, $g(z_t)$, defined by,

$$g\left(z_t \middle| I_{t-\Delta}\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta}(\lambda_J \Delta)^j}{j\,!} \frac{1}{\sqrt{\left(h_t + j\sigma_J^2\right)/\sigma_t^2}} e^{\frac{-\left[z_t - (j-\lambda_J\Delta)\theta_J/\sigma_t\right]^2}{2\left(h_t + j\sigma_J^2\right)/\sigma_t^2}}$$

$$= f\left(x_{\text{smooth},t}\right) + \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta}(\lambda_J \Delta)^j}{j\,!} f\left(x_{\text{jump},t}\right) \quad (23)$$

$$= \mathrm{N}\left(0, \frac{h_t}{\sigma_t^2}\right) + \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta}(\lambda_J \Delta)^j}{j\,!} \mathrm{N}\left(\frac{(j-\lambda_J\Delta)\theta_J}{\sigma_t}, \frac{j\sigma_J^2}{\sigma_t^2}\right)$$

where $\theta_J = \ln(1+\mu_J) - \sigma_J^2/2$. By allowing the risk-neutral mean relative jump size μ_J^* to be different from its data-generating counterpart, we accommodate a premium for jump-size uncertainty. Similarly, a premium for jump timing risk can be incorporated if we allow the coefficient λ_J^* for the risk-neutral jump-arrival intensity to be different from its data-generating counterpart λ_J. In this paper, however, we follow Pan's (2002) work to concentrate mainly on the risk premium for jump-size uncertainty, while ignoring the risk premium for jump-timing uncertainty by supposing $\lambda_J^* = \lambda_J$. With this assumption, all jump risk premiums will be artificially absorbed by the jump-size risk premium coefficient $\mu_J - \mu_J^*$. By the relative entropy principle in Eq.(1), the risk-neutral density for z_t, i.e. $g^*(z_t)$, is given by,

$$g^*\left(z_t \middle| I_{t-\Delta}\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta}(\lambda_J \Delta)^j}{j\,!} \frac{1}{\sqrt{\left(h_t^* + j\sigma_J^2\right)/\sigma_t^{*2}}} e^{\frac{-\left[z_t - (j-\lambda_J\Delta)\theta_J/\sigma_t^* - \lambda_{\text{smooth}}^* - \lambda_{\text{jump}}^*\right]^2}{2\left(h_t^* + j\sigma_J^2\right)/\sigma_t^{*2}}}$$

$$= \mathrm{N}\left(\lambda_{\text{smooth}}^*, \frac{h_t^*}{\sigma_t^{*2}}\right) + \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta}(\lambda_J \Delta)^j}{j\,!} \mathrm{N}\left(\frac{(j-\lambda_J\Delta)\theta_J}{\sigma_t^*} + \lambda_{\text{jump}}^*, \frac{j\sigma_J^2}{\sigma_t^{*2}}\right) \quad (24)$$

$$= g\left(z_t - \lambda_{\text{smooth}}^* - \lambda_{\text{jump}}^*\right)$$

where $\lambda_{\text{smooth}}^*$ and λ_{jump}^* are associated with the premiums for price and jump risk. $\{h_t^*, \sigma_t^*\}$ are the quantities under the risk-neutral probability measure, given by,

$$h_{t+\Delta}^* = \mu_h + B\left(h_t^* - \mu_h\right) + C\sqrt{h_t^*}\; x_{S,t}^* + D\sqrt{h_t^*}\;\left[\,\left|x_{S,t}^*\right| - E\left|x_{S,t}^*\right|\,\right] \quad (25)$$

$$\sigma_t^* = \frac{\theta_J \lambda_{\text{jump}}^* \lambda_J \Delta}{1-\lambda_{\text{jump}}^{*2}} + \left[\left(\frac{\theta_J \lambda_{\text{jump}}^* \lambda_J \Delta}{1-\lambda_{\text{jump}}^{*2}}\right)^2 + \frac{h_t^* + \left(\theta_J^2 + \sigma_J^2\right)\lambda_J \Delta}{1-\lambda_{\text{jump}}^{*2}}\right]^{\frac{1}{2}} \quad (26)$$

with

$x_{S,t}^* = x_{S,t} + \sigma_t^*\, \lambda_{\text{smooth}}^* \big/ \sqrt{h_t^*}$ with $x_{S,t} = \left(R_t - \mu_S\Delta + 0.5h_t - \lambda_J\theta_J\Delta\right)\big/\sqrt{h_t}$ and

$E\left|x_{S,t}^*\right| = E\left|x_{S,t}\right| + \sigma_t^*\lambda_{\text{smooth}}^* \big/ \sqrt{h_t^*} = \sqrt{2/\pi} + \sigma_t^*\lambda_{\text{smooth}}^* \big/ \sqrt{h_t^*}$;

$$\sigma_t^{*2} = \mathrm{var}\left(R_t^* \middle| I_{t-\Delta}\right) \text{ with } R_t^* \cong \mu_S \Delta - 0.5 h_t^* + \sqrt{h_t^*}\, x_{S,t}^* + \left(\sum_{j=0}^{N_t} \ln Y_{j,t} + \lambda_{\text{jump}}^* \sigma_t^* \right).$$

In the spirit of Eq.(3), the risk premium-like parameters of $\lambda_{\text{smooth}}^*$ and λ_{jump}^* corresponding to smooth and jump innovations , respectively, are solved via the following Eq.(27),

$$\begin{aligned} e^{(r-d)\Delta} &= \int_{-\infty}^{\infty} e^{\mu_t + \sigma_t \Phi^{-1}[G(z_t)]}\, g\left(z_t - \lambda_{\text{smooth}}^* - \lambda_{\text{jump}}^*\right) dz_t \\ &= \int_{-\infty}^{\infty} e^{R_t}\, \mathrm{N}\left(\lambda_{\text{smooth}}^*, \frac{h_t^*}{\sigma_t^{*2}} \right) dz_t \\ &+ \int_{-\infty}^{\infty} e^{R_t} \sum_{j=0}^{\infty} \frac{e^{-\lambda_J \Delta} (\lambda_J \Delta)^j}{j\,!}\, \mathrm{N}\left(\frac{(j - \lambda_J \Delta)\theta_J}{\sigma_t^*} + \lambda_{\text{jump}}^*, \frac{j \sigma_J^2}{\sigma_t^{*2}} \right) dz_t \end{aligned} \tag{27}$$

where $G(\cdot)$ denotes the cumulative density function of z_t, driven by a compounded Poisson process. We solve $\lambda_{\text{smooth}}^*$ and λ_{jump}^* numerically and choose a kernel density function scheme to generate the risk-neutral density $g^*(S_\tau)$ of index prices according to the following system,

$$S_t^* = S_{t-\Delta} e^{\mu_t^* + \sigma_t^* \Phi^{-1}\left[G\left(z_t^*\right); \hat{\Xi}\right]} = S_{t-\Delta} e^{\mu_t^* + \sigma_t^* x_t^*} = S_{t-\Delta} e^{R_t^*} \tag{28}$$

where $z_t^* \sim g(z_t - \lambda_{\text{smooth}}^* - \lambda_{\text{jump}}^*)$; $\mu_t^* = (\mu_S \Delta - 0.5 h_t^* + \lambda_J \theta_J \Delta)$;

$$x_t^* = \frac{\sqrt{h_t^*}}{\sigma_t^*} x_{S,t} + \frac{\left(\sum_{j=0}^{N_t} \ln Y_{j,t} - \lambda_J \theta_J \Delta \right)}{\sigma_t^*} + \lambda_{\text{smooth}}^* + \lambda_{\text{jump}}^* ;$$

$$R_t^* = R_t - 0.5\,(h_t^* - h_t) + (\sqrt{h_t^*} - \sqrt{h_t})\, x_{S,t} + \left(\lambda_{\text{smooth}}^* + \lambda_{\text{jump}}^*\right)\sigma_t^* \ .$$

