Strategic Market Making and Risk Sharing

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SEPTEMBER 2005

EFM Classification: 360

[0]We thank Jean Charles Rochet, Jean Claude Gabillon, Yacine Ait-Sahalia, Sophie Moinas, Thierry Foucault, Jacques Hamon, Bruno Biais, Tom Flavin and the seminar participants at the TBS Finance Workshop, at the AFFI conference, at the 5th Toulouse Workshop at IDEI, and at Maynooth.

Abstract

We analyze the result of allowing risk averse traders to split their orders among markets when market makers are assumed to be risk averse.

We prove that linear symmetric equilibria exist in that setting. We find that market makers' aggregate expected utility of profit may increase with the number of market markers despite the fact that the aggregate liquidity always increases with it. This implies that the cost of trading for the traders may increase with the number of market makers. The larger the market makers' risk aversion, the bigger that cost is. We also find that when the number of market makers tends to infinity, their aggregate expected utility of profit tends to zero. We offer a potential answer to the ongoing debate concerning the dealers' competitiveness. Indeed, risk aversion reduces competition between market makers as it acts as a commitment for market makers to set higher prices. This commitment is higher the higher the risk aversion.

JEL Classification: G14, D82

1 Introduction

A large body of papers analyze the formation and properties of price and liquidity in financial markets.[1] In order to study both, three assumptions are commonly made. First, market makers behave competitively. Second, traders cannot split their orders among market makers.[2] Third, market makers are risk neutral. As a result of the first and second assumptions, risk neutral market makers set a price equal to the expected value of the asset given market maker's information and aggregate order flow. This implies that market makers earn zero expected profit. Both the second and the third assumptions are more simplifying assumptions than realistic ones. Indeed, traders have now a wide range of possibilities to trade a given asset. In addition, Lyons (1995) proves that, in FX markets, dealers closely control their inventory position showing the fact they are risk averse. It is likely to be also true for equity and bond markets.

In the present model we remove the three aforementioned assumptions. This enables us to combine the assumptions of imperfect competition and risk aversion from the side of both the market makers and the traders. We then analyze the effects of these assumptions on prices, liquidity and the level of expected profit market makers achieve in a situation where traders split their orders among market makers.

In the recent years, the number of markets where traders have the possibility to trade a given asset has increased, due to the emergence of "New Markets" as well as the introduction of "Crossing Network" within existing dealer markets. Parallelly, after the recent crashes, market participants' attitude toward risk has changed, this has implied an increase in both market participants' risk aversion as well as market volatility. As a consequence, some natural questions arise: How is the cost of trading affected by the level of the market makers' risk aversion?[3] How is that same cost influenced by the number of market makers with whom the traders can exchange? How is the overall liquidity of an asset affected by both the number of market makers and their risk aversion? How is the degree of competition between market makers influenced by the number of traders competing for the exchange of an asset? How is the trading behavior of risk averse traders affected by the possibility to trade the same asset on different markets or with multiple dealers? These questions need to be answered in order to shed some light on the facts observed in financial markets such as, wider bid-ask spreads (high transaction costs), for instance.

We propose to answer these questions in a setting close to Kyle (1985). The price schedule of a market maker is contingent on the aggregate order flow for that particular market maker only and not contingent on the order flow received

[1] Liquidity is defined as the volume necessary to move the price by one unit. See Kyle (1985).

[2] See Kyle (1984), Kyle (1985), Subrahmanyam (1991), Foster and Viswanathan (1994) and Vives (1995) among others.

[3] Lyons (2001) raises the fact that too few models study the situation where market makers are risk averse.

by the other market makers. Each market maker determines the price maximizing her expected utility taking as given the price set by her competitors and taking into account its impact on the market orders submitted by the traders.[4] Prior to knowing the price schedule, traders receive (*i*) different signals concerning the fundamental value of the asset and (*ii*) different endowments of the risky asset. When deciding the size of their orders for each market maker, traders know the different market makers' price schedule. Each trader determines the size of each order submitted to the different markets by maximizing his conditional expected utility taking into account the impact of his orders on the price for each market and taking as given the quantity submitted by the other traders. We find a counterintuitive result that increasing the number of market makers, N, with whom traders exchange, can adversely affect the traders' overall cost of trading and this despite the fact that the aggregate liquidity increases with N.[5] Indeed, the market makers' aggregate expected utility of profit may increase with the number of market makers which implies an increase of the investors' trading costs. This is only true if the market makers' risk aversion, ρ_m, is positive. The interpretation of these results is as follows and depends on the size of the market N (number of market makers) and the level of the market makers' risk aversion ρ_m. Firstly, increasing N has the following effects: (*i*) it increases the aggregate risk tolerance of the market makers and increases risk sharing, (*ii*) it reduces the individual liquidity in each market, and finally (*iii*) it reduces the volume handled by market makers. The first and the second effect clearly increase aggregate expected utility of profit. However, the reduction in volume has two opposite effects on aggregate expected utility of profit. Secondly, increasing ρ_m has the following implications: (*i*) it decreases the aggregate risk tolerance of the market makers, (*ii*) it reduces the individual liquidity in each market, and finally (*iii*) it reduces the volume received by market makers. Effect (*i*) decreases aggregate expected utility of profit whereas effect (*ii*) increases it. The reduction in volume has again two opposite effects. In fact we show that when $\rho_m > 0$, the positive effects (those which increase the aggregate expected utility of profit) dominate for a small number of market makers while the negative effect dominates for a large number of market makers. As a result risk aversion can magnify the transaction costs paid by investors. To the best of our knowledge this is the first time this result has been found, as our model looks at the most general situation where both traders and market makers are strategic and risk averse. This finding has important implications for the regulation of financial markets. Our result could be regarded as an answer to the ongoing debate about the implications of market fragmentation on traders' welfare. We find that increasing market fragmentation seen as increasing the number of market makers can damage the traders' welfare. Having more market makers or markets is not always desirable from the point of view

[4] In our context as the price is linear function with the aggregate order flow it is equivalent to find the level of liquidity maximizing her expected utility.

[5] Since the game is a zero sum game, that cost is defined as minus the market makers' aggregate expected utility of profit.

of investors' trading costs.
Other results include that, for a finite number of market makers, the level of aggregate liquidity is below its competitive level implying that market makers earn positive expected profits.[6] The explanation of that result is as follows: by increasing her price, a market maker reduces the volume received without modifying the proportion of the trader's market order due to hedging needs.[7] However, the increase in price may still compensate for the effect of the decrease in volume on the market maker's expected utility of profit. In fact, despite a higher price, the trader is willing to exchange on that market, as by splitting his order he reduces its overall impact on the price. This implies that all market makers have an incentive, due to their risk aversion, to set less competitive price schedules. Nevertheless, when the number of market makers tend to infinity, both the market makers' expected utility of profit and the aggregate liquidity tend to their competitive level.

Our work is linked to research focusing on dealers' competition. There is a strong evidence that dealers behave strategically and earns monopoly rents. Christie and Schultz (1994) and Christie et al. (1994) show that market makers on the NASDAQ may exhibit a non-competitive behavior. This is also confirmed by latest studies such as Weston (2000) and Simaan, Weaver and Whitcomb (2003). Lamoureux and Schnitzlein (2004) find, in an experimental study, similar results. They compare the size of the bid-ask spread and of the dealers' profit for two scenario: (*i*) three competing dealers in a single asset (i.e. direct competition) and (*ii*) three assets with a monopolistic dealer in each (indirect competition). They find that bid-ask spreads are wider and that per-trade dealer profits are larger for the first scenario.[8] Theoretical papers have looked at the effect of the competition among market makers on their expected profits and their price schedule.[9] Glosten (1994) and Biais, Martimort and Rochet (2000) study competition in limit orders. In Biais, Martimort and Rochet (2000) when the number of market makers is finite, market makers earn positive expected profits. They also show that as the number of market makers tends to infinity, market makers earn zero expected profits and the price schedule converges to the competitive one obtained in Glosten (1994). Biais et al. (1998) and Viswanathan and Wang (2002) consider risk averse market makers. The former compares the cost of trading across markets organized differently, i.e. floors, dealer markets and limit orders. The latter looks at dealership markets, limit order markets and a hybrid market mixing the two preceding structures. They do not provide an analysis of the model we study here. In addition, they

[6] The competitive level is computed in a situation where traders cannot split their orders and market makers face competition.

[7] Due the traders' CARA utility framework, an increase in price only alters the size of the market order without changing the proportion of hedging motives within the order.

[8] An important difference between the two scenarios lies in the fact that in the three asset case, liquidity traders as well as having the possibility to time their trade have the choice of which asset to trade. This is the main driving force for their result.

[9] Less recent papers [Admati and Pfleiderer (1988) and Glosten (1989)] focus on the extreme case where the market maker or specialist has a monopolistic position over a particular asset.

look at the case where a unique liquidity trader is present in the market. Vogler (1997) and Lyons (1997) look at risk averse market makers, however, their main focus is on an inter-dealer markets. Finally, Bernhardt and Hughson (1997) are closer to our analysis. They study the competition between market makers for the duopoly case. Their setting is similar to Kyle (1985) with market makers setting price schedules as a function of the aggregate order flow before traders submit their orders. They show that in equilibrium market makers cannot earn zero expected profits. For the duopoly case, the existence and the form of the equilibrium is shown. However, for the oligopolistic case they show that an equilibrium cannot be such that market makers earn zero expected profit but do not prove its existence. We depart from their analysis on two important points. First, we consider the case of risk averse market makers. Second, heterogeneously informed traders also possessing heterogenous endowments of the risky asset compete between each other.

The contribution of our paper is twofold. Firstly, on a purely theoretical basis, we generalize Bernhardt et al. (1997) and Biais, Martimort and Rochet (2000) to the cases where there are $N > 2$ risk averse market makers and more than 1 trader. To the best of our knowledge, the dealers' risk aversion has not been incorporated in any analysis for the type of model we are dealing with, i.e. models with asymmetry of information with splitting orders, an exception being Subrahmanyam (1991) and Spiegel and Subrahmanyam (1992) for the case where traders cannot split their orders. Secondly, we offer a potential answer to the ongoing debate concerning the dealers' competitiveness. Indeed, risk aversion reduces competition between market makers as it acts as a commitment for market makers to set higher prices. This commitment is higher the higher the risk aversion.

