The Pricing of Finite Maturity Corporate Coupon Bonds with Rating-Based Covenants

Preliminary and incomplete

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JEL classification: G12, G33

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This paper models the price of finite maturity corporate coupon bonds with a rating-based covenant. Bhanot (2003) already addressed this issue, but the corresponding model embeds two features that in some way could be considered as significant limitations. In the first place, it does not take into consideration any kind of payment to the bondholders when the rating-based covenant is triggered. In the second place, and more importantly, Bhanot (2003) incorporates an inconsistency regarding the payment the bondholders are entitled to at maturity: the model considers this payment (when the firm has not previously entered in bankruptcy) to correspond always to the principal of the debt outstanding and this can only be true if the value of the debtor's assets is greater than or equal to the principal, which obviously is not always the case. In this way Bhanot's model overprices the bond. Following Bhanot and Mello (2005), we propose to overcome the former weekness considering two alternative procedures: *i)* an increase in the coupon rate or *ii)* a partial amortization of the principal. The latter limitation is addressed by explicitly modeling the possibility of a partial payment of the principal if at maturity the market value of the assets in place is not enough to allow for full repayment. The values of the equity, tax benefits, bankruptcy costs and the leveraged firm are also obtained.

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1. Introduction

Since Merton's (1974) seminal paper which pioneered the so called structural approach to pricing corporate bonds, we have seen an increasing growth of the literature in this field.

At an early stage, an important contribution was provided by Black and Cox (1976) who, in defining default as a trigger event that may happen at any moment of a bond's life instead of occurring only at the maturity, relaxed one of the simplifying assumptions present in Merton's model, and established a feature common to almost all structural models published thereafter. In this type of modelling exercise, default is triggered when the value of the firm's assets[1] reaches some specified value, the barrier level. The way in which this barrier level is set, either endogenously or exogenously, has been a distinguishing factor between different models. One way is to consider that the barrier level is determined by the shareholders, in order to maximize the equity value (e.g. Black and Cox (1976), Leland (1994), Leland and Toft (1996), Goldstein, Ju and Leland (2001), Ericsson and Reneby (2003)). The alternative takes into consideration a barrier level set exogenously, reflecting the presence of some kind of covenant in the bond indenture (e.g Black and Cox (1976), Kim, Ramaswamy and Sundaresan (1993), Longstaff and Schwartz (1995), Briys and de Varenne (1997), Ericsson and Reneby (1998), Schobel (1999), Hsu, Saá-Requejo and Santa-Clara (2003), Hui, Lo and Tsang (2003), Taurén (1999), Collin-Dufresne and Goldstein (2001), Ju and Ou-Yang (2004), Huang et al. (2003)).

Besides different extensions concerning the interest rate process, recovery values, debt structure, bond characteristics, definition of barriers and dynamic capital structure, to name a few, the research in this field has also focused on aspects related to the asset substitution problem (Mello and Parsons (1992), Leland (1998), Ericsson (2000), Bhanot and Mello (2005)), the equity/bond holders strategic behavior (Anderson and Sundaresan (1996), Anderson, Sundaresan and Tychon (1996), Mella-Barral and Peraudin (1997), Mella-Barral (1999), Fan and Sundaresan (2000)), and bankruptcy codes (François Morrelec (2004), Moraux (2002), Galai, Raviv and Wiener (2003) and Yu (2003)).

The structural model proposed here aims to price corporate bonds whose indenture incorporates a rating trigger based covenant, which links the pay-offs to bondholders with the credit rating of the firm. As put by Bhanot and Mello (2005): "a "rating trigger clause" in a corporate bond indenture requires a firm to prepay its debt or to change the coupon rate on its debt if the firm's credit rating reaches a specified level."

Although the Bhanot and Mello (2005) framework does take into consideration this kind of bond[2], and considers two types of rating trigger covenants (partial amortization of the debt's

[1] Or other state variable related to the firm such as cash-flow or debt ratio.

[2] The primary focus of Bhanot and Mello (2005) is the asset substitution problem. Specifically, the authors analyze the effects of such covenants on the asset substitution problem.

principal[3] or an accrued coupon rate), their model only applies to perpetual debt. By contrast, the present paper proposes a framework capable of dealing with finite maturity bonds.

The finite maturity case was addressed by Bhanot (2003), whose model assumes the existence of two possible credit events[4]: namely, a downgrade in the credit rating of the firm and bankruptcy, which implies the liquidation of the firm. These two events were modeled through the specification of two barriers levels, V_{B1} and V_{B2} respectively (with $V_{B1} > V_{B2}$), established exogenously. However, there are two limitations in Bhanot (2003) that we will try to overcome in the current paper. In the first place, albeit the purpose of Bhanot (2003) is to price bonds with rating based covenants, the model does not explicitly assume any kind of change in bondholder pay-offs, when the covenant is triggered. In other words, when the value of the firm's assets reaches the first barrier (V_{B1}), the rating change only affects some parameter values associated with the diffusion process governing the value dynamics of the firm's assets[5]. Additionally, even if the previous remark is not taken into consideration, the price formula developed by Bhanot (2003) assumes that the payment to bondholders at maturity (admitting that, in the mean time, the firm has not entered in bankruptcy and so has not been liquidated, which is equivalent to admiting that the second barrier V_{B2} as not been reached) always corresponds the bond principal. This final cash-flow only makes sense in a scenario where the value of the firm's assets is enough to cover it. Otherwise, if the value of these assets is insufficient to cover the face value of the bond, at most the bondholders will only receive the value of the assets, since the equity holders will not be willing to pay the difference. In this sense, we may say that Bhanot (2003) overvalues the bondholders expected cash-flows, resulting in an overpricing of the bond.

Using the same base structure of Bhanot's model, which defines both credit events (rating change and bankruptcy) through the barrier levels, V_{B1} and V_{B2}, we propose to obtain a bond pricing formula that takes into account those two remarks. Besides obtaining the bond value before the rating change takes place, we also derive value expressions at the moment of the rating change

[3] To be exact, Bhanot and Mello (2005) assume that debt holders receive a fraction of the initial market value of debt when the rating of the firm changes. This amount corresponds to a fraction of the principal, since the analysis is restricted to debt sold at par.
[4] These are also assumed in Bhanot and Mello (2005).
[5] "Recall that we have assumed a change in credit rating causes two characteristics to change: the volatility of firm assets, and the outflow to shareholders (and an increased interest expense)", Bhanot (2003), page 62.

and immediately after that. The same is done for equity, bankruptcy costs, tax benefits and leveraged firm value.

As a final note, concerning the partial prepayment of the bond's principal rating trigger, Bhanot and Mello (2005) analyzed this case taking into account two different financing sources – either by cash infusion or by selling assets. The same will be done here, but instead of considering them separately, they will be jointly modelled, which will allow the simultaneous use of both sources.

The model developed in the current paper fills the gap between Bhanot (2003) and Bhanot and Mello (2005).

The paper is organized as follows: section 2 establishes the valuation framework, in section 3 the bond pricing formulas are derived, in section 4 the value of equity, bankruptcy costs, tax benefits and the leveraged firm are obtained, in section 5 a comparative analysis between our model and Bhanot's model results is performed, in section 6 we compare the two types of rating trigger covenant and, finally, section 7 concludes the paper.

2. The Valuation Framework

It is assumed that the value of the firm's assets, V_t, is described by the following continuous diffusion process, under the risk neutral probability:

$$dV_t/V_t = (r - \alpha_i)dt + \sigma_i dW_t$$

Where r is the (constant) risk free interest rate, α_i is the cash payout rate, σ_i the (constant) assets return volatility and dW_t an increment of a standard Brownian motion. As in Bhanot (2003), we allow the parameters of the diffusion process to alter after a rating change so the subscript i takes the value 1 (before) or 2 (after)[6]. Notice that this change, once occurred is assumed to be irreversible and permanent.

[6] Specifically, Bhanot (2003) assumes that $\alpha_2 > \alpha_1$ and $\sigma_2 > \sigma_1$.

The debt of the firm is characterized by a single coupon bond with principal F, coupon rate c and finite maturity T. Thus, the bondholders receive a continuous payment flow cFdt at least until the rating change.

We consider the existence of two credit events, namely a firm's rating downgrade and the occurrence of bankruptcy. Each of these will be modeled through an exogenous specification of two thresholds, which is two barrier levels, designated by V_{B1} and V_{B2} respectively. In addition, we also assume that the bankruptcy event is always preceded by the rating downgrade so: $V_{B1} > V_{B2}$.

The rating change will occur when the value of the assets intersects for the first time the first trigger level (V_{B1}). As a result, we define the time of the rating change as:

$$\tau_1 = \inf\{t \geq 0: V_t \leq V_{B1}\}$$

Once V_{B1} is reached, two possible outcomes may result, depending on the formulation of the rating-based covenant of the bond, namely:

1. An increase in the coupon rate keeping the principal at the initial level. In this case, the continuous coupon flow to bondholders changes to: $\Delta cFdt$, with $\Delta > 1$ (Δc, corresponds to the new coupon rate).
2. A partial refund of the principal, keeping the coupon rate at the initial level. This will lead to a reduction in interest payments: $c\Delta Fdt$, with $\Delta < 1$, ($(1-\Delta)$ is the fraction of the nominal debt value that is redeemed).

In both cases, the new coupon payments will occur until the bond matures or until the firm goes bankrupt, which will happen when the value of the assets reaches the second (constant) barrier V_{B2} , ($V_{B2} < F$). Thus, the bankruptcy time is defined as:

$$\tau_2 = \inf\{t \geq \tau_1: V_t \leq V_{B2}\}$$

At τ_2, the firm is liquidated, and the bondholders receive the value of the firm's assets net of bankruptcy costs: $\rho_1 V_{\tau_2} = \rho_1 V_{B2}$, ($0 < \rho_1 \leq 1$). Consequently, we assume that bankruptcy costs are a constant fraction $(1-\rho_1)$ of the value of the assets. Notice that this implies the verification of the absolute priority rule in bankruptcy, since $V_{B2} < F$) and that, in case of bankruptcy, the shareholders get nothing.