3.4 In- and Out-of-Sample Performance of Time-Series Models

Table 2 shows the results from estimating likelihood functions of constant-volatility, GARCH-SV and GARCH-SVJ models using index returns of various investment horizons, as well as the daily-average absolute percentage forecast errors. In the GARCH-SV model the skewness and kurtosis levels of index returns are respectively controlled, for the most part, by correlation $\rho \equiv C/\sigma_V \Delta$ and volatility variation coefficient $\sigma_V \equiv \sqrt{C^2 + D^2(1 - 2/\pi)}/\Delta$. The GARCH-SVJ model relies on the same flexibility, with the additional features to allows price jumps to occur, which can internalize more negative skewness and higher kurtosis without making other parameters unreasonable. The implied speed-of-volatility-adjustment $\kappa_V \equiv (1-B)/\Delta$ is the highest for the 30-day GARCH-SV model. The implied long-run mean variance ($\theta_V \equiv \mu_h/\Delta$) in the GARCH-SV is 4.38%, 9.61% and 9.96%, respectively, for the 30-, 100- and 300-day returns, whereas 6.07%, 5.51% and 1.84%, respectively, for the

GARCH-SVJ. The variation coefficient σ_v and the magnitude of ρ are the lowest for the GARCH-SVJ. These estimates together present the picture that, to the extent that the pricing structure of the index returns can be explained respectively by each model, the GARCH-SVJ model's demand on the $v_t = h_t/\Delta$ process is the least stringent as it requires both the lowest σ_v and the lowest ρ (in magnitude), whereas the GARCH-SV requires σ_v and ρ to be respectively as high as 0.42 and –076. The GARCH-SVJ model attributes part of the implicit negative skewness and excess kurtosis to the possibility of a jump occurring with an average frequency of 0.23, 0.28 and 0.28 times per year and an average jump size of –0.13, –0.16 and –0.16 (with the jump size uncertainty estimated at 0.21, 0.33 and 0.33) for the 30-, 100- and 300-day returns.[3] The estimated instantaneous conditional (or "spot") variances $v_{jump} + v_t$ for the GARCH-SVJ are generally smoother than spot variance v_t for the GARCH-SV, where $v_{jump} = \lambda_J \left\{ \left[\ln(1+\mu_J) - 0.5\sigma_J^2 \right]^2 + \sigma_J^2 \right\}$ is the (constant) variance per year attributable to jumps. However, the sample path for spot variance estimated under the GARCH-SVJ model involves a reflection off the minimum value of v_{jump} = 0.0010%, 3.6135% and 3.5503% for 30-, 100- and 300-day returns, whereas the path estimated under the GARCH-SV model approaches the reflecting barrier at $v_t = 0$. Given estimated slow mean reversion, the estimated expected average variance for index returns is close to the spot variance.

Table 2 Time-Series Parameters and Market Prices of Volatility and Jump Risks

The table shows the quasi-maximum likelihood results for three time-series models. Included are the risk premium-like parameters $\lambda^*_{\text{smooth}}$ (also represented for λ^*_t on the constant-volaitlity and GARCH-SV models) and λ^*_{jump}. The *t*-statistics are in parentheses; *, ** denote significance at the 5% and 1% levels, respectively. The parameters *B*, *C* and *D* correspond to continuous-time parameters by $\kappa_v \equiv (1-B)/\Delta$, $\theta_v \equiv \mu_h/\Delta$, $\sigma_v \equiv \sqrt{C^2 + D^2(1-2/\pi)}/\Delta$ and $\rho \equiv C/(\sigma_v \Delta)$. κ_v, θ_v and σ_v are respectively the speed of adjustment, long-run mean and variation coeffiecint of the diffusion volatility $v_t \equiv h_t/\Delta$. v_t is the diffusion component of return variance (conditional on no jump occurring). ρ denotes the correlation between return and variance innovations, defined by $\rho \equiv Cov_t(dw_{S,t}, dw_{v,t})$ whereas $w_{S,t}$ and $w_{v,t}$ are each a standard Brownian motion. The parameter μ_J represents the mean jump size, λ_J the frequency of the jumps per year, and σ_J the standard deviation of the logarithm of one plus the percentage jump size. $v_{jump} \equiv \left\{ \left[\ln(1+\mu_J) - 0.5\,\sigma_J^2 \right]^2 + \sigma_J^2 \right\} \lambda_J$ is the instantaneous variance of the jump components. Expected average variances are a maturity-dependent weighted average of spot and steady-state variances computed by $E(\bar{v}) = v_{jump} + \omega(\tau) v_t + (1-\omega(\tau))\theta_v$ with $\omega(\tau) = (1 - e^{-k_v \tau})/k_v \tau$ for the GARCH-SVJ and $E(\bar{v}) = \omega(\tau) v_t + (1-\omega(\tau))\theta_v$ for the GARCH-SV. $v_t^* = h_t^*/\Delta$ and h_t^* is risk-adjusted conditional heteroskedasticity,

$$h^*_{t+\Delta} = \mu_h + B\,(h^*_t - \mu_h) + C\sqrt{h^*_t}\, x^*_{S,t} + D\sqrt{h^*_t} \left[\left| x_{S,t} + \sigma^*_t \lambda^*_{\text{smooth}} / \sqrt{h^*_t} \right| - \sqrt{2/\pi} - \lambda^*_{\text{smooth}} \right], x_{S,t} \sim \text{N}(0,1).$$

$\text{APE}_{OFS} = \left| \left(R_t - \hat{R}_{t|t-\Delta}\right)/R_t \right|$ shows absoluate percentage forecast errors between actual returns R_t and

[3] Bates (1996, 2000) and Bakshi et al. (1997) also finds that the SVJ is less demanding than the SV on the volatility process and its correlation with stock price changes. For the post-1987 crash years, Bates identifies an infrequent negative price jump implicit in SPX futures options of a magnitude similar to Bakshi et al.'s (1997) findings.

predicted returns $\hat{R}_{t|t-\Delta}$ given the information set at time $t-\Delta$. All mean values are reported in the table.

	Constant Volatility			GARCH-SV			GARCH-SVJ		
Parameters	30	100	300	30	100	300	30	100	300
μ_S	0.0781**	0.0747**	0.0852**	0.0798	0.0911**	0.0986**	0.1054	0.0657	0.0677**
	(74.38)	(73.08)	(83.32)	(1.35)	(4.14)	(15.84)	(0.38)	(0.60)	(13.84)
σ_S	0.1177**	0.1101**	0.0790**						
	(114.62)	(107.61)	(75.85)						
μ_h				0.0052	0.0381	0.1186	0.0072	0.0219	0.0219
				(0.65)	(0.80)	(0.85)	(1.18)	(0.84)	(0.55)
B				0.6888*	0.8028**	0.5815	0.782**	0.7227**	0.7286**
				(2.21)	(3.99)	(1.42)	(2.92)	(4.84)	(4.12)
C				−0.0369	−0.0782**	−0.2715	−0.0212	−0.0105	−0.0105
				(−0.59)	(−34.97)	(−0.99)	(−1.11)	(−0.37)	(−0.23)
D				0.0559	0.1158	0.3843	0.0370	0.0247**	0.0383**
				(0.63)	(1.10)	(1.08)	(0.71)	(7.25)	(3.09)
startup				0.0774**	0.0066	0.8815**	0.0811**	1.0000	1.0000*
				(5.81)	(0.45)	(30.04)	(6.57)	(0.92)	(2.25)
$\sqrt{v_t}$ (%)	11.7743	11.0068	7.8980	12.9982	14.5649	15.8881	14.3464	14.4941	8.6916
λ_J							0.2332**	0.2768**	0.2768**
							(11.85)	(2.79)	(78.11)
μ_J							−0.1257*	−0.1594	−0.1574**
							(−2.22)	(−0.49)	(−12.22)
σ_J							0.2073**	0.3313	0.3293
							(4.08)	(0.25)	(0.89)
$\sqrt{v_{jump}}$ (%)							16.0227	20.1117	19.9396
$\sqrt{v_t + v_{jump}}$ (%)							23.8946	21.5435	19.8574
$\sqrt{E(\overline{v})}$ (%)	11.7743	11.0068	7.8980	14.0805	16.3436	19.5105	22.9643	25.6064	22.1046
κ_v				2.6141	0.4970	0.3515	1.8311	0.6987	0.2280
θ_v				0.0438	0.0961	0.0996	0.0607	0.0551	0.0184
σ_v				0.4201	0.2641	0.2998	0.2604	0.0474	0.0221
ρ				−0.7366	−0.7441	−0.7606	−0.6643	−0.4183	−0.3286
$\lambda^*_{\text{smooth}}$	0.0542	0.1443	−0.1378	0.0063**	−0.0004**	−0.0742**	0.0524**	0.0354**	−0.1378**
	(0.00)	(0.01)	(0.01)	(4.67)	(5.74)	(6.21)	(8.85)	(6.21)	(7.93)
λ^*_{jump}							−0.0120**	−0.0188**	−0.1054**
							(8.10)	(6.31)	(10.83)
$\sqrt{v^*_t}$ (%)				12.8131	14.6079	16.8226	11.3170	13.6622	10.8941
APE_{OFS}	1.7573	2.4750	5.2299	1.5636	1.8908	3.0676	1.3987	1.4899	1.3674

