OPTIMAL ASSET ALLOCATION BASED ON UTILITY MAXIMIZATION IN THE PRESENCE OF MARKET FRICTIONS

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Abstract

We develop a model of optimal asset allocation based on a utility framework. This applies to a more general context than the classical mean-variance paradigm since it can also account for the presence of constraints in the portfolio composition. Using this approach, we study the distribution of a measure of wealth compensative variation, we propose a benchmark and portfolio efficiency test and a procedure to estimate the implicit risk aversion parameter of a power utility function. Our empirical analysis makes use of the S&P 500 and industry portfolios time series to show that, although the market index cannot be considered an efficient investment in the standard mean-variance metric, in our framework the wealth loss associated with such an investment is rather small (lower than 0.5%), and is not statistically different from zero when the risk aversion is small. The wealth loss is at its minimum for a representative agent with a constant risk aversion index not higher than 5. Furthermore we show that, for reasonable levels of risk aversion, the use of an equally weighted portfolio is surprisingly consistent with an expected utility maximizing behavior.

JEL classification codes: C15, D14, G11

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1. Introduction

The efficiency of an investment is usually assessed by means of a standard mean-variance approach. In the simplest case of no restrictions on portfolio shares, such a framework implies that the performance of any investment is measured in terms of its Sharpe ratio, i.e., the expected return over the standard deviation of its excess returns. Using such a measure, several statistical tests have been developed to establish the efficiency of an investment; among others, the tests proposed by Jobson and Korkie (1982), Gibbons et al. (1989), and Gourieroux and Jouneau (1999) are noteworthy.

The use of the Sharpe ratio is relatively simple and rather intuitive but lacks some important features. The most important being that, by acting this way, it is not possible to take account of market imperfection when building the optimal portfolio weights. The widespread use of Sharpe ratios depends on the well-known fact that their upper limit is reached by any portfolio in the mean-variance efficient frontier built as a combination of the market portfolio and the risk free asset. Such a frontier is derived disregarding market imperfections, but in their presence it would take a different shape.

In particular, two kinds of constraints are relevant: transaction costs and inequality constraints. Transaction costs are costs incurred when buying or selling assets. These include brokers' commissions and spreads, i.e., the difference between the price paid for an asset and the price it can be sold. Transaction costs may be negligible in the case of financial assets, but several authors (among others Grossman and Laroque, 1990, Flavin, 2002, and Pelizzon and Weber, 2003) point out how they are instead relevant for real assets such as housing[1]. Following Gourieroux and Jouneau (1999) we know that, when equality constraints on some portfolio weights are taken into account, it is however possible to translate the original plane in another mean-variance frontier, conditional on the constrained assets.

Another important market imperfection is represented by inequality constraints. In actual stock markets, for instance, short sales are not prohibited, but discouraged by the fact that the proceeds are not normally available to be invested elsewhere; this is enough to eliminate a private investor with just mildly negative beliefs (Figlewski, 1981). On the contrary, mutual fund constraints are widespread and may be seen as one component of the set of monitoring mechanisms that reduce the costs arising from frictions in the principal-agent relation (Almazan et al., 2004). Considering these constraints, we would be faced with a different frontier of feasible portfolios, of unknown shape, whose relationship with the Sharpe ratio is not clear. With only short-sale restrictions in particular, there may be switching points along the mean-variance frontier corresponding to changes in the set of assets held. Each switching

[1] to such an extent that real assets become illiquid in the presence of transaction costs. In other words, their investment is kept as fixed in the short run, and an optimizing investor chooses the composition of her financial portfolio conditional on the stock held in real assets.

point corresponds to a kink (Dybvig, 1984), and the mean-variance frontier consists then of parts of the unrestricted mean-variance frontiers computed on subsets of the primitive assets.

Notwithstanding this evidence, empirical works often come out with optimal portfolio weights in a standard mean-variance framework that take extreme values (both negative and positive) in some assets. Green and Hollifield (1992) state that: *«[…] The extreme weights in efficient portfolios are due to the dominance of a single factor in the covariance structure of returns, and the consequent high correlation between naively diversified portfolios. With small amounts of cross-sectional diversity in asset betas, well-diversified portfolios can be constructed on subsets of the assets with very little residual risk and different betas. A portfolio of these diversified portfolios can then be constructed that has zero beta, thus eliminating the factor risk as well as the residual risk*». This portfolio is unfeasible in practice and, unjustifiably, gets compared with observed investments in terms of Sharpe ratios[2]. This way, we relate actual investments with unrealistic ones, which ensure an even better performance than the optimal feasible portfolios. Hence, the comparison is erroneous since it tends to overestimate the inefficiency of any observed investment.

The problem is dealt with in Basak et al. (2002) and Bucciol (2003); following a mean-variance approach, these authors develop an efficiency test in which the discriminating measure is no longer based on a Sharpe ratio comparison, but on a variance comparison instead, for a given expected return. Such a technique, nevertheless, circumvents the above mentioned problem at the cost of neglecting some information: it just fixes the value of the expected return, and does not take into account how it could affect the importance of deviations in risk.

In this paper we try, instead, to cope with inequality constraints in a model that pays attention to expected returns as well as variance of investment returns. In lieu of working with efficient frontiers, we concentrate on the expected utility paradigm. Quoting Gourieroux and Monfort (2005), *«the main arguments for adopting the mean-variance approach and the normality assumption for portfolio management and statistical inference are weak and mainly based on their simplicity of implementation»*. It is well known (Campbell and Viceira, 2002), however, that the two procedures provide the same results, under several assumptions. Already Brennan and Torous (1999), Das and Uppal (2004) and Gourieroux and Monfort (2005) consider an agent who maximizes her expected utility in order to get an optimal portfolio. Brennan and Torous (1999), in particular, define a performance measure, based on the concept of compensative variation, which compares the utility from an optimal investment with that resulting from a given investment. Drawing inspiration from this strand

[2] Any portfolio is indeed proportional to the zero-beta portfolio since the two fund separation theorem holds.

of literature we will subsequently show that, using a specific utility function, this procedure boils down to maximizing a function of mean and variance of a portfolio, for a given risk aversion; furthermore, the measure of compensative variation has the intuitive economic interpretation of the amount of wealth wasted or generated by the investment, relative to the optimal portfolio. The main contribution of this paper is to characterize the asymptotic probability distribution and confidence intervals of this measure of compensative variation; this will permit us to conduct statistically valid inference, and therefore to test for portfolio or benchmark efficiency. This task is made difficult, nevertheless, by the presence of inequality constraints.

The paper is organized as follows: section 2 compares the standard mean-variance approach with our method based on expected utility maximization. It shows the underlying algebra of the agent's problem, and introduces a measure of wealth compensative variation. Section 3 specifies the efficiency test, by means of a weak version of the central limit theorem and the delta method. This procedure does not permit to run the test for extreme null hypotheses (e.g., all the wealth is wasted), but is enough to construct confidence intervals. Section 4 describes the statistic in a closed-form expression when there are no inequality constraints, and examines analogies with optimal portfolios derived in a mean-variance framework. Section 5 presents a fruitful way to estimate the relative risk aversion parameter using the data. In the absence of constraints, the expression can be derived in a clear closed-form expression; otherwise it can be obtained numerically. In section 6 we describe the data used in the empirical exercise, the S&P 500 index and 10 industry portfolios for the U.S. market. We further run some tests to assess the efficiency of the S&P index and a equally weighted portfolio; we also compute the optimal risk aversion parameters and assess the importance of any single constraint. In section 7 we study the empirical distribution of our test, running several block bootstrap simulations. Lastly, section 8 summarizes the results and concludes.

2. Agent's behavior

Disregarding constraints, we may assess the efficiency of an investment by comparing its Sharpe ratio with the optimal, as shown in figure 1[3]. It is the case, for instance, of the test proposed by Jobson and Korkie (henceforth JK, 1982) in a portfolio setting. The optimal Sharpe ratio depicts the slope of the efficient frontier which includes a risk free asset within the endowment. The greater the difference between the two ratios, the greater the inefficiency of the observed investment (figure 1).

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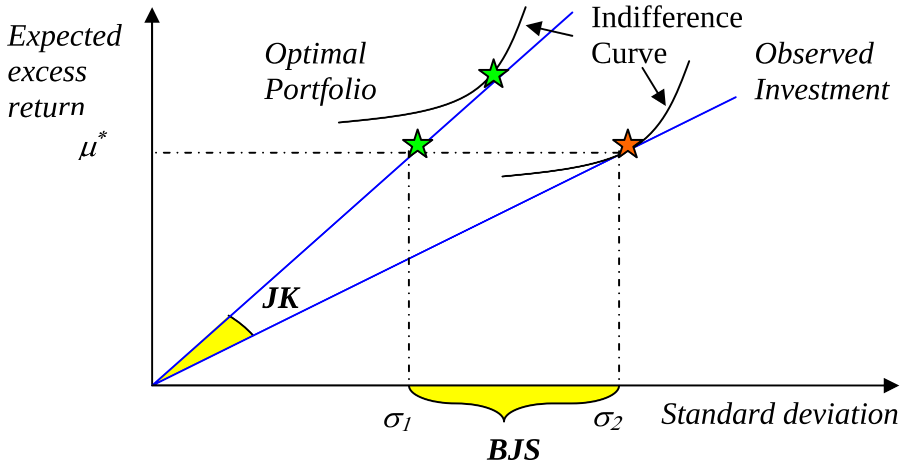
[3] Although in the figure we draw an optimal portfolio with the same expected excess return as the observed investment, there are infinite optimal portfolios with the same Sharpe ratio; they differ only in the share invested in the risk free asset.

Figure 1.

Measures of efficiency – mean-variance framework



Some other tests, such as the one in Basak, Jagannathan and Sun (BJS, 2002), fix the level of expected return μ^* and consider the difference between the two variances, σ_1^2 and σ_2^2, namely the lowest achievable variance minus the observed variance. The smaller this difference (negative by construction), the higher the inefficiency of the observed investment. A caveat of this approach is that one dimension of the problem, the expected excess return, is kept fixed and therefore completely neglected by the efficiency analysis. It is however difficult to think of different ways to face this problem, since the shape of the efficient frontier does not admit a closed-form representation in the presence of inequality constraints.

A reasonable alternative is to consider an expected utility framework instead of a mean-variance approach. It is well known that the two methods are equivalent under several assumptions; Campbell and Viceira (2002), for instance, argue that a power (or CRRA) utility function and log-normally distributed asset returns produce results that are consistent with those of a standard mean-variance analysis. The property of constant relative risk aversion, moreover, is attractive and helps explain the stability of financial variables over time.

We then draw inspiration from Gourieroux and Monfort (2005) and study the economic behavior of a rational agent who maximizes her expected utility of future wealth. The authors explain that such an approach is appropriate even when return distributions do not seem normal; in our context, this framework also takes account of constraints in portfolio composition.

In figure 1 the indifference curves for observed and optimal portfolios is drawn. The optimal portfolio does not need to be the same as the one in the mean-variance framework; we know (see §4) that, in the absence of constraints, it differs only in the fraction invested in the risk free component. Our test, then, accounts for the distance between the two indifference curves; the greater the distance, the greater the inefficiency. The reason why we base our work on this measure is that, in the presence of market frictions, it is no longer true that the Sharpe ratio is an adequate quantity to assess the efficiency and, at the same time, the simple difference between variances considers just part of the available information.

Brennan and Torous (1999) analyze the same problem in a portfolio choice framework with a power utility function and come up with a measure of compensative variation that calculates the amount of wealth wasted when adopting a suboptimal portfolio allocation strategy; the same concept is used in Das and Uppal (2004) when assessing the relevance of systemic risk in portfolio choice.

In the following sections we show how this measure of compensative variation can be used to develop an efficiency test whose validity is not affected by the presence of equality and/or inequality constraints on the portfolio asset shares.

2.1. An approach based on utility comparison

According to Brennan and Torous (1999), an investor is concerned with maximizing the expected value of a power utility function defined over her wealth at the end of the next period:

$$U\left(W_{t+dt}\right)=\frac{W_{t+dt}^{1-\gamma}-1}{1-\gamma}$$

where $\gamma>0$ is the relative risk aversion (RRA) coefficient and W_{t+dt} the wealth at time $t+dt$.

Our investor holds a benchmark b[4]. We assume that the price P_t^b at time t of the benchmark follows the stochastic differential equation

(1) $$\frac{dP_t^b}{P_t^b}=\mu_b dt+\sigma_b d\beta_t^b=\left(\eta_b+r_0\right)dt+\sigma_b d\beta_t^b$$

where μ_b (expected return) and σ_b (standard deviation) are constants, and $d\beta_t^b$ is the increment to a univariate Wiener process. In this framework, the overall wealth W_t evolves with P_t^b:

$$\frac{dW_t}{W_t}=\frac{dP_t^b}{P_t^b}$$

[4] A standard against which the performance of a security, index or investor can be measured. We use this term according to Basak et al. (2002), but we could instead consider a mutual fund, a pension fund etc.

Using a property of the geometric Brownian motion, equation (1) implies that, over any finite interval of time $[t,t+dt]$

$$W_{t+dt} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)(t+dt-t)+\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt+\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}$$

with $\beta_t^b \sim N(0,t)$. In turn this implies that W_{t+dt} is conditionally log-normally distributed:

$$W_{t+dt} \mid W_t \sim LN\left(\log(W_t) + \left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt, \sigma_b^2 dt\right)$$

with expectation

$$E\left[W_{t+dt} \mid W_t\right] = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} E\left[e^{\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}\right] =$$

$$= W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} e^{\frac{1}{2}\sigma_b^2(t+dt-t)} = W_t e^{\mu_b dt}$$

Therefore, the expected utility associated with the benchmark is given by

$$E\left[U\left(W_{t+dt}\right) \mid \mu_b, \sigma_b, \gamma, W_t\right] = \frac{1}{1-\gamma}\left(E\left[W_{t+dt}^{1-\gamma} \mid W_t\right] - 1\right) = \frac{1}{1-\gamma}\left(E\left[\left(e^{\log W_{t+dt}}\right)^{1-\gamma} \mid W_t\right] - 1\right) =$$

$$= \frac{1}{1-\gamma}\left(E\left[e^{(1-\gamma)\log W_{t+dt}} \mid W_t\right] - 1\right) = \frac{1}{1-\gamma}\left(e^{(1-\gamma)\log W_t + (1-\gamma)\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt + \frac{1}{2}\sigma_b^2(1-\gamma)^2 dt} - 1\right) =$$

$$= \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\mu_b dt - \frac{1}{2}\sigma_b^2\gamma(1-\gamma)dt} - 1\right) = \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt} - 1\right)$$

In order to study the efficiency of such an investment, an investor compares its performance with that of the best alternative: a portfolio of primitive assets. The endowment is given by one risk free asset (with return r_0) and a set of n risky assets (with return $r_i, i = 1,...,n$).