Thus, we have:

For $t < \tau_1$, the assets value diffusion process (under the risk-neutral probability measure) is given by: $dV_t/V_t = (r - \alpha_1)dt + \sigma_1 dW_t$ and the coupon flow: cFdt.

For $t \geq \tau_1$, the assets value diffusion process changes to: $dV_t/V_t = (r - \alpha_2)dt + \sigma_2 dW_t$, and the coupon flow to ΔcFdt.

To value the bond, Bhanot (2003) takes into account the payments to the bondholders in three distinct situations:

1. The value of the assets remains above V_{B1} until the maturity of the bond ($\tau_1 > T$);
2. The value of the assets crosses V_{B1} but remains above V_{B2} until the maturity of the bond ($\tau_2 > T$);
3. The value of the assets reaches V_{B2} before the maturity of the bond ($\tau_2 < T$).

The distinguishing feature of our model relative to Bhanot's (besides the formulation of the coupon payments after V_{B1} has been reached) derives from the fact that it takes into account a possible default at the maturity of the bond. This is absent in Bhanot (2003), which only considers the possibility of bankruptcy when V_{B2} is crossed.

Consider the following numerical example: $V_{B1} = 100$ and $V_{B2} = 50$, $F = 100$ (values taken from Bhanot (2003)). In Bhanot's formulation, if the value of the firm's assets never crosses the bankruptcy barrier, the bondholders' pay-off at maturity is always the principal of the bond. In the example 100, which is possible only if the value of the firm's assets at maturity, V_T, is enough to cover the corresponding payment ($V_T > 100$). If not, the firm defaults, since it is impossible to honour the payment. The situation where Bhanot's framework is valid corresponds to the restricted case where $V_{B2} \geq F$.

A general formulation would lead to distinguishing five situations:

1. The value of the assets remains above V_{B1} until the maturity of the bond ($\tau_1 > T$);

1.1. and the value of the assets at maturity is greater than or equal to the principal ($V_T \geq F$);

1.2. or the value of the assets is insufficient to repay the principal ($V_T < F$)

2. The value of the assets crosses V_{B1} but remains above V_{B2} until the maturity of the bond ($\tau_2 > T$):

   2.1. and the value of the assets at maturity is greater than or equal to the principal ($V_T \geq F$);

   2.2. or the value of the assets is insufficient to repay the principal ($V_T < F$)

3. The value of the assets reaches V_{B2} before the maturity of the bond ($\tau_2 \leq T$).

Notice that, the scenario 1.2 is only possible if the face value of the bond, F, is greater than V_{B1}. Considering $V_{B1} > F$, figure 1 shows four sample paths, I, II, III and IV for the assets values associated with the four possible scenarios, 1, 2.1, 2.2 and 3 respectively.

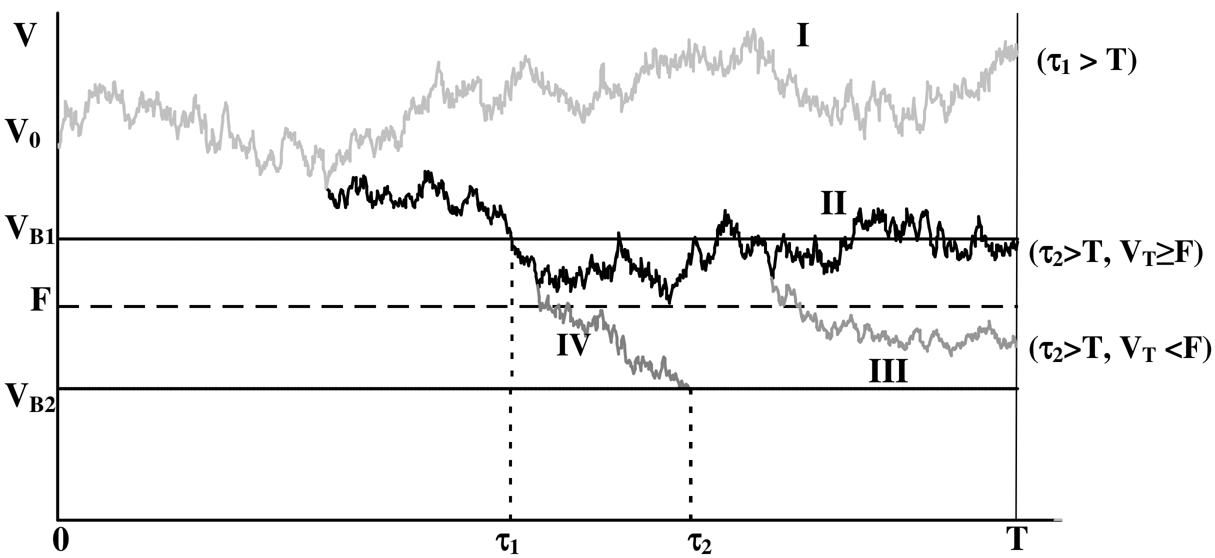


Figure 1 - Four sample path for the assets values associated to the four possible scenario, considering V_{B1}>F. Assets value remains above V_{B1} until the maturity of the bond - I; assets value crosses V_{B1} but remains above V_{B2}, and at maturity the value of assets is sufficient (insufficient) to repay the bond face value - II (III) and finally the assets value reaches V_{B2} before the maturity of the bond – IV.

It will be assumed that, in the event of default at maturity (τ_1>T, V_T <F and τ_2>T, V_T <F), the pay-off to bondholders will be a fraction of the market value of the assets: $\rho_2 V_T$, and (0< ρ_2 <1).

3. The Bond Value

3.1 Rating-based covenant: increase in the coupon rate

In the first place we will consider the case of an increase in the coupon rate, when the asset value hits V_{B1}. Remember that the new coupon, after the rating trigger, is given by $\Delta cFdt$, with $\Delta>1$. The value, at time t, of the bond will be given by the expected value, under the risk neutral probability (Q_B), of all payments discounted at the risk free rate:

$$B(V, t, T, c, F, \Delta) = E^{Q_B}\left[\int_t^T cFe^{-r(s-t)}1_{s<\tau_1}\,ds + e^{-r(T-t)}\left(F1_{\tau_1>T,V_T\geq F} + \rho_2 V_T 1_{\tau_1>T,V_T<F}\right)\middle|\mathcal{F}_t\right] +$$

$$+\ E^{Q_B}\left\{e^{-r(\tau_1-t)}1_{\tau_1<T}E^{Q_B}\left[\int_{\tau_1}^T \Delta cFe^{-r(s-\tau_1)}1_{s<\tau_2}\,ds + e^{-r(T-\tau_1)}\left(F1_{\tau_2>T,V_T\geq F} + \rho_2 V_T 1_{\tau_2>T,V_T<F}\right)\middle|\mathcal{F}_{\tau_1}\right]\middle|\mathcal{F}_t\right\}$$

$$+\ E^{Q_B}\left\{e^{-r(\tau_1-t)}1_{\tau_1<T}E^{Q_B}\left[e^{-r(\tau_2-\tau_1)}\rho_1 V_{\tau_2}1_{\tau_2<T}\middle|\mathcal{F}_{\tau_1}\right]\middle|\mathcal{F}_t\right\} \qquad (1)$$

Where 1_A is the indicator function, which assumes the value of one if the event A is true and zero otherwise.

The first line of the second member represents the discounted expected value of the coupon flow until the first barrier (V_{B1}) is reached and the payment at maturity if the assets value remains above V_{B1} prior to the maturity date. The second line refers to the coupon flow after the rating change and the payment at maturity when the asset value crosses the first trigger level (V_{B1}) but remains above the second barrier (V_{B2}) prior to the maturity date. Finally the last line refers to the recovery value that accrues to bondholders when bankruptcy is triggered prior to the maturity date.

Notice that the sum of the two expected values inside the cotter in the second and third line represents the value of the bond at τ_1, that is to say at the time of the rating change:

$B(V_{\tau_1}, \tau_1, T, c, F, \Delta) =$

$$E^{Q_B}\left\{\left[\int_{\tau_1}^{T} \Delta cFe^{-r(s-\tau_1)} 1_{s<\tau_2}\, ds + e^{-r(T-\tau_1)}\left(F1_{\tau_2>T,V_T\geq F} + \rho_2 V_T 1_{\tau_2>T,V_T<F}\right) + e^{-r(\tau_2-\tau_1)}\rho_1 V_{\tau_2} 1_{\tau_2<T}\right]\middle|\mathcal{F}_{\tau_1}\right\} \qquad (2)$$

Thus, given the continuity of the assets value process, $\left(V_{\tau_i} = V_{Bi}\right)$, we can rewrite expression (1) as:

$$B(V_t, t, T, c, F, \Delta) = E^{Q_B}\left[\int_t^T cFe^{-r(s-t)} 1_{s<\tau_1}\, ds + e^{-r(T-t)}\left(F1_{\tau_1>T,V_T\geq F} + \rho_2 V_T 1_{\tau_1>T,V_T<F}\right)\middle|\mathcal{F}_t\right] +$$

$$+\ E^{Q_B}\left[e^{-r(\tau_1-t)} 1_{\tau_1<T} B(V_{B1}, \tau_1, T, c, F, \Delta)\middle|\mathcal{F}_t\right] \qquad (3)$$

Whose value is given by (derivation in the appendix A.1):

$$B(V_t, t, T, c, F, \Delta) = \frac{cF}{r}\left[1 - e^{-r(T-t)}\left[1 - Q_B\left(\tau_1 \leq T\middle|\mathcal{F}_t\right)\right] - \left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m-\mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \leq T\middle|\mathcal{F}_t\right)\right]$$

$$+\ Fe^{-r(T-t)}Q_B\left(\tau_1 > T, V_T \geq F\middle|\mathcal{F}_t\right) + \rho_2 V_t e^{-\alpha_1(T-t)} Q_V\left(\tau_1 > T, V_T < F\middle|\mathcal{F}_t\right)$$

$$+\int_t^T e^{-r(\tau_1-t)} B(V_{B1}, \tau_1, T, c, F, \Delta)\, g_B(\tau_1, V_t, V_{B1})\, d\tau_1 \qquad (4)$$