Finally, the fact that incorporating jumps seems to enhance the GARCH-SV model's fit is further illustrated by each model's absolute percentage forecast errors (APE_{OFS}) on an average day. The out-of-sample forecasts rely on previous month's index returns to back out the required parameter values and then use them as input to compute current day's model-based index returns. The absolute percentage pricing error is computed based on subtracting the model-determined price ($\hat{R}_{t|t-\Delta}$) from its observed counterpart (R_t). For the constant-volatility,

$$\hat{R}_{t|t-\Delta} = \hat{\mu}_S \Delta .$$

For the GARCH-SV,

$$\hat{R}_{t|t-\Delta} = \hat{\mu}_S \Delta - \frac{1}{2} E_{t-\Delta}(h_t) \text{ with } E_{t-\Delta}(h_{t+\Delta}) = \hat{\mu}_h + \hat{B}\left[E_{t-\Delta}(h_t) - \hat{\mu}_h\right].$$

For the GARCH-SVJ,

$$\hat{R}_{t|t-\Delta} = \hat{\mu}_S \Delta - \frac{1}{2} E_{t-\Delta}(h_t) + \hat{\lambda}_J \left[\ln(1+\hat{\mu}_J) - \frac{1}{2}\hat{\sigma}_J^2\right]\Delta \text{ with } E_{t-\Delta}(h_{t+\Delta}) = \hat{\mu}_h + \hat{B}\left[E_{t-\Delta}(h_t) - \hat{\mu}_h\right]$$

This procedure is repeated for each horizon and each day in the sample, to obtain the average absolute percentage pricing errors, $\text{APE}_{OFS} = \left| (R_t - \hat{R}_{t|t-\Delta})/R_t \right|$. These steps are separately followed for the constant-volatility, the GARCH-SV and the GARCH-SVJ models. The APE_{OFS} across 30-, 100- and 300-day index returns is 1.56 (1.76), 1.89 (2.48) and 3.07 (5.23) for the GARCH-SV (constant-volatility), while 1.40, 1.49 and 1.37 for the GARCH-SVJ. Allowing jumps to occur does improve the GARCH-SV model's out-of-sample fit further.

3.5 Volatility and Jump Risk Premiums of Time-Series Models

Volatility and jump risks are priced via the terms $\lambda^*_{\text{smooth}}$ and λ^*_{jump}, respectively, in the risk-neutral dynamics of index returns. For a positive (negative) coefficient $\lambda^*_{\text{smooth}}$ or λ^*_{jump}, the mean growth rate of the return process R_t is, therefore, $\lambda^*_{\text{smooth}}\sigma^*_t$ or $\lambda^*_{\text{jump}}\sigma^*_t$ higher (lower) under the risk-neutral measure than under the data-generating measure.[4] Table 2 shows that the volatility-risk premium-like coefficient is estimated to be significant from zero for the GARCH-SV and GARCH-SVJ, which is consistent with the findings of Guo (1998), Benzoni (2000), Poteshman (2000), Bakshi and Kapadia (2003) and Chernov and Ghysels (2000). The role of jump-risk premium-like coefficient for the GARCH-SVJ implies positive and significant premiums for the jump-size uncertainty. The constant-volatility model has an insignificant risk-premium structure. One possible explanation is that investors' aversion to volatility uncertainty or jump risks is not incorporated. Figure 1 displays implied risk-neutral probability density functions across models from index returns using the canonical valuation approach. The implied probability distributions in the 300-day period are somewhat leftskewed and platykurtic. That is, the mean of the distribution tends to be to the right of the mode and the mode tends to be less

[4] The discrepancy between R_t and R^*_t also depends on the difference between h_t and h^*_t.

pronounced than the mode of the corresponding lognormal distribution. In contrast, the 30-day distributions become more left-skewed and change from platykurtic to leptokurtic. This adjustment is pronounced by the GARCH-SVJ. Thereafter, we observe consistent levels for both skewness and kurtosis. The short-term distributions are significantly more left-skewed than in the long-term period and the mode is persistently more pronounced than the mode of the corresponding lognormal distribution.

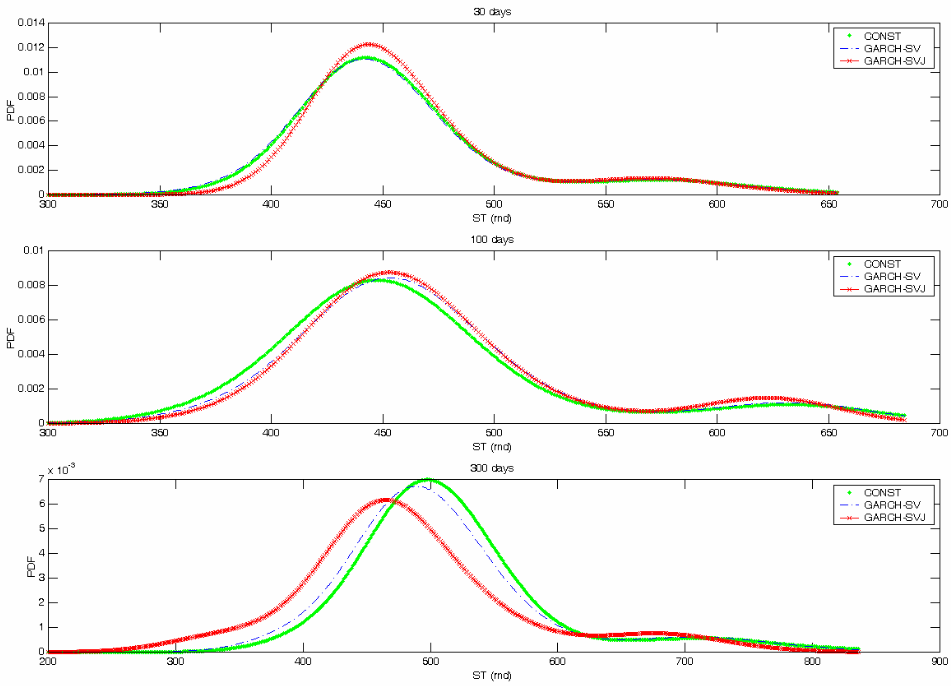

Figure 1. Canonical risk-neutral densities of S&P 500 indexes with 30-, 100- and 300-day investment horizons over the period January 1993–December 1995. The canonical probability distributions in the 300-day period are somewhat left skewed and platykurtic. That is, the mean of the distribution tended to be to the right of the mode and the mode tends to be less pronounced than the mode of the corresponding lognormal distribution. In contrast, the 30-day distributions become more left-skewed and change from platykurtic to leptokurtic. This adjustment is pronounced by the GARCH-SVJ.