Calling w_i the fraction of wealth allocated to the *i*-eth risky asset, w the vector of w_i 's and $(1 - w'\iota)$ the residual fraction invested in the risk free asset, the overall wealth evolves as

$$\frac{dW_t}{W_t} = \left(w'\left(\mu_p - r_0\iota\right) + r_0\right)dt + \left(w'\Sigma_p w\right)^{1/2} d\beta_t =$$

$$= \left(w'\eta_p + r_0\right)dt + \left(w'\Sigma_p w\right)^{1/2} d\beta_t$$

where $d\beta_t$ is the increment to an univariate Wiener process, and μ_p and Σ_p are the vector of the expected returns and the covariance matrix of the primitive assets.

Following the computation already made for the benchmark case, the expected utility is

$$E\left[U\left(W_{t+1}\right)|\,\mu_p,\Sigma_p,\gamma,W_t\right]=\frac{1}{1-\gamma}\left(W_t^{1-\gamma}e^{(1-\gamma)\left(\left(w'\left(\mu_p-r_0\iota\right)+r_0\right)-\frac{1}{2}\gamma w'\Sigma_p w\right)dt}-1\right)$$

We consider a "buy & hold" strategy in which the investor observes the asset returns at time t and makes her choice once and forever; it is intended to represent the type of inefficiency in portfolio allocations induced by the *status quo* bias described in Samuelson and Zeckhauser (1988).

The optimal portfolio w^* is defined as

(2) $$w^*=\arg\max_w E\left[U\left(W_{t+1}\right)|\,\mu_p,\Sigma_p,\gamma,W_t\right]$$

subject to several constraints (equality, inequality, sum to one etc.) on its composition:

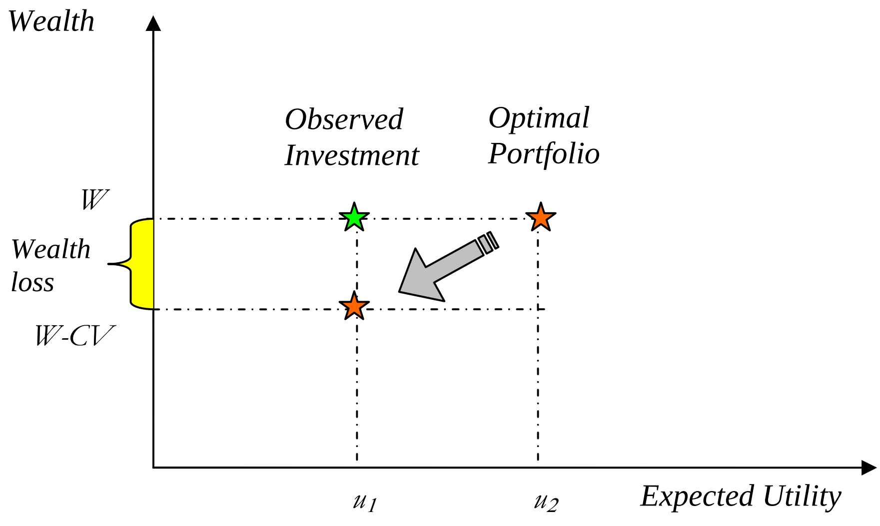
$$Aw=a$$

$$lb\le w\le ub$$

A natural way to assess the performance of the benchmark, then, is to compare its expected utility with that resulting from the optimal portfolio. In accordance with Brennan and Torous (1999) and Das and Uppal (2004), we establish this comparison by means of a compensative variation metric. In other words, we pose the question of what level of initial wealth W_t^* is needed to obtain with the optimal portfolio the same expected utility as with the benchmark and initial wealth W_t; this technique is graphically described in figure 2.

Figure 2.

Measures of efficiency – expected utility framework



Note: the figure shows the case $CV>0$ *only.*

In formulae, we impose that

$$E\left[U\left(W_{t+1}\right) \mid \mu_p, \Sigma_p, \gamma, W_t^*\right] = E\left[U\left(W_{t+1}\right) \mid \mu_b, \sigma_b, \gamma, W_t\right]$$

where we want to derive $W_t^* = W_t - CV$, with CV amount of wealth wasted (if positive) or generated (if negative) by the benchmark instead of using the best alternative.

Therefore,

$$\frac{\left(W_t - CV\right)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\left(\left(w^{*\prime}\left(\mu_p - r_0 \iota\right) + r_0\right) - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt} = \frac{W_t^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt}$$

so that

$$CV = W_t\left[1 - \exp\left\{\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt - \left(\left(w^{*\prime}\left(\mu_p - r_0\iota\right) + r_0\right) - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right]$$

or, in relative terms,

$$\begin{aligned}
cv_b &= cv\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = \frac{CV}{W_t} = \\
&= \left[1 - \exp\left\{\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt - \left(\left(w^{*\prime}\left(\mu_p - r_0\iota\right) + r_0\right) - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right] \\
&= \left[1 - \exp\left\{\left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right)dt - \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right]
\end{aligned}$$

with $cv_b \in \left(-\infty, 1\right]$. This function has a clear economic interpretation: it measures the amount of wealth that the agent wastes (if positive) or generates (if negative) with respect to the initial level of wealth, when using the benchmark instead of the best alternative. $cv_b = 1$ means that the benchmark is completely inefficient (the agent is wasting 100 percent of her wealth); $cv_b \to -\infty$, instead, means that the benchmark is totally efficient (the agent is generating infinite new wealth).

In case we want to assess the efficiency of a portfolio ω, instead of a benchmark, against the optimal portfolio w, it is easily shown that the relative wealth loss is

$$cv_p = cv\left(\eta_p, \Sigma_p, \gamma\right) = \frac{CV}{W_t} = \left[1 - \exp\left\{\left(\omega'\eta_p - \frac{1}{2}\gamma\omega'\Sigma_p\omega\right)dt - \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right]$$

with $cv_p \in \left[0,1\right]$ since the observed portfolio ω comes from the *same* space of primitive assets as the optimal portfolio w^*. When $cv_p = 0$ the agent is investing in a portfolio that does not waste any wealth; it is, in other words, efficient.

We are able to associate to cv_b and cv_p a standard error, a confidence interval and an efficiency test. This will be shown in the next section, referring primarily to the benchmark case. Before proceeding with the algebra, it turns useful to define a simpler expression:

$$\lambda_b = \lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = -\frac{1}{dt}\log\left(1 - cv_b\right) =$$
$$= \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right) - \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right) =$$

(3)
$$= \max_w \left\{\left(w'\eta_p - \frac{1}{2}\gamma w'\Sigma_p w\right) - \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right)\right\}$$

in the case of a benchmark, and likewise λ_p for the portfolio.

It is worth pointing out that the optimal weights w^* in the agent's problem (2) are the same as we would get by maximizing λ_b or λ_p in (3) subject to the same constraints. From the investor's point of view, therefore, maximizing the expected utility or its transformation is equivalent.

Below we ignore the constant term that involves dt[5], for the sake of simplicity and since it disappears when computing the test statistic.

3. Development of an efficiency test

The function $\lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)$ depends on unknown moments[6] and has to be replaced with a consistent sample estimate, defined as

(4)
$$\ell_b = \ell\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{\left(w'e_p - \frac{1}{2}\gamma w'S_p w\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)\right\}$$

subject to the constraints

$$Aw = a$$
$$lb \le w \le ub$$

We solve, therefore, the maximization problem using a function of *sample* moments instead of *true* moments. As a consequence, we need to take account of sampling errors and derive a statistical distribution for the ℓ_b function. Yet establishing its exact distribution is both cumbersome and useless. It is cumbersome because the presence of inequality constraints hinders the recourse to standard statistical procedures; it is useless as, even if we knew the exact distribution, it would in the end be a

[5] The reader can assume that $dt = 1$.
[6] Let us assume for now to know the relative risk aversion coefficient γ.

mixture of different distributions. De Roon et al. (2001), dealing with inequality constraints, conclude that their statistic is asymptotically distributed as a mixture of χ^2 distributions. Therefore, even if we computed the exact distribution, this could be used only through numerical simulation. It would in fact be exactly the same procedure we should follow in the case of not knowing the exact distribution of ℓ_b.

Another possibility is to approximate the exact distribution by means of the delta method. Following Basak et al. (2002), we can use a weak central limit theorem to establish that the first and second moments of returns are asymptotically normally distributed; we can then calculate the derivative of ℓ_b relative to $\left(e_b, s_b^2, e_p, S_p\right)$, obtaining a first-order approximation of the exact distribution of $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$. The procedure is described in detail below.

First of all, we recognize that the only source of randomness in $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is given by the non-central first and second moments of the primitive assets and the benchmark. Since working with vectors is more convenient than with matrices, once we define

$$e_b = \frac{1}{T}\sum_{t=1}^{T} e_{bt}\ ; \qquad M_b = \frac{1}{T}\sum_{t=1}^{T} M_{bt} = \frac{1}{T}\sum_{t=1}^{T} e_{bt}^2 = s_b^2 + e_b^2$$

$$e_p = \frac{1}{T}\sum_{t=1}^{T} e_{pt} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} e_{p1t} \\ \vdots \\ e_{pnt}\end{bmatrix}; \qquad M_p = S + e_p e_p' = \frac{1}{T}\sum_{t=1}^{T} M_{pt} = \frac{1}{T}\sum_{t=1}^{T} e_{pt} e_{pt}'$$

we consider the vector $\overline{X}_T$ as

(5) $$\overline{X}_T = \begin{bmatrix} \bar{e}_T \\ \overline{M}_T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} e_p \\ e_b \end{bmatrix} \\ \begin{bmatrix} vech\left(M_p\right) \\ M_b \end{bmatrix} \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} e_t \\ M_t \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \begin{bmatrix} e_{pt} \\ e_{bt} \end{bmatrix} \\ \begin{bmatrix} vech\left(M_{pt}\right) \\ M_{bt} \end{bmatrix} \end{bmatrix}$$

where the operator $vech$ takes all the distinct elements in a symmetric matrix:

$$vech\left(e_{pt}e_{pt}'\right) = \begin{bmatrix} e_{p1t}^2 & e_{p2t}e_{p1t} & \cdots & e_{pnt}e_{p1t} & e_{p2t}^2 & \cdots & e_{pnt}e_{p2t} & \cdots & e_{pnt}^2 \end{bmatrix}'$$

It is worth stressing one more time that the benchmark returns come from a different, although possibly correlated, parametric space than those for the primitive assets. As a consequence the benchmark could be either more or less efficient than the portfolio.

We require (i) $\left\{X_t, t \ge 1\right\}$ to be a sequence of stationary and ergodic random vectors with mean $E\left[X_t\right] = X = \begin{bmatrix} \eta \\ M \end{bmatrix}$ and covariance matrix $\operatorname{cov}\left(X_t\right) = \Lambda$ with Λ non-singular; this is commonly assumed in the financial economics literature.

The expected value on $\overline{X}_T$ is, therefore, $E\left[\overline{X}_T\right] = X$ and its variance is

$$
\begin{aligned}
Var\left(\overline{X}_T\right) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right) = E\left[\left(\frac{1}{T}\sum_{t=1}^{T} X_t - X\right)\left(\frac{1}{T}\sum_{t=1}^{T} X_t - X\right)'\right] = \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[\left(X_t - X\right)\left(X_s - X\right)'\right] = \\
&= \frac{1}{T^2}\left(\sum_{t=1}^{T} E\left[\left(X_t - X\right)\left(X_t - X\right)'\right] + \sum_{t=1}^{T}\sum_{s\neq t} E\left[\left(X_t - X\right)\left(X_s - X\right)'\right]\right) = \\
&= \frac{1}{T^2}\left(T\Lambda + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(E\left[\left(X_t - X\right)\left(X_s - X\right)'\right] + E\left[\left(X_s - X\right)\left(X_t - X\right)'\right]\right)\right) = \\
&= \frac{1}{T^2}\left(T\Lambda + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(Cov\left(X_t, X_s\right) + Cov\left(X_s, X_t\right)\right)\right) = \\
&= \frac{1}{T}\left(\Lambda + \sum_{t=1}^{T-1}\left(1 - \frac{t}{T}\right)\left(Cov\left(X_t, X_0\right) + Cov\left(X_0, X_t\right)\right)\right)
\end{aligned}
$$

from which the long-run covariance matrix Λ_0 is

$$\Lambda_0 = \lim_{T\to\infty} T \; Var\left(\overline{X}_T\right) = \Lambda + 2\sum_{t=1}^{\infty} Cov\left(X_t, X_0\right).$$

Note, in particular, that we do not exclude *a priori* the possibility of a correlation between the benchmark and the primitive asset returns. Since the benchmark comes from a different parametric space than the primitive assets we do, however, exclude *a priori* a perfect correlation (± 1) between the benchmark and the portfolio. The benchmark, in other words, can only be *partially* tracked by a portfolio.

We require, furthermore, that (ii) $\lim_{T\to\infty} E\left[\left|X_t\right|^{2+\delta}\right] < \infty$ and $\lim_{T\to\infty} Var\left(I'S_T\right) = \infty \quad \forall t \ge 1,\ \forall I \in \Re^n$ and $\forall \delta \in (0,1)$, where $S_T = \sum_{t=1}^{T} X_t$; (iii) $\rho_I(t) = \lim_{t\to\infty} \max\left\{Corr\left(I'Y, I'Z\right)\right\} = 0 \quad \forall Y \in \sigma\left\{X_k : k \le s\right\}$, $\forall Z \in \sigma\left\{X_k : k \ge t + s\right\}$ and $\forall I \in \Re^n$; (iv) $\sum_{t=1}^{\infty}\left\|Cov\left(X_t, X_0\right)\right\| < \infty$ and $\Lambda_0 = \Lambda + \sum_{t=1}^{\infty} Cov\left(X_t, X_0\right)$ is non-singular. Condition (ii) is similar to the Lyapounov condition, and is used to show the uniform asymptotic negligibility condition of Lindeberg for the Central Limit Theorem to hold. The total variability of the sum, S_T, on the other hand, is always required to grow to infinity. Condition (iii) ensures asymptotic independence; it is required for applying the Central Limit Theorem for non-i.i.d. random variables or random vectors. The first part of condition (iv), the finiteness condition, implies that Λ_0 exists and is finite, and that S_T in condition (ii) grows at the same rate as T. Finally the second part – the non-singularity of Λ_0 – is required to get a non-degenerate asymptotic distribution when applying the Central Limit Theorem. All these (weak) conditions are necessary to apply the result 1 in Basak et al. (2002, p. 1203) and identify a distribution for the vector $\overline{X}_T$:

$$\sqrt{T}\left(\overline{X}_T - X\right) \xrightarrow{d} N\left(0, \Lambda_0\right)$$

The second step is to obtain the asymptotic distribution of $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right) = f\left(\overline{X}_T, \gamma\right)$. In order to apply the delta method, the optimal solution λ_b in (3) has to be a smooth function of the parameters. For this to be satisfied, whenever an inequality constraint is binding, the corresponding Lagrange multiplier should be strictly positive. Following our previous notation we need, in other words, the vectors of constraints plus the vectors of Lagrange multipliers δ_2 and δ_3 associated with the inequality constraints to be strictly positive:

$$\left(w^* - lb\right) + \delta_2 > 0$$

$$\left(ub - w^*\right) + \delta_3 > 0$$

The following conditions (v) and (vi) ensure that this is the case. In order to state the assumptions, however, we need the following additional notation. Let $\{i_1, \dots, i_k, i_{k+1}, \dots i_n\}$ be any permutation of $(1, \dots, n)$. Let $\left(\Sigma^{-1}\right)_2$ be the $(n-k)\times n$ matrix consisting of the $\{i_{k+1}, \dots i_n\}$ rows of the Σ_p^{-1}; $\left(\Sigma^{-1}\right)_{22}$ be the $(n-k)\times(n-k)$ principal minor matrix which consists of $\{i_{k+1}, \dots i_n\}$ rows and columns of the Σ_p^{-1}; Σ_{11}^{-1} be the inverse of the $k \times k$ principal minor matrix corresponding to the $\{i_1, \dots i_k\}$ rows and columns of the Σ_p. Lastly, let lb_2 be the vector consisting of the $\{i_{k+1}, \dots i_n\}$ rows of lb, and ub_2 be the corresponding vector of the $\{i_{k+1}, \dots i_n\}$ rows of ub.