An outline of the paper is as follows. In Section 2, we present the general model allowing traders to split their orders. In Section 3, we solve the model for the linear symmetric equilibrium. We look at the properties of the liquidity and the market makers' aggregate expected profit in section 4. Section 5 presents our conclusions and summarizes our results. Finally all proofs and some of the graphs are gathered in the Appendix.

2 The model

Consider a market where a risky asset and a riskless asset are traded among K traders and N market makers. For convenience, the riskless asset has its interest rate normalized to zero. The liquidation value of the risky asset, $\widetilde{v}$, is normally distributed with mean 0 and variance σ_v^2 (precision $\tau_v = \frac{1}{\sigma_v^2}$).

All agents, i.e. traders and market makers, are risk averse and have preferences described by a CARA utility function of the following form

$$U(W_k) = -\exp(-\rho W_k), \text{ for each trader } k,$$

$$U(W_n) = -\exp(-\rho_m W_n), \text{ for each market maker } n,$$

where ρ and ρ_m represent the parameter of risk aversion and W_k and W_n represent the final wealth.

All traders, before trading, receive heterogenous signals about the future value of the risky asset and heterogenous endowments of both the risky and the riskless assets. Each trader k's signal, s_k, is a realization of a normally distributed random variable $\tilde{s}_k = \tilde{v} + \tilde{\varepsilon}_k$ where $\tilde{\varepsilon}_k$ is normally distributed with mean 0 and variance σ_ε^2 (precision τ_ε). Trader k's endowment of the risky asset, w_k, is a realization of a normally distributed random variable, $\tilde{w}_k$ with zero mean and variance σ_w^2. If w_k is positive (negative), the trader holds a long (short) position in the risky asset. Trader k's endowment of the riskless asset is denoted by c_k. The traders exchange for two reasons: hedging motives and informational reasons. Indeed, on the one hand, they trade for pure risk-sharing reasons as they receive an endowment shock to the risky asset. On the other hand, as they receive private information they will exploit their informational advantage by trading on that private information, they are then informed speculators. In the present model, we do not require noise traders as part of the orders submitted to the market makers are due to risk sharing motives.

All random variables $\widetilde{v}$, $\tilde{\varepsilon}_k$, $\tilde{w}_j$ for $k = 1, ..., K$ and $j = 1, ..., K$ are independent.

The timing unfolds as follows:

1. Each trader $k = 1, ..., K$, simultaneously observes his private signal s_k as well as his endowments, w_k and c_k for the risky and the riskless asset, respectively;

2. Each market maker $n = 1, ..., N$, simultaneously, posts a price schedule depending, solely, on her own order flow. The price schedule is not contingent on the order flow received by the other market makers as it is not observed;

3. Given the market makers' price schedules, each trader, simultaneously, determines how much to trade with each market maker;

4. Each market maker observes her own aggregate order flow and then clears it at the price previously posted;

5. The value of the asset is revealed and payoffs are realized.

It is assumed that traders submit market orders.

3 Characterization of the equilibrium

As in Kyle (1985), the model is solved for linear symmetric equilibria.

We assume that the market order submitted by trader k to market maker n, is linear in both the signal and the endowment of the risky asset, i.e.,

$$x_{nk} = a_n s_k - b_n w_k, \; \forall n = 1, ..., N \text{ and } \forall k = 1, ..., K. \tag{1}$$

The price schedule set by market maker n is linear in the anticipated aggregate order flow, y_n, in her own market,

$$p_n = \lambda_n y_n, \; \forall n = 1, ..., N \text{ with } y_n = \sum_{k=1}^{K} x_{nk}. \tag{2}$$

Definition (Equilibrium) $(\lambda_1, ..., \lambda_N) \in \Re^N$ *and* $(X_1^*, ..., X_K^*) \in L_2^{N(1+K)}$ *with* $X_k^* = (X_{1k}^*, ..., X_{nk}^*, ..., X_{Nk}^*)$ *is an equilibrium if, given the market orders submitted by the other traders and the liquidity set by each market maker, the market orders submitted by trader k,* X_k^**, to the different market makers are such that*

$$\begin{aligned} X_k^* &\in \arg\max_{x_{nk}\in\Re} E\left[W_k | \, s_k, w_k\right] - \tfrac{\rho}{2} var\left[W_k | \, s_k, w_k\right] \\ \textit{with } W_k &= w_k \tilde{v} + \sum_{n=1}^{N} x_{nk}\tilde{v} - \sum_{n=1}^{N} p_n x_{nk} + c_k. \end{aligned}$$

and given every market orders submitted to market maker n and the liquidity set by the other market makers, the liquidity set by market maker n, λ_n*, is such that*

$$\lambda_n \in \arg\max E\left[\left(p_n\left(y_n^*\right) - \tilde{v}\right) y_n^*\right] - \tfrac{\rho_m}{2} var\left[\left(p_n\left(y_n^*\right) - \tilde{v}\right) y_n^*\right], \text{ with } y_n^* = \sum_{k=1}^{K} x_{nk}^*.$$

Each trader determines the size of each order, x_{nk}^*, submitted to the different markets by maximizing his conditional expected utility taking into account the impact of his orders on the price for each market and taking as given the quantity submitted by the other traders. Each market maker determines the level of liquidity maximizing her expected utility taking as given the liquidity set by her competitors and taking into account its impact on the market orders submitted by the traders.

Given the linearity assumption of the price schedule, computing the price level maximizing the market maker's expected profit is equivalent to computing the liquidity parameter, λ, maximizing the expected profit. This is used in order to write the above definition. The price being linear in the aggregate order flow is a function of y_n^*, the aggregate order flow arising from the traders' maximization program.

The model is solved by backward induction.

We dedicate the next proposition to the resolution of the trader's maximization program.

Proposition 1 *There exists a unique solution to the trader's maximization program. The quantity submitted to market* n, with $n = 1, ..., N$, *is the positive root of the following third degree equation*

$$a_n = \frac{(1-\lambda_n a_n (K-1))\tau_\varepsilon \prod\limits_{\substack{i=1 \\ i \neq n}}^{N} \lambda_i}{\rho \sum\limits_{j=1}^{N} \prod\limits_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i \left((1-\lambda_n a_n (K-1))^2 + (K-1)\lambda_n^2 a_n^2 \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon} \right)^2 \right) (\tau_\varepsilon + \tau_v) \right) + 2(\tau_\varepsilon + \tau_v) \prod\limits_{i=1}^{N} \lambda_i},$$

with $a_n = \frac{\tau_\varepsilon}{\rho} b_n.$

The quantity submitted to any other market, $j \neq n$, *is such that* $\lambda_n a_n = \lambda_j a_j$.

Proof: See Appendix.

The trader splits his market order across the different markets in such a way that the marginal cost of trading across markets is equalized. Suppose that a particular market is less liquid than the other markets. The trader still submits an order to that market, as by splitting the order the trader reduces the overall impact of his order on the price. However, because the price impact of the order in that particular market is higher, the trader will reduce the size of the order in such a way that the marginal cost of trading is the same across markets.

We now give the expression of the market makers' expected profit.

Lemma 1 *Given the linearity of both the market orders and the price schedule, given by (1) and (2) respectively, the expected utility of profit for market maker* n *is given by*

$$\Pi_n = K a_n \sigma_v^2 (\lambda_n a_n K - 1) + \lambda_n K \left(a_n^2 \sigma_\varepsilon^2 + b_n^2 \sigma_w^2 \right) - \frac{\rho_m}{2} \Big[2\sigma_v^4 K^2 a_n^2 (\lambda_n a_n K - 1)^2 + K \left(a_n^2 \sigma_\varepsilon^2 + b_n^2 \sigma_w^2 \right) \left(\sigma_v^2 (2\lambda_n a_n K - 1)^2 + 2K\lambda_n^2 \left(a_n^2 \sigma_\varepsilon^2 + b_n^2 \sigma_w^2 \right) \right) \Big]. \quad (3)$$

Proof. See Appendix. ■

The term multiplied by ρ_m is the reduction in the market maker's expected utility of profit due to her risk aversion. Two elements contribute to that reduction: payoff uncertainty and size uncertainty. Firstly, any given size of the order flow implies a departure from her optimal inventory position implying a direct cost of providing liquidity to the market and therefore a decrease in her expected profit. Secondly, the timing of the game has also an impact on her expected profit. Indeed when she decides her level of liquidity, the aggregate order flow she will have to clear is unknown to her. As she is risk averse this increase in uncertainty adversely affects her expected utility of profit and therefore increases the premium required to provide liquidity to the market.

The next proposition states the existence of the equilibrium.

Proposition 2 *If* $\rho\tau_w^{-1} > \tau_\varepsilon \left(1 + \tau_v^{-1}\tau_\varepsilon\right)$, *a unique linear symmetric equilibrium exists.*

The price set by each market maker $n = 1, ..., N$ is

$$p_n = \lambda(N)\ y_n, \forall n = 1, ..., N,$$

each trader $i = 1, ..., K$ submits to the different market makers a market order of the following form

$$x(s_i, w_i) = a(N)\left(s_i - \rho\tau_\varepsilon^{-1} w_i\right),$$

where $a(N)$ and $\lambda_n(N)$ are defined in the Appendix.

Proof. See Appendix. ■

The model studied here is very general. A drawback of such a general model is that closed form solutions cannot be found. However, the proposition is proved using numerical procedures.

The sufficient condition for the existence of the equilibrium can be interpreted as follows. It states that the hedging motives must outweigh the informational motives for the existence of a linear equilibrium price schedule. Indeed the hedging motives must be large enough to induce, with a linear price schedule, a non-negative expected profit for the market makers.[10]

The trader's risk aversion as well as the precision of the private information affect both the size of the market order and its composition. Intuitively and keeping constant the size of the market order, an increase in the trader's risk aversion has a direct effect of increasing the proportion of the market order due to hedging motives whereas an increase of the precision τ_ε increases the proportion of the market order due to private information. All other parameters affect the size of the order, without changing its composition.