And,

$B(V_{B1}, \tau_1, T, c, F, \Delta) =$

$$\frac{\Delta cF}{r}\left\{1 - e^{-r(T-\tau_1)}\left[1 - Q_B\left(\tau_2 \leq T\middle|\mathcal{F}_{\tau_1}\right)\right]\right\} + \left(\rho_1 V_{B2} - \frac{\Delta cF}{r}\right)\left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \leq T\middle|\mathcal{F}_{\tau_1}\right)$$

$$+\ Fe^{-r(T-\tau_1)}Q_B\left(\tau_2 > T, V_T \geq F\middle|\mathcal{F}_{\tau_1}\right) + \rho_2 V_{B1} e^{-\alpha_2(T-\tau_1)} Q_V\left(\tau_2 > T, V_T < F\middle|\mathcal{F}_{\tau_1}\right) \qquad (5)$$

Where:

$Q_X\left(\tau_1 \leq T\middle|\mathcal{F}_t\right)$ - stands for the probability, under measure X, of having a rating change until the maturity of the bond (period T-t);

$Q_X\left(\tau_2 \leq T \middle| \mathcal{F}_{\tau_1}\right)$ - the probability, under measure X, of firm entering into bankruptcy, from the moment of the rating change until the maturity of the bond (period T - τ_1);

$Q_X\left(\tau_1 > T, V_T \geq F \middle| \mathcal{F}_t\right)$ $\left(Q_X\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right)\right)$ - the probability, under measure X, of the firm never having a rating change, and the value of the assets at maturity being greater than or equal to (lesser than) the face value of the bond;

$Q_X\left(\tau_2 > T, V_T \geq F \middle| \mathcal{F}_{\tau_1}\right)$ $\left(Q_X\left(\tau_2 > T, V_T < F \middle| \mathcal{F}_{\tau_1}\right)\right)$ - the probability under measure X, from the moment of the rating change, of the firm never entering into bankruptcy, and the value of the assets at maturity being greater than or equal to (lesser than) the face value of the bond;

Note that: $Q_X\left(\tau_i > T, V_T < F \middle| \cdot\right) = 1 - Q_X\left(\tau_i \leq T \middle| \cdot\right) - Q_X\left(\tau_i > T, V_T \geq F \middle| \cdot\right)$

$$Q_X\left(\tau_1 \leq T \middle| \mathcal{F}_t\right) = N\left[-a_1 - b_1(\mu_1^X, t)\right] + \left(\frac{V_t}{V_{B1}}\right)^{\frac{-2\mu_1^X}{\sigma_1^2}} N\left[-a_1 + b_1(\mu_1^X, t)\right]$$

$$Q_X\left(\tau_1 > T, V_T \geq F \middle| \mathcal{F}_t\right) = N\left[a_1 - c_1(t) + b_1(\mu_1^X, t)\right] - \left(\frac{V_t}{V_{B1}}\right)^{\frac{-2\mu_1^X}{\sigma_1^2}} N\left[-a_1 - c_1(t) + b_1(\mu_1^X, t)\right]$$

$$Q_X\left(\tau_2 \leq T \middle| \mathcal{F}_{\tau_1}\right) = N\left[-d - b_2(\mu_2^X, \tau_1)\right] + \left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{-2\mu_2^X}{\sigma_2^2}} N\left[-d + b_2(\mu_2^X, \tau_1)\right]$$

$$Q_X\left(\tau_2 > T, V_T \geq F \middle| \mathcal{F}_{\tau_1}\right) = N\left[d - c_2(\tau_1) + b_2(\mu_2^X, \tau_1)\right] - \left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{-2\mu_2^X}{\sigma_2^2}} N\left[-d - c_2(\tau_1) + b_2(\mu_2^X, \tau_1)\right]$$

$$a_i = \frac{\ln\left(V_t / V_{Bi}\right)}{\sigma_i \sqrt{T - t}};\ b_i(\mu_i^X, s) = \frac{\mu_i^X \sqrt{T - s}}{\sigma_i};\ c_i(s) = \frac{\ln\left(F / V_{Bi}\right)}{\sigma_i \sqrt{T - s}};\ d = \frac{\ln\left(V_{B1} / V_{B2}\right)}{\sigma_2 \sqrt{T - \tau_1}}$$

Where μ_i^X is the drift of the logarithm of the assets value diffusion process, under probability measure Q_X, $\mu_i^B = r - \alpha_i - \sigma_i^2/2$; $\mu_i^V = \mu_i^B + \sigma_i^2$ and $\mu_i^m = \sqrt{\left(\mu_i^B\right)^2 + 2r\sigma_i^2}$, where the subscript "i" identifies the sate of the firm (i =1 before the rating change, i = 2 after the rating change). And N[.] –cumulative standard normal density function

If $V_{B1} > F$, substitute $Q_X\left(\tau_1 > T, V_T \geq F \middle| \mathcal{F}_t\right)$ by $Q_X\left(\tau_1 > T \middle| \mathcal{F}_t\right)$ and $Q_X\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right)$ by 0.

After a rating change, for $t > \tau_1$, the bond value reduces to[7]:

$$B_2(V_t,t,T,c,F,\Delta) = \frac{\Delta cF}{r}\left\{1-e^{-r(T-t)}\left[1-Q_B\left(\tau_2 < T\middle|\mathcal{F}_t\right)\right]\right\} + \left(\rho_1 V_{B2} - \frac{\Delta cF}{r}\right)\left(\frac{V_t}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \le T\middle|\mathcal{F}_t\right)$$

$$+\ Fe^{-r(T-t)}Q_B\left(\tau_2 > T, V_T \ge F\middle|\mathcal{F}_t\right) + \rho_2 V_t e^{-\alpha_2(T-t)} Q_V\left(\tau_2 > T, V_T < F\middle|\mathcal{F}_t\right) \qquad (5')$$

Where:

$$Q_X\left(\tau_2 \le T\middle|\mathcal{F}_t\right) = N\left[-a_2 - b_2(\mu_2^X,t)\right] + \left(\frac{V_t}{V_{B2}}\right)^{\frac{-2\mu_2^X}{\sigma_2^2}} N\left[-a_2 + b_2(\mu_2^X,t)\right]$$

$$Q_X\left(\tau_2 > T, V_T \ge F\middle|\mathcal{F}_t\right) = N\left[a_2 - c_2(t) + b_2(\mu_2^X,t)\right] - \left(\frac{V_t}{V_{B2}}\right)^{\frac{-2\mu_2^X}{\sigma_2^2}} N\left[-a_2 - c_2(t) + b_2(\mu_2^X,t)\right]$$

3.2 Rating-based covenant: repayment of a fraction of the principal.

Consider now a case where the rating trigger covenant, instead of generating an increase in coupon rate, leads to a partial repayment. Let (1- Δ) be the fraction of the principal that is repaid ($\Delta<1$). This implies that, at the rating change date (τ_1), the bondholders receive an additional cash-flow given by: (1-Δ)F, being ΔF the new face value of the bond which will generate a new coupon flow of $\Delta cFdt$. As in Bhanot and Mello (2005), we will consider two sources of financing for this repayment: either through new equity or through the sale of assets. But instead of treating them separately (as done by those authors), we will integrate both approaches, giving rise to a more general framework where the refund can be funded by a combination of the two sources. Specifically, if we define θ as the fraction of the payment made through the sale of assets, ($0 \le \theta \le 1$), the repayment of (1-Δ)F will be funded by (1- θ)(1-Δ)F generated through a new equity issue and θ(1-Δ)F generated through the sale of assets.

The different source of funding influences the bond value only through the probability of reaching the liquidation level (V_{B2}), when the value of the assets reaches the rating trigger level (V_{B1}). Specifically, at τ_1, when the rating of the firm is changed, if the sale of assets is used to

[7] We will use the notation B_2(.) as the t value bond after the rating downgrade to differentiates from B(.) which is the t value bond before the rating downgrade.

finance the refund (for $\theta > 0$), the assets value jumps immediately from V_{B1} to $V_{B1} - \theta(1-\Delta)F$ (see figure 2), raising the probability of liquidation and reducing in this manner the value of the bond. So, the greater the θ, the lower the bond value will be. Notice also that, if the value of the assets after the jump is equal or lower than V_{B2} ($[V_{B1} - \theta(1-\Delta)F] \leq V_{B2}$), than, the firm is immediately liquidated ($\tau_1 = \tau_2$)[8].

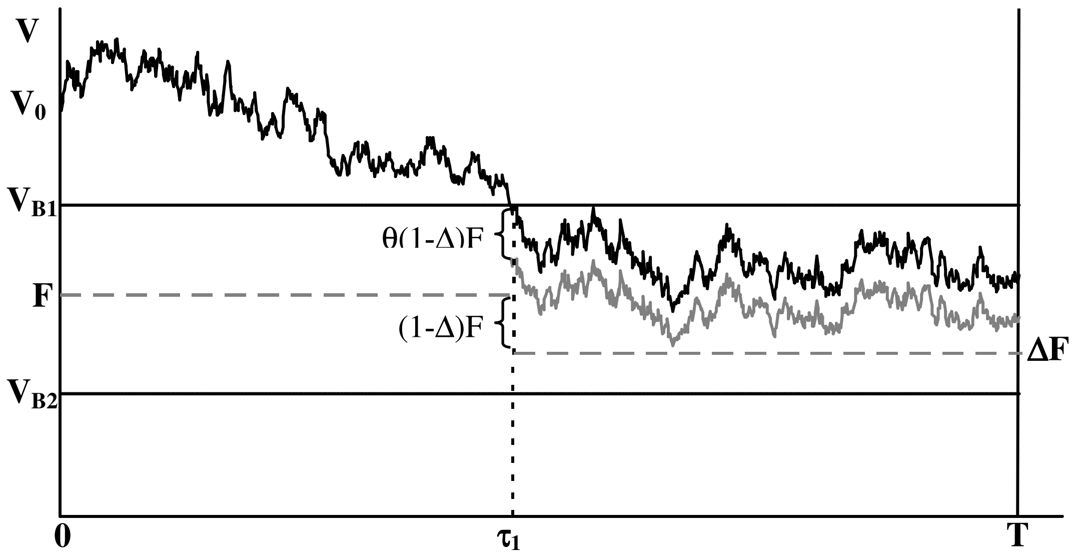


Figure 2 - Sample paths of the assets value considering different sources of funding. Black line - fully financed by new equity ($\theta = 0$); Gray line – when the selling of assets is used ($\theta > 0$).