We thus modify assumption (7) in Basak et al. (2002) and require that (v) all the elements of the $(n-k)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2-\left(\Sigma^{-1}\right)_2\left(\eta_p-A'\delta_1\right)\right)$ are strictly positive. It turns out that this condition (v) is sufficient to ensure that $\left(w^*-lb\right)+\delta_2$ is a strictly positive vector.

Note indeed that the first order condition to the maximization problem implies that

$$\gamma\Sigma_p w=\eta_p-A'\delta_1+\delta_2-\delta_3 .$$

Hence

$$w^*=\frac{1}{\gamma}\left(\Sigma_p^{-1}\left(\eta_p-A'\delta_1\right)+\Sigma_p^{-1}\left(\delta_2-\delta_3\right)\right). \tag{6}$$

To see how assumption (v) works, suppose now that only some elements of w^* given by the subvector $\left[w^*_{i_{k+1}},\dots,w^*_{i_n}\right]$, are such that the constraint $w^*\geq lb$ holds with equality. We need to show that the corresponding subvector of Lagrange multipliers, $\left(\delta_{2i_{k+1}},\dots,\delta_{2i_n}\right)$ is positive. Without loss of generality assume that the last $(n-k)$ elements of w^* are such that the lower-inequality constraint is binding. Denote the vector of these elements as w_2^* and the rest as w_1^*, i.e., $w^*=\begin{bmatrix}w_1^{*\prime} & w_2^{*\prime}\end{bmatrix}'$; therefore $w_2^*=lb_2$. Let δ_{21} and δ_{22}, δ_{31} and δ_{32} denote the corresponding partition of the Lagrange multiplier vector, i.e., $\delta_2=\begin{bmatrix}\delta_{21}' & \delta_{22}'\end{bmatrix}'$ and $\delta_3=\begin{bmatrix}\delta_{31}' & \delta_{32}'\end{bmatrix}'$. From the constraint $w^*-lb\geq 0$, δ_{21} is a zero vector. Furthermore, $\delta_{32}=0$. Now partition Σ_p, Σ_p^{-1}, η_p and A, similarly, as

$$\Sigma_p=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}; \qquad \Sigma_p^{-1}=\begin{bmatrix}\Sigma^{11} & \Sigma^{12}\\ \Sigma^{21} & \Sigma^{22}\end{bmatrix}$$

$\eta_p=\begin{bmatrix}\eta_1' & \eta_2'\end{bmatrix}'$, and $A=\begin{bmatrix}A_1 & A_2\end{bmatrix}$, with Σ_{11} and Σ^{11} $k\times k$ matrices, η_1 and $A_1'\delta_1$ are $k\times 1$ vectors. This gives $w_2^*=\frac{1}{\gamma}\left(\begin{bmatrix}\Sigma^{21} & \Sigma^{22}\end{bmatrix}\left(\eta_p-A'\delta_1\right)+\Sigma^{22}\delta_{22}\right)=lb_2$. Thus,

$$\delta_{22}=\left(\Sigma^{22}\right)^{-1}\left(\gamma lb_2-\begin{bmatrix}\Sigma^{21} & \Sigma^{22}\end{bmatrix}\left(\eta_p-A'\delta_1\right)\right)=\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2-\left(\Sigma^{-1}\right)_2\left(\eta_p-A'\delta_1\right)\right)$$

which is positive by the assumption.

Analogously, to conclude that $\delta_{32} > 0$ when $w_2^* = ub_2$ we need the following assumption (vi): all the elements of the $(n-k)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\left(\Sigma^{-1}\right)_2\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right)$ are strictly positive. If so, $\delta_{32} = \left(\Sigma^{22}\right)^{-1}\left(\begin{bmatrix}\Sigma^{21} & \Sigma^{22}\end{bmatrix}\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right) + \delta_{22} = \left(\Sigma^{22}\right)^{-1}\left(\begin{bmatrix}\Sigma^{21} & \Sigma^{22}\end{bmatrix}\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right) > 0$.

In case conditions (v) and (vi) hold true, therefore, $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is a continuous function with a continuous first derivative in any point except for its boundaries. By means of the delta method we thus obtain

$$\sqrt{T}\left(\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right) - \lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)\right) \xrightarrow{d} N(0, V)$$

with $V = \nabla(\gamma)' \Lambda_0 \nabla(\gamma)$, where

$$\nabla(\gamma) = \frac{\partial \lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)}{\partial X} = \frac{\partial f(X, \gamma)}{\partial X}.$$

Define $\lambda\left(w, \delta \mid \overline{X}_T\right)$ as the Lagrangian and $\delta = \begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}$ as the set of Lagrange multipliers:

$$\lambda\left(w, \delta \mid \overline{X}_T\right) = \left(w'e_p - \frac{1}{2}\gamma w' S_p w\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) + \\ -\delta_1'(Aw - a) - \delta_2'(l - w) - \delta_3'(w - u)$$

By making use of the envelope theorem, the gradient $\nabla(\gamma)$ is consistently estimated by

$$D(\gamma) = \frac{\partial f\left(\overline{X}_T, \gamma\right)}{\partial \overline{X}_T} = \frac{\partial \lambda\left(w, \delta \mid \overline{X}_T\right)}{\partial \overline{X}_T}\Bigg|_{\substack{w = w^* \\ \delta = \delta^*}}.$$

The derivative is worth

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e_p' w^*\right) w^*\right)' \quad -1 - \gamma e_b \quad -\frac{1}{2}\gamma\begin{bmatrix} w_1^{*2} & 2w_1^* w_2^* & \cdots & 2w_1^* w_n^* & w_2^{*2} & \cdots & 2w_2^* w_n^* & \cdots & w_n^{*2}\end{bmatrix} \quad \frac{1}{2}\gamma\right]$$

Lastly, we replace Λ_0 with its standard heteroskedasticity and autocorrelation consistent estimate L_0 as proposed by Newey and West (1987) and make use of Bartlett-type weights:

$$L_0 = \hat{\Omega}_0 + \sum_{j=1}^{m}\left(1 - \frac{j}{m+1}\right)(\hat{\Omega}_j + \hat{\Omega}_j')$$

with

$$\hat{\Omega}_j = \frac{1}{T}\sum_{t=j+1}^{T}\left(X_t - \overline{X}_T\right)\left(X_{t-j} - \overline{X}_T\right)'$$

and m the number of lags to be considered. As suggested by Newey and West (1994), good asymptotic properties can be achieved by using the automatic lag selection rule

(7) $$m = \text{int}\left(4\left(\frac{T}{100}\right)^{2/9}\right).$$

Consider therefore the statistic

(8) $$t = T^{1/2}\frac{\ell_b - \lambda_b}{\hat{V}^{1/2}} = T^{1/2}\frac{\ell_b - \lambda_b}{\left(D(\gamma)' L_0 D(\gamma)\right)^{1/2}}$$

Under the null hypothesis $H_0 : \lambda_b = \lambda_0$, $t \overset{a}{\sim} N(0,1)$. Notice that the null can be equivalently written as $H_0 : cv_b = cv_0 = 1 - \exp\{-\lambda_0\}$. This second specification highlights a shortcoming of this procedure: since $cv_b \in (-\infty, 1]$, we are not able to test whether $cv_b = 1$. A similar issue arises in Snedecor and Cochran (1989), when trying to test a null hypothesis on a variance $\sigma^2 = 0$. In their framework, a statistic with an exact distribution exists for any value of the variance, except for $\sigma^2 = 0$, i.e., on the boundary of the feasible set. An analogous situation is reported in Kim et al. (2005) when dealing with Sharpe-style regressions, used to investigate issues such as style composition, style sensitivity and style change over time. The method employed to obtain the distribution and confidence intervals of the style coefficients are statistically valid only when none of the true style weights are zero or one. In practice, it seems to be quite plausible to have zero or one as the values of some style weights. In our framework, nevertheless, such a hypothesis is not economically relevant: it is, indeed, hard to imagine a benchmark, however badly managed, able to dissipate *all* the wealth. We can, however, test any other hypothesis, and in particular if $cv_b = 0$, that is, if the benchmark can perfectly replicate the performance of the optimal portfolio.

Since we know the large sample distribution for $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$, a confidence interval for λ_b is derived by

$$\begin{aligned}\alpha &= P\left(z_{\frac{\alpha}{2}} \le T^{1/2}\frac{\ell_b - \lambda_0}{\hat{V}^{1/2}} \le z_{1-\frac{\alpha}{2}}\right) = \\ &= P\left(\ell_b - z_{1-\frac{\alpha}{2}}\frac{\hat{V}^{1/2}}{T^{1/2}} \le \lambda_0 \le \ell_b + z_{1-\frac{\alpha}{2}}\frac{\hat{V}^{1/2}}{T^{1/2}}\right)\end{aligned}$$

where $z_{1-\frac{\alpha}{2}}$ is the $1-\frac{\alpha}{2}$-eth percentile of a standard normal distribution.

Since $cv_0 = 1 - \exp\{-\lambda_0\}$, a confidence interval for the wealth loss is

$$CI(cv_0) = \left\{ cv_b : cv_b \in \left[1 - \exp\left\{ -\left(\ell_b - z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\}, 1 - \exp\left\{ -\left(\ell_b + z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\} \right] \right\}.$$

If we are interested in testing portfolio efficiency, once we define

$$\overline{X}_T = \frac{1}{T} \sum_{t=1}^{T} X_t = \begin{bmatrix} e_p \\ vech(M_p) \end{bmatrix} = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e_{pt} \\ vech(M_{pt}) \end{bmatrix}$$

it is straightforward to see that

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e_p' w^* \right) w^* \right)' \quad -\frac{1}{2}\gamma \left[w_1^{*2} \quad 2w_1^* w_2^* \quad \cdots \quad 2w_1^* w_n^* \quad w_2^{*2} \quad \cdots \quad 2w_2^* w_n^* \quad \cdots \quad w_n^{*2} \right] \right] +$$
$$- \left[\left(\omega + \left(\gamma e_p' \omega \right) \omega \right)' \quad -\frac{1}{2}\gamma \left[\omega_1^2 \quad 2\omega_1\omega_2 \quad \cdots \quad 2\omega_1\omega_n \quad \omega_2^2 \quad \cdots \quad 2\omega_2\omega_n \quad \cdots \quad \omega_n^2 \right] \right]$$

and that a confidence interval for the wealth loss cv_0 is

$$CI(cv_0) = \left\{ cv_p : cv_p \in \left[1 - \exp\left\{ -\left(\ell_p - z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\}, 1 - \exp\left\{ -\left(\ell_p + z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\} \right] \right\}$$

This specification of the test does not hold true for cv_p equal to 0 or 1; in this context, therefore, we are not allowed to test either $H_0 : cv_0 = 1$ or $H_0 : cv_0 = 0$. In particular, we cannot test whether the observed portfolio is efficient or not. As in Snedecor and Cochran (1989), however, we may rely on the confidence interval to cv_0 and check how far its lower (upper) boundary is from zero (one).

4. Closed-form solutions with no inequality constraints

The expression of the test derived in §3 still depends on the optimal portfolios. We are able to establish their closed-form expression only in the simplest settings, with no inequality constraints; otherwise we have to rely on numerical solutions. For instance, a Matlab® code which implements the function `quadprog` can solve the problem numerically.

The closed-form solution is feasible when i) there are no constraints at all or ii) there are only equality constraints. Below we consider the two cases separately. We establish, moreover, that a strong relationship between standard mean-variance and utility paradigms exists; the link is provided by deriving the optimal portfolios. In the next part we show the results taking into account only the benchmark case; analogous results apply in the portfolio framework.

4.1. No constraints

We call $\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ the difference between utilities in the case of no constraints:

$$\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{\left(w'e_p - \frac{1}{2}\gamma w'S_p w\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)\right\}$$

Deriving $\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ with respect to w we get:

$$\frac{\partial \ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)}{\partial w} = e_p - \gamma S_p w = 0$$

so that the optimal weights are

(9) $$w_{NO}^* = \frac{1}{\gamma} S_p^{-1} e_p$$

Replacing expression (9) into $\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ we have then

$$\begin{aligned}\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right) &= \left(w_{NO}^{*}{}' e_p - \frac{1}{2}\gamma w_{NO}^{*}{}' S_p w_{NO}^*\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\ &= \left(\frac{1}{\gamma} e_p{}' S_p^{-1} e_p - \frac{1}{2\gamma} e_p{}' S_p^{-1} e_p\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\ &= \frac{1}{2\gamma} e_p{}' S_p^{-1} e_p - \left(e_b - \frac{1}{2}\gamma s_b^2\right)\end{aligned}$$

We recognize in equation (9) an expression similar to that in the standard mean-variance analysis with no restrictions, where the optimal portfolio can be any of the infinite ones with the highest Sharpe ratio. The weights of the optimal portfolio with the same excess return r_b as the benchmark are then given by

(10) $$w_{NO}^{BJS} = \frac{r_b S_p^{-1} e_p}{e_p' S_p^{-1} e_p}$$

This expression identifies the optimal portfolio used in Basak et al. (2002), where an agent aims at minimizing the variance of her investment given the expected return r_b.