IV. SPD Inferred from Cross-Sectional Options

This section extracts the risk-neutral density $f^*(S_\tau)$ implicit in option prices. Avoiding parametric assumptions on the density driving the observable asset's price, this study follows Bahra's (1997) work to adopt Breeden and Litzenberger's (1978) butterfly-spread strategy to back out the risk-neutral density that is directly comparable to the one inferred from the time-series model. This method can also extend to incorporate the information of volatility or jump risk embedded by the prices of SPX options. To implement the butterfly-spread strategy, an approach for interpolating and extrapolating option prices of arbitrary strikes and expirations is required. In practice, traders often quote option prices by means of the Black-Scholes implied volatilities corresponding to given strike and expiry. Hence, one may end up interpolate and extrapolate implied volatilities, as opposed to option prices. These volatilities are then converted to option prices through the Black-Scholes formula. Similarly, one could use a more sophisticated option pricing formula such as the SV (Heston, 1993)[5] or SVJ (Bakshi et al., 1997)[6] model, to obtain implied volatilities given a set of model parameters. More comparable to standard Black-Scholes implicit variances, the expected average variance of the SV and SVJ is a maturity-dependent weighted average of the spot variance and the steady-state variance. For the SVJ, $E^*(\bar{v}) = v^*_{jump} + \omega(\tau)v_t + [1-\omega(\tau)]\theta^*_v$ with $v^*_{jump} = \lambda^*_J \left\{ \left[\ln(1+\mu^*_J) - 0.5\sigma^2_J\right]^2 + \sigma^2_J \right\}$ $\omega(\tau) = \left(1-e^{-\kappa^*_v\tau}\right)/\kappa^*_v\tau \in (0,1)$, while for the SV $E^*(\bar{v}) = \omega(\tau)v_t + [1-\omega(\tau)]\theta^*_v$. For a given maturity, calls and puts imply smirk-shaped volatility patterns across strike prices (see Figure 1 for demonstrating observed implied volatilities from specific option pricing models across moneyness and maturity.). Following the work of Bates (1991, 1996, 2000), Dumas et al. (1998), Longstaff (1995), Madan et al. (1998), and Nandi (1998), this study minimizes the sum of squared dollar option pricing errors to obtain an estimate of the implied spot variance v_t for date *t* and the structural parameter values Θ_{SV} or Θ_{SVJ} for month *i*, January 1993–December 1995. Table 3 presents the average of each parameter/volatility series as well as the averaged sum of squared in-sample pricing errors (SSE), respectively for the BS, SV and SVJ models. The implied spot volatility is on average close across the models, of which 13.86% (implied volatility), 13.88% (= $\sqrt{v_t}$) and 13.22% (= $\sqrt{v_t + v^*_{jump}}$), respectively, for the BS, SV and SVJ, which are close across the models. The expected average volatility, however, is 14.11% and 14.29% for the SV and SVJ models, compared to 13.86% of the BS model. The relative pattern of estimated structural parameters for the spot volatility process across the SV and SVJ models is generally consistent with their time-series counterparts. In the SV (GARCH-SV) model the skewness and kurtosis levels of index returns respectively controlled, for the most part, by correlation ρ and volatility variation coefficient σ_v are greater in magnitude than the SVJ

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[5] Based on the risk-neutral dynamics of Eq.(10), Heston (1993) provides a closed-form formula for a time-*t* European call option on the underlying *S* with strike *X* and expiry *T* shown as follows,

$$C_t(S, X, T; \Theta_{\text{SV}}) = S_t P_1 - Xe^{-r(T-t)}P_2$$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi \ln X} f_j}{i\phi}\right] d\phi,\ j = 1, 2$$

f_j = the characteristic function corresponding to P_j

where $\Theta_{\text{SV}} = \{ \kappa^*_v, \theta^*_v, \sigma_v, \rho \}$ being the risk-neutral model parameter.

[6] Given an assumption that the underlying S_t follows a jump-diffusion with stochastic volatility as shown in Eq.(17), Bakshi et al. (1997) provide a closed-form formula for a time-*t* European call option with strike *X* and expiry *T*,

$$C_t(S, X, T; \Theta_{\text{SVJ}}) = S_t P_1 - Xe^{-r(T-t)}P_2$$

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi \ln X} f_j}{i\phi}\right] d\phi,\ j = 1, 2$$

f_j = the characteristic function with respective to the probability function P_j

where the risk-neutral model parameter is $\Theta_{\text{SVJ}} = \{ \kappa^*_v, \theta^*_v, \sigma_v, \rho, \lambda^*_J, \mu^*_J, \sigma_J \}$.

(GARCH-SVJ) model, and so do the implied speed-of-volatility-adjustment κ_v^* (κ_v) and long-run mean variance θ_v^* (θ_v). The GARCH-SVJ model attributes part of the risk-neutral index returns to the possibility of a jump occurring with an average jump component of –0.0357, –0.0540 and –0.0617, i.e. $E\left(R_{jump}^*\right)/\Delta = \left(\theta_J + \lambda_{\text{jump}}^* \sigma_t^*\right)\lambda_J$, for 30-, 100- and 300-day index returns, while –0.0257 of the SVJ model, i.e. $E\left(R_{jump}^*\right)/dt = \theta_J^* \lambda_J^*$, for the SPX options.

Table 3 Implicit Parameters and In-Sample Fit of Option Pricing Models

Each Wednesday in the sample, the structural parameters of a given option pricing model are estimated by minimizing the sum of squared pricing errors between the market price and the model price for each option. The average of the estimated parameters is presented. For each model, SSE denotes the average sum of squared errors. The structural parameters κ_v^*, θ_v^* and σ_v are respectively the speed of adjustment, the long-run mean, and the variation coefficient of the diffusion volatility v_t. The parameter μ_J^* represents the mean jump size, λ_J^* the frequency of the jumps per year, and σ_J the standard deviation of the logarithm of one plus the percentage jump size. v_{jump}^* is the instantaneous variance of the jump component. BS, SV and SVJ, respectively, represent the Black-Scholes, Heston's (1993) stochastic-volatility model and Bakshi et al.'s (1997) stochastic-volatility model with random jumps. More comparable to BS implied variances, the expected average variance of the SV and SVJ is reported.

Parameters	BS	SV	SVJ
κ_v^*		4.6136	3.2991
θ_v^*		0.0288	0.0219
σ_v		0.7747	0.4769
ρ		–0.6639	–0.6887
$\sqrt{v_t}\,(\%)$		13.8759	10.9115
λ_J^*			0.3784
μ_J^*			–0.0776
σ_J			0.0665
$\sqrt{v_{jump}^*}\,(\%)$			6.7508
$\sqrt{v_t + v_{jump}^*}\,(\%)$			13.2247
Implied Volatility (%)	13.8625	14.1076	14.2934
SSE	2.0175	0.1550	0.1430