The bond value expressions for this case, are similar to the previous one, with the exception of the definition of $B(V_{\tau_1}, \tau_1, T, c, F, \Delta)$ where it considers now the partial redemption $(1-\Delta)F$, the new face value ΔF, and the fact that at τ_1 the assets value is given by $[V_{B1} - \theta(1-\Delta)F]$ instead of V_{B1}, thus expression (2) is replaced by:

$B(V_{\tau_1}, \tau_1, T, c, F, \Delta, \theta) =$

$$E^{Q_B}\left\{\left[(1-\Delta)F + \int_{\tau_1}^{T} \Delta c F e^{-r(s-\tau_1)} 1_{s<\tau_2}\, ds + e^{-r(T-\tau_1)}\left(\Delta F 1_{\tau_2>T, V_T \geq \Delta F} + \rho_2 V_T 1_{\tau_2>T, V_T<\Delta F}\right) + e^{-r(\tau_2-\tau_1)} \rho_1 V_{\tau_2} 1_{\tau_2<T}\right] \Big| \mathcal{F}_{\tau_1}\right\}$$

Whose value is:

[8] This scenario will not be taken into account in the derivation of the bond value formula since we will assume that $V_{B1} - \theta(1-\Delta)F) > V_{B2}$.

$$B\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \theta\right) =$$

$$(1-\Delta)F + \frac{\Delta cF}{r}\left\{1 - e^{-r(T-\tau_1)}\left[1 - Q_B^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)\right]\right\} + \left(\rho_1 V_{B2} - \frac{\Delta cF}{r}\right)\left(\frac{V_{B1} - \theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)$$

$$+ \Delta F e^{-r(T-\tau_1)} Q_B^*\left(\tau_2 > T, V_T \ge \Delta F \middle| \mathcal{F}_{\tau_1}\right) + \rho_2\left[V_{B1} - \theta(1-\Delta)F\right] e^{-\alpha_2(T-\tau_1)} Q_V^*\left(\tau_2 > T, V_T < \Delta F \middle| \mathcal{F}_{\tau_1}\right) \qquad (6)$$

Where $Q_X^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)$ is interpreted as the probability under measure Q_X that the assets value reaches V_{B2} from $[V_{B1} - \theta(1-\Delta)F]$ by the maturity and is defined as $Q_X\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)$ replacing V_{B1} by ($V_{B1} - \theta(1-\Delta)F$). The same is true for $Q_X^*\left(\tau_2 > T, V_T \ge \Delta F \middle| \mathcal{F}_{\tau_1}\right)$ in relation to $Q_X\left(\tau_2 > T, V_T \ge F \middle| \mathcal{F}_{\tau_1}\right)$ where additionally F is replaced by ΔF.

Notice that the previous expression assumes that the firm is not liquidated at the rating change - that is: $(V_{B1} - \theta(1-\Delta)F) > V_{B2}$ and additionally that $\Delta F > V_{B2}$.

After the rating change, for $t > \tau_1$, the bond value is given by expression (5'), after adjusting for the new face value, thus the second line turns to:

$$\Delta F e^{-r(T-t)} Q_B\left(\tau_2 > T, V_T \ge \Delta F \middle| \mathcal{F}_t\right) + \rho_2 V_t e^{-\alpha_2(T-t)} Q_V\left(\tau_2 > T, V_T < \Delta F \middle| \mathcal{F}_t\right)$$

It is worthwhile pointing out that both previous models work on the premise that the relationship between the value of the firm's assets and liabilities is the single driver of its credit stance. This is equivalent to assuming that any exceptional situation of instantaneous insolvency, potentially leading to the inability of the firm to honour a coupon payment, would be solved with a temporary inflow of funds, provided by the shareholders.

4. The Leveraged Firm Value

In the previous section we derived the valuation formulae for debt. In this section, we obtain the valuation formulae for equity, tax benefits and bankruptcy costs.

4.1 Equity value

The equity value at $t < \tau_1$, will be given by the expected present value of all cash flows arising from the different future scenarios the firm is facing. Applying the same reasoning used in the valuation of debt claims, we may use the following expression (similar to expression (3) for the debt):

$$E(V_t, t, T, c, F, \Delta) = E^{Q_B}\left[\int_t^T (\alpha_1 V_s - cF(1-\iota))e^{-r(s-t)} 1_{s<\tau_1}\, ds + e^{-r(T-t)}(V_T - F)1_{\tau_1>T, V_T \geq F} \middle| \mathcal{F}_t\right] +$$
$$+ E^{Q_B}\left[e^{-r(\tau_1 - t)} 1_{\tau_1 < T} E(V_{\tau_1}, \tau_1, T, c, F, \Delta) \middle| \mathcal{F}_t\right] \qquad (7)$$

Where $E(V_t, t, T, c, F, \Delta)$ and $E(V_{\tau_1}, \tau_1, T, c, F, \Delta)$ denotes the equity value at t ($t < \tau_1$) and τ_1 respectively. So, conditioning in the firm not suffering a rating downgrade until maturity of the debt (first line), the value of equity takes into account the stream of dividends (defined as the cash payout ($\alpha_1 V_s$) minus the coupon payments adjusted for the tax benefit of debt ($cF(1- \iota)$), where ι is the corporate tax rate) and the residual value of the firm at maturity after the repayment of the principal of the debt. Notice that if $\tau_1 > T$ and $V_T < F$, the shareholders receive nothing since we are assuming the verification of the absolute priority rule. Otherwise, if the value of the assets reaches the first barrier before the bond matures (second line), the value of the equity at τ_1 is $E(V_{\tau_1}, \tau_1, T, c, F, \Delta)$, which in turn is defined as follows:

- For the accrued coupon rate rating trigger covenant ($\Delta>1$):

$$E(V_{\tau_1}, \tau_1, T, c, F, \Delta) = E^{Q_B}\left[\int_{\tau_1}^T (\alpha_2 V_s - c\Delta F(1-\iota))e^{-r(s-\tau_1)} 1_{s<\tau_2}\, ds \middle| \mathcal{F}_{\tau_1}\right]$$
$$+ E^{Q_B}\left[(V_T - F)e^{-r(T-\tau_1)} 1_{\tau_2>T, V_T\geq F} \middle| \mathcal{F}_{\tau_1}\right] \qquad (8)$$

- For the partial repayment of the principal rating trigger covenant ($\Delta<1$):

$$E(V_{\tau_1}, \tau_1, T, c, F, \Delta) = E^{Q_B}\left[\int_{\tau_1}^T (\alpha_2 V_s - c\Delta F(1-\iota))e^{-r(s-\tau_1)} 1_{s<\tau_2}\, ds \middle| \mathcal{F}_{\tau_1}\right]$$
$$+ E^{Q_B}\left[(V_T - \Delta F)e^{-r(T-\tau_1)} 1_{\tau_2>T, V_T\geq \Delta F} \middle| \mathcal{F}_{\tau_1}\right] -$$
$$- (1-\theta)(1-\Delta)F \qquad (8')$$

So, at τ_1, the value of equity is given by the expected present value of:

- The stream of dividends from τ_1 to T or τ_2 whichever comes first (first line of the expressions). These are defined by the new cash payout (α_2Vs) deducted from the new coupon payment adjusted for the fiscal benefit ($c\Delta F(1-\iota)$). Remember that for the accrued coupon rate rating trigger case, $\Delta > 1$, where (Δ-1) is the relative increase in the coupon rate and for expression (8'), $\Delta < 1$, where (1 - Δ) corresponds to the fraction of the principal that is redeemed.
- The residual value of the firm at maturity, if the firm hasn't been liquidated in the meantime; which is defined by V_T - F, (expression (8)) since the debt principal remains unchanged, and by $V_T - \Delta F$, in the second case (expression (8') since a fraction (1- Δ) of the principal has been paid at the moment of the rating change.
- For the second type of rating trigger covenant, we still have to take into account the fraction of the principal amortized through a new cash infusion from shareholders (last line in expression (8')).

The final expressions for equity value, for t < τ_1 are (derivation in the appendix A.2):

$E(V_t, t, T, c, F, \Delta) =$

$$V_t\left[1-e^{-\alpha_1(T-t)}Q_V\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right)\right] - \left(V_{B1} - \frac{cF(1-\iota)}{r}\right)\left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m-\mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \le T \middle| \mathcal{F}_t\right) -$$

$$-\frac{cF(1-\iota)}{r}\left\{1-e^{-r(T-t)}\left[1-Q_B\left(\tau_1 \le T \middle| \mathcal{F}_t\right)\right]\right\} - Fe^{-r(T-t)}Q_B\left(\tau_1 > T, V_T \ge F \middle| \mathcal{F}_t\right) +$$

$$+\int_t^T e^{-r(\tau_1-t)}E\left(V_{B1},\tau_1,T,c,F,\Delta\right)g_B\left(\tau_1,V_t,V_{B1}\right)d\tau_1 \qquad (9)$$

As for the debt value, if $V_{B1} > F$, substitute $Q_X\left(\tau_1 > T, V_T \ge F \middle| \mathcal{F}_t\right)$ by $Q_X\left(\tau_1 > T \middle| \mathcal{F}_t\right)$ and $Q_X\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right)$ by 0.