We look at some of the important properties of both the liquidity and the expected profit of the market makers.

4 Properties of the Equilibrium

4.1 Liquidity

We look at some of the properties of both the individual liquidity, or market depth, i.e. the liquidity set by each market maker, and aggregate liquidity defined as being the sum of all liquidities. In our case, aggregate liquidity is $\sum_{n=1}^{N} \frac{1}{\lambda_n}$.

Proposition 3 (Liquidity) *Individual liquidity decreases with the number of market makers (N) whereas aggregate liquidity increases with N.*

Proof. See Appendix. ■

[10]That condition is similar to the one obtained in Glosten (1989). Spiegel and Subrahmanyam (1992) also obtain a sufficient condition for the existence of a linear equilibrium. Their condition is a function of the number of hedgers. However, in our case, the condition is not as the traders are hedgers and informed speculators at the same time.

The proposition is proved using numerical procedures.

Individual liquidity is decreasing with the number of competing market makers. This result is also present in *BH.* However, the intuition is different. As stated in Proposition 1, the trader splits his order across markets in such a way that the marginal trading cost is equalized across markets. As a consequence, the trader submits a smaller quantity to markets with lower liquidity. By setting a higher price, the market maker does not modify the ratio of hedging to informed trading received. Indeed the trader reduces the size of his order without altering its composition.[11] Hence, by increasing her price, a market maker reduces the volume received, however, the increase in price may still compensate for the decrease in volume implying higher expected payoff. This implies that all market makers have an incentive to set less competitive price schedules. This increase in prices can be understood as an increase in the bid-ask spread set by each market maker.

Figure 1 compares our findings with the ones of Bernhardt and Hughson (1997) (*BH* on the graph). [12] For an initial low number of market makers, the decrease in individual liquidity is sharper for risk neutral than for risk averse. In addition, as the risk aversion increases, the impact of increasing the number of market makers decreases. In the Appendix, we show a corresponding graph with 100 traders.

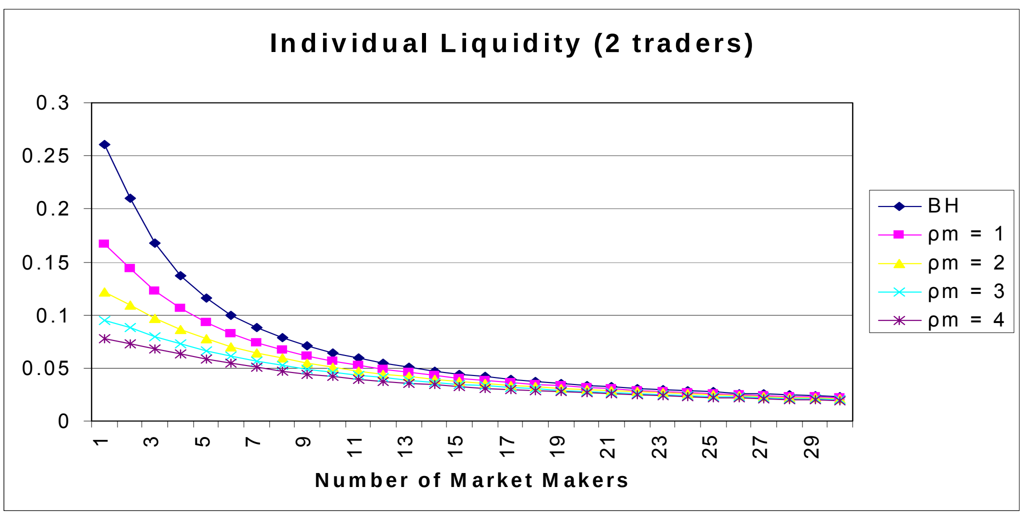

Figure 1: Individual Liquidity with 2 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

The following simulations (Figures 2 and 3) show the levels of aggregate

[11]This property is implied by the CARA setting used here. In a different setting, increasing the price (due to a high level of risk aversion of the market makers, for instance) may induce traders to reduce the hedging to information trading ratio implying a closure of the market. In the present setting, this does not happen.

[12]They assume that the market makers are strategic and risk neutral and trade with N risk neutral informed traders and some traders facing hedging needs.

liquidity for risk neutral as well as for risk averse market makers. They compare our results with those of Bernhardt and Hughson (1997), *BH* in the graph, Subrahmanyam (1991), *S* in the graph, and finally with those of Kyle (1985), *KYLE* in the graph. Case *S* is computed given a market maker's risk aversion of 4. In Kyle (1985) market makers are risk neutral. For the latter two papers, perfect competition between market makers is assumed and traders cannot split orders. The decrease in individual liquidity is shown below and is true for risk neutral market makers as well as for risk averse.

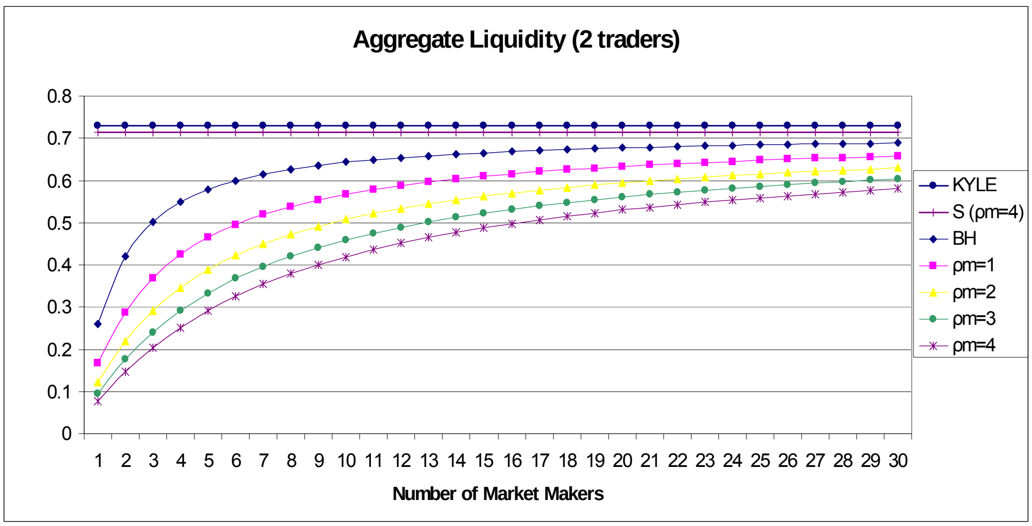

Figure 2: Aggregate Liquidity with 2 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

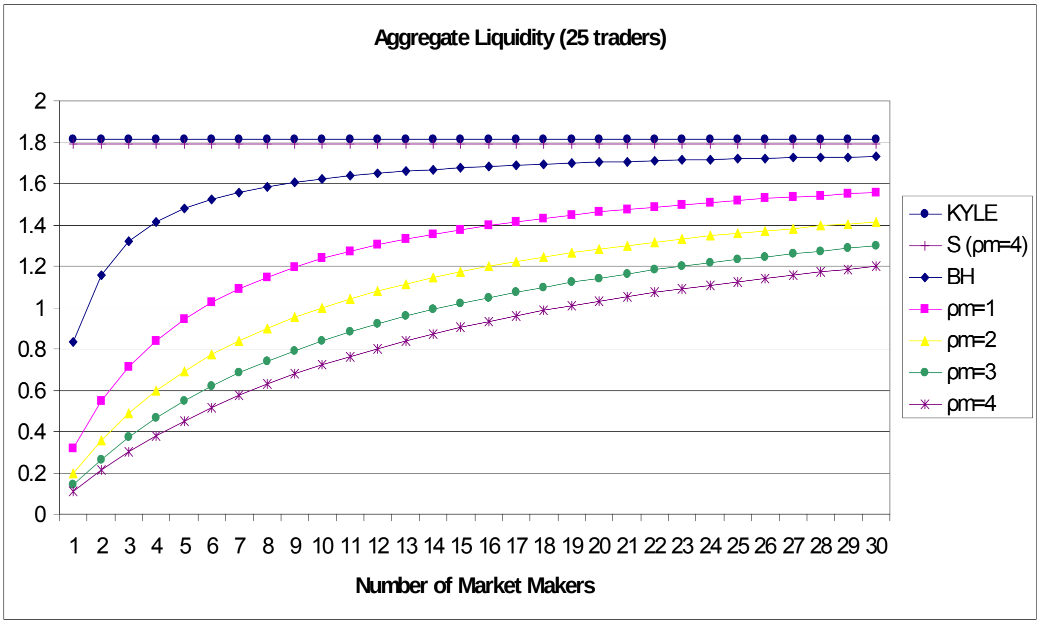

Figure 3: Aggregate Liquidity with 25 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

In order to compute the competitive level of liquidity when the market makers are risk averse, referred to as $S(\rho_m = 4)$ in the graphs above, we assume that market makers earn the autarky level of utility. Competition between market makers and the inability of traders to split their order among market makers, lead the market makers to be indifferent between making the market or not making it (autarky level). For convenience, we normalize the autarky level of utility to zero.

From the figures obtained, aggregate liquidity increases with the number of market makers and converges to the competitive level.[13] The effective price schedule faced by the traders decreases due to more competition. It is also the case, that the aggregate liquidity decreases with the market makers' risk aversion. This comparative static is very intuitive. Indeed, as the market maker's risk aversion increases, the cost of handling a given size of the order flow increases. The market maker then requires more compensation which decreases liquidity. However, as can be seen, increasing the market makers' risk aversion reduces the positive impact of competition on the aggregate liquidity level. This can be understood as follows. Risk aversion acts as a commitment device for market makers to set high prices. As their risk aversion increases, their commitment is even stronger reducing the positive impact of competition.

Our model displays some properties consistent with *BMR* and *BH* regarding aggregate liquidity and volume traded. They both increase with the number of market makers. It should be pointed out that in *BMR* the measure of liquidity is the Bid-Ask spread, they show that it decreases with n.

[13]This result can be proved analytically for the case of one trader splitting his orders among N risk neutral market makers.