4.1 Kernel-Regression Implied Volatilities

This study adopts a nonparametric kernel regression to smooth the implied volatility (σ) across the spot index price (*S*), strike price (*X*) and time to expiration (*T*).[7] Observed implied volatilities $\sigma_i\left(S_i, X_i, T_i; \hat{\Theta}\right)$, backed out from either the Black-Scholes, SV or SVJ option pricing model given a set of model parameter estimates $\hat{\Theta}$,[8] multiplied by a density-like weight, called the kernel function, are summed to give an estimate of unobserved volatility $\hat{\sigma}(S, X, T)$. A Nadaraya-Watson kernel estimator[9] is used as follows,

$$\hat{\sigma}(S,X,T)=\frac{\sum_{i=1}^{n}k_S\left(\frac{S-S_i}{h_S}\right)k_X\left(\frac{X-X_i}{h_X}\right)k_T\left(\frac{T-T_i}{h_T}\right)\sigma_i}{\sum_{i=1}^{n}k_S\left(\frac{S-S_i}{h_S}\right)k_X\left(\frac{X-X_i}{h_X}\right)k_T\left(\frac{T-T_i}{h_T}\right)} \tag{29}$$

where *n* is the sample size of observed implied volatilities; $k_S(\cdot)$, $k_X(\cdot)$ and $k_T(\cdot)$ are the kernel functions of the spot index price (*S*), strike price (*X*) and time to expiration (*T*), respectively, defined on the Gaussian-family density function. As the risk-neutral density is related to the second derivative of a call option pricing formula with respect to the strike price (see the following section for the details), a Gaussian kernel function with order 2, $k_{(2)}(z)=1/\sqrt{2\pi}\exp\left(-z^2/2\right)$, is used as $k_X(\cdot)$, whereas a Gaussian kernel function with order 4 serves for both $k_S(\cdot)$ and $k_T(\cdot)$, i.e. $k_{(4)}(z)=3/\sqrt{8\pi}\left(1-z^2/3\right)\exp\left(-z^2/2\right)$. The bandwidths, h_S, h_X and h_T, determine the impact of the kernel functions on each observed implied volatility. Given a larger bandwidth and thus an increase in smoothness of interpolated implied volatility surface, the kernel estimator induces a greater bias but a smaller variance. The tradeoff between bias and variance can determine an optimal value for the bandwidth.[10] The resultant kernel-regression volatilities $\hat{\sigma}(S, X, T)$ are the basic inputs to the butterfly-spread state price density. The kernel-regression volatility surface, inferred from SV (SVJ) implied volatilities, absorbs the premium for volatility (and jump) risk.

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[7] Campa et al. (1998) use cubic spline to interpolate the implied volatility surface. The cubic spline interpolation only allows for one point of option price on the same moneyness and maturity, whereas the kernel regression provides additional flexibility allowing for multiple option quotes on the same option contract.

[8] There is no extra model parameter for the Black-Scholes formula while the estimate of model parameters for the SV model is $\hat{\Theta}_{\text{SV}}=\left\{\hat{\kappa}_v^*, \hat{\theta}_v^*, \hat{\sigma}_v, \hat{\rho}\right\}$ and $\hat{\Theta}_{\text{SVJ}}=\left\{\hat{\kappa}_v^*, \hat{\theta}_v^*, \hat{\sigma}_v, \hat{\rho}, \hat{\lambda}_J^*, \hat{\mu}_J^*, \hat{\sigma}_J\right\}$ for the SVJ model.

[9] Härdle (1990), Aït-Sahalia and Lo (1998, 2000), and Härdle et al. (2002) use a Nadaraya-Watson kernel estimator to explain the regression relationship between dependent and independent variables.

[10] Aït-Sahalia and Lo (1998) use a rule for the *x*-variable bandwidth: $h=c\ s(z)\ n^{-1/(d+2p)}$ where $z=(x-x_i)/h$; *d*=the number of kernel regressors; *p*=the differentiate order of the Nadaraya-Watson kernel estimator; $s(z)=$ the standard deviation of z ; $c=c_0/\ln(n)$ with constant c_0 and the sample size *n*.

4.2 Inferring Risk-Neutral Probability Densities Directly from the Implied Volatilities

Building on Ross' (1976) insight that options can be used to create pure Arrow-Debreu state-contingent securities, Banz and Miller (1978) and Breeden and Litzenberger (1978) provide an elegant method for obtaining an explicit expression for the risk-neutral density from option prices: the risk-neutral density is the second derivative (normalized to integrate to unity) of a call option pricing formula with respect to the strike price, multiplied by the risk-free return, shown as follows,

$$f_t(S_T) = \exp\left(r_{t,T}(T-t)\right) \times \left.\frac{\partial^2 C_t(S,X,T)}{\partial X^2}\right|_{S_T = X} \tag{30}$$

where $C_t(S,X,T)$ is the market price of a call option at time t with strike price X, expiry T, and the underlying asset price S; $r_{t,T}$ is the annualized risk-free interest rate for the period of time to expiry. The implementation of Eq.(30) can be accomplished by selling two call options struck at X and buying one struck at $X - \Delta X$ and one at $X + \Delta X$, often called a "butterfly-spread" strategy. By construction, the price of a butterfly- spread strategy is given by,

$$\left.\frac{\partial^2 C_t(S,X,T)}{\partial X^2}\right|_{S_T = X_i} = \frac{C_{i+1} + C_{i-1} - 2C_i}{\Delta X^2}\ ,\ i = 1,\ldots,N \tag{31}$$

where $X_{i-1} < X_i < X_{i+1}$; $\Delta X = X_{i+1} - X_i = X_i - X_{i-1}$; N is the sample size of kernel-regression implied volatilities; $C(\cdot)$ is the model prices for calls based on kernel-regression implied volatilities. The risk-neutral density at expiry T can be thus obtained after multiplying Eq.(31) by the risk-free growth return. Figure 2 displays the resultant price distributions of the S&P 500 without assuming any parametric density function, which are in turn the implied probability distribution from market data (i.e. the kernel-regression volatilities). Features of skewnesss and leptokurtosis in the implied distribution for the S&P 500 are found to be consistent with the volatility smile in Table 1, indicating that large price drops are more likely than the geometric Brownian motion model predicts (or, at least the market's opinion, expressed through options trading). The extent of the convexity of the smile curves indicates the degree to which the market risk-neutral density function differs from the corresponding BS, SV or SVJ risk-neutral distribution. In addition, the direction in which the smile curve slopes reflects the skew of the market risk-neutral density function: a positively (negatively) sloped implied volatility smile curve across moneyness results in a risk-neutral density function that is less (more) positively skewed than the corresponding BS, SV or SVJ risk-neutral distribution that would result from a flat smile curve. As a result, the existence of the volatility smile curve indicates that market participants make more complex assumptions than BS, SV and SVJ about the path of the underlying index price.