In expression (9), $E\left(V_{B1},\tau_1,T,c,F,\Delta\right)$ is defined as (derivation in the appendix A.2):

$E\left(V_{B1},\tau_1,T,c,F,\Delta\right) =$

$$V_{B1}\left[1-e^{-\alpha_2(T-\tau_1)}Q_V\left(\tau_2>T,V_T<F\middle|\mathcal{F}_{\tau_1}\right)\right]-\left(V_{B2}-\frac{cF\Delta(1-\iota)}{r}\right)\left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}}Q_m\left(\tau_2\le T\middle|\mathcal{F}_{\tau_1}\right)-$$

$$-\frac{cF\Delta(1-\iota)}{r}\left\{1-e^{-r(T-\tau_1)}\left[1-Q_B\left(\tau_2\le T\middle|\mathcal{F}_{\tau_1}\right)\right]\right\}-Fe^{-r(T-\tau_1)}Q_B\left(\tau_2>T,V_T\ge F\middle|\mathcal{F}_{\tau_1}\right) \quad (10)$$

for the accrued coupon rate rating trigger covenant case, and for the partial repayment of the principal rating trigger covenant:

$$E\left(V_{\tau_1},\tau_1,T,c,F,\Delta\right)=\left[V_{B1}-\theta(1-\Delta)F\right]\left[1-e^{-\alpha_2(T-\tau_1)}Q_V^*\left(\tau_2>T,V_T<\Delta F\middle|\mathcal{F}_{\tau_1}\right)\right]\text{ -}$$

$$-\left(V_{B2}-\frac{cF\Delta(1-\iota)}{r}\right)\left(\frac{V_{B1}-\theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}}Q_m^*\left(\tau_2\le T\middle|\mathcal{F}_{\tau_1}\right)\text{ -}$$

$$-\frac{cF\Delta(1-\iota)}{r}\left\{1-e^{-r(T-\tau_1)}\left[1-Q_B^*\left(\tau_2\le T\middle|\mathcal{F}_{\tau_1}\right)\right]\right\}\text{ -}$$

$$-\Delta Fe^{-r(T-\tau_1)}Q_B^*\left(\tau_2>T,V_T\ge\Delta F\middle|\mathcal{F}_{\tau_1}\right)-(1-\theta)(1-\Delta)F \quad (10')$$

Where the probabilities $Q_X(.)$ and $Q_X^*(\cdot)$ are defined as above in section 3. Recall that expression (10') is only valid for V_{B1} - θ (1-Δ)F > V_{B2}.

After the rating change, for t > τ_1, the value of equity is obtained through expressions (10) and (10') after replacing τ_1 for t, V_{B1} and (V_{B1} - θ (1-Δ)F) for V_t in (10) and (10') respectively. The last term in the fourth line of (10') also disappears and in (10') $Q_X^*(\cdot)$ are substituted by $Q_X(.)$.

It is worth noting that, for high face value bonds (face values higher than the barrier level: F > V_{B1}) with a partial refund rating trigger covenant, when an equity issue is used to fund the partial repayment (θ < 1), the expression (10') may turn negative from a certain value of τ_1 onwards. This reflects the fact that, from the equity holders point of view, in this kind of situation, an additional cash infusion might have a negative expected net present value, consequently, in this case, the shareholders' rational decision making process will lead them to shun the corresponding payment. If that happens, the equity value must be zero and the firm will enter into bankruptcy.

For example, assume the following parameters value[9]: $V_0 = 150$, $V_{B1} = 100$, $V_{B2} = 50$, $\alpha_1 = 0{,}07$, $\alpha_2 = 0{,}10$, $\sigma_1 = 0{,}30$, $\sigma_2 = 0{,}45$, $\iota = 0{,}35$. Considering a bond with F = 130, c = 13%, a time to maturity of 10 years and a partial refund of 20% ($\Delta = 0{,}8$) partially financed by equity ($\theta = 0{,}5$), the firm immediately enters into bankruptcy if the barrier is hit after 8 years (if $\tau_1 \geq 8$). This level of τ_1 (8 years, in the example) will tend to be lower the lower θ and Δ and the higher F and c. Thus, for face values greater than V_{B1}, in the integral of expression (9), $E(V_{\tau_1}, \tau_1, T, c, F, \Delta)$ must be replaced by max[$E(V_{\tau_1}, \tau_1, T, c, F, \Delta)$;0][10,11].

4.2 Tax benefits and bankruptcy cost values

The final expressions for the value of tax benefits (TB) and bankruptcy costs, at $t < \tau_1$ are as follows (derivation in the appendix A.3):

$$TB(V_t, t, T, c, F, \Delta, \iota) = \frac{cF\iota}{r}\left[1 - e^{-r(T-t)}\left[1 - Q_B\left(\tau_1 \leq T \middle| \mathcal{F}_t\right)\right] - \left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m - \mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \leq T \middle| \mathcal{F}_t\right)\right] +$$

$$+ \int_t^T e^{-r(\tau_1 - t)} TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) g_B\left(\tau_1, V_t, V_{B1}\right) d\tau_1 \qquad (11)$$

Where:

$$TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) = \frac{c\Delta F\iota}{r}\left[1 - e^{-r(T-\tau_1)}\left[1 - Q_B\left(\tau_2 \leq T \middle| \mathcal{F}_{\tau_1}\right)\right] - \left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \leq T \middle| \mathcal{F}_{\tau_1}\right)\right] \qquad (12)$$

For the first case ($\Delta > 1$), and for the second ($\Delta < 1$):

[9] From Bhanot (2003)

[10] If this adjustment is not taken into account, although expression (9) with (10') will undervalue the equity, the difference will be insignificant for high values of V_0 and time to maturity. For the example of the text, the difference in value is less than 0,3%.

[11] Note further that, in this situation we also have to adjust the bond pricing formula; thus, if we assume that in the event of bankruptcy at τ_1 (when shareholders are not willing to realize the cash infusion), the same recovery rate to bond holders as in τ_2, expression (6) is replaced by $\rho_1 V_{B1}$, which is the pay-off to bond holders conditioned on bankruptcy at τ_1 (that is conditioned on max[$E(V_{\tau_1}, \tau_1, T, c, F, \Delta)$;0] = 0) .

$TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) =$

$$\frac{c\Delta F\iota}{r}\left[1-e^{-r(T-\tau_1)}\left[1-Q_B^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)\right]-\left(\frac{V_{B1}-\theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}} Q_m^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right)\right] \quad (12')$$

Turning to the bankruptcy costs:

$$BC\left(V_t, t, T, c, F, \Delta\right) = \left(1-\rho_2\right)V_t e^{-\alpha_1(T-t)} Q_V\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right) +$$

$$+\int_t^T e^{-r(\tau_1 - t)} BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right) g_B\left(\tau_1, V_t, V_{B1}\right) d\tau_1 \quad (13)$$

Where, for the accrued coupon rate case (Δ>1):

$$BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) = \left(1-\rho_1\right)V_{B2}\left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right) +$$

$$+\left(1-\rho_2\right)V_{B1} e^{-\alpha_2(T-\tau_1)} Q_V\left(\tau_2 > T, V_T < F \middle| \mathcal{F}_{\tau_1}\right) \quad (14)$$

and for the partial amortization of principal (Δ<1) :

$$BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) = \left(1-\rho_1\right)V_{B2}\left(\frac{V_{B1}-\theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}} Q_m^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right) +$$

$$+\left(1-\rho_2\right)\left[V_{B1}-\theta(1-\Delta)F\right] e^{-\alpha_2(T-\tau_1)} Q_V^*\left(\tau_2 > T, V_T < \Delta F \middle| \mathcal{F}_{\tau_1}\right) \quad (14')$$

Once again, if $V_{B1} > F$, substitute $Q_X\left(\tau_1 > T, V_T < F \middle| \mathcal{F}_t\right)$ by 0, and in (12') and (14'), it is assumed that $V_{B1} - \theta\,(1-\Delta)F > V_{B2}$.

After the rating change, for $t > \tau_1$, the values of tax benefits and bankruptcy costs are obtained through expressions (12) and (14) after replacing τ_1 for t and V_{B1} for V_t in the accrued coupon rate case. Similarly, for the partial amortization of principal, the corresponding values are obtained using expressions (12') and (14') after substituting τ_1 and ($V_{B1} - \theta\,(1-\Delta)F$) by t and V_t respectively and $Q_X^*(\cdot)$ by $Q_X(.)$[12].

[12] It is worth noting that if the face value is greater than the barrier level, an adjustment, similar to the one stated in the footnote 10 for the bond value, would have to be made in expressions (11) and (13).

4.3 Leveraged firm value

The value of the leveraged firm, defined as v(.) is now obtained either by the sum of bond and equity values or by the sum of the values of firm assets and tax benefits less the value of bankruptcy costs:

$$v(\cdot) = B(\cdot) + E(\cdot) \quad \text{or} \quad v(\cdot) = V + TB(\cdot) - BC(\cdot)$$

Notice that for the partial repayment of principal rating trigger covenant case, the value of some variables will jump with the change of the firm´s rating (when the barrier level V_{B1} is reached - at τ_1). Specifically: i) the bond value, irrespective of the funding source used to realize the partial redemption of the principal, declines immediately after τ_1, by an amount equal to the value amortized; ii) the equity value, when the payment to bondholders is fully funded through a cash infusion, rises by the same amount, immediately after τ_1. In contrast, when the sale of assets is used, the equity value remains unchanged immediately after τ_1; iii) the leveraged firm value, immediately after τ_1 remains the same with the issue of equity, and drops with the sale of assets, by the same amount of the asset sale which is $(1-\Delta)F$. The simultaneous use of both sources of funding will, naturally, lead to intermediate jump observed for equity and firm value.

5. Comparison with Bhanot's Model

As previously noted, the main shortcoming of Bhanot's model is the absence of default at maturity or, putting it differently, the implicit assumption that, at maturity, bondholders always receive the full amount of the principal. Such an assumption leads to an overprice of the bond. Following we compare Bhanot's (2003) results with ours, using the same parameter values to highlight the differences, namely: $V_0 = 150$; $V_{B1} = 100$; $V_{B2} = 50$; $\alpha_1 = 0{,}07$; $\alpha_2 = 0{,}10$; $\sigma_1 = 0{,}30$; $\sigma_2 = 0{,}45$; $r = 0{,}075$; $\rho_1 = 0{,}5$[13]; $\iota = 0{,}35$; $F = 100$. For the coupon rate we will assume $c = 0{,}09$.