4.2 Aggregate Expected Utility of Profit

We now look at the properties concerning the market makers' aggregate expected utility of profit.

Proposition 4 (Aggregate Expected Utility of Profit) *In equilibrium, the market makers' aggregate expected utility of profit is not monotonic with* ρ_m*, and* N*. For* $\rho_m > 0$*, it is inversely U-shaped with* N*. For small (large)* N*, the market makers' aggregate expected utility of profit increases (decreases) with* ρ_m*.*

Proof. See Appendix. ■

The proposition is proved using numerical procedures.

Figures 4-6 show the relationship between the aggregate expected utility of profit and N, the number of market makers, ρ_m, and the number of traders present in the auction.

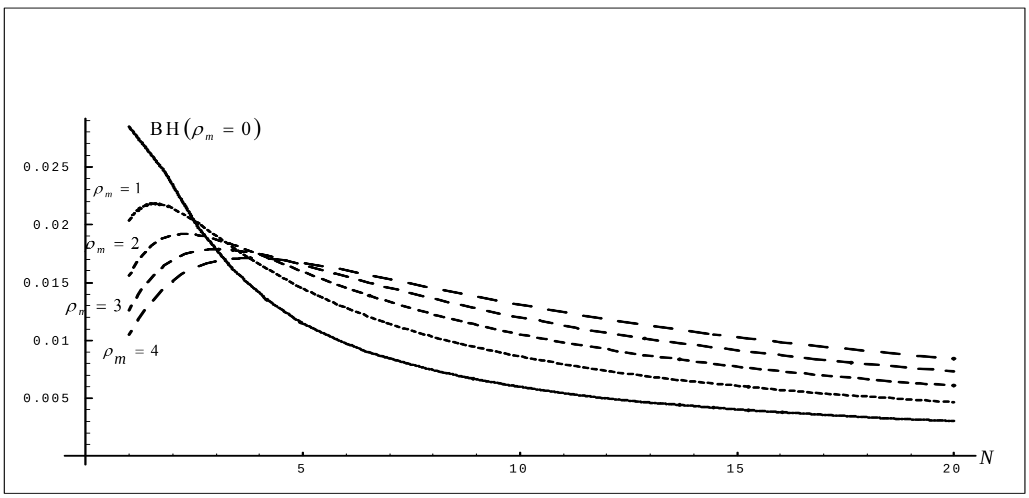

Figure 4: Market makers' aggregate expected utility of profit with one trader and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$ and $\rho = 7$.

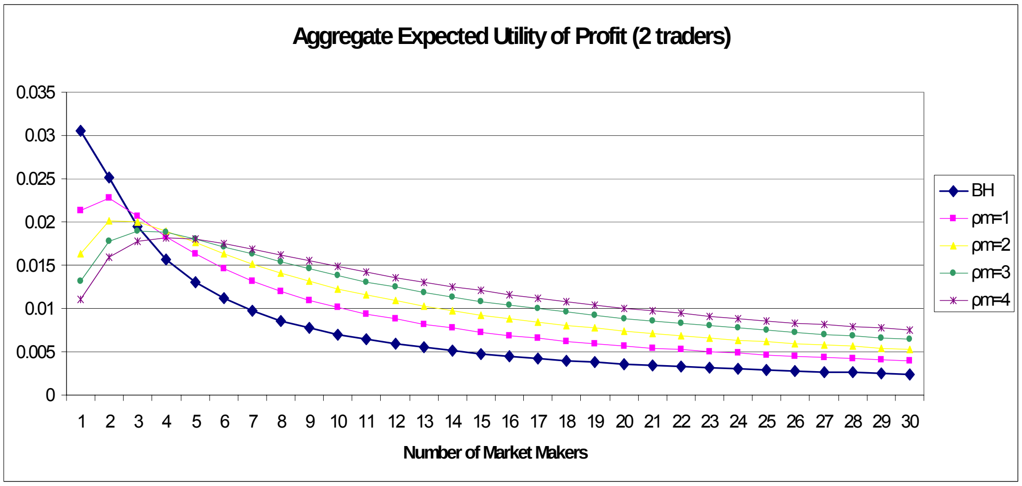

Figure 5: Market makers' aggregate expected utility of profit with 2 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

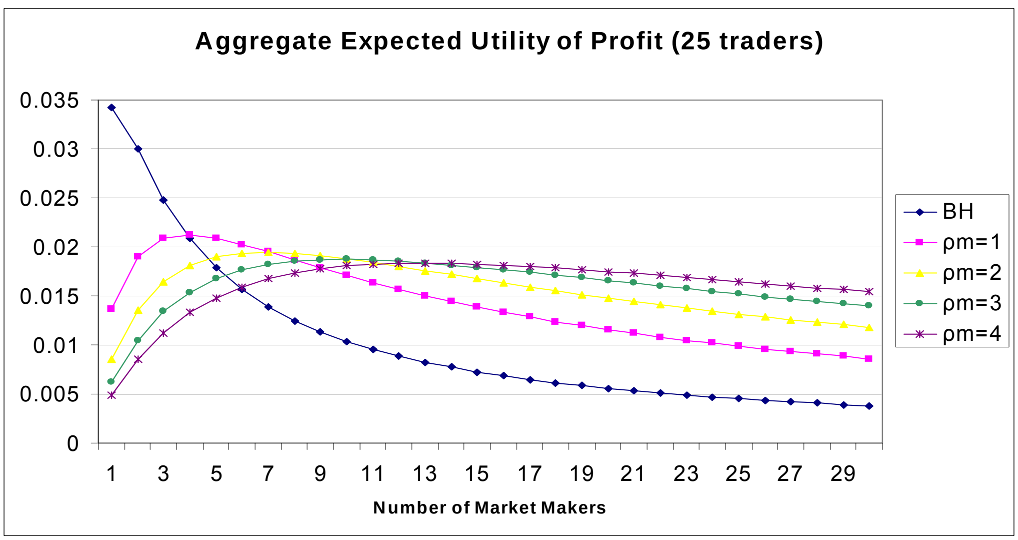

Figure 6: Market makers' aggregate expected utility of profit with 25 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

The market makers' aggregate expected utility of profit always decreases with the number of market makers when their risk aversion is 0 and tends to zero when the number of market makers is infinite.[14] However, whenever the market makers are strictly risk averse ($\rho_m > 0$), their aggregate expected utility of profit may increase as a result of increasing their number. The range for which this is true also increases with the market makers' risk aversion. For large values of the market makers' risk aversion ($\rho_m = 4$), their aggregate

[14] This result is the same as Bernhardt and Hughson (1997).

expected utility of profit is higher with 30 market makers than with 1 if at least 10 traders exchange the asset (See figure 8 in Appendix). It can also be seen that the market maker's aggregate expected utility of profit might be non-monotonic with ρ_m (inversely U-shaped) for intermediate values of N the number of market makers. For extreme values, it appears to be monotonic: decreasing with ρ_m for small values of N whereas increasing with ρ_m for large values of N.

In order to understand these results we have to understand all the basic effects on the market makers' aggregate expected utility of profit of varying N and ρ_m. Firstly, increasing N has the following effects: (i) it increases the aggregate risk tolerance of the market makers and increases risk sharing, (ii) it reduces the individual liquidity in each market, and finally (iii) it reduces the volume handled by market makers. The first and the second effect clearly increase aggregate expected utility of profit. However, the reduction in volume has two opposite effects on aggregate expected utility of profit. Indeed, the reduction in volume has an obvious effect of reducing them but at the same time it reduces the uncertainty faced by market makers increasing them. Secondly, increasing ρ_m has the following implications: (i) it decreases the aggregate risk tolerance of the market makers, (ii) it reduces the individual liquidity in each market, and finally (iii) it reduces the volume received by market makers. Effect (i) decreases aggregate expected utility of profit whereas effect (ii) increases it. The reduction in volume has again two opposite effects described earlier. Obviously, the magnitude of all these effects is also influenced by the number of traders present in the auction.

When $\rho_m = 0$, the negative effect of raising N dominates the positive ones, and the expected profit is diminishing with N. When $\rho_m > 0$, the positive effects dominate for a small number of market makers while the negative effect dominates for a large number of market makers.

The same effects can explain the behavior of the aggregate expected utility of profit with respect to the market makers' risk aversion.

This result has important implications for the traders' cost of trading. Indeed, in our model, this aggregate expected utility of profit provides a indirect measure of the overall and true cost of trading for the investors as a group. The following corollary states a result concerning that cost.

Corollary 1 *When $\rho_m > 0$, the overall traders' cost of trading may be increasing with the number of market makers.*

Increasing the number of market makers adversely affects the cost of trading. Looking at figure 6, we can see that there is a range of values of N for which the overall cost of trading increases with N. When $\rho_m = 1$, a trader's overall cost is lower with one market maker than with up to 15 market makers. This range increases with ρ_m. For instance when $\rho_m = 4$, the trader's cost is lower with 1 market maker than with 30. Paradoxically, from the point of view of the traders it may not be desirable to increase the number of market makers providing liquidity in the market. This is true only if the market makers are risk averse.

The above result implies that the widely used measure of traders' welfare, i.e. market depth or liquidity, is an inappropriate measure. Indeed, the traders' cost of trading increases with the number of market makers despite the fact that aggregate liquidity increases.

In *BMR*, the mark-ups above the competitive or efficient price schedule are shown to decrease with the number of market markers. As market makers are risk neutral this results in a decrease of their expected profit when their number increases. Their result is identical to *BH*.

5 Conclusion

This paper looks at the case where traders can split their orders among different market makers. Our model combines the assumptions of imperfect competition and risk aversion from the perspective of both market makers and traders. This study is conducted for a financial market organized as a batch auction. Each market maker commits to a level of liquidity and to a price form, in our case the price is a linear function of the order flow. At that price, each market maker clears the market, i.e., takes a position that balances supply and demand. The risk averse traders receive both heterogenous private information of the liquidation value of the traded risky asset and heterogenous endowment of the same asset. As a consequence, the traders trade for informational as well as hedging motives.