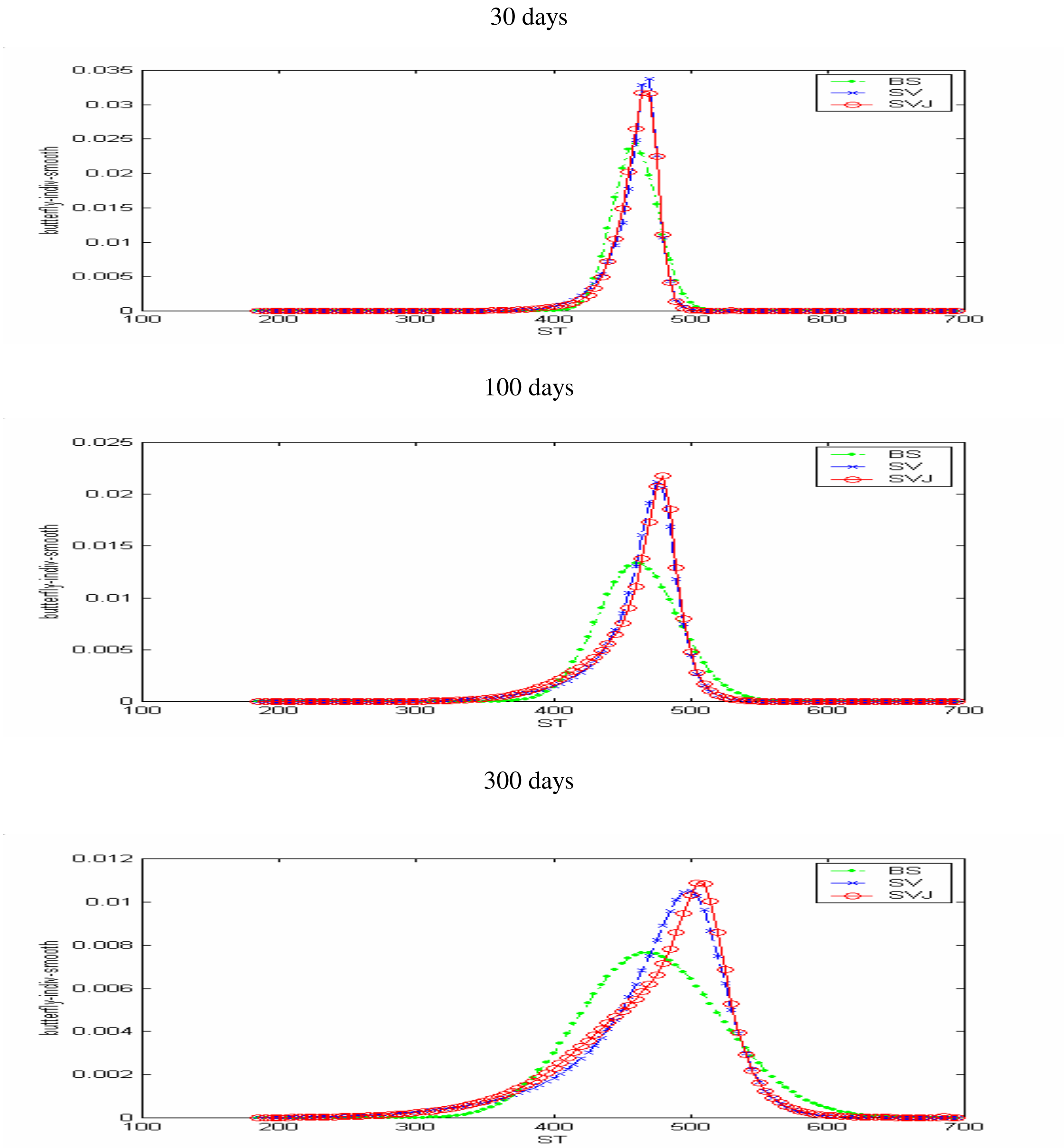

Figure 2. Implied risk-neutral densities of SPX options with 30, 100 and 300 days to expiry traded in January 1995. The way of estimating the implied risk-neutral density function is by direct application of the Breeden and Litzenberger (1978) result to the call option pricing function of either Black-Scholes, SV or SVJ. This requires an interpolated call pricing function, $c(X,\tau)$, that is consistent with the monotonicity and convexity conditions, and that can be differentiated twice. This can be achieved nonparametrically, by applying a statistical technique called nonparametric kernel regression.

V. Statistical Tests of Distribution Similarities

Do stochastic volatility and/or jumps help improve the consistency of risk-neutral distributions implicit in the S&P 500 index option market and the cash index market? This study answers this question by comparing the risk-neutral density estimated from cross-section of S&P 500 option prices to the risk-neutral density inferred from the time series density of the S&P 500 index. One (denoted as $g^*(S_\tau), \tau = 30, 100, 300$ days to horizon) is estimated, using a canonical valuation approach, from a sequence of observed index returns assumed to follow constant volatility and stochastic volatility with/without jumps. And the other (denoted as $f^*(S_\tau), \tau = 30, 100, 300$ days to maturities) is inferred from option prices based on three option pricing models of Black-Scholes (1973), Heston's (1993) SV and Bakshi et al.'s (1997) SVJ. If the difference between $f^*(S_\tau)$ and $g^*(S_\tau)$ happens to be smaller for the SV model than for the SVJ model, then we will conclude that an underlying asset imposing stochastic volatility without jumps will bring much closer to the corresponding underlying dynamic relative to the SVJ case. This will implicate that the superiority of the SV model in option pricing relative to the SVJ model. Alternative statistics are proposed for comparisons of the resultant risk-neutral densities, consisting of Kolmogorov-Smirnov goodness-of-fit test and Kullback-Leibler distance measure.

5.1 Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov (K-S) test considers the goodness-of-fit between two empirical distribution functions. The first empirical distribution function is given in terms of the order statistics, u_i, which is an observed value of the cumulative distribution function for each expiration at $S_{T,i}$ to produce a probability u_i. The cumulative function $C(u_i)$, which is the proportion of the u_i equal to or less than u_i, is calculated from the second empirical distribution function. The K-S statistic is defined as the maximum value of $\left| C(u_i) - u_i \right|$. Given the significant level of 5% (1%), the acceptance rates of the K-S statistics to measure the "closeness" of risk-neutral densities between the butterfly-spread strategy and the canonical valuation approach across alternative models of constant volatility/stochastic volatility/jump-diffusion under stochastic volatility model equal 97%/ 100%/ 94% (100%/ 100%/ 100%) for the 30 days to expiration, 94%/ 75%/ 69% (100%/ 94%/ 81%) for the 100-day expiration, and 0%/ 3%/ 81% (17%/ 17%/ 89%) for the 300 days to expiry. The results suggest distribution similarities between index- and option-implied densities only for 30 and 100 days to expiry for the constant- and stochastic-volatility models, while incorporating a jump component improves the internal consistency for 300 days to expiry. The rejection of density similarities indicates mispricing of the constant- and stochastic-volatility models for long-dated SPX options. For robust comparison purposes, K-S test results over 72 one-month subperiods are graphed in Figure 3.

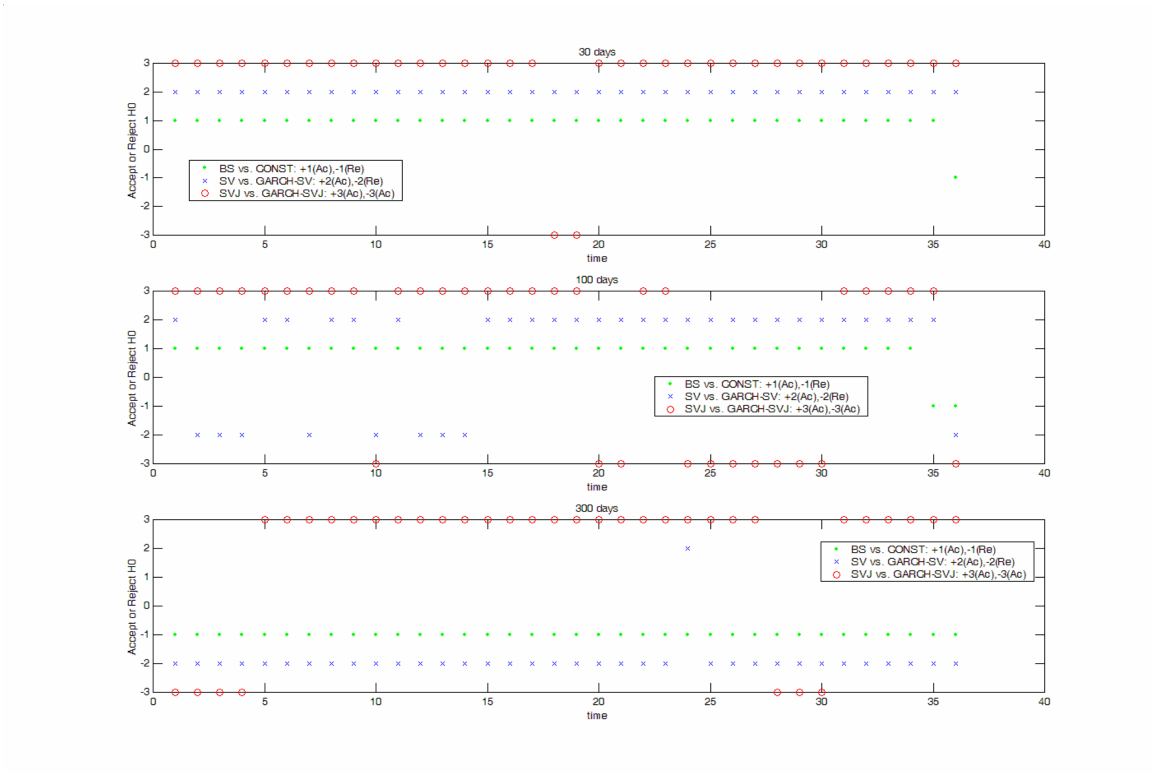

Figure 3. Given the significance level of 0.05, Kolmogorov-Smirnov (K-S) goodness-of-fit tests over the period 1993–1995. The K-S test is used to decide if option-implied densities come from a population with an index-implied empirical distribution. The K-S test is based on the maximum distance between these two cumulative distribution functions. The dots (•) with value of 1 (−1) indicate the acceptance (rejection) of the null hypothesis of distribution similarity for the constant volatility, while the symbol of cross (×) with values of +2 or −2 and the symbol of round (o) with values of +3 or −3 are, respectively, for the stochastic volatility and jump-diffusion with stochastic volatility.