[13] Although on the legend of exhibit 2 in Bhanot (2003), page 61, it is written that the bankruptcy costs represent 50%, leading to a recovery fraction of 50% ($\rho_1 = 0{,}5$), the graph assumes that $\rho_1 = 1$ (no bankruptcy costs). That same value ($\rho_1 = 1$) is assumed thereafter on the other exhibits in Bhanot. By default, in the current section, it will be assumed that $\rho_1 = 0{,}5$. Even so, in order to allow a comparison with Bhanot (2003), some numerical results were calculated assuming $\rho_1 = 1$.

Since Bhanot's model does not explicitly assume any kind of change in bondholders pay-offs, either at or after the rating change of the firm, and to isolate only the maturity default effect, we shall assume that in our model $\Delta = 1$.

We start by defining the excess price (EP) of Bhanot's model in relation to ours as:

$$EP = (B_B/B - 1)100$$

Where, B_B and B stand for the price of the bond in Bhanot's model and in our model, respectively.

Figure 3, shows the excess price as a function of time to maturity. Figure 3a, considers different values for the recovery fraction of the asset values at maturity. As we can see, even in the case where there is no default cost at maturity ($\rho_2 = 1$, the bondholders receive V_T if $V_T < F$), the price difference can reach 3%. As expected, a decrease in the recovery fraction lowers the "true" price of the bond and so the excess price rises. Figure 3b, highlights the effects of the asset value on price differences (considering $\rho_2 = 0{,}9$) . The lower the value of the assets, the greater the probability of the firm entering in default at maturity, resulting in higher excess prices.

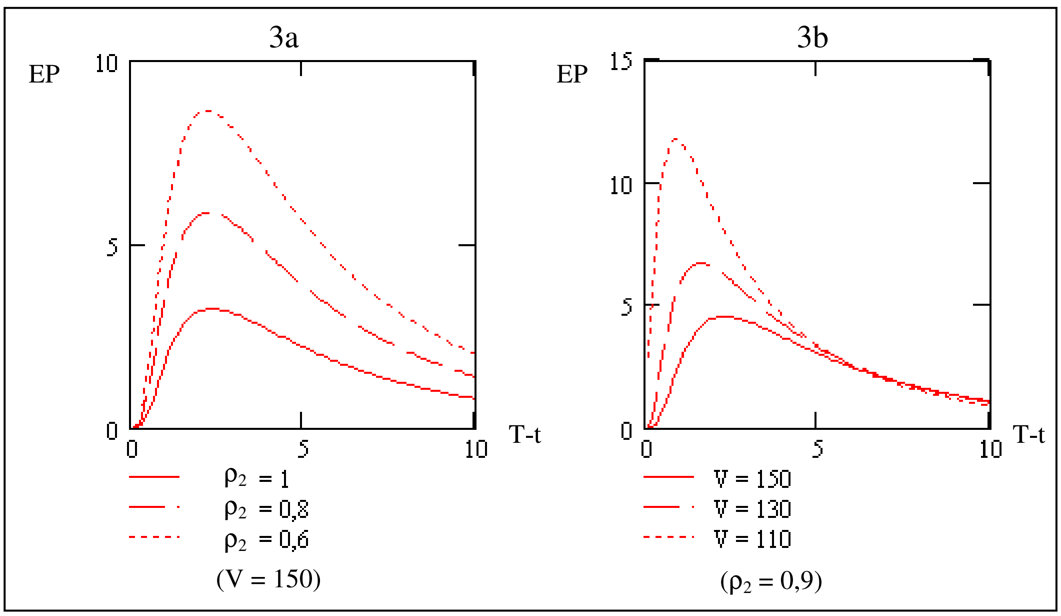


Figure 3 - Excess price as a function of time to maturity, with: $V_{B1} = 100$; $V_{B2} = 50$; $\alpha_1 = 0{,}07$; $\alpha_2 = 0{,}10$; $\sigma_1 = 0{,}30$; $\sigma_2 = 0{,}45$; $r = 0{,}075$; $\rho_1 = 0{,}5$; $F = 100$ and $c = 0{,}09$. Figure 3b – considers three different values for ρ_2 with V = 150, and figure 3b considers three different values for V with ρ_2=0,9.

In fact, as figure 4 illustrates, overpricing rises substantially when the value of the firm's assets reaches V_{B1} (figure 4a), and then approaches the liquidation level, V_{B2} (figure 4b).

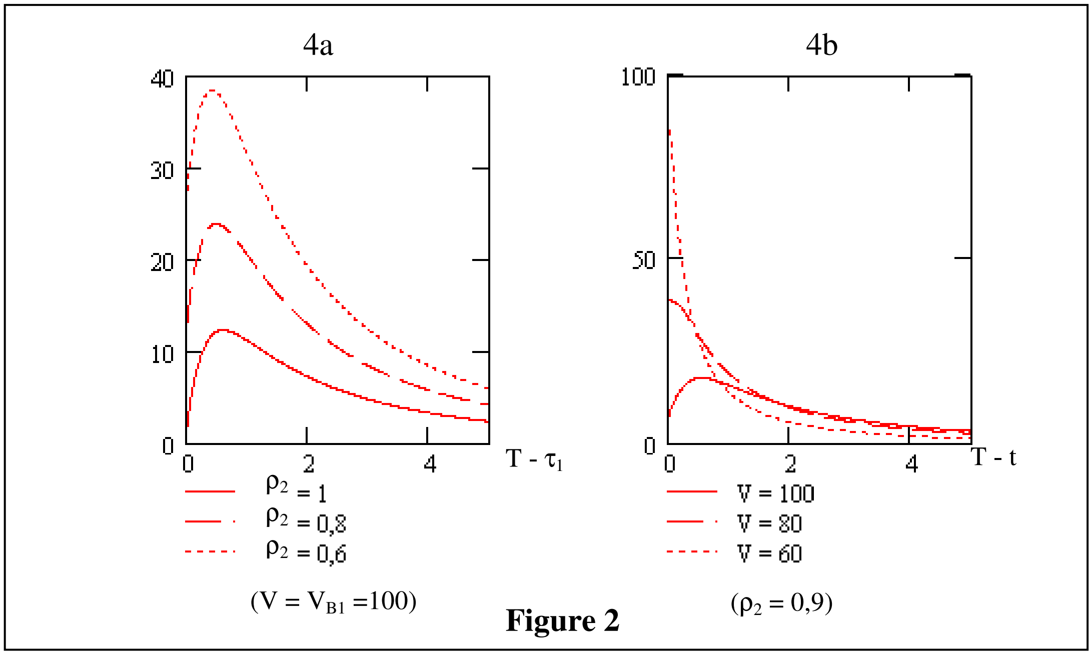
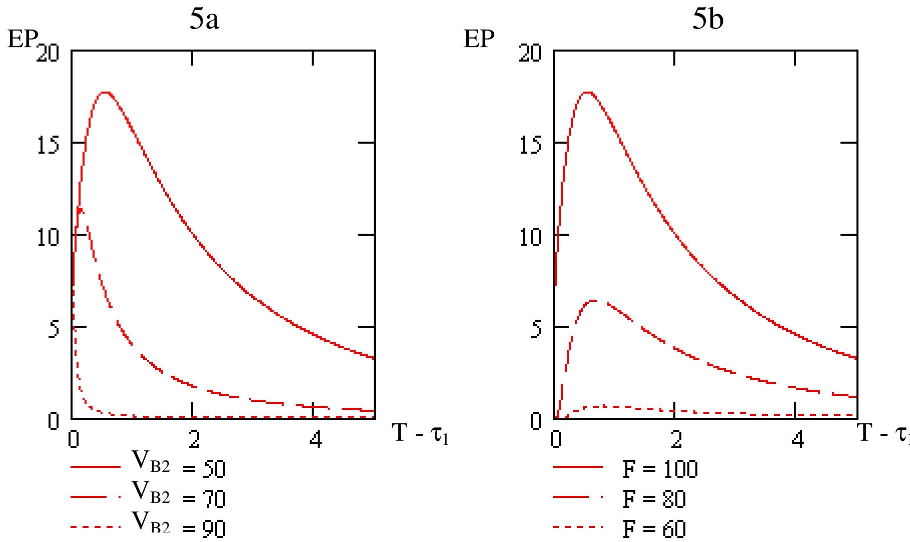


Figure 4 - Excess price as a function of time to maturity, with: $V_{B1} = 100$; $V_{B2} = 50$; $\alpha_1 = 0{,}07$; $\alpha_2 = 0{,}10$; $\sigma_1 = 0{,}30$; $\sigma_2 = 0{,}45$; $r = 0{,}075$; $\rho_1 = 0{,}5$; $F = 100$ and $c = 0{,}09$. Figure 4a – considers three different values for ρ_2 with $V = 100$ (at τ_1) , and figure 4b considers three different values for V (approaching the liquidation threshold) with $\rho_2 = 0{,}9$.

Remember that since we are assuming that $\Delta=1$, the price difference is only attributable to a possible default at maturity which presumes that the face value of the bond is greater than the second barrier level ($F > V_{B2}$). Otherwise, if V_{B2} is greater than F, then the value of the assets at maturity (in conditioning of never having reached V_{B2} in the meantime) will always be sufficient to repay the principal and, in that case, default at maturity will never happen. So, given a recovery fraction at maturity (ρ_2), the smaller the gap between F and $V_{B2,}$ the smaller the excess in price will be. If $V_{B2} \geq F$, our model (assuming $\Delta = 1$) converges to Bhanot's model. These remarks are illustrated in figure 5. In figure 5a, three different values for V_{B2} are considered (with $\rho_2 = 0{,}9$ and $F = 100$), and in figure 5b, three different face values of debt are also taken into consideration (keeping the coupon rate at 9%, $\rho_2 = 0{,}9$ and $V_{B2} = 50$). In both figures, it is assumed that $t = \tau_1$, that is, the value of the assets equals V_{B1}.