It can be shown that, when we impose that the portfolio weights sum to one then:

- the Sharpe ratio of the portfolio (10) is equivalent to the Sharpe ratio of the tangency portfolio (TP),

$$w_{TP}^{BJS} = \frac{S_p^{-1} e_p}{\iota' S_p^{-1} e_p}$$

- the optimal portfolio that maximizes the expected utility of the agent with the highest optimal expected utility is the tangency portfolio, i.e., exactly the same portfolio we have in a standard mean-variance setting.

The optimal portfolio resulting in our expected utility framework in the case of no constraint is: i) equivalent to the optimal one in the mean-variance framework if there are no risk free assets and ii) is otherwise proportional. Indeed, both equations (9) and (10) share the same numerator $S_p^{-1} e_p$; the different denominators just normalize the weights. In other words, the importance of the two quantities

$$\frac{e_p' S_p^{-1} e_p}{r_b}; \qquad \gamma$$

is in defining what fraction of wealth, if any, should be invested in the risky assets and consequently in the risk free; the relationship between risky shares is instead kept fixed. This implies that the two portfolios are on the same efficient frontier; see for instance the two optimal portfolios in figure 1. According to the two fund separation theorem, they could be seen as a combination of the tangency risky portfolio and a risk free asset.

4.2. Equality constraints only

If, instead, we define the function $\ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ that takes account of equality constraints on some of the optimal portfolio weights,

$$\ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{ \left(w' e_p - \frac{1}{2}\gamma w' S_p w \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right) \right\}$$

subject to

$$Aw = a$$

the Lagrangian is

$$\lambda\left(w, \delta_1 \mid \overline{X}_T\right) = w' e_p - \frac{1}{2}\gamma w' S_p - w e_b + \frac{1}{2}\gamma s_b^2 - \delta_1' \left(Aw - a\right)$$

If we take the derivative with respect to w,

$$\frac{\partial \lambda\left(w, \delta_1 \mid \overline{X}_T\right)}{\partial w} = 0 \Rightarrow -e_p + \gamma S_p w - A' \delta_1 = 0$$

(11) $$w = \frac{1}{\gamma} S_p^{-1} \left(e_p + A' \delta_1\right)$$

and with respect to δ_1,

$$\frac{\partial \lambda\left(w,\delta_1 \mid \overline{X}_T\right)}{\partial \delta_1} = 0 \Rightarrow Aw = a$$

we face a system of two equations that can be solved premultiplying (10) by A,

$$\begin{aligned} Aw = a &= \frac{1}{\gamma} A S_p^{-1}\left(e_p + A'\delta_1\right) \\ \Rightarrow \delta_1^* &= \left(AS_p^{-1}A'\right)^{-1}\left(\gamma a - AS_p^{-1}e_p\right) \end{aligned}$$

from which

$$\begin{aligned} w_{EQ}^* &= \frac{1}{\gamma} S_p^{-1}\left(e_p + A'\delta_1^*\right) = \\ &= \frac{1}{\gamma}\left(I - S_p^{-1}A'\left(AS_p^{-1}A'\right)^{-1}A\right)S_p^{-1}e_p + S_p^{-1}A'\left(AS_p^{-1}A'\right)^{-1}a = \frac{1}{\gamma}Q + q \end{aligned}$$

Replacing this expression in the objective function we have

$$\begin{aligned} \ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right) &= \left(w_1^{*\prime}e_p - \frac{1}{2}\gamma w_1^{*\prime}S_p w_1^*\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\ &= \left(\frac{1}{\gamma}Q'e_p + q'e_p - \frac{1}{2}\gamma\left(\frac{1}{\gamma^2}Q'S_pQ + \frac{1}{\gamma}Q'S_pq + \frac{1}{\gamma}q'S_pQ + q'S_pq\right)\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\ &= \left(\frac{1}{\gamma}Q'e_p + q'e_p - \frac{1}{2\gamma}Q'S_pQ - \frac{1}{2}Q'S_pq - \frac{1}{2}q'S_pQ - \frac{1}{2}\gamma q'S_pq\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) \end{aligned}$$

In order to make a comparison with the existing literature, it turns helpful to split the primitive assets in two groups[7]:

$$w = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}; \qquad e_p = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}; \qquad S_p = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix}$$

and to deal with the constraint

$$\tilde{w}_2 = \tilde{\omega}_2 .$$

After some algebra we obtain

$$\tilde{w}_1^* = \frac{1}{\gamma} S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \tag{12}$$

In selecting the optimal values, an agent has then to take into account a hedge term against the constrained assets. It is interesting to deal with an equality constraint because it allows us to model the

[7] This setting was used in Gourieroux and Jouneau (1999). Their statistic stems from a restricted mean-variance space, where the unconstrained portfolio shares are normalized by the constrained shares.

presence of transaction costs in some assets that, for this reason, are illiquid. For instance, using Italian data and the Gourieroux and Jouneau (GJ, 1999) test, Pelizzon and Weber (2003) observe that housing is an important part (nearly 80%) of the overall wealth of Italian households, and the efficiency greatly improves when real assets are taken as a fixed component of the overall portfolio. Bucciol (2003) bears out their results and shows that the efficiency improves further when inequality constraints are also taken into account.

In a setting à la Gourieroux and Jouneau (1999), we would be given the optimal portfolio as

$$w_{EQ}^{GJ} = w_{EQ}^{BJS} = \begin{cases} \dfrac{\left(r_b - \tilde{\omega}_2'\left(\tilde{e}_2 - S_{12}'S_{11}^{-1}\tilde{e}_1\right)\right)}{\tilde{e}_1'S_{11}^{-1}\tilde{e}_1} S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

where r_b is the expected excess return on the observed portfolio. Given the expected return r_b the optimal portfolio is exactly the same when computed with the test of Basak et al. (2002).

Moreover, with the restriction on the sum of weights

$$w_{TP}^{BJS} = \begin{cases} \left(\dfrac{1 - \iota'\tilde{\omega}_2 + \iota'S_{11}^{-1}S_{12}\tilde{\omega}_2}{\iota'S_{11}^{-1}\tilde{e}_1}\right) S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

In our utility framework, instead, extending equation (11) to all the primitive assets, the optimal portfolio is given by

$$w_{EQ}^{*} = \begin{cases} \dfrac{1}{\gamma} S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

and, in the case we require the sum to one,

$$w_{EQ}^{**} = \begin{cases} \left(\dfrac{1 - \iota'\tilde{\omega}_2 + \iota'S_{11}^{-1}S_{12}\tilde{\omega}_2}{\iota'S_{11}^{-1}\tilde{e}_1}\right) S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

i.e., exactly the same equation obtained in a setting à la Basak et al. (2002). Without imposing the sum to one, the only difference with GJ and BJS tests is, as before, in the normalization term: on the one hand, we have the expression

$$\frac{\left(r_b - \tilde{\omega}_2'\left(\tilde{e}_2 - S_{12}'S_{11}^{-1}\tilde{e}_1\right)\right)}{\tilde{e}_1'S_{11}^{-1}\tilde{e}_1}$$

whereas, on the other, we have only the term γ. The same remarks made in §4.1 apply here.

In summary, despite slight differences the behavior in a setting with no inequality constraints is similar to the mean-variance framework. If we add inequality constraints, instead, we do not have any closed-form solution for the optimal portfolios, and therefore we are not able to make any analytical comparison.

5. The relative risk aversion parameter

The knowledge of the relative risk aversion parameter γ is critical to asset allocation choice since it is decisive in determining the level of investment in risky assets, as we see for example in equation (9).

By definition, γ depends neither on time nor wealth:

$$\gamma = -W_t \frac{U''(W_t)}{U'(W_t)}$$

It is well known, however, (see Stutzer, 2004, for a review) that its exact value for an investor is as hard to know as it is to estimate it through an *ad hoc* question. Rabin and Thaler (2001) believe that any method used to measure a coefficient of relative risk aversion is doomed to failure, since *«the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of relative risk aversion, so it is a waste of time to try to measure it».*

In this section we show that it is possible to provide an estimate of the relative risk aversion parameter γ within this framework. Our procedure is closely related to that in Gourieroux and Monfort (2005); they test their hypothesis using a statistic which depends on an exogenous preference parameter. Should the parameter not *a priori* be given, they obtain an estimate by minimizing the statistic with respect to such a parameter. In our setting, the role of the preference parameter is played by γ, the risk aversion coefficient. By solving a similar problem for the objective function we can empirically find the implied risk aversion parameter, the one for which the welfare loss is minimized. Under the hypothesis that the portfolio is managed in order to maximize the expected utility function, the estimator $\hat{\gamma}$ then provides a consistent estimate for the utility function.

It is straightforward to develop a procedure for deriving $\hat{\gamma}$ in a portfolio setting. Since the function $\ell\left(e_p, S_p, \gamma\right)$ is always non-negative, we can estimate γ by choosing the value that makes the objective function as small as possible, i.e., leads to the lowest inefficiency. In formulae, we solve

$$\hat{\gamma} = \arg\min_{\gamma} \max_{w} \left\{ \left(w' e_p - \frac{1}{2} \gamma w' S_p w \right) - \left(\omega' e_p - \frac{1}{2} \gamma \omega' S_p \omega \right) \right\}$$

subject to several constraints.

5.1. No constraints

If there are no restrictions, the optimal γ is chosen by

$$\hat{\gamma}_{NO} = \arg\min_{\gamma} \left\{ \frac{1}{2\gamma} e_p{}' S_p^{-1} e_p + \frac{1}{2} \gamma \omega' S_p \omega - \omega' e_p \right\}$$

It leads us to the first order condition

$$\frac{1}{2} \omega' S_p \omega - \frac{1}{2\gamma^2} e_p{}' S_p^{-1} e_p = 0$$

which implies

(13) $$\hat{\gamma}_{NO} = \left(\frac{e_p{}' S_p^{-1} e_p}{\omega' S_p \omega} \right)^{1/2}$$

Knowing its analytical expression, we can also derive a standard error and a confidence interval for $\hat{\gamma}_{NO}$, making use of the same results in §3.

Let us define

$$\overline{Y}_T = \begin{bmatrix} e_p \\ vech\left(S_p\right) \end{bmatrix} = g\left(\begin{bmatrix} e_p \\ vech\left(S_p + e_p e_p'\right) \end{bmatrix} \right) = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right) = \frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T} = \begin{bmatrix} I_n & \mathbf{0}_{n \times \frac{n(n+1)}{2}} \\ \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} & I_{\frac{n(n+1)}{2}} \end{bmatrix}$$

and

$$G_i = -\begin{bmatrix} \mathbf{0}_{(n-i+1)\times(i-1)} & \begin{bmatrix} e_{pi} \\ e_{pi+1} \\ \vdots \\ e_{pn} \end{bmatrix} & \mathbf{0}_{(n-i+1)\times(n-i)} \end{bmatrix} - e_{pi} \begin{bmatrix} \mathbf{0}_{(n-i+1)\times(i-1)} & I_{n-i+1} \end{bmatrix}$$

Moreover,

$$Z\left(\overline{Y}_T\right)' = \left(\frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T}\right)' =$$

$$= \left[\underset{1\times n}{\mathbf{0}} \quad \frac{1}{2\left(\omega' S_p \omega\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}} OT\right] - \left[\frac{e_p' S_p^{-1}}{\left(\omega' S_p \omega\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}} \quad \frac{\left(e_p' S_p^{-1} e_p\right)^{1/2}}{2\left(\omega' S_p \omega\right)^{3/2}} WT\right]$$

where

$$OT = \left[\omega_1^2 \quad 2\omega_1\omega_2 \quad \cdots \quad 2\omega_1\omega_n \quad \omega_2^2 \quad \cdots \quad 2\omega_2\omega_n \quad \cdots \quad \omega_n^2\right]$$

$$WT = \left[e_p' DS_{11} e_p \quad e_p' DS_{12} e_p \quad \cdots \quad e_p' DS_{1n} e_p \quad e_p' DS_{22} e_p \quad \cdots \quad e_p' DS_{23} e_p \quad \cdots \quad e_p' DS_{nn} e_p\right]$$

and DS_{ij} denotes the derivative of the (i, j)-eth element of S_p^{-1}.

Therefore, the standard error for $\hat{\gamma}_{NO}$ is

$$s.e.\left(\hat{\gamma}_{NO}\right) = \frac{1}{T^{1/2}}\left(Z\left(\overline{Y}_T\right)' V\left(\overline{X}_T\right) L_0 V\left(\overline{X}_T\right)' Z\left(\overline{Y}_T\right)\right)^{1/2}$$

and, applying the central limit theorem, a confidence interval for $\hat{\gamma}_{NO}$ is

$$CI\left(\hat{\gamma}_{NO}\right) = \left\{\gamma > 0 : \gamma \in \left[\hat{\gamma}_{NO} - z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right), \hat{\gamma}_{NO} + z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right)\right]\right\}$$

5.2. Other constraints

Analogously, in the case of equality constraints only, it is necessary to solve

$$\hat{\gamma}_{EQ} = \arg\min_{\gamma}\left\{\left(\frac{1}{\gamma} Q' e_p + q' e_p - \frac{1}{2\gamma} Q' S_p Q - \frac{1}{2} Q' S_p q - \frac{1}{2} q' S_p Q - \frac{1}{2}\gamma q' S_p q\right) - \left(\omega' e_p - \frac{1}{2}\gamma \omega' S_p \omega\right)\right\}$$

Deriving with respect to γ,

$$\frac{1}{2}\omega' S_p \omega - \frac{1}{\gamma^2} Q' e_p + \frac{1}{2\gamma^2} Q' S_p Q - \frac{1}{2} q' S_p q = 0$$

(14) $$\hat{\gamma}_{EQ} = \left(\frac{2Q' e_p - Q' S_p Q}{\omega' S_p \omega - q' S_p q}\right)^{1/2}$$

although it is not always true that a real solution exists (it should be $\omega' S_p \omega < q' S_p q$)

When inequality constraints are also present, it is no longer possible to find an exact expression for the estimate of the risk aversion parameter γ; we know, nevertheless, that the function

$$\min_{\gamma} \ell\left(e_p, S_p, \gamma\right) = \min_{\gamma} \max_{w} \left\{ \left(w' e_p - \frac{1}{2}\gamma w' S_p w \right) - \left(\omega' e_p - \frac{1}{2}\gamma \omega' S_p \omega \right) \right\}$$

determines the first order condition

$$\frac{\partial \ell\left(e_p, S_p, \gamma\right)}{\partial \gamma} = \frac{\partial \lambda\left(e_p, S_p, \delta\right)}{\partial \gamma} \Big| w = w^* = \frac{1}{2}\omega' S_p \omega - \frac{1}{2} w^*\left(\gamma\right)' S_p w^*\left(\gamma\right) = 0$$

The optimal γ, therefore, is implicitly defined by the equation

$$\omega' S_p \omega = w^*\left(\gamma\right)' S_p w^*\left(\gamma\right)$$

The same argument does not hold for the benchmark case, where the function $\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ can indeed take on both positive and negative values. In this case we might consider either the value that maximizes the objective function (i.e., the benchmark gets the highest efficiency), or the value that makes the objective function null (i.e., the benchmark is as efficient as the optimal portfolio).