The main findings of the paper are the following. We prove the existence of a linear symmetric equilibrium. We obtain that aggregate liquidity increases with the number of market makers. For a finite number of market makers, they earn positive expected utility of profit. However, it is shown that, aggregate expected utility of profit may increase with the number of market makers whenever they are risk averse. This implies that the investors' cost of trading may increase with the number of market makers. As a result the traders' welfare may be adversely affected by increasing the number of market makers when these market makers are risk averse. A direct implication of that finding is that market liquidity or market depth is an inappropriate measure of investors' trading costs. As in various other papers, it is also shown that market makers' aggregate expected profit tends to zero whenever the number of market makers is infinite.

Empirical papers such as Christie and Schultz (1994), Christie et al. (1994), Weston (2000) and Simaan et al. (2003) find that market makers on the NASDAQ exhibit a non-competitive behavior. Our paper brings a new perspective to this non-competitive behavior. We find that their non-competitive behavior is exacerbated by their risk aversion. The more risk averse the market makers, the more market makers it takes for the aggregate liquidity to converge to its competitive level. In other words, risk aversion decreases the benefits of competition on the level of aggregate liquidity.

Our results could be regarded as an answer to the ongoing debate about the implications of market fragmentation on traders' welfare. We find that increasing market fragmentation, seen as increasing the number of market makers, can damage traders' welfare. Having more market makers or markets is not desirable from the point of view of investors' trading costs.

6 Bibliography

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7 Appendix

7.1 Proofs

Proof of Proposition 1:

Given the different prices set by each market maker $n = 1, ..., N$, $p_n = \lambda_n y_n$, each investor, $k = 1, ..., K$, submits a quantity $x_{nk} = a_n s_k - b_n w_k$ to each market. That quantity maximizes the expected profit for trader k taking into account its effect on the price,

$$\begin{aligned} \max_{x_{1k}\cdots,x_{Nk}} & E\left[\tilde{W}\middle|\, \Phi_k\right] - \frac{\rho}{2} var\left[\tilde{W}\middle|\, \Phi_k\right] \\ \text{with } \tilde{W} &= w\tilde{v} + \sum_{n=1}^{N} x_{nk}\tilde{v} - \sum_{n=1}^{N} p_n x_{nk} + c_k. \end{aligned}$$

This leads to

$$\max_{x_{1k}\cdots,x_{Nk}} w_k E\left[\tilde{v}\middle|\, \Phi_k\right] + \sum_{n=1}^{N} x_{nk} E\left[\tilde{v}\middle|\, \Phi_k\right] - \sum_{n=1}^{N} p_n x_{nk} + c - \frac{\rho}{2}\left(w_k + \sum_{n=1}^{N} x_{nk}\right)^2 var\left[\tilde{v}\middle|\, \Phi_k\right].$$

Differentiating the above expression with respect to x_{nk}, we get $\forall n = 1, ...N$

$$\begin{aligned}\frac{\partial}{\partial x_{nk}} &= E\left[\tilde{v}\right|\Phi_k\left]\left(1-\lambda_n a_n \left(K-1\right)\right) - 2\lambda_n x_{nk} \\ -\rho\left(K-1\right)&\left[\sigma_\varepsilon^2 \lambda_n a_n \sum_{j=1}^N \lambda_j a_j x_{jk} + \sigma_w^2 \lambda_n b_n \sum_{j=1}^N \lambda_j b_j x_{jk}\right] \\ -\rho\left(1-\lambda_n a_n \left(K-1\right)\right)&\left(w_k + \sum_{j=1}^N x_{jk}\left(1-\lambda_j a_j\left(K-1\right)\right)\right) var\left[\tilde{v}\right|\Phi_k\left] = 0.\end{aligned} \tag{4}$$

The entire system of first order conditions is given by

$$D_N \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{Nk} \end{pmatrix} = \begin{pmatrix} 1-\lambda_1 a_1 \left(K-1\right) \\ 1-\lambda_2 a_2 \left(K-1\right) \\ \vdots \\ 1-\lambda_N a_N \left(K-1\right) \end{pmatrix} \left(E\left[v\right|\Phi_k\left] - \rho w_k var\left[v\right|\Phi_k\right]\right) \tag{5}$$

with

$$\begin{aligned} D_N &= \begin{pmatrix} C_1 & D_{12} & \cdots & D_{1N} \\ D_{12} & C_2 & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1N} & D_{2N} & \cdots & C_N \end{pmatrix}, \\ D_{ij} &= \rho\left(\left(1-\lambda_i a_i\left(K-1\right)\right)\left(1-\lambda_j a_j\left(K-1\right)\right) var\left[v\right|\Phi_k\right]) \\ &\quad +\rho(K-1)\left(\lambda_i a_i \lambda_j a_j \sigma_\varepsilon^2 + \lambda_i b_i \lambda_j b_j \sigma_w^2\right), \\ C_i &= 2\lambda_i + D_{ii}. \end{aligned}$$

We first prove that the above system admits a unique solution as a maximum using a sequence of steps.

In step 1, we prove a useful property of the above system, i.e. trader k chooses his quantity such that the marginal cost of trading is equal across markets. In step 2, we prove that D_N can be inverted, i.e. its determinant is different from zero. In step 3, we prove the existence and unicity of a positive solution. In step 4, we show that the solution is indeed a maximum.

Step 1**:**

Lemma 2 *$\forall (n, j) \in [1, N] \times [1, N]$ and $n \neq j$, we have that $\lambda_n a_n = \lambda_j a_j$ and $\lambda_n b_n = \lambda_j b_j$.*

Proof. Using the expressions of the market orders as well as $E\left[v\right|\Phi_k\left] = \frac{\tau_\varepsilon s_k}{\tau_\varepsilon + \tau_v}\right.$ and $var\left[v\right|\Phi_k\left] = \frac{1}{\tau_\varepsilon + \tau_v}\right.$, the above system (5) can be rewritten as

$$\begin{pmatrix} C_1 & \cdots & D_{1N} \\ \vdots & \ddots & \vdots \\ D_{1N-1} & \cdots & D_{N-1N} \\ D_{1N} & \cdots & C_N \end{pmatrix} \begin{pmatrix} a_1 s_k - b_1 w_k \\ \vdots \\ \vdots \\ a_N s_k - b_N w_k \end{pmatrix} = \begin{pmatrix} 1-\lambda_1 a_1 \left(K-1\right) \\ \vdots \\ \vdots \\ 1-\lambda_N a_N \left(K-1\right) \end{pmatrix} \left(\frac{\tau_\varepsilon s_k - \rho w_k}{\tau_\varepsilon + \tau_v}\right). \tag{6}$$

Looking at the jth line of the above system and identifying the multiplicative parameters for s_k and w_k respectively, we get

$$\sum_{i=1}^{N} a_i D_{ij} + 2\lambda_j a_j = \left(1 - \lambda_j a_j \left(K-1\right)\right) \frac{\tau_\varepsilon}{\tau_\varepsilon + \tau_v},$$

$$\sum_{i=1}^{N} b_i D_{ij} + 2\lambda_j b_j = \left(1 - \lambda_j a_j \left(K-1\right)\right) \frac{\rho}{\tau_\varepsilon + \tau_v}.$$

Factorizing all terms with $\lambda_j a_j$ and $\lambda_j b_j$ for both equations we have

$$\begin{aligned} \lambda_j a_j t\left(a_i\right) + \lambda_j b_j z\left(a_i\right) &= A, \\ \lambda_j a_j t\left(b_i\right) + \lambda_j b_j z\left(b_i\right) &= A', \end{aligned} \tag{7}$$

with

$$\begin{aligned} t\left(q_i\right) &= 2 - \tfrac{\rho(K-1)}{\tau_\varepsilon+\tau_v} \sum_{i=1}^{N} q_i \left(1 - \lambda_i a_i \left(K-1\right)\right) + \rho\left(K-1\right)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i q_i + \tfrac{\tau_\varepsilon(K-1)}{\tau_\varepsilon+\tau_v}, \\ z\left(q_i\right) &= \rho\left(K-1\right)\sigma_w^2 \sum_{i=1}^{N} \lambda_i b_i q_i, \\ A &= \tfrac{\tau_\varepsilon}{\tau_\varepsilon+\tau_v} - \tfrac{\rho}{\tau_\varepsilon+\tau_v} \sum_{i=1}^{N} a_i \left(1 - \lambda_i a_i \left(K-1\right)\right), \\ A' &= \tfrac{\rho}{\tau_\varepsilon+\tau_v} - \tfrac{\rho}{\tau_\varepsilon+\tau_v} \sum_{i=1}^{N} b_i \left(1 - \lambda_i a_i \left(K-1\right)\right). \end{aligned}$$

Solving the system (7) for $\lambda_j a_j$ and $\lambda_j b_j$, we get

$$\lambda_j b_j = \tfrac{A'}{z(b_i)} - \lambda_j a_j \tfrac{t(b_i)}{z(b_i)},$$

$$\lambda_j a_j \left(t\left(a_i\right) - t\left(b_i\right) \tfrac{z(a_i)}{z(b_i)}\right) = A - A' \tfrac{z(a_i)}{z(b_i)}. \tag{8}$$

In order to prove that $\lambda_j a_j$ is indeed equal to a constant, we still have to prove that its multiplicative term is different from zero. We prove it by contradiction.