($V = V_{B1} = 100$; $\rho_2 = 0{,}9$; $F = 100$)   ($V = V_{B1} = 100$; $\rho_2 = 0{,}9$; $V_{B2} = 50$)

Figure 5 - Excess price as a function of time to maturity, with: $V = V_{B1} = 100$; $\alpha_1 = 0{,}07$; $\alpha_2 = 0{,}10$; $\sigma_1 = 0{,}30$; $\sigma_2 = 0{,}45$; $r = 0{,}075$; $\rho_1 = 0{,}5$; $\rho_2 = 0{,}9$ and $c = 0{,}09$. Figure 5a – considers three different values for V_{B2} with $F = 100$, and figure 5b considers three different values for F and $V_{B2}=50$.

Fixing the time to maturity, we can have an infinite number of combinations of coupon rates and face values to which a single value for the bond issue corresponds. If we consider only bonds issued at par (face value equal to emission value), we are able to compare, for the different models, the par coupon rate c_{par} (known as par yield), defined as follows:

$$c_{par}: \quad B(V_t, t, T, c_{par}, F, .) = F \qquad (15)$$

Since Bhanot´s model overprices the bond, for a given value of the issue, ceteris paribus, it is expected that the par coupon rate inherent in Bhanot's model will be lower than that in ours. Figure 6 shows exactly these results. It graphs the par coupon rate (in percentage), as a function of the face value of the bond (equal to the emission value) considering a 5-year (figure 6a) and a 10-year (figure 6b) time period to maturity. In each of these figures, two different values are considered for the recovery fraction upon liquidation of the firm ($\rho_1 = 0{,}5$ and 1). Relating to our model, we still continue to assume that $\Delta = 1$, so after the rating change of the firm, the coupon and principal of the bond do not undergo any changes, and we also consider $\rho_2 = 1$ (no bankruptcy costs when default occurs at maturity).

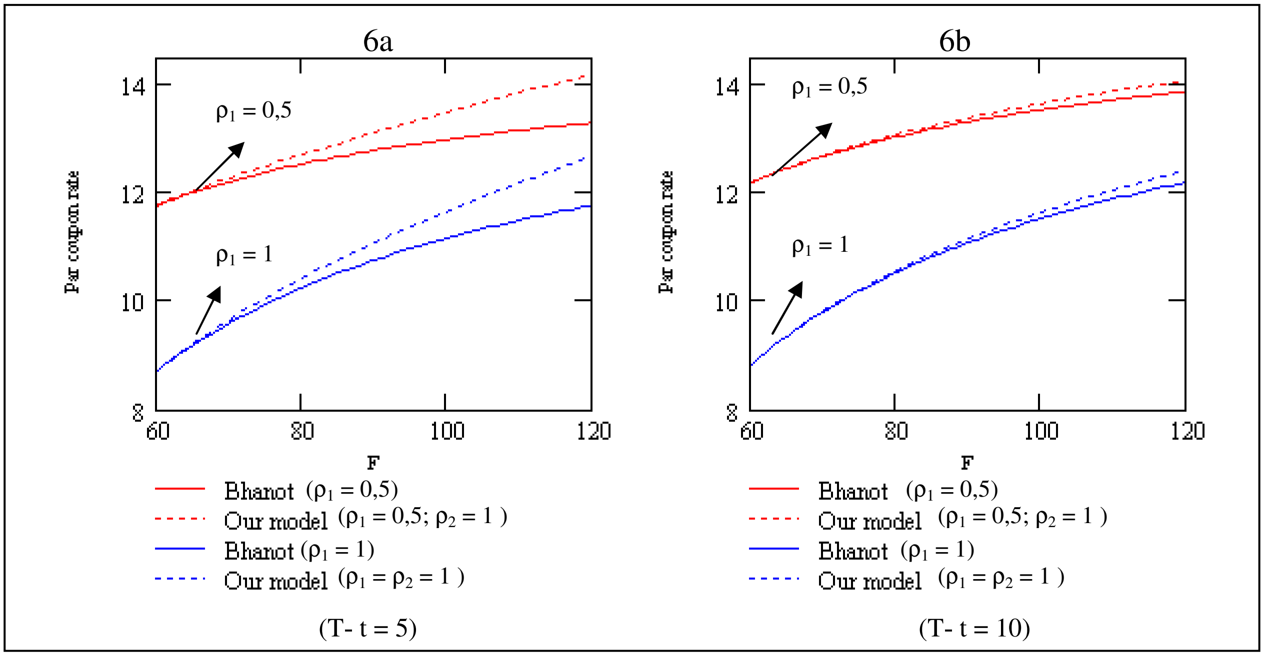


Figure 6 – Comparison of par coupon rates (in percentage), as a function of face value, in Bhanot´s model and in our model with: V=150; V_{B1} = 100; V_{B2} = 50; α_1 = 0,07; α_2 = 0,10; σ_1 = 0,30; σ_2 = 0,45; r = 0,075, ρ_2 = 1, for different values of ρ_1 (0,5 and 1) considering a five year time lapse to bond maturity (figure 6a) and a ten-year time to bond maturity (figure 6b).

As we can see, the greater the principal (meaning a greater gap between F and V_{B2} and consequently a greater excess price), the greater the difference will be between par coupon rates derived from our model and those calculated on the basis of Bhanot´s model. Furthermore, this difference tends to shrink with time to maturity, since our model (with Δ = 1) only differs from Bhanot's model in respect to cash-flow at maturity: the greater the maturity the lower the relative importance of the cash-flow (specifically, the expected present value) will be, in terms of price.

If, instead of varying the face value, we vary the time to maturity (T – t) in (15) for a given face value for the bond, we obtain the par yield curve. Bhanot (2003) calculates the par yields for six maturities (second column of exhibit 6, page 63 in Bhanot (2003)) assuming F = 100 and ρ_1 = 1. We conduct a similar analysis, using the same parameter values, in comparing our model with Bhanot's model, but instead of relying on the par yield curves, we analyze the resulting credit spreads. Notice that the analysis is similar since the framework relies on a constant risk-free interest rate with a flat risk-free yield curve.

In figure 7a below we graph the credit spread curves (in basis points) resulting from Bhanot's model and our model. The hump-shape is present in both models, although more pronounced in our model. As we can see, and as expected, even considering the absence of bankruptcy costs, in the case of default at maturity ($\rho_2 = 1$ in our model), the differences are expressive especially in short maturities. To highlight these differences we have plotted them in a separate graph (figure 7b).

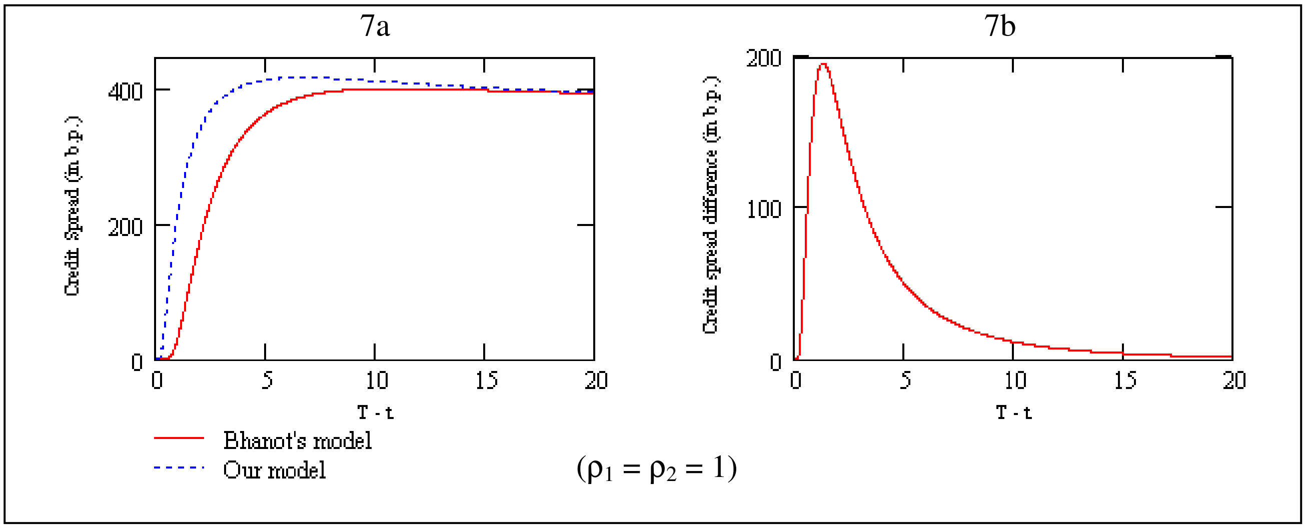


Figure 7 – Comparison of credit spreads (in basis point), as a function of time to maturity, in Bhanot´s model and in our model (figure 7a); Credit spread differences between the models (in basis point), as a function of time to maturity, (figure 7b). With V_0=150; V_{B1} = 100; V_{B2} = 50; α_1 = 0,07; α_2 = 0,10; σ_1 = 0,30; σ_2 = 0,45; r = 0,075, $\rho_1 = \rho_2 = 1$ and considering a face value of 100 issued at par.

It is worth pointing out that these credit spread differences are independent of the recovery value ρ_1. In fact, the influence of ρ_1 is the same in both models. It only influences the bondholders' payoff when the firm is liquidated (before maturity). So, although different recovery values lead to different bond prices, these changes in price will be the same in both models, which in turn will lead to an equal change in the par yields leaving the difference on credit spreads unchanged.