When we consider the value of γ that maximizes the objective function, we simply need to adjust the formulae already derived in the portfolio context:

$$\hat{\gamma}_{NO} = \left(\frac{e'_p S_p^{-1} e_p}{s_b^2} \right)^{1/2} \quad ; \quad \hat{\gamma}_{EQ} = \left(\frac{2Q' e_p - Q' S_p Q}{s_b^2 - q' S_p q} \right)^{1/2}$$

In the presence of inequality constraints, the same situation arises.

For $\hat{\gamma}_{NO}$ we can also define the standard error, corresponding to the one obtained in a portfolio setting. What changes is that

$$\overline{Y}_T = \begin{bmatrix} e_p \\ e_b \\ vech\left(S_p\right) \\ s_b^2 \end{bmatrix} = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right)=\frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T}=\begin{bmatrix} I_n & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times\frac{n(n+1)}{2}} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 & \mathbf{0}_{1\times\frac{n(n+1)}{2}} & 0 \\ \begin{bmatrix} G_1 \\ \vdots \\ G_k \end{bmatrix} & \mathbf{0}_{\frac{n(n+1)}{2}\times 1} & I_{\frac{n(n+1)}{2}} & \mathbf{0}_{\frac{n(n+1)}{2}\times 1} \\ \mathbf{0}_{1\times n} & -2e_b & \mathbf{0}_{1\times\frac{n(n+1)}{2}} & 1 \end{bmatrix}$$

Finally,

$$Z\left(\overline{Y}_T\right)=\frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T}=\begin{bmatrix} \dfrac{S_p^{-1}e_p}{\left(s_b^2\right)^{1/2}\left(e_p'S_p^{-1}e_p\right)^{1/2}} \\ 0 \\ \dfrac{1}{2\left(s_b^2\right)^{1/2}\left(e_p'S_p^{-1}e_p\right)^{1/2}}WT \\ -\dfrac{\left(e_p'S_p^{-1}e_p\right)^{1/2}}{2\left(s_b^2\right)^{3/2}} \end{bmatrix}$$

6. Empirical analysis

We perform two separate empirical analyses on the efficiency of a benchmark and of a portfolio. As a benchmark we use the S&P 500 index[8] against a set of ten industry portfolios representative of the U.S. market[9]. The industry is divided into non-durable, durable, manufacturing, energy, hi-tech, telecommunication, shops, health, utilities and other sectors. We consider monthly returns that cover the period February 1950 through May 2005 (664 observations).

Table 1 reports some descriptive statistics for our sample; we notice from panel A that the expected return of the benchmark is lower than that of any other primitive asset. This fact has a critical impact on obtaining the optimal portfolio when several constraints are required. In a Basak et al. (2002) framework, for instance, the efficient portfolio must have the same mean as the benchmark. If using these data we also impose short-sale constraints, the problem cannot be solved, since it is not possible to obtain any portfolio with such a low mean.

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[8] Downloaded from http://www.yahoo.com.
[9] Average value-weighted returns, taken from Kenneth French's website:
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

In panel B we notice, moreover, that the utilities industry sector guarantees a lower variance than the benchmark. This asset therefore *dominates* the benchmark. We consequently expect the benchmark to be an inefficient financial instrument and that our test will detect a high wealth loss.

Table 1.

Descriptive statistics for industry portfolios and benchmark returns

Panel A: Mean

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Benchmark
	1.0836	1.0241	1.0144	1.1960	1.1993	0.9088	1.0342	1.1782	0.9312	1.0836	0.7274

Panel B: Covariance (normal) and correlation (italic) of percentage returns

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Benchmark
NoDur	17.892	*0.64166*	*0.81769*	*0.49454*	*0.57554*	*0.62724*	*0.83830*	*0.74980*	*0.63453*	*0.82366*	*0.82558*
Durbl	14.968	30.414	*0.78544*	*0.46413*	*0.62017*	*0.57020*	*0.74695*	*0.49183*	*0.45877*	*0.75108*	*0.78984*
Manuf	16.379	20.512	22.425	*0.62428*	*0.74151*	*0.61882*	*0.82430*	*0.72566*	*0.54838*	*0.89333*	*0.91494*
Enrgy	10.578	12.943	14.948	25.569	*0.41925*	*0.39057*	*0.44976*	*0.44836*	*0.54592*	*0.60415*	*0.68284*
HiTec	15.791	22.185	22.777	13.751	42.075	*0.59744*	*0.6874*	*0.63587*	*0.31595*	*0.71070*	*0.80680*
Telcm	11.334	13.434	12.518	8.4369	16.555	18.250	*0.6568*	*0.54124*	*0.53258*	*0.67322*	*0.74720*
Shops	17.389	20.201	19.142	11.153	21.866	13.759	24.049	*0.66010*	*0.51009*	*0.84001*	*0.84336*
Hlth	15.757	13.475	17.072	11.263	20.491	11.487	16.082	24.681	*0.47911*	*0.71912*	*0.76917*
Utils	10.262	9.6734	9.9285	10.554	7.8355	8.6987	9.5639	9.1005	14.618	*0.61203*	*0.60763*
Other	17.037	20.256	20.687	14.939	22.543	14.064	20.144	17.471	11.443	23.914	*0.91609*
Benchmark	14.430	18.000	17.904	14.268	21.625	13.190	17.090	15.790	9.600	18.512	17.076

Using these data, we compute the optimal portfolios for our t test with different levels of risk aversion, imposing different constraints (nothing, non-negativity constraints, equality constraints, both kinds of constraints). As equality constraints, we require the following:

$$\begin{aligned} w_{Hlth} &= 0.1 \\ w_{Enrgy} + w_{Utils} &= 0.2 \end{aligned}$$

to represent a commitment to invest a fixed amount of wealth in “socially useful” industries, disregarding their performance. This choice is also motivated by the evidence that, in most cases, these two constraints would be binding in the optimal portfolios with no equality constraints (see table 2). Such constraints are also compatible with a naive investment strategy.

In table 2 we thus report the optimal portfolios for different objective functions and different constraints. For each portfolio it is necessary for the weights to sum to one, i.e., there is no risk free asset. Therefore, when we refer to the unconstrained case, we mean that one equality constraint (the sum to one of the weights) actually holds. Without inequality constraints, the optimal portfolios hold several short positions (1 to 3, according to the level of γ). Such portfolios provide the best performance, but are typically unfeasible in reality, and to compare them with an observed benchmark or an observed portfolio would be misleading. By imposing non-negativity constraints, the optimal portfolios turn out to be composed of only a subset of assets; four primitive assets in particular (durable, manufacturing, shops, other sectors) are never in the investment decisions. Not surprisingly, these are the assets which offer the lowest return/risk profiles, or that correlate highly with other assets.

Table 2.

Optimal portfolios

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
No constraints										
μ−Σ	48.5868	14.8428	-43.7188	35.5916	12.9201	13.0905	-3.4854	23.8141	25.7680	-27.4096
BJS*	-30.3313	-6.1700	89.4892	-18.7369	-22.8688	63.9575	19.3286	-8.6478	79.3282	-65.3488
γ=1	267.2189	73.0559	-412.7541	186.1017	112.0687	-127.8298	-66.6888	113.7455	-122.6137	77.6959
γ=2	144.1036	40.2751	-204.9444	101.3470	56.2365	-48.4753	-31.0980	63.1037	-39.0576	18.5093
γ=5	70.2345	20.6067	-80.2585	50.4942	22.7372	-0.8626	-9.7435	32.7186	11.0761	-17.0027
γ=10	45.6114	14.0505	-38.6966	33.5433	11.5708	15.0083	-2.6253	22.5902	27.7873	-28.8400
γ=20	33.2999	10.7725	-17.9156	25.0678	5.9876	22.9438	0.9338	17.5260	36.1429	-34.7586
Non-negativity constraints										
γ=1	0	0	0	52.1807	10.9537	0	0	36.8656	0	0
γ=2	0	0	0	49.8126	7.4591	0	0	42.7282	0	0
γ=5	23.4332	0	0	33.9227	3.5890	0.0102	0	24.2651	14.7799	0
γ=10	17.0364	0	0	23.0100	0	14.6349	0	16.4329	28.8858	0
γ=20	13.5149	0	0	17.1718	0	20.9590	0	11.6415	36.7128	0
Equality constraints (Hlth= 10%, Enrgy + Utils = 20%)										
BJS*	-26.1586	-1.2672	99.2919	-31.5658	-34.3632	79.0682	11.4805	10	51.5658	-58.0516
γ=1	355.2869	54.8451	-373.2211	173.0075	126.1979	-104.9604	-93.0068	10	-153.0075	104.8585
γ=2	199.0107	31.8562	-179.6351	89.1949	60.4168	-29.5649	-50.1989	10	-69.1949	38.1151
γ=5	105.2451	18.0628	-63.4835	38.9074	20.9482	15.6724	-24.5142	10	-18.9074	-1.9309
γ=10	73.9898	13.4651	-24.7663	22.1449	7.7920	30.7515	-15.9526	10	-2.1449	-15.2796
γ=20	58.3622	11.1662	-5.4077	13.7636	1.2139	38.2911	-11.6718	10	6.2364	-21.9539
Non-negativity and Equality constraints (Hlth= 10%, Enrgy + Utils = 20%)										
γ=1	27.8857	0	0	20	42.1143	0	0	10	0	0
γ=2	48.3044	0	0	20	21.6956	0	0	10	0	0
γ=5	52.5340	0	0	20	6.6560	10.8100	0	10	0	0
γ=10	42.7189	0	0	16.6937	0	27.2811	0	10	3.3063	0
γ=20	36.0484	0.7459	0	9.8075	0	33.2058	0	10	10.1925	0

* *Optimal portfolio in a mean-variance setting with the same expected return as the benchmark.*

With the combination of all restrictions, the equality constraint on the sum of the weights associated with the energy and utilities sectors leads to hold a null position on the utilities sector when γ is small.

In table 3 we summarize the first two moments of returns on optimal portfolios. We observe that, once γ increases, the expected return and the standard deviation of optimal portfolios in a t test setting decrease, but in such a way that the Sharpe ratio grows. On the other hand, the Sharpe ratio for the optimal portfolio in a BJS setting is always much lower, meaning that, when fixing the level of expected utility, we neglect important information for optimal portfolio choice.

Notice, moreover, that there is little difference in performance when equality constraints are added. When inserting non-negativity constraints, instead, the portfolio shares are completely different, so is their performance. We should expect the Sharpe ratio for an optimal portfolio under t test to be lower with more constraints; yet our test compares utility levels, not Sharpe ratios. For this reason it may happen that a Sharpe ratio is higher in a world with more constraints.

In the same table we then provide a numerical value for the utility loss, computed as

$$\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \left(w^{*\prime} e_p - \frac{1}{2}\gamma w^{*\prime} S_p w^*\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)$$

The utility loss indeed decreases when we add more constraints.

Table 3.

First two moments of the optimal portfolio returns

%	Mean	Std. Dev.	Sharpe	Utility Loss	Mean	Std. Dev.	Sharpe	Utility Loss
	No constraints				Non-negativity constraints			
μ–Σ	1.1221	3.5463	31.6414	-	-	-	-	-
BJS	0.7278	4.0471	17.9832	-	-	-	-	-
γ=1	2.2155	11.5833	19.1267	0.9025	1.1898	4.2841	27.7725	0.4559
γ=2	1.5998	6.4665	24.7398	0.6247	1.1886	4.2638	27.8765	0.4499
γ=5	1.2303	3.9944	30.8006	0.5303	1.1263	3.8108	29.5555	0.4621
γ=10	1.1072	3.5016	31.6198	0.6193	1.0554	3.5204	29.9795	0.5608
γ=20	1.0456	3.3671	31.0534	0.8895	1.0213	3.4470	29.6287	0.8107
	Equality constraints				Non-negativity and equality constraints			
BJS	0.7274	4.2342	17.1786	-	-	-	-	-
γ=1	2.1218	11.2532	18.8551	0.8465	1.1643	4.5114	25.8071	0.4204
γ=2	1.5505	6.3986	24.2322	0.5842	1.1406	4.0998	27.8219	0.4157
γ=5	1.2077	4.1168	29.3372	0.4829	1.1043	3.8465	28.7101	0.4333
γ=10	1.0935	3.6770	29.7389	0.5426	1.0591	3.6701	28.8568	0.5107
γ=20	1.0364	3.5585	29.1233	0.7477	1.0300	3.6104	28.5301	0.7042

Note: the benchmark has a mean of 0.72738, a standard deviation of 4.1292 and a Sharpe ratio of 0.17616.

Equality constraints:Health = 10%, Energy + Utilities = 20%.

In figure 3 we plot the optimal portfolios for the t test and their indifference curves against the benchmark; figure 4 shows the same plots for only $\gamma = 5$ and with the efficient frontier. Our test makes a comparison between the indifference curves of the benchmark and the optimal portfolio.

Figure 3.