5.2 Kullback-Leibler distance measure

The Kullback-Leibler distance (Kullback, 1959; Kullback and Leibler, 1951) is perhaps the most frequently used information-theoretic "distance" measure from a viewpoint of theory. The relative entropy of option-implied distributions $f_k^*(S_\tau)$ with respect to index-implied densities $g_k^*(S_\tau)$, called the Kullback-Leibler (KL) distance, is defined by

$$d\left(f_k^* \| g_k^*\right) := \sum_{k=1}^{n} f_k^* \ln\left(\frac{f_k^*}{g_k^*}\right) \tag{32}$$

While $d\left(f_k^* \| g_k^*\right)$ is often called a distance, it is not a true metric because it does not satisfy the triangle inequality. However, the properties of the relative entropy equation make it a convex function of $f_k^*(S_\tau)$, always nonnegative, and equal zero only if $f_k^*(S_\tau) = g_k^*(S_\tau)$. The smaller the relative entropy, the more similar the distributions of the index and option data. Note that the measure is asymmetrical and the distance

$d\left(f_k^* \| g_k^*\right)$ is not equal to $d\left(g_k^* \| f_k^*\right)$. If the distributions are not too dissimilar, the difference between $d\left(f_k^* \| g_k^*\right)$ and $d\left(g_k^* \| f_k^*\right)$ is small. Figure 4 displays the relative entropy between index- and option-implied risk-neutral densities across models and days to expiration. The SV with respect to GARCH-SV has a close KL distance resemblance to the BS with respect to time-series constant volatility, for 30 days to horizon. The SVJ in general rates its underlying 300-day return level the same as the GARCH-SVJ. Finally, the relative entropy shows an appropriate measure of the similarity of the underlying distribution for the BS with respect to the constant volatility for 100 days to expiration.

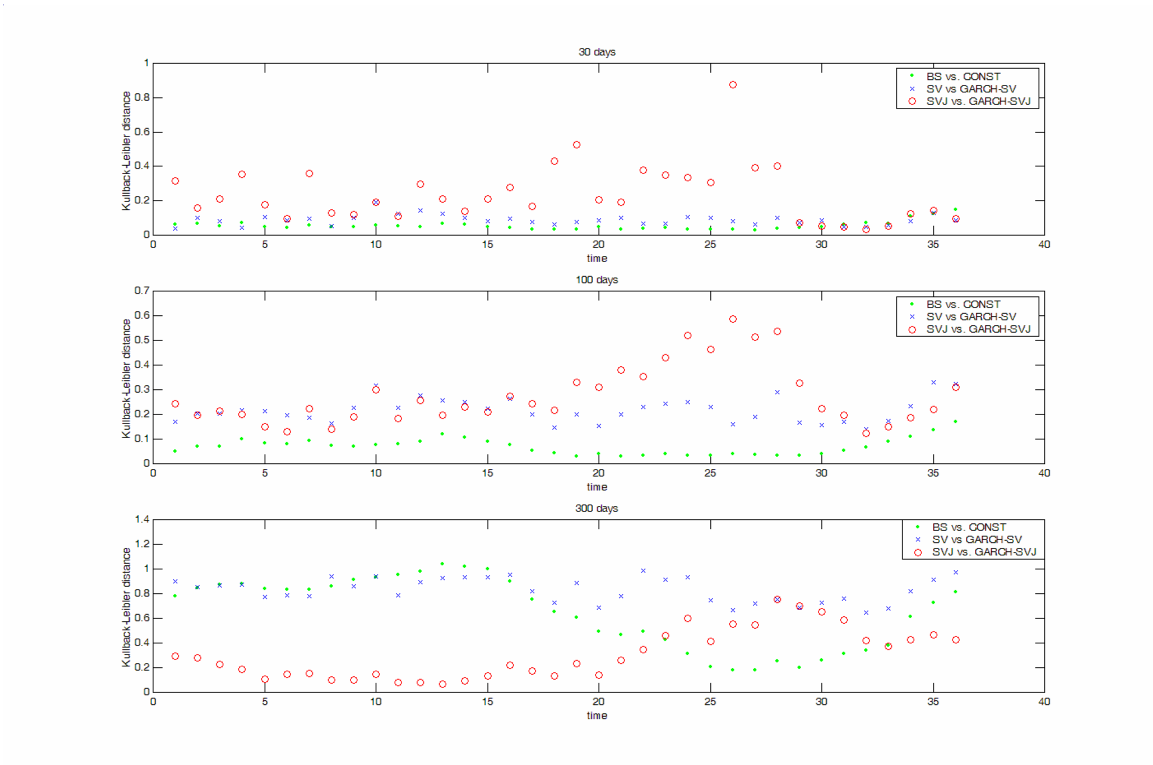

Figure 4. Kullback-Leilber (KL) distance measure over the period 1993–1995. The KL distance quantifies the degree of divergence between two distributions. The KL distance is given by their relative entropy, $d\left(f_k^*(S_\tau) \| g_k^*(S_\tau)\right) = \sum_{k=1}^{n} f_k^*(S_\tau) \ln\left(f_k^*(S_\tau)/g_k^*(S_\tau)\right)$, where $f_k^*(S_\tau)$ and $g_k^*(S_\tau)$ are, respectively, option- and index-implied risk-neutral distributions with size n. The KL distance is zero if both distributions are equivalent ($f_k^*(S_\tau) = g_k^*(S_\tau)$). The smaller the relative entropy, the more similar the distributions of these two data sets. Three models are considered here: constant volatility (BS vs. its corresponding time-series constant-volatility model), stochastic volatility (SV vs. GARCH-SV), and jump-diffusion under stochastic volatility (SVJ vs. GARCH-SVJ).

VI. Implied Risk Aversion

To provide an economic interpretation for the differences between option- (f^*) and index-implied (g^*) risk-neutral densities, this study infers the relative preferences that are compatible with the pair of option and index values. The risk-neutral probabilities are an MRS (marginal rate of substitution)-weighted probability density function, which are adjusted for risk aversion, time preferences, and other variations in economic valuation (Aït-Sahalia and Lo, 2000). Thus, f^* and g^* carry information that is relevant for the risk aversion between option and stock investors. Subject to certain conditions such as complete and frictionless markets and a single asset, the absolute risk aversion (ARA) is related to the risk-neutral density function, $h^*(S_T)$, and the physical density function, $h(S_T)$, by the representative investor's utility function, $U(S_T)$, as follows,

$$\text{ARA} = -\frac{U''(S_T)}{U'(S_T)} = \frac{h'(S_T)}{h(S_T)} - \frac{h^{*\prime}(S_T)}{h^*(S_T)} \tag{33}$$