Suppose that $\exists\left(j,n\right) \in [1,N] \times [1,N]$ with $j \neq n$ such that $\lambda_j a_j \neq \lambda_n a_n$. Equation (8) is also true for $n = 1, ..., N$. We then get $\forall\left(j,n\right) \in [1,N] \times [1,N]$ with $n \neq j$

$$\begin{aligned} \lambda_j a_j G &= F, \\ \lambda_n a_n G &= F. \end{aligned}$$

with $G = \left(t(a_i) - t(b_i)\frac{z(a_i)}{z(b_i)}\right)$ and $F = A - A'\frac{z(a_i)}{z(b_i)}$. This implies that $G = 0$ leading to $F = 0$. We now prove that there is a contradiction. We can rewrite G as

$$G = (K-1)F + \rho(K-1)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i^2 + 2 - \tfrac{z(a_i)}{z(b_i)+2}\left(\rho(K-1)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i b_i\right).$$

If $G = 0$, this would imply that

$$-F = \rho(K-1)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i^2 + 2 - \tfrac{z(a_i)}{z(b_i)+2}\left(\rho(K-1)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i b_i\right) = 0.$$

Factorizing the term $\frac{1}{z(b_i)+2}$, we can rewrite the above expression as

$$4+2\rho(K-1)\sigma_\varepsilon^2 \sum_{i=1}^{N} \lambda_i a_i^2 + 2z(b_i) + \rho(K-1)\sigma_\varepsilon^2\left(z(b_i)\left(\sum_{i=1}^{N} \lambda_i a_i^2\right) - z(a_i)\left(\sum_{i=1}^{N} \lambda_i a_i b_i\right)\right).$$

We now look at the sign of the last term $z(b_i)\left(\sum\limits_{i=1}^{N} \lambda_i a_i^2\right) - z(a_i)\left(\sum\limits_{i=1}^{N} \lambda_i a_i b_i\right)$. Using the definition of $z(.)$, this can be expressed as

$$\rho(K-1)\sigma_w^2\left(\left(\sum_{i=1}^{N} \lambda_i b_i^2\right)\left(\left(\sum_{i=1}^{N} \lambda_i a_i^2\right) - \left(\sum_{i=1}^{N} \lambda_i a_i b_i\right)^2\right)\right).$$

Using some algebra, we can write the above expression as follows

$$\rho(K-1)\sigma_w^2\left(\left(\sum_{i=1}^{N} \lambda_i^2 b_i^2 a_i^2 + \sum_{j=1}^{N}\sum_{i\neq j} \lambda_i b_i^2 \lambda_j a_j^2 - \sum_{i=1}^{N} \lambda_i^2 b_i^2 a_i^2 - \sum_{j=1}^{N}\sum_{i\neq j} \lambda_i b_i a_i \lambda_j b_j a_j\right)\right),$$

which is equal to

$$\rho(K-1)\sigma_w^2\left(\sum_{j=1}^{N}\sum_{i\neq j} \lambda_i b_i \lambda_j a_j (b_i a_j - a_i b_j)\right).$$

The expression between brackets can be split into two terms: the terms such that $i < j$ and the terms such that $i > j$. Again using some basic algebra we can show that $\sum\limits_{j=1}^{N}\sum\limits_{i>j} \lambda_i b_i \lambda_j a_j (b_i a_j - a_i b_j) = \sum\limits_{i=1}^{N}\sum\limits_{j>i} \lambda_i b_i \lambda_j a_j (b_i a_j - a_i b_j)$. Using the latter and proceeding of a change of variable whereby $j' = i$ and $i' = j$, we obtain

$$\rho(K-1)\sigma_w^2\left(\sum_{j=1}^{N}\sum_{i<j} \lambda_i b_i \lambda_j a_j (b_i a_j - a_i b_j) + \sum_{j'=1}^{N}\sum_{i'<j} \lambda_{j'} b_{j'} \lambda_{i'} a_{i'} \left(b_{j'} a_{i'} - a_{j'} b_{i'}\right)\right),$$

which after some computations can be written as

$$\rho\left(K-1\right)\sigma_w^2\left(\sum_{j=1}^{N}\sum_{i<j}\lambda_i\lambda_j\left(b_ia_j-b_ja_i\right)^2\right).$$

As $-F$ is a sum of positive terms, F is different from zero. That leads to a contradiction. As a conclusion we have that $\forall\left(n,j\right)\in[1,N]\times[1,N]$ and $n\neq j$, $\lambda_n a_n=\lambda_j a_j$. Moreover this also implies that $\forall\left(n,j\right)\in[1,N]\times[1,N]$ and $n\neq j$, $\lambda_n b_n=\lambda_j b_j$. ■

Step 2: We now prove that D_N can be inverted.

Given step 1, D_N can be written as follows

$$D_N=\begin{pmatrix} 2\lambda_1+D & D & \cdots & D \\ D & 2\lambda_2+D & \cdots & D \\ \vdots & \vdots & \ddots & \vdots \\ D & D & \cdots & 2\lambda_N+D \end{pmatrix}$$

with $D_{ij}=D$.

Lemma 3 $\det D_N=2^{N-1}D\left(\sum\limits_{i=1}^{N}\prod\limits_{\substack{j\neq i\\ j=1}}^{N}\lambda_j\right)+2^N\prod\limits_{j=1}^{N}\lambda_j.$

Proof. The proof is done by iteration.

For $N=1$ and $N=2$, the determinants are given by

$$\begin{aligned}\det D_1 &= D+2\lambda_1,\\ \det D_2 &= 2D\left(\lambda_1+\lambda_2\right)+4\lambda_1\lambda_2.\end{aligned}$$

It is straightforward to show that both determinants verify the form set in the lemma.

We now show that the form is also true for N, assuming that it is true for $N-2$ and $N-1$. We rewrite D_N as

$$D_N=\begin{pmatrix} 2\lambda_1+D & D & \cdots & 0 \\ D & 2\lambda_2+D & \cdots & 0 \\ \vdots & \vdots & 2\lambda_{N-1}+D & -2\lambda_{N-1} \\ 0 & 0 & -2\lambda_{N-1} & 2\lambda_{N-1}+\lambda_N \end{pmatrix}$$

where the last column of D_N was replaced by the last column minus the $N-1th$ column. The same change was performed for the last row.

The determinant by developing from the last line and then from the last column gives

$$\det D_N=2\left(\lambda_N+\lambda_{N-1}\right)\det D_{N-1}-4\lambda_{N-1}^2\det D_{N-2}.$$

Using the form of $\det D_{N-1}$ and $\det D_{N-2}$, and reorganizing the resulting expression we get

$$\begin{aligned}\det D_N &= 2^{N-1}D\left(\sum_{i=1}^{N-1}\prod_{\substack{j\neq i\\ j=1}}^{N-1}\lambda_j\left(\lambda_{N-1}+\lambda_N\right)-\sum_{i=1}^{N-2}\prod_{\substack{j\neq i\\ j=1}}^{N-2}\lambda_j\lambda_{N-1}^2\right)\\ &\qquad +2^N\left(\prod_{j=1}^{N-1}\lambda_j\left(\lambda_{N-1}+\lambda_N\right)-\prod_{j=1}^{N-2}\lambda_j\lambda_{N-1}^2\right).\end{aligned}$$

After some algebra on both the first and the second term in brackets respectively, we can rewrite them as follows

$$\sum_{i=1}^{N-1}\prod_{\substack{j\neq i\\ j=1}}^{N-1}\lambda_j\left(\lambda_{N-1}+\lambda_N\right)-\sum_{i=1}^{N-2}\prod_{\substack{j\neq i\\ j=1}}^{N-2}\lambda_j\lambda_{N-1}^2=\sum_{i=1}^{N}\prod_{\substack{j\neq i\\ j=1}}^{N}\lambda_j,$$

$$\prod_{j=1}^{N-1}\lambda_j\left(\lambda_{N-1}+\lambda_N\right)-\prod_{j=1}^{N-2}\lambda_j\lambda_{N-1}^2=\prod_{j=1}^{N}\lambda_j.$$

Using the latter expressions, the determinant of D_N is equal to

$$\det D_N=2^{N-1}D\sum_{i=1}^{N}\prod_{\substack{j\neq i\\ j=1}}^{N}\lambda_j+2^N\prod_{j=1}^{N}\lambda_j,$$

which is the form we were looking for. Moreover the determinant is strictly positive as the λ's are positive. We can then conclude that the matrix can be inverted. ■

Step 3: Existence and Unicity.

Given step 1 and step 2, it is straightforward to show that $a_n = \frac{\tau_\varepsilon}{\rho}b_n$ for $n = 1, ..., N$.

Moreover given step 1, step 2 and the above, the first order condition (4) can be written as

$$x_{nk} = \frac{A_k}{2\lambda_n},$$

with

$$\begin{aligned}A_k &= E\left[\tilde{v}|\,\Phi_k\right]\left(1-\lambda_n a_n\left(K-1\right)\right)-\rho\left(K-1\right)\left[\sigma_\varepsilon^2\lambda_n a_n\sum_{j=1}^{N}\lambda_j a_j x_{jk}+\sigma_w^2\lambda_n b_n\sum_{j=1}^{N}\lambda_j b_j x_{jk}\right]\\ &\qquad -\rho\left(1-\lambda_n a_n\left(K-1\right)\right)\left(w_k+\sum_{j=1}^{N}x_{jk}\left(1-\lambda_j a_j\left(K-1\right)\right)\right)var\left[\tilde{v}|\,\Phi_k\right].\end{aligned}$$

Given step 1, A_k is independent of n and is therefore a constant. In the expression defining A_k, we replace all x_{ik} $\forall i = 1, ..., N$ by $\frac{A_k}{2\lambda_i}$ and all b_n by $a_n \frac{\rho}{\tau_\varepsilon}$ and put in factor the term A_k and simplify

$$A_k \left\{ 1 + \rho \left(1 - \lambda_n a_n (K-1)\right) var\left[\tilde{v} | \Phi_k\right] \left(\sum_{j=1}^{N} \frac{(1-\lambda_j a_j (K-1))}{2\lambda_j} \right) \right. \tag{9}$$

$$\left. + \rho (K-1) \lambda_n a_n \sum_{j=1}^{N} \frac{a_j \lambda_j}{2\lambda_j} \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon}\right)^2 \right) \right\} = \left(1 - \lambda_n a_n (K-1)\right) \left(E\left[\tilde{v} | \Phi_k\right] - \rho var\left[\tilde{v} | \Phi_k\right] w_k \right).$$

The term multiplied by A_k, henceforth called H, can be simplified as follows. We multiply that term by $\prod_{i=1}^{N} \lambda_i$ as the following simplification can be done

$$\prod_{i=1}^{N} \lambda_i \sum_{j=1}^{N} \frac{a_j \lambda_j}{2\lambda_j} = \frac{a_n \lambda_n}{2} \prod_{i=1}^{N} \lambda_i \sum_{j=1}^{N} \frac{1}{\lambda_j} = \frac{a_n \lambda_n}{2} \sum_{j=1}^{N} \frac{\prod_{i=1}^{N} \lambda_i}{\lambda_j} = \frac{a_n \lambda_n}{2} \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i.$$

The first equality sign is due to step 1, the rest is just some basic algebra. The same can be done for the term $\prod_{i=1}^{N} \lambda_i \sum_{j=1}^{N} \frac{(1-\lambda_j a_j (K-1))}{2\lambda_j}$, we then get that it is equal to $\frac{(1-\lambda_n a_n (K-1))}{2} \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i$. Using the above, we can rewrite $\prod_{i=1}^{N} \lambda_i H$ as

$$\sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i \left(\frac{\rho}{2(\tau_\varepsilon + \tau_v)} \left(1 - \lambda_n a_n (K-1)\right)^2 + \frac{\rho(K-1)}{2} \lambda_n^2 a_n^2 \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon}\right)^2 \right) \right) + \prod_{i=1}^{N} \lambda_i.$$

It is straightforward to see that H is positive.