Efficient portfolios in a mean-standard deviation plan

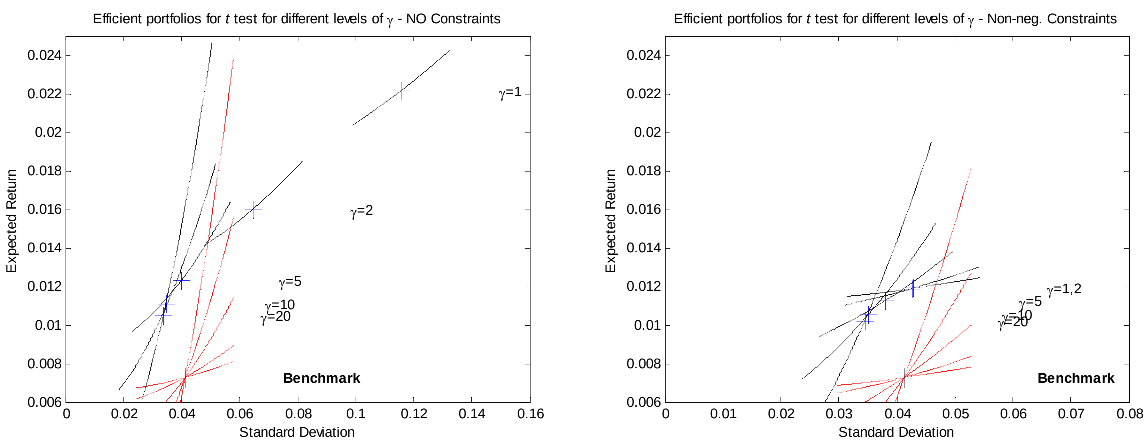
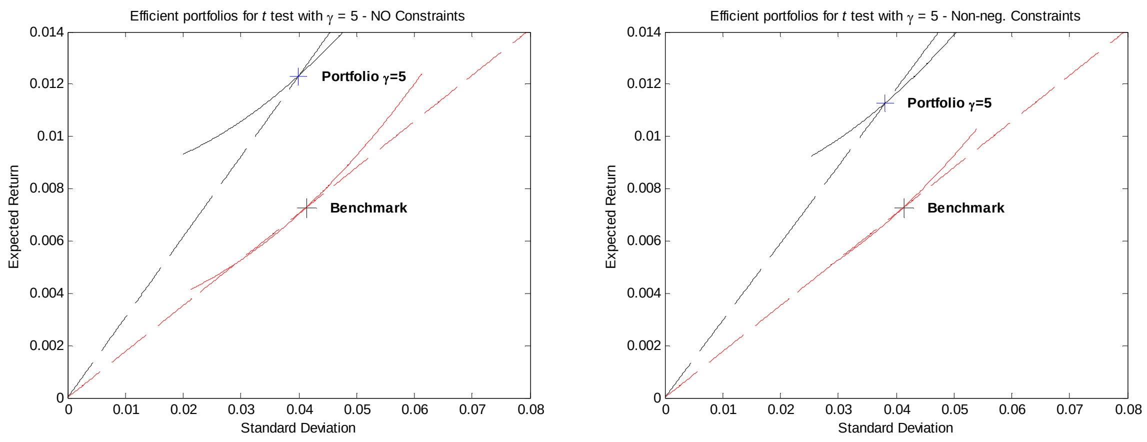


Figure 4.

Efficient portfolios in a mean-standard deviation plan, case γ =5.

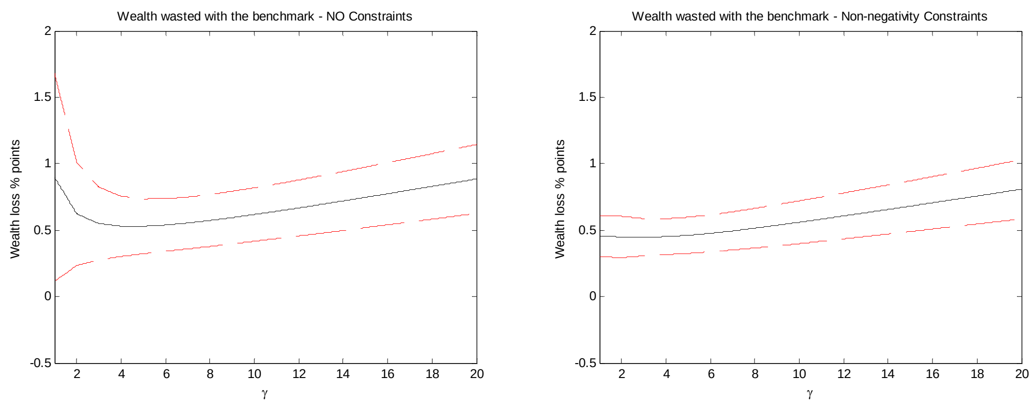


6.1. Benchmark case

We already know that, by construction, the benchmark is suboptimal in a standard mean-variance metric[10]. Its inefficiency, however, decreases as we add more constraints; in particular, it decreases appreciably when we impose non-negativity constraints. Figure 5 plots the amount of wealth wasted against the level of risk aversion, for the cases of no constraints and only non-negativity constraints. The inefficiency is always lower in the second situation; in many cases, we observe that the benchmark wastes less than 0.5% of wealth. The dashed lines represent the confidence intervals for the wealth wasted; such an interval is smaller with constraints.

Figure 5.

Wealth wasted by the benchmark for different levels of relative risk aversion (%)



[10] An unconstrained BJS test, however, would not reject the null of efficiency for the benchmark, obtaining a statistic equal to -0.1037 with an associated p-value of 0.9174. The benchmark would actually provide a risk (4.13%) only slightly higher than the one (4.05%) of the optimal portfolio with the same expected return (0.73%).

In table 4 we show the result of an efficiency test on the benchmark, where the null hypothesis is

$$H_0 : \lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = 0$$

The wealth loss does not seem to change a great deal when adding further (especially equality) constraints; it is, instead, much more sensitive to the risk aversion parameter, which affects the magnitude of the difference between the two variances. Still in table 4, we see that a t test of efficiency rejects the null hypothesis for any risk aversion value and for any combination of constraints, even if its realization is smaller in the absence of inequality constraints and when γ is small.

Table 4.

Test statistics and hypothesis testing - benchmark

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Non-Negativity Constraints				
Wealth Loss	0.8984	0.6228	0.5289	0.6173	0.8855	0.4548	0.4489	0.4610	0.5593	0.8074
Std. error	0.3995	0.1987	0.1054	0.1027	0.1326	0.0782	0.0791	0.0697	0.0821	0.1137
Lower Conf. Int.	0.1123	0.2326	0.3222	0.4158	0.6254	0.3015	0.2937	0.3243	0.3982	0.5843
Upper Conf. Int.	1.6784	1.0114	0.7352	0.8185	1.1450	0.6079	0.6039	0.5976	0.7201	1.0301
Test*	2.2386	3.1248	5.0065	5.9905	6.6505	5.8056	5.6601	6.5968	6.7904	7.0718
P-Value	0.0252	0.0018	0	0	0	0	0	0	0	0
	Equality Constraints					Non-Negativity and Equality Constraints				
Wealth Loss	0.8429	0.5825	0.4818	0.5411	0.7449	0.4195	0.4148	0.4324	0.5094	0.7017
Std. error	0.3791	0.1838	0.0886	0.0820	0.1046	0.0586	0.0503	0.0571	0.0663	0.0880
Lower Conf. Int.	0.0970	0.2215	0.3080	0.3802	0.5397	0.3046	0.3162	0.3205	0.3794	0.5291
Upper Conf. Int.	1.5833	0.9422	0.6552	0.7018	0.9496	0.5342	0.5133	0.5442	0.6393	0.8740
Test*	2.2139	3.1594	5.4253	6.5787	7.0974	7.1481	8.2308	7.5601	7.6640	7.9472
P-Value	0.0268	0.0016	0	0	0	0	0	0	0	0

* *Null hypothesis: wealth loss equal to zero.*

We can also derive the optimal relative risk aversion coefficient, i.e., the coefficient that makes the performance of the benchmark as good as possible. Table 5 shows that the optimal γ amounts to a reasonable 4.5227[11]. To understand γ, consider the following experiment. An investor is given a choice of a fixed sum of money in the next period or a lottery that pays $800 with a probability of 0.5 and $1,200 with a probability of 0.5. A risk neutral investor would be indifferent between the actuarial value of the lottery, $1,000, and the lottery. An investor with $\gamma = 3$ is indifferent between $940 and the

[11] It would be equal to 4.8050 with the equality constraint, 2.7075 with short-sale constraints and 1.8332 with short-sale and equality constraints. The last two values can be obtained only numerically.

lottery, and an investor with $\gamma = 5$ is indifferent between $900 and the lottery. Gollier (2002), furthermore, observes that γ levels higher than 10 are implausible.

Table 5.

Optimal RRA coefficient - benchmark

BENCHMARK NO CONSTRAINTS	OPTIMAL RRA	S.E.	LOWER CONF. INT.	UPPER CONF. INT.	P-VALUE
RRA	4.5227	1.1153	2.3366	6.7087	-
WEALTH LOSS (%)	0.5275	0.1093	0.3130	0.7416	-
TEST	4.8129	-	-	-	0

The estimated 95 percent confidence interval is acceptable too. Using this coefficient, there is a wealth loss of 0.53%, and it is significantly different from zero; a statistical test, indeed, rejects the null hypothesis of $cv_0 = 0$; this implies that there is no risk aversion coefficient for which the benchmark is at least as efficient as the optimal portfolio.

In general, we conclude that the benchmark is inefficient, but its inefficiency turns out to be unexpectedly small, even if the benchmark is dominated by one of the primitive assets.

6.2. Portfolio case

In the following section we consider an application of the portfolio version of our statistic. We analyze two cases; we first consider equally-weighted portfolios, to establish how costly naïve strategies are. We then analyze how each single constraint can affect our measure of wealth loss.

NAÏVE STRATEGY

Let us suppose that an agent follows a naïve strategy, i.e., invests exactly the same amount of wealth in each of the ten assets. Such a portfolio is inefficient under a mean-variance analysis; a JK test run using the 10 industry portfolios, indeed, is worth 17.5876 with a p-value of 0.0403. We wonder, therefore, if this portfolio is still inefficient under the framework in this paper.

There are several reasons for studying a naïve portfolio. First, it is easy to implement because it does not require any estimation or optimization. Second, it is empirically proven (Benartzi and Thaler, 2001) how investors often continue to use such simple rules for allocating their wealth across assets. The literature deals with this portfolio, then, since it is simple to use and reasonably easy to implement assuming difficulty in diversifying (DeMiguel et al., 2005). In order to empirically arrange the portfolio composition as suggested by the theoretical models we need to know, indeed, the parameters of the model for a particular set of asset returns and then to solve for the optimal portfolio weights.

Figure 6 shows that the point estimate of wealth loss is always below 0.6%, and well below 0.2% for most levels of risk aversion – except for the smallest and the largest – and with several constraints, especially short-sales constraints. In all cases, however, the lower bound of the confidence interval reaches the zero point for any $\gamma \leq 8$: for reasonable levels of risk aversion, therefore, we cannot reject the null of efficiency of a naïve portfolio. This suggests us that more financially educated agents could take just a slightly higher profit by investing in a portfolio different from the naïve.

Figure 6.

Wealth wasted with the naïve portfolio

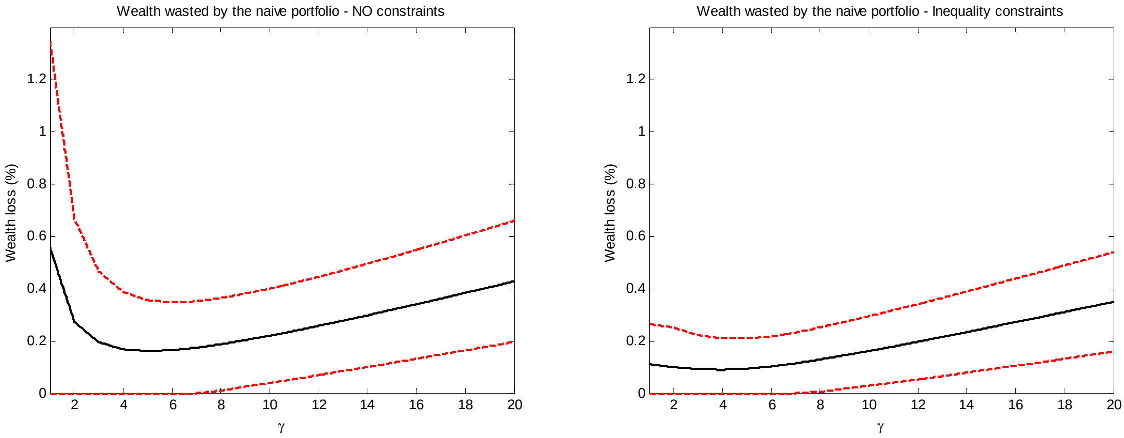


Table 6 provides us with the wealth wasted when holding this portfolio instead of the optimal under several constraints. The wealth however wasted is smaller (below 0.2%) for γ between 5 and 10; it grows for more elevate and small risk aversions. Notice, however, that when we consider in the analysis inequality constraints even in the case of $\gamma = 1$ we get a very small amount of wealth loss. Given the lower boundary of the confidence interval, however, when $\gamma = 1$, 2 or 5 in no case we have enough evidence to conclude that the naïve strategy is inefficient.

We also report the results of the t test in which the null hypothesis assumes that the amount of wealth loss is equal to that obtained with no restrictions. Using the theoretical distribution, just in a few cases we have enough evidence for concluding that imposing more restrictions the naïve strategy becomes less inefficient: in particular when we consider both constraints. This makes us believe that the effect of a combination of constraints may be stronger than the sum of the effects of single constraints, because of their interrelations. Adding equality constraints, in particular, does not seem to cause relevant differences. We thus argue that, for reasonable levels of risk aversion, according with this model it is not possible to conclude that a naïve strategy is inefficient, and that, as we add more constraints, the point estimate of its wealth loss decreases significantly for low risk-averse individuals.

In this case, therefore, accounting for market frictions helps explain much of the rationale behind the recourse to this strategy.

Table 6.

Test statistics and hypothesis testing – equally weighted naïve portfolio

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Non-negativity Constraints				
Wealth f	0.5568	0.2741	0.1616	0.2198	0.4280	0.1117	0.0997	0.0935	0.1615	0.3496
Std. error	0.4050	0.2006	0.0996	0.0921	0.1182	0.0779	0.0777	0.0590	0.0680	0.0972
Lower Conf. Int.	0	0	0	0.0391	0.1960	0	0	0	0.0281	0.1589
Upper Conf. Int.	1.3475	0.6666	0.3569	0.4001	0.6595	0.2643	0.2519	0.2091	0.2947	0.5399
Test*	-	-	-	-	-	-5.7259	-2.2471	-1.1545	-0.8575	-0.8076
P-value	-	-	-	-	-	0	0.0246	0.2483	0.3911	0.4193
	Equality Constraints					Non-negativity and Equality Constraints				
Wealth f	0.5012	0.2337	0.1143	0.1433	0.2867	0.0763	0.0654	0.0647	0.1114	0.2433
Std. error	0.3854	0.1869	0.0839	0.0714	0.0898	0.0570	0.0387	0.0419	0.0510	0.0701
Lower Conf. Int.	0	0	0	0.0032	0.1106	0	0	0	0.0115	0.1059
Upper Conf. Int.	1.2536	0.5995	0.2785	0.2831	0.4625	0.1878	0.1412	0.1468	0.2112	0.3806
Test*	-0.1444	-0.2160	-0.5645	-1.0723	-1.5754	-8.4570	-5.4016	-2.3142	-2.1288	-2.6382
P-value	0.8852	0.8290	0.5724	0.2836	0.1152	0	0	0.0207	0.0333	0.0083

* *Null hypothesis: wealth loss equal to the one in case of no restrictions.*

The overall impression, therefore, is that a naïve strategy is not a bad investment at all. This conclusion is not new in the literature: Brennan and Torous (1999), for instance, infer from a simulation analysis that even an equally weighted portfolio of as few as five randomly chosen firms can provide the same level of expected utility as the market portfolio. In DeMiguel et al. (2005), furthermore, this strategy performs quite well too. The authors argue then that, even if naïve diversification results in a lower performance than optimal diversification, the loss is smaller than the one arising from having to use as inputs for the optimizing models parameters that are estimated with error rather than known precisely. In most cases, in our analysis we do not have enough information to reject the null of efficiency. We are led to believe, furthermore, that even the point estimate of the wealth loss is so small that it is actually cheaper than any cost of information search and, thus, many investors would prefer this solution to a theoretically more efficient portfolio.