The Arrow-Pratt measure of relative risk aversion can be inferred by $\text{RRA} = S_T \times \text{ARA}$ (see Leland, 1980; Bliss and Panigirtzoglou, 2004). The economic theory suggests that the ARA and RRA functions should be positive and monotonically downwards sloping for risk-averse investors.[11] However, some studies such as Aït-Sahalia and Lo (2000), Jackwerth (2000), Aït-Sahalia et al. (2001), and Rosenberg and Engle (2002) show that the resulting option-implied risk-aversion functions are somewhat inconsistent with theory: either U-shaped or generally declining but not monotonically so. In contrast, the discrepancy in ARA and RRA between option and stock investors provides a new method of inferring risk aversion implied by security market prices, together with the informativeness of options, to present unique evidence of the term structure of risk preferences, shown as follows,

$$\text{ARA}_{\text{option}} - \text{ARA}_{\text{index}} = -\frac{f^{*\prime}(S_T)}{f^*(S_T)} + \frac{g^{*\prime}(S_T)}{g^*(S_T)} \tag{34}$$

$$\text{RRA}_{\text{option}} - \text{RRA}_{\text{index}} = -S_T\left[\frac{f^{*\prime}(S_T)}{f^*(S_T)} - \frac{g^{*\prime}(S_T)}{g^*(S_T)}\right] \tag{35}$$

The difference allows us to infer the degree of risk aversion of the option and stock traders. A comparison of these two risk-neutral probabilities shows that the discrepancies in absolute and relative risk aversions are not constant across states or maturity dates, but changes in important nonlinear ways (see Table 4). In most cases, option traders are more risk averse than stock traders, with some exceptions in risk-neutral densities between the SVJ and GARCH-SVJ models for 30 days to horizon.

[11] Some well-behaved functional form for the underlying utility function such as the power function has constant RRA, while the exponential-utility function exhibits constant absolute risk aversion.

Table 4 Discrepancy in Option- and Index-Implied Risk Aversion Functions across States

Underlying data are 36×3×3 option- and index-implied risk-neutral distributions across maturities (30, 100 and 300 days to horizon) and models (BS/constant-volatility, SV/GARCH-SV, and SVJ/GARCH-SVJ), from January 1993 to December 1995. Values for the representative investors "RRA (ARA) Diff" obtained by subtracting the index-implied RRA (ARA) from the option-implied RRA (ARA).

	30 days			100 days			300 days		
RRA Diff (ARA Diff)	BS vs. CONST	SV vs. GARCH-SV	SVJ vs. GARCH-SVJ	BS vs. CONST	SV vs. GARCH-SV	SVJ vs. GARCH-SVJ	BS vs. CONST	SV vs. GARCH-SV	SVJ vs. GARCH-SVJ
M	4.8640	4.3977	−2.7216	2.6957	2.5730	4.6985	9.5784	7.3235	7.6142
	(0.0107)	(0.0101)	(−0.0059)	(0.0059)	(0.0061)	(0.0106)	(0.0228)	(0.0123)	(0.0182)
Mdn	7.4469	3.6051	−2.5078	4.5045	2.9771	5.1670	15.8014	11.5553	12.8274
	(0.0165)	(0.0071)	(−0.0058)	(0.0111)	(0.0070)	(0.0125)	(0.0323)	(0.0208)	(0.0246)
Max	29.5697	29.7412	29.8701	29.8430	29.2726	29.7688	29.7405	29.9507	29.6815
	(0.0672)	(0.0692)	(0.0674)	(0.0682)	(0.0714)	(0.0726)	(0.0733)	(0.0728)	(0.0700)
Min	−26.9062	−29.3136	−29.9333	−29.8770	−29.9195	−29.0645	−29.6075	−29.8089	−29.0154
	(−0.0641)	(−0.0666)	(−0.0613)	(−0.0738)	(−0.0672)	(−0.0610)	(−0.0467)	(−0.0670)	(−0.0570)
Std	16.9216	16.5405	17.2311	18.1824	15.2693	13.8055	16.5879	16.0666	16.2086
	(0.0375)	(0.0376)	(0.0381)	(0.0407)	(0.0335)	(0.0303)	(0.0323)	(0.0326)	(0.0322)
Skewness	−0.3888	−0.4254	0.3533	−0.1830	−0.2741	−0.2788	−0.8073	−0.6397	−0.7130
	(−0.4007)	(−0.3767)	(0.3863)	(−0.2306)	(−0.2245)	(−0.2370)	(−0.5493)	(−0.7040)	(−0.4563)
Kurtosis	2.1134	2.3093	2.2305	1.6829	2.3641	2.2991	2.3965	2.3160	2.3470
	(2.1868)	(2.3320)	(2.2398)	(1.7564)	(2.4073)	(2.1497)	(2.0841)	(2.6289)	(2.1763)

VII. Conclusions

The dynamics of the underlying fundamental asset cannot be related to option prices without additional assumptions or information. One possibility is to assume that the risks associated with stochastic volatility or jumps are diversifiable and not priced by the market. However, most recent empirical work clearly indicates there are prices for volatility and jump risk (Anderson et al., 2002; Chernov and Ghysels, 2000; and Pan, 2002). Thus, there is evidence of jumps and stochastic volatility in the underlying stock index process. In lights of the phenomena, this study incorporates stochastic volatility and jumps into the underlying dynamics and adopts a canonical valuation method proposed by Stutzer (1996) and Duan (2002) to recover a risk-neutral distribution from stock index prices only to avoid making any assumptions on investors' preferences. In particular, while the timer interval between the data shrinks to zero, the underlying dynamics constructed as GARCH-like models with/without jumps will converge to continuous-time processes under which Heston (1993) and Bakshi et al. (1997) build option pricing formulas. The feature enables us to clarify to what extent of degrees in the contribution of stochastic volatility and jumps to the consistency between option- and index-implied risk-neutral distributions. We extend the kernel regression approach to incorporate stochastic volatility and jumps so that resultant implied volatilities from option prices contain the information of volatility and jump risks. It is of interest to see whether these two risk-neutral distributions inferred from either option prices or index returns are consistent after taking into account stochastic volatility with/without jumps. The positive answer gives rise to strength the improvement and importance in the incorporation of stochastic volatility and/or jumps into both the underlying dynamics and option pricing formulas. This is the goal that we expect to achieve. Even though these models turn out to be misspecified in terms of the criteria discussed above, results from the tests of those models are still informative and could be used on how well those models work for practitioners. We are to learn the degree to which cross-sectional option pricing patterns are quantitatively consistent with the time series properties of the underlying asset and of option prices. The stochastic volatility is found to reconcile the spot and option risk-neutral densities for 30 days to expiration, while additionally adding a jump component brings densities closer for 300 days to horizon. We propose a peso-problem interpretation of this evidence: cross-sectional option prices capture a premium as compensation for the risk of a market crash, but actual realizations of that jump are too infrequent to be consistently observed and reflected in estimates drawn from the time series of asset returns. The option- and index-implied risk-neutral probabilities are also compared to gauge the magnitudes of option and stock investors' differences in preferences. Our results show that the option investors possess more risk averse than stock traders except for the 30-day jump-diffusion with stochastic volatility. The finding may help illustrate the stylized puzzle arising in empirical options research that implicit volatilities are greater than subsequent realized volatilities. This can be achieved a negative volatility risk premium that implies slower volatility mean reversion under the risk-neutral than under the physical probability measure or by a positive jump risk premium that increases risk-neutral jump variance and total variance relative to objective jump variance (Chernov, 2003; Pan, 2002; and Broadie et al., 2004). However, our finding more or less provides a support for the existence of market segmentation in spot and option-market risk aversions. The 'representative agent' construction and calibration associated with implicit pricing kernels and risk premiums, which implies risk dispersal across all agents in the economy until all agents are indifferent at the margin to the systematic risks they bear given the prices of those risks, appear inconsistent with the industrial organization of stock index option markets, which appears to be functioning as an insurance market. Meanwhile, judging appropriate prices of risks is also complicated by the empirical rejections of standard asset pricing models such as the CAPM and CCAPM.

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