Using all the simplifications, equation (9) leads to

$$2\lambda_n x_{nk} = \frac{(1-\lambda_n a_n (K-1))(\tau_\varepsilon s_k - \rho w_k) \prod_{i=1}^{N} \lambda_i}{(\tau_\varepsilon + \tau_v) \prod_{i=1}^{N} \lambda_i H},$$

Given the expression of x_{nk}, by identification we have

$$a_n = \frac{(1-\lambda_n a_n (K-1)) \tau_\varepsilon \prod_{\substack{i=1 \\ i \neq n}}^{N} \lambda_i}{\rho \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i \left((1-\lambda_n a_n (K-1))^2 + (K-1) \lambda_n^2 a_n^2 \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon}\right)^2 \right) (\tau_\varepsilon + \tau_v) \right) + 2(\tau_\varepsilon + \tau_v) \prod_{i=1}^{N} \lambda_i}.$$

This expression can be written as a third degree polynomial of the following form

$$a_n^3 t + a_n^2 \left(-2\rho\lambda_n (K-1) \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i \right) + a_n \left(\rho \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i + 2(\tau_\varepsilon + \tau_v) \prod_{i=1}^{N} \lambda_i \right) - \tau_\varepsilon \prod_{\substack{i=1 \\ i\neq n}}^{N} \lambda_i = 0,$$

$$\text{with } t = \rho (K-1) \lambda_n^2 \left(K - 1 + \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon} \right)^2 \right) (\tau_\varepsilon + \tau_v) \right) \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i.$$

The last term being negative and t being positive, the third degree equation admits at least one positive solution. The existence of a solution is then guaranteed.

The proof of the unicity follows Subrahmanyam (1991). Let us define the following functions

$$\begin{aligned}
f(a_n) &= a_n,\ g(a_n) = \tfrac{g_1(a_n)}{g_2(a_n)}, \\
\text{with } g_1(a_n) &= (1 - \lambda_n a_n (K-1)) \tau_\varepsilon \prod_{\substack{i=1 \\ i\neq n}}^{N} \lambda_i, \\
g_2(a_n) &= \rho \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i \left((1 - \lambda_n a_n (K-1))^2 + (K-1) \lambda_n^2 a_n^2 \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\tfrac{\rho}{\tau_\varepsilon} \right)^2 \right) (\tau_\varepsilon + \tau_v) \right) \\
&\quad + 2(\tau_\varepsilon + \tau_v) \prod_{i=1}^{N} \lambda_i.
\end{aligned}$$

Let us point out that $g_2(a_n)$ is positive when $a_n > 0$. We show the unicity of the solution by proving that the derivative of $g(.)$ is strictly smaller than 1 (derivative of $f(.)$) at points such that $f(a_n) = g(a_n)$. After some algebra, we have that

$$\tfrac{\partial g(a_n)}{\partial a_n} = \tfrac{h(a_n)}{g_2(a_n)},$$

with

$$\begin{aligned}
h(a_n) &= 2\rho\lambda_n \left(\sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i \right) (K-1) \left(-\lambda_n a_n^2 \left(K - 1 + \left(\sigma_\varepsilon^2 + \sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon} \right)^2 \right) (\tau_\varepsilon + \tau_v) \right) + a_n \right) \\
&\quad + (K-1) \prod_{i=1}^{N} \lambda_i.
\end{aligned}$$

The 2nd degree polynomial h reaches its maximum at $a_n = \frac{1}{2\lambda_n \left(K-1+\left(\sigma_\varepsilon^2+\sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon} \right)^2 \right)(\tau_\varepsilon+\tau_v) \right)}$

at this point the value of the function is

$$\frac{\rho(K-1) \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i}{2\left(K-1+\left(\sigma_\varepsilon^2+\sigma_w^2 \left(\frac{\rho}{\tau_\varepsilon} \right)^2 \right)(\tau_\varepsilon+\tau_v) \right)} - (K-1) \prod_{i=1}^{N} \lambda_i < \frac{\rho}{2} \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq j}}^{N} \lambda_i.$$

As $g_2(a_n) > \rho \sum_{j=1}^{N} \prod_{\substack{i=1\\ i\neq j}}^{N} \lambda_i$, we get that $\forall a_n \in \Re^{+}$, $g'(a_n) < 1$. The unicity of a positive solution is then proved.

Step 4**:** In order to prove that the solution is a maximum we prove that $-D_n$ is negative semidefinite. That matrix is given by

$$D_N = \begin{pmatrix} -2\lambda_1 - D & -D & \cdots & -D \\ -D & -2\lambda_2 - D & \cdots & -D \\ \vdots & \vdots & \ddots & \vdots \\ -D & -D & \cdots & -2\lambda_N - D \end{pmatrix}.$$

It can be seen that $\det(-D_N) = (-1)^N \det D_N$. From the Lemma proved in step 2, we know that $\det D_N > 0$, which implies for uneven N that $\det(-D_N) < 0$, whereas for even N $\det(-D_N) > 0$. This proves that the matrix $-D_N$ is negative semidefinite which in turn proves that the solution is a maximum.

Proof of Lemma 1:

In the price schedule (2) replace x_{nk} by its expression given in (1) and after some rearranging, the price schedule can be written as

$$p_n = \lambda_n y_n = \lambda_n a_n K \tilde{v} + \lambda_n a_n \sum_{k=1}^{K} \tilde{\varepsilon}_k - \lambda_n b_n \sum_{k=1}^{K} \tilde{w}_k.$$

Replace the above expression into the market maker's expected utility, we get

$$\Pi_n = E\left[\left(\tilde{v}(\lambda_n a_n K - 1) + \lambda_n a_n \sum_{k=1}^{K} \tilde{\varepsilon}_k - \lambda_n b_n \sum_{k=1}^{K} \tilde{w}_k\right)\left(a_n K \tilde{v} + a_n \sum_{k=1}^{K} \tilde{\varepsilon}_k - b_n \sum_{k=1}^{K} \tilde{w}_k\right)\right]$$
$$-\tfrac{\rho_m}{2} var\left[\left(\tilde{v}(\lambda_n a_n K - 1) + \lambda_n a_n \sum_{k=1}^{K} \tilde{\varepsilon}_k - \lambda_n b_n \sum_{k=1}^{K} \tilde{w}_k\right)\left(a_n K \tilde{v} + a_n \sum_{k=1}^{K} \tilde{\varepsilon}_k - b_n \sum_{k=1}^{K} \tilde{w}_k\right)\right].$$

Developing and using the fact that all random variables are independent and have zero mean leads to

$$\begin{aligned}\Pi_n = {} & K a_n (\lambda_n a_n K - 1)\sigma_v^2 + \lambda_n K\left(a_n^2 \sigma_\varepsilon^2 + b_n^2 \sigma_w^2\right) - \frac{\rho_m}{2}\left[(K a_n (\lambda_n a_n K - 1))^2 var\left[\tilde{v}^2\right]\right. \\ & + \lambda_n^2 b_n^4 var\left[\left(\sum_{k=1}^{K} \tilde{w}_k\right)^2\right] + (2\lambda_n a_n K - 1)^2 \left(a_n^2 var\left[\tilde{v}\sum_{k=1}^{K} \tilde{\varepsilon}_k\right] + b_n^2 var\left[\tilde{v}\sum_{k=1}^{K} \tilde{w}_k\right]\right) \\ & \left. + 4(\lambda_n a_n b_n)^2 var\left[\left(\sum_{k=1}^{K} \tilde{\varepsilon}_k\right)\left(\sum_{k=1}^{K} \tilde{w}_k\right)\right] + \lambda_n^2 a_n^4 var\left[\left(\sum_{k=1}^{K} \tilde{\varepsilon}_k\right)^2\right]\right].\end{aligned} \tag{10}$$

We need to compute all individual variances, using some basic statistics techniques, we have

$$\begin{aligned}
var\left[\tilde{v}^2\right] &= E\left[\tilde{v}^4\right] - E^2\left[\tilde{v}^2\right] = 2\sigma_v^4, \\
var\left[\tilde{v}\sum_{k=1}^{K}\tilde{\varepsilon}_k\right] &= \sum_{k=1}^{K} var\left[\tilde{v}\tilde{\varepsilon}_k\right] = K\sigma_v^2\sigma_\varepsilon^2, \\
var\left[\tilde{v}\sum_{k=1}^{K}\tilde{w}_k\right] &= \sum_{k=1}^{K} var\left[\tilde{v}\tilde{w}_k\right] = K\sigma_v^2\sigma_w^2, \\
var\left[\left(\sum_{k=1}^{K}\tilde{\varepsilon}_k\right)\left(\sum_{k=1}^{K}\tilde{w}_k\right)\right] &= \sum_{k=1}^{K}\sum_{k=1}^{K} var\left[\tilde{\varepsilon}_k\tilde{w}_k\right] = K^2\sigma_\varepsilon^2\sigma_w^2, \\
var\left[\left(\sum_{k=1}^{K}\tilde{\varepsilon}_k\right)^2\right] &= \sum_{k=1}^{K} var\left[\tilde{\varepsilon}_k^2\right] + \sum_{k=1}^{K}\sum_{j\neq k}^{K} var\left[\tilde{\varepsilon}_k\tilde{\varepsilon}_j\right] = 2K^2\sigma_\varepsilon^4, \\
var\left[\left(\sum_{k=1}^{K}\tilde{w}_k\right)^2\right] &= \sum_{k=1}^{K} var\left[\tilde{w}_k^2\right] + \sum_{k=1}^{K}\sum_{j\neq k}^{K} var\left[\tilde{w}_k\tilde{w}_j\right] = 2K^2\sigma_w^4.
\end{aligned}$$