Lastly, we see in table 7 that the optimal γ derived using equation (11) takes a value of 5.0680[12] and however not higher than 7.3688 in a 95 percent confidence interval. In other words, the agent who gets the smaller wealth loss has a risk aversion roughly equal to about $\gamma = 5$. The

[12] 3.8974 with short-sale constraints, 5.7577 with the equality constraint and 3.9223 with short-sale and equality constraints.

corresponding wealth loss is 0.1616 percent, and its confidence interval produces a lower bound just equal to zero. The wealth loss, therefore, is not significantly different from zero.

Table 7.

Optimal RRA coefficient – equally weighted naïve portfolio

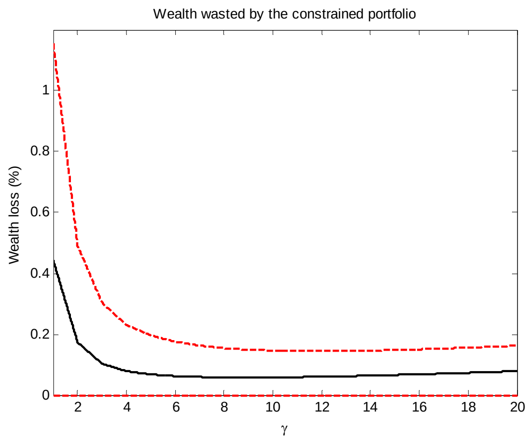
PORTFOLIO NO CONSTRAINTS	OPTIMAL RRA	S.E.	LOWER CONF. INT.	UPPER CONF. INT.
RRA	5.0680	1.1739	2.7672	7.3688
WEALTH LOSS (%)	0.1616	0.0990	0	0.3554

COST OF ADDITIONAL CONSTRAINTS

In figure 7 we show the pattern of the wealth wasted when comparing the optimal portfolio subject to short-sale constraints with the unconstrained optimal portfolio. The level of inefficiency decreases sharply after $\gamma = 2$, stabilizing soon below 0.1 percent. The lower confidence interval, however, is always equal to zero; it means that there is no evidence that adding non-negativity constraints worsens the efficiency. We have not considered, nevertheless, the constraints separately yet. We wonder, in particular, if some constraints are able to explain more wealth loss then others.

Figure 7.

Wealth wasted by the constrained portfolio



The experiment that we run in this section is thus the following. Starting from the unconstrained optimal portfolio, we add separately each constraint into the analysis, and compute the resulting wealth loss given by the recourse to the constrained optimal portfolio instead of the unconstrained optimal portfolio. This approach gives an idea of the cost of imposing additional constraints to a portfolio, and could in principle be used to compare any pair of nested portfolios, in which one portfolio is optimal under more restrictions than the other. Table 8 reports the point estimate of the wealth loss, its standard error and its confidence interval for the sole case of $\gamma = 5$.

Table 8.

Effect of any constraint on the wealth loss

$\gamma = 5$	NoDur	Durbl	Manuf	HiTec	Telcm	Shops	Other	Hlth	Enrgy + Utils
%	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	$= 0.1$	$= 0.2$
Wealth Loss	0	0	0.0481	0	0	0.0010	0.0029	0.0112	0.0372
Std. error	0	0	0.0536	0	0.0005	0.0079	0.0132	0.0260	0.0483
Lower Conf. Int.	0	0	0	0	0	0	0	0	0
Upper Conf. Int.	0	0	0.1531	0	0.0010	0.0164	0.0288	0.0622	0.1318

Note: wealth loss is computed by comparing the optimal unconstrained portfolio with the optimal constrained portfolio.

We see that three constraints (non-durable, durable and hi-tech) do not produce any reduction in wealth. They are, indeed, largely positive in the optimal unconstrained portfolio. All other constraints that cause a wealth loss refer to weights that assume negative values in the unconstrained portfolio, or that do not respect the equality. In particular, the most relevant restrictions seem to be the non-negativity constraint on the manufacturing sector and the equality constraint on the combination of energy and health sector. It is worth pointing out, however, that we are talking about tiny amounts, and the fact that the lower bound of the confidence interval is equal to zero does not exclude the possibility that they are actually equal to zero.

The picture that emerges is that no restriction is relevant by itself. A proper combination of constraints, instead, may change much more the performance of the optimal portfolio, because of the interrelation between constraints.

7. Empirical distribution of the test

In this section we study how the test performs in small samples. The statistic $\ell_b = \ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is a highly non-linear function of the random variables and for this reason the small sample distribution of the test may significantly differ from its normal asymptotic distribution. The knowledge of the test small sample properties may have relevant implication in the empirical analysis. For instance, in Bucciol (2003) the author makes use of a statistic closely related to the one in Basak et al. (2002); partly because of a small sample size, he obtains a generalized efficiency of Italian household portfolios, apparently too wide to be explained only with the addition of inequality constraints.

To establish the small-sample properties of our test we then perform a block boostrap (Kunsch, 1989) simulation. Given the time series of observed data,

$$e_t = \begin{bmatrix} e_{pt} \\ e_{bt} \end{bmatrix}$$

for $t = 1,\dots,T$, we adopt the following algorithm for a benchmark test:

1. Compute the sample moments e_p, e_b, S_p, s_b^2 and consequently the statistics $\ell_0 = \left(w^{*\prime} e_p - \frac{1}{2}\gamma w^{*\prime} S_p w^* \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right)$, $cv_0 = 1 - \exp\{-\ell_0\}$ and $V_0 = \hat{V}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$;
2. Define the block size $b = \mathrm{int}\left(T^{1/5}\right)$ according with Hall et al. (1995)[13]; as a consequence the number of blocks is $k = \mathrm{int}\left(\frac{T}{b}\right)$ and the length of any bootstrap sample is $L = kb \le T$;
3. Repeat the following a number N of times, with $j = 1,\dots,N$:
   a. Generate a random i.i.d. sample $\{i_0, i_1, \dots i_{k-1}\}$ from a discrete uniform distribution on $\{1, 2, \dots, T - b + 1\}$;
   b. Construct a bootstrap pseudo-series $\{e_t^j, t = 1,\dots,L\}$ as $e_{mb+h}^j = e_{i_m + h - 1}$ for $h = 1,\dots,b$;
   c. Compute the statistics $\ell^j = \left(w^{*\prime} e_p^j - \frac{1}{2}\gamma w^{*\prime} S_p^j w^* \right) - \left(e_b^j - \frac{1}{2}\gamma s_b^{2j} \right)$ and $cv^j = 1 - \exp\{-\ell^j\}$;

The block bootstrap distribution of the statistic $\ell_b = \ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is, therefore, $L^{1/2}\left(\ell^j - \ell_0\right)$ for $j = 1,\dots,N$, with associated a variance $V^{BB} = Var\left(L^{1/2}\left(\ell^j - \ell_0\right)\right)$. Such distribution may be used to perform any hypothesis testing. If the null is $H_0 : \ell = \ell_{H_0}$, a comparison is made between the bootstrap realizations and the statistic $T^{1/2}\left(\ell_0 - \ell_{H_0}\right)$; an analogously algorithm may be implemented for a portfolio test.

In this paper we show results using $N = 1000$; smaller or larger values of N do not seem to provide significant differences. Below we show the results for portfolios relative to $\gamma = 5$, with no constraint or only non-negativity constraints. With other risk aversion parameters and other constraints

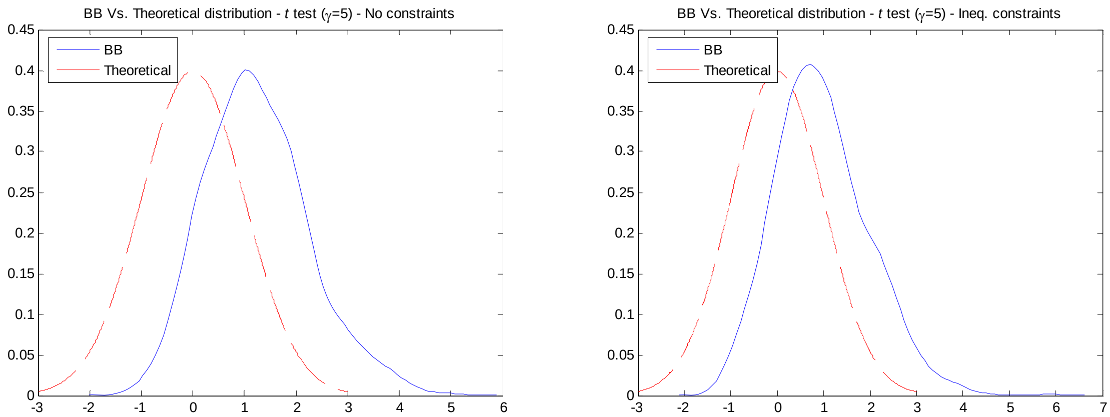
[13] In this case then $b = 3$, $k = 221$ and $l = 663$. Hall et al. (1995) show that following this rule it is possible to determine the optimal block size in the case of estimation of a two-sided distribution function. The optimality is meant in terms of minimization of the mean square error of the block bootstrap estimator.

the test does not show to behave differently, and neither does with a variable number of primitive assets (5 or 30); the following simulation results may apply then to a broad range of data.

Figure 8 compares the theoretical (dashed line) with the simulated (solid line) distribution for the benchmark test; the statistic have been rescaled according with their variance, V_0 for the theoretical distribution and V^{BB} for the simulated distribution. Using this technique we notice that 1) the empirical statistic actually appears to be normally distributed, 2) the estimated variance correctly replicates the true variance, especially in the constrained case, but 3) the empirical distribution is not centered around zero.

Figure 8.

Theoretical (dashed line) Vs. simulated (solid line) distribution for the benchmark test



The bias

$$bias = L^{1/2}\left(\frac{1}{N}\sum_{j=1}^{N}\ell^{j} - \ell_0\right)$$

is always higher than zero, with an average value that tends to disappear as $L \to \infty$. See indeed table 9 for the benchmark test; it shows results from a Monte Carlo simulation, assuming normality in the asset returns.

Table 9.

Bias of the benchmark test, $\gamma = 5$

L	500	664	1000	1500	2000
No	4.2761	3.6002	3.0366	2.4737	2.0779
Inequality	1.6333	1.4077	1.2919	1.0786	0.9318
Equality	3.3044	2.8402	2.4121	1.9639	1.6278
Both	0.9766	0.8293	0.6969	0.5836	0.4765

Note: $bias$ *in percentage scale.*

With time series of length $L = 2000$ we get a bias roughly equal to half of the bias with $L = 500$. The same results apply to a portfolio test; see indeed figure 9 for a graphical comparison. The portfolio underlying this analysis is the equally-weighted portfolio.

Figure 9.

Theoretical (dashed line) Vs. simulated (solid line) distribution for the portfolio test

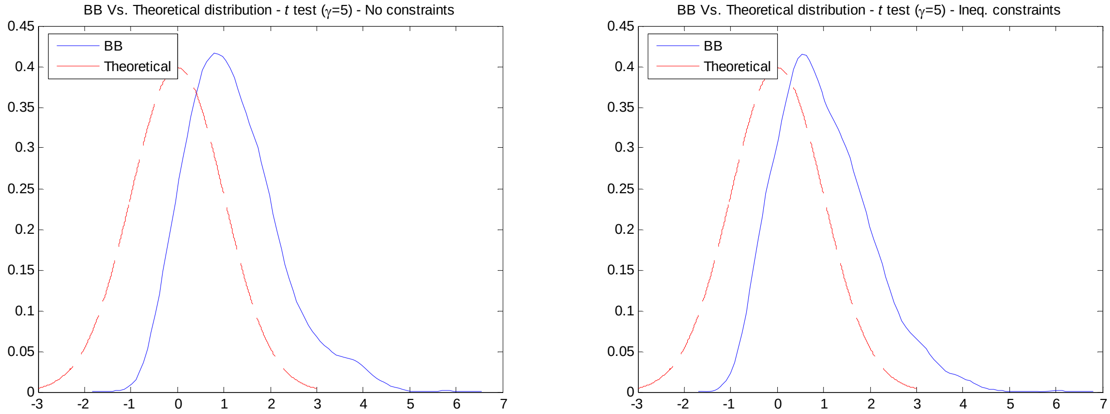


Table 10 reports the magnitude of this bias for a portfolio test with $\gamma = 5$ and shows that, also here, the bias is less relevant as L increases. Note here, as well as above, that the bias is always higher when the portfolio allocation problem has a closed form solution.

Table 10.

Bias of the portfolio test, $\gamma = 5$

L	500	664	1000	1500	2000
No	4.2032	3.6577	2.9276	2.3572	2.0241
Inequality	1.6535	1.4842	1.2124	1.0692	0.9546
Equality	3.2445	2.8980	2.3195	1.8604	1.5154
Both	0.9799	0.8638	0.6857	0.5848	0.4702

Note: bias in percentage scale.

Bias apart, the two distributions look alike. Correcting the simulated distribution for the bias in a Monte Carlo sample, we could indeed replicate almost perfectly the asymptotic distribution of both the benchmark and the portfolio test.

The bias could, however, alter the conclusion of a hypothesis testing. Its presence in small samples depends on the application of the delta method to the highly non-linear function $\ell_b = \ell\left(e_b, s_b^2, e_p, S_p, \gamma\right)$. The delta method, indeed, makes use of a first-order Taylor expansion, from which

(15) $$\ell_b = f\left(\overline{X}_T, \gamma\right) \cong f\left(X, \gamma\right) + \nabla\left(\gamma\right)'\left(\overline{X}_T - X\right)$$

and $\nabla(\gamma)$ is the gradient of $f\left(\overline{X}_T,\gamma\right)$, following the notation of §3. From equation (15) the expectation is worth

$$E[\ell_b] \cong f(X,\gamma)+E\left[\nabla(\gamma)'\left(\overline{X}_T-X\right)\right]=f(X,\gamma)=\lambda_b$$

since $\overline{X}_T$ is an unbiased estimator of X. If ℓ_b were actually linear, i.e., equation (15) were true with no approximation, the statistic would be unbiased and the asymptotic theory developed on §3 would be accurate, even in small samples. To ascertain this we perform a Tapered Block Bootstrap (TBB) as developed in Paparoditis and Politis (2001, 2002). TBB represents an improvement of the standard block bootstrap for approximately linear statistics since it produces a mean squared error of order $O\left(N^{-4/5}\right)$ compared with the $O\left(N^{-2/3}\right)$ rate of a standard block bootstrap. The main difference with the standard block bootstrap is that it pays less importance to the observations at the extremes of any block; doing so, the dependency structure underlying the original data is removed more than with a non-tapered algorithm.

Given the original time series $\{e_t, t=1,\ldots,T\}$, we define the influence function $IF(e_t,F)$ as

$$IF(e_t,F)=\nabla(\gamma)'(X_t-X)$$

with X_t defined in (5), X its expectation and F the underlying distribution function. According with (15), therefore,

$$\ell_b \cong f(X,\gamma)+\frac{1}{T}\sum_{t=1}^{T} IF(e_t,F)$$

Since F is unknown, we replace $IF(e_t,F)$ with a sample estimate $IF\left(e_t,\hat{F}_T\right)$ based on the empirical distribution $\hat{F}_T$:

$$IF\left(e_t,\hat{F}_T\right)=D(\gamma)'\left(X_t-\overline{X}_T\right).$$

The simulation is performed according with the following algorithm:

1. Compute the time series $Y_t = IF\left(e_t,\hat{F}_T\right)$ for $t=1,\ldots,T$. By construction, the sample average $\overline{Y}_T=\frac{1}{T}\sum_{t=1}^{T} Y_t$ is centered at zero;

2. Choose an appropriate data-tapering window $z_n(.)$, with $z_n(.) \in [0,1]$ and the block size b, such that the number of blocks is equal to $k = \text{int}\left(\frac{T}{b}\right)$ and the length of any bootstrap sample is $L = kb \le T$;
3. Repeat the following a number N of times, with $j = 1,\dots,N$:
   a. Generate a random i.i.d. sample with replacement $\{i_0, i_1, \dots i_{k-1}\}$ from a discrete uniform distribution on $\{1, 2, \dots, T-b+1\}$;
   b. Construct a bootstrap pseudo-series $\{Y_t^j, t = 1,\dots,L\}$ as $Y_{mb+h}^j = \frac{b^{1/2}}{\|z_b\|} z_b(h) Y_{i_m+h-1}$ for $h = 1,\dots,b$, with $\|z_b\| = \left(\sum_{h=1}^{h} z_b^2(h)\right)^{1/2}$
   c. Compute the bootstrap sample mean $\overline{Y}_L^j = \frac{1}{L}\sum_{l=1}^{L} Y_l^j$

The tapered block bootstrap distribution of the statistic $\overline{Y}_T$ is, therefore, $L^{1/2}\overline{Y}_L^j$ for $j = 1,\dots,N$, with associated a variance $V^{TBB} = Var\left(L^{1/2}\overline{Y}_L^j\right)$. We use the TBB sample of statistics $\overline{Y}_L^j$ to simulate the distribution of $\overline{Y}_T$. Note that, if equation (15) held without approximation, the distribution of $T^{1/2}\overline{Y}_T$ and the distribution of $T^{1/2}(\ell_b - \lambda_b)$ would be exactly the same. We interpret, therefore, the distribution of $T^{1/2}\overline{Y}_T$ as the distribution of $T^{1/2}(\ell_b - \lambda_b)$ if ℓ_b were a linear statistic. We then compare the distribution of

$$t_{TBB}^j = L^{1/2} \frac{\overline{Y}_L^j}{\left(V^{TBB}\right)^{1/2}}$$

for $j = 1,\dots,N$, with a standard normal distribution, i.e., our theoretical distribution.

Following the suggestions in Paparoditis and Politis (2002), we use a trapezoidal shape,

$$z_c^{TRAP}(t) = \begin{cases} \frac{t}{c} & t \in [0,c] \\ 1 & t \in [c, 1-c] \\ \frac{1-t}{c} & t \in [1-c, 1] \end{cases}$$

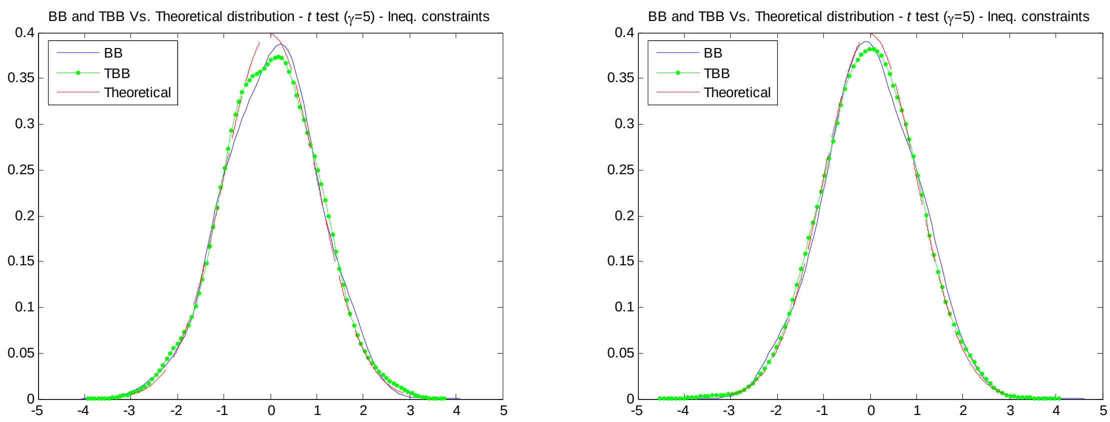
with $c = 0.43$ to minimize the mean squared error of the resulting bootstrap statistic, from which the data-tapering window is $z_b(h) = z_{0.43}^{TRAP}\left(\frac{h-0.5}{b}\right)$. The block size is also chosen according with equation (20) in Paparoditis and Politis (2002); such equation is, roughly speaking, a function of the lagged covariance of the original data X_t, $t = 1, \ldots, T$. According with this procedure, it turns out that the optimal block size with these data is equal to $b = 5$, from which $k = 132$ and $L = kb = 660 < 664$.

We take advantage of this frame and follow the same recipe while running a block bootstrap sample. The difference with the TBB, in this context, is that we consider a smaller block size ($b = 3$ instead of $b = 5$), and we do not weigh the simulated data. The reason for doing so relies on our concern that a block bootstrap simulation, compared with a TBB sample, could provide a worse approximation of the true distribution; since ℓ_b is non-linear, a parallel between the two techniques was not possible in the previous simulation. The comparison is, then, made between a standard normal distribution and the simulated distribution of the statistics t^j_{TBB} and t^j_{BB}, where t^j_{BB} is the equivalent of t^j_{TBB} in a block bootstrap setting.

Figure 10 shows the density of the simulated distribution, using either a BB or a TBB algorithm, and the theoretical normal distribution of a benchmark test with either no constraints or short-sales constraints. We notice that, indeed, they are almost overlapped; this means that, were the statistic ℓ_b linear, its asymptotic distribution would actually be a standard normal. Also observe how the two simulated distributions obtain analogous values, especially along the tails; this result makes us believe that the use of a standard block bootstrap technique in this context may be as accurate as the recourse to a TBB approach.

Figure 10.

Simulated (BB and TBB) Vs. theoretical distribution for the benchmark test



Yet the statistic ℓ_b is non-linear; with the delta method we just operate a first-order approximation. This simplification may result poor in small samples, as the above block bootstrap simulation attests. If we considered a second-order Taylor expansion, indeed, we would get that

$$\ell_b = f\left(\overline{X}_T,\gamma\right) \cong f\left(X,\gamma\right) + \nabla\left(\gamma\right)'\left(\overline{X}_T - X\right) + \frac{1}{2}\left(\overline{X}_T - X\right)'\nabla^2\left(\gamma\right)\left(\overline{X}_T - X\right)$$

with $\nabla^2\left(\gamma\right)$ hessian of $f\left(\overline{X}_T,\gamma\right)$. Taking its expectation,

$$E\left[\ell_b\right] \cong f\left(X,\gamma\right) + \frac{1}{2}E\left[\left(\overline{X}_T - X\right)'\nabla^2\left(\gamma\right)\left(\overline{X}_T - X\right)\right] \quad (16)$$

where the second-order term does not disappear; ℓ_b is, then, biased. We do not go further and make $\nabla^2\left(\gamma\right)$ explicit since it is the derivative of $\nabla\left(\gamma\right)$, a complicated function of X and the optimal weights w^*, whose relation with X is in most cases unknown.

Given this evidence, and noticed how the block bootstrap technique works well in estimating the theoretical distribution, we repeat the analysis in §6 using the block bootstrap instead of the standard normal distribution. Table 12 reports the simulated rejection rates and confidence intervals of the benchmark test, that can be compared with the results in table 4 with a theoretical distribution.

Table 12.

BB rejection rates average for the benchmark test

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Non-Negativity Constraints				
Lower Conf. Int.	-1.0297	-0.3384	0.0983	0.2842	0.5258	0.1786	0.1989	0.2558	0.3324	0.5364
Upper Conf. Int.	1.0187	0.6812	0.5843	0.7262	1.0738	0.4959	0.4945	0.5147	0.6583	0.9832
Rej. Rates	0.6180	0.3440	0.0060	0	0	0.0020	0	0	0	0
	Equality Constraints					Non-Negativity and Equality Constraints				
Lower Conf. Int.	-0.9338	-0.2873	0.1102	0.2742	0.4619	0.2185	0.2384	0.2782	0.3341	0.4902
Upper Conf. Int.	1.0210	0.6433	0.5409	0.6225	0.8889	0.4668	0.4532	0.4866	0.5958	0.8532
Rej. Rates	0.5080	0.2360	0.0100	0	0	0	0	0	0	0

* *Null hypothesis: wealth loss equal to zero.*

Note that, with no constraints or with just equality constraints and for small levels or risk aversion, using this distribution we cannot exclude the possibility that the S&P 500 index is efficient. The lower bound of the confidence interval, furthermore, is negative, meaning that such index could even be *more* efficient than the optimal portfolio. In any other case, instead, the lower bound is strictly positive, meaning that the S&P 500 is not efficiently managed. The fact that the index could be efficient in a world with no restrictions and inefficient otherwise seems counterintuitive; the point estimate of the wealth loss, indeed, decreases as we add more constraints. The conclusion, however, depends on the standard error of the estimate; with more restrictions, the portfolio allocation is more limited, and therefore not many options are available to the investor. For this reason, even a smaller amount of wealth loss may be considered inefficient; in other words, our test gets more precise as we add more constraints. Also with the theoretical distribution, indeed, in these few cases we obtained smaller values for the test statistic.

Table 13 reports the rejection rates and the confidence interval of the portfolio test run on a naïve portfolio; the table is comparable with table 6. The estimated lower bound in the confidence interval is here equal to zero also when $\gamma = 10$, not only when $\gamma = 1,2,5$; this implies that, even when the coefficient of risk aversion is as high as 10, we cannot reject the hypothesis that a naïve portfolio is efficient. Note that, in general, the confidence intervals are much smaller than before; for this reason, the rejection rates are always lower than the theoretical p-values. In three cases (highlighted) with higher values of γ we reject the null hypothesis, overturning our conclusion from table 6. Adding constraints to the standard case, therefore, seems to provide a significant reduction in the wealth loss, at least with small levels of risk aversion and for short-sales constraints. The overall impression is that, using the theoretical distribution, we accept too much the null hypothesis; this could prove as evidence for the high fraction of efficient portfolios observed in Bucciol (2003).

Table 13.

BB rejection rates average for the naïve portfolio test

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Non-negativity Constraints				
Lower Conf. Int.	0	0	0	0	0.1071	0	0	0	0	0.0779
Upper Conf. Int.	0.6733	0.3297	0.2143	0.3064	0.6011	0.1418	0.1244	0.1263	0.2311	0.4970
Rej. Rates	-	-	-	-	-	0	0	0	0.0780	0.2520
	Equality Constraints					Non-negativity and Equality Constraints				
Lower Conf. Int.	0	0	0	0	0.0194	0	0	0	0	0.0716
Upper Conf. Int.	0.6772	0.3071	0.1598	0.2058	0.4062	0.0945	0.0839	0.0886	0.1637	0.3489
Rej. Rates	0.148	0.0900	0.0440	0.0280	0.0220	0	0	0	0	0.0020

* *Null hypothesis: wealth loss equal to the one in case of no restrictions.*

To summarize, using the block bootstrap distribution we draw conclusion just seldom different from those using the asymptotic distribution. In particular, we reject the efficiency of the S&P 500 for almost any combination of constraints and levels of risk aversion higher than 2, and we accept the efficiency of a naïve portfolio for risk aversion parameters below 10. Adding constraints to the analysis, furthermore, seems to decrease significantly the wealth loss, at least with risk aversion coefficients lower or equal to 10 and with short-sales constraints.

8. Conclusion

In this paper we study the efficiency of a benchmark or a portfolio in an expected utility framework, dealing with complex problems in which the optimal portfolio depends on weight constraints. We consider a measure of compensative variation which reads as the wealth loss between optimal and sub-optimal portfolios. We provide its asymptotic distribution and discuss the related inefficiency test. We suggest an estimation strategy for the risk aversion parameter based on the parameter value that minimizes the wealth loss with respect to the optimal portfolio. This estimate turns out to be useful when establishing, for instance, the implicit risk aversion adopted by fund managers when building their fund portfolio. The statistic can flexibly deal with equality and inequality constraints on portfolio composition, even if the presence of inequality constraints makes it impossible to derive a closed-form solution.

Although we depart from the classical literature of mean-variance analysis, we show that the two frameworks are comparable and to some extent provide analogous results; in particular, the optimal portfolios without inequality constraints differ only for a normalizing factor.

We find the asymptotic distribution for the test and discuss its small sample properties: given the evidence from our simulations, we believe that a better way to make use of this statistic is to consider its simulated distribution obtained through a block bootstrap (Kunsch, 1989) technique.

Our empirical application, based on ten industry portfolios for the U.S. market, shows that there is not enough evidence to reject the null of efficiency for a naïve investment strategy with reasonable values of the risk aversion coefficient; This conclusion confirms the results in Brennan and Torous (1999) and Das and Uppal (2004). The point estimates of the wealth loss, furthermore, are rather small, often about 0.10%, and it seems that considering inequality constraints into the analysis really helps explain such an apparently inefficient behavior when the risk aversion parameter is reasonably low. When using a benchmark, such as the S&P 500, that in the relevant period is dominated by at least one industry portfolio, our test does not find enough evidence to conclude for its inefficiency when the relative risk aversion parameter is small. In any other case, however, the inefficiency is unexpectedly small; the wealth loss is never higher than 0.5 point percentage, and the optimal risk aversion is reasonably equal to 5.

In our agenda we have set two goals for the future. We first plan to focus more in depth on the size of the implicit risk aversion parameter used by fund managers when choosing the composition of their fund portfolio. We further aim at accounting for a long term perspective within this framework and then analyzing the behavior of forward-looking agents with regards to their lifetime portfolios.

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