Replace all the individual variances into the expression of the expected utility of profit (10) and do some rearranging, that leads to the desired result.■

Proof of Proposition 2: Due to the complexity of the case, the market makers' maximization program is solved using numerical procedures. In order to perform it, we use the form given in the previous lemma for the market maker's expected utility of profit where we replace a_n by the solution obtained when solving the third degree equation of proposition 1. As a consequence, market maker n's expected utility of profit is a function of all the liquidities set by the $n-1$ other competitors. We use numerical procedures to find market maker n best reply to the conjectured level of liquidity set by her competitors. As all market makers are identical, we look for a symmetric equilibrium where we assume that all her competitors set an identical level of liquidity $\frac{1}{\lambda}$. Given that, we find a fixed point, i.e. a level of liquidity equal to the level of her competitors that maximizes her level of expected utility of profit. The solution is then called $\lambda(N)$. However, for the solution to exist the following condition is required $\rho\tau_w^{-1} > \tau_\varepsilon\left(1+\tau_v^{-1}\tau_\varepsilon\right)$. That condition is easily checked for the case where we have risk neutral market makers or when only one trader is present. This condition arises for the more complex case.

Once we have found the level of liquidities, we retrieve the values of $a(N)$ and $b(N)$.■

Proof of Proposition 3: The proof is done by numerical applications. We reproduce the process by which we find the expression of the liquidity parameter from proposition 2 for the different values of N, ρ_m and K.■

Proof of Proposition 4: The proof is done by numerical applications. Once the liquidity value is found from proposition 3 , we then compute the value of both $a(N)$ and $b(N)$. Once all the values are computed, we plug them into the expression the market maker's expected utility of profit given by (10). The aggregate expected utility of profit is then computed as the sum of all individual market maker's expected utility of profit. We reproduce the above process for the different values of N, ρ_m and K.■

7.2 Figures

7.2.1 Liquidity

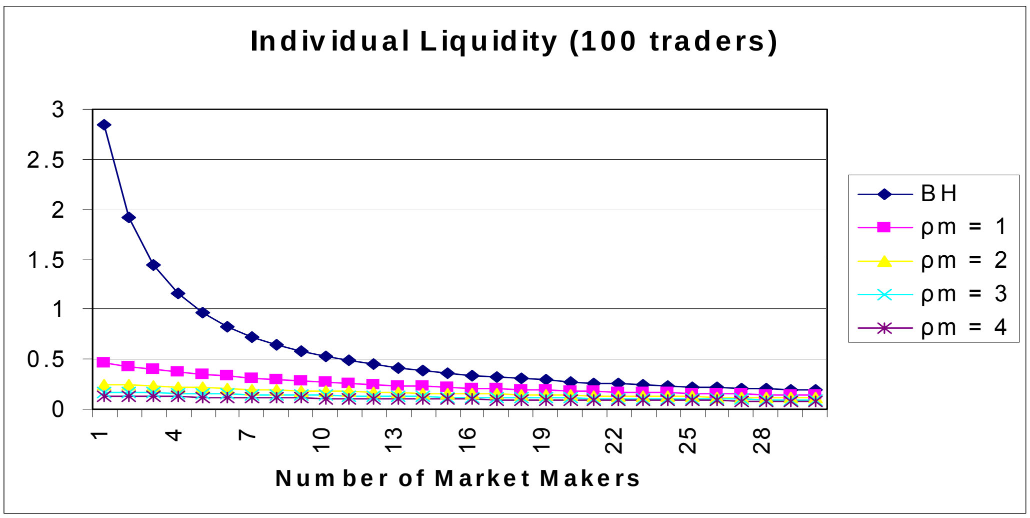

Figure 7: Individual Liquidity with 100 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

7.2.2 Aggregated Expected Utility of Profit

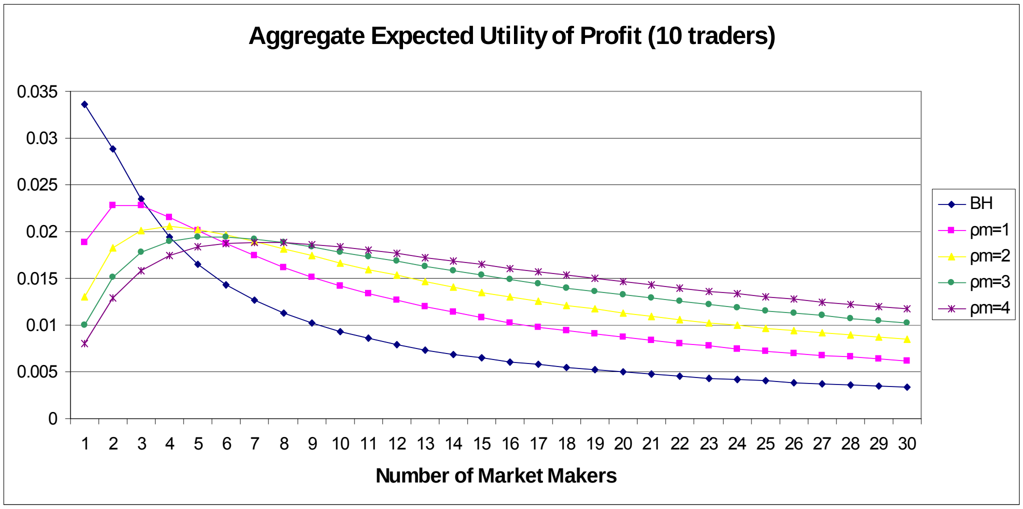

Figure 8: Market makers' aggregate expected utility of profit with 10 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

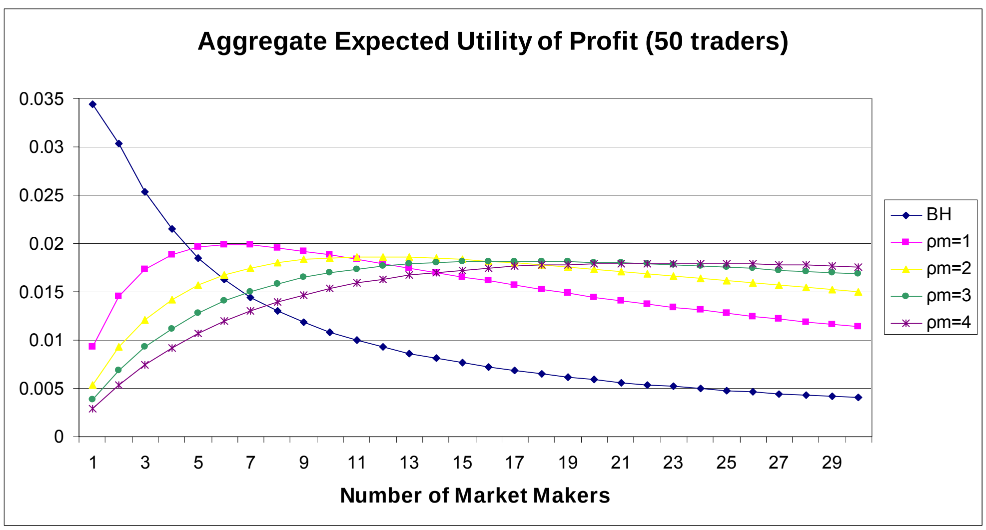

Figure 9: Market makers' aggregate expected utility of profit with 50 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.

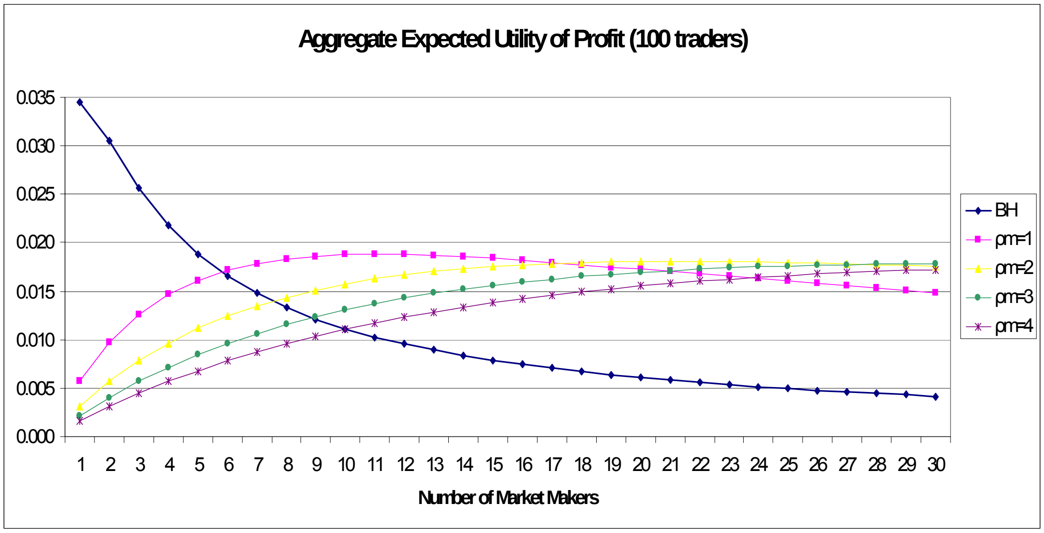

Figure 10: Market makers' aggregate expected utility of profit with 100 traders and $\tau_v = 1$, $\tau_w = 3$, $\tau_\varepsilon = 2$, $\rho = 7$.