6. The Influence of the Rating Trigger Covenant.

In the previous section, since the focus was to highlight the overpricing inherent in Bhanot's model, we have assumed, in our model, for the purpose of comparison, that $\Delta = 1$. In the current section, this assumption will be relaxed for two main reasons: to analyze in more detail the effects on bond prices and credit spreads of a rating trigger covenant specified as an increase in coupon rate or a partial redemption of the principal, and to allow comparisons with bonds issued without this kind of covenant. This comparison is also present in Bhanot (2003) but our analysis is distinct in several aspects. In the first place, as stated previously, the bond price formula presented by this author does not take into account any kind of change on bondholders' cash flow after the rating change. In essence, the only influence of the rating based covenant on Bhanot's bond price is the consideration of a greater payout ratio and asset value volatility after the downgrade of the firm. Additionally, in pricing bonds without covenants, those changes are not taken into account[14]; instead Bhanot (2003) uses an implied volatility value[15]. On the contrary, relying on the fact that irrespective of the debt type (with or without rating trigger covenants), the firm is equally subject to rating notation (and so to rating changes), we always assume the existence of the two barriers (the downgrade level and the liquidation level). Thus, for the case of bonds without covenants, when the rating of the firm is changed, notwithstanding the fact that the bondholders' cash flow remains unchanged, the firm may alter its cash payout rate and risk in the same way as is assumed in the case of bonds with rating trigger covenants. In short, the prices of the different bonds are obtained as follow:

- using expression (4) and (5), with $\Delta = 1$, for bonds without rating trigger covenants;
- using expressions (4) and (5), with $\Delta > 1$, for bonds with rating trigger covenant of the type accrued coupon rate, where $(\Delta-1)$ is the relative change in the coupon rate;
- using expressions (4) and (6), with $\Delta < 1$, for bonds with rating trigger covenant of the type partial prepayment of the principal, where $(1-\Delta)$ is the fraction of the facial value that is redeemed;

Remember from section 3, where in this last case, the partial amortization of the debt can be financed either through selling assets, cash infusions from shareholders or a combination of both.

In what follows, we will assume for the parameter values: $V_0 = 150$; $V_{B1} = 100$; $V_{B2} = 50$; $\alpha_1 = 0{,}07$; $\alpha_2 = 0{,}10$; $\sigma_1 = 0{,}30$; $\sigma_2 = 0{,}45$; $r = 0{,}075$; $\rho_1 = 0{,}8$; $\rho_2 = 1$.

To compare the three types of bonds, we restrict our analysis to par issued debt.

[14] If they were, there would be no distinction between the two types of bonds.
[15] "I compute the implied volatility of the asset process that makes the model bond price without covenants equal the true price." Page 61 in Bhanot (2003)

6.1 Par yields

Since the existence of a covenant in the bond indenture aims to protect bondholders' interest, we would expect that, for a given face value of debt, bondholders would require a lower coupon rate in this kind of bond as compared to that required from unprotected debt. Such a result is provided in our model, as illustrated in figure 8a. The figure plots the (par) coupon rate as a function of bond's face value (emission value), assuming a time to maturity of 10 years, for three types of debt: a bond without covenant, a bond with accrued coupon rate rating trigger covenant (with Δ = 1,2, so a 20% increase in the coupon), and a bond with partial refund rating trigger covenant (with Δ = 0,8, so a 20% reduction in the face value). For this last type, a distinction is also made regarding the financing source (fully financed by new equity - $\theta = 0$, or fully financed through the sale of assets, $\theta = 1$).

Comparing both covenants, accrued coupon rate and partial amortization of principal when financed through cash infusion ($\theta = 0$), for the same par value, and a same percentage change (positive in the former, negative in the latter case), the required par coupon rate for the first case is always greater than that for the second case. In other words, for the two types of bonds to have the same required par coupon rate (thus being equivalent at the issue date), the percentage increase in the coupon rate must be greater than the face value percentage decrease. For example, considering a time to maturity of 10 years and a par emission value of 100, the par coupon rate of a bond with a 20% reduction of principal rating trigger covenant is 10,994%, while for the accrued coupon rate rating trigger covenant case, the percentage increase in the coupon that returns the same par coupon rate is 37,53%. Note however, that after the debt is in place, the respective par yields evolve differently ways as time passes.

Notice that what differentiates these two types of bonds is the coupons received by bondholders and the way the principal is redeemed, after the rating downgrade. Specifically, in one case, bondholders receive higher coupons but are exposed to a greater loss if bankruptcy occurs (they receive $\rho_1 V_{B2}$ instead of F). In the other case, although bondholders receive a lower coupon, the loss incurred in the bankruptcy scenario is also lower ($\rho_1 V_{B2}$ instead of ΔF[16]), since part of the

[16] It is important to remember that we are assuming that ΔF (the new face value after the partial redemption of principal) is greater than V_{B2}.

principal was already received when the firm's rating changed. Note also that, in either case, the probabilities of the firm entering into bankruptcy (V_{B2} be crossed) are the same[17].

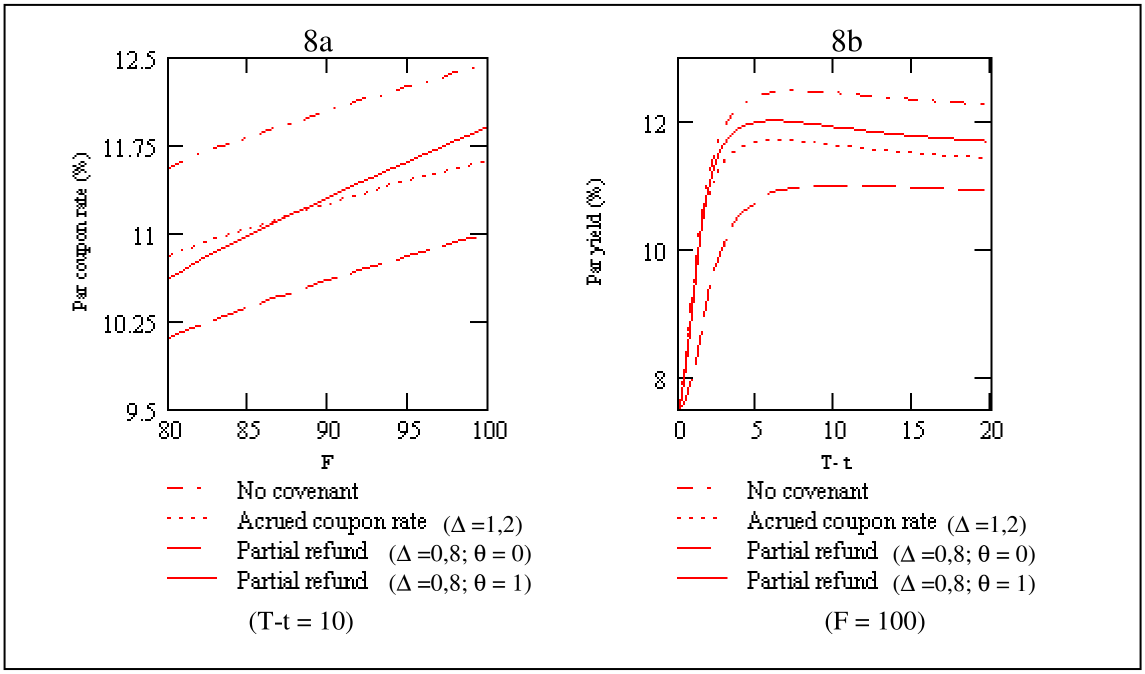


Figure 8 –Par coupon rate (in percentage) as a function of the face value considering a 10 years maturity bond (figure 8a). Par yields (in percentage) as a function o time to maturity, considering F=100 (figure 8b). V_0=150; V_{B1}=100; V_{B2} = 50; α_1 = 0,07; α_2 = 0,10; σ_1 = 0,30; σ_2 = 0,45; r = 0,075, ρ_1 = 0,8 and ρ_2 = 1.

On the other hand, when the partial refund is funded through the sale of assets (θ =1), the downgrade in the firm's credit rating leads to a downward jump in the value of the assets (by the amount of the refund) which in turn will cause an increase in the probability of bankruptcy. Thus, this kind of bond is riskier or less protected, when compared with the equity issue case, and that fact is reflected in the higher par coupon rate required by bondholders (at the issue date) as illustrated in figure 8a. The greater the face value, the greater the downward jump (for a fixed Δ) would be, and also the greater the difference would be between the required coupon rate on the two bonds. In effect, for high values of principal, the partial refund rating trigger covenant bond, when financed by the selling of assets, can be riskier than the accrued coupon rate rating trigger covenant bond.

[17] Yet, the probability of default at maturity is different since $\Delta F < F$.

The par yield curves for the four bonds are also plotted (figure 8b), assuming a 100 face value. As we can see, in conformity with what was stated in the previous paragraphs, the bond without the covenant yields the highest credit spreads (reaching to almost 500 basis points for a time lapse to maturity of 7,5 years given the values for the parameters in the example) while the lowest credit spreads are generated by the partial refund rating trigger covenant when financed by cash infusion (where the credit spreads do not exceed 250 basis points).

It must be emphasised that these numerical results rely upon the specific parameter values used. Specifically it was assumed that after the rating downgrade, the payout ratio of the firm changed in the same manner irrespective of the bond type. If instead, depending on what happens to the coupon value after the rating change, we had considered different changes in the payout ratio, the results would also be different. For example, if we had assumed no change for the no covenant case (since the coupon remains unchanged) and a decrease for the partial amortization case (since the reduction in the principal reduces the coupon), the par yields of these bonds would have been lower[18]. In particular, the par yields of the bond with the partial refund covenant when fully financed by the selling of assets, could even be lower than those from the accrued coupon rate case.

6.2 Equity and leveraged firm value

It is interesting to compare the values of equity and the whole leveraged firm associated with the various kinds of debt. Table 1 reports those values considering a 100 face value bond issued at par with a maturity of 10 years. For bonds without the rating trigger covenant and accrued coupon rate rating trigger covenant type, the values are obtained for the issue date ($t = 0$), and for two hypothetical downgrade dates, $\tau_1 = 6$ and $\tau_1 = 8$ (the barrier V_{B1} is hit when the time to maturity of the bond is 4 and 2 years respectively). For bonds with a partial refund rating trigger covenant, besides the issued date, values are obtained for $\tau_1 = 6$, and immediately after (τ_1^+).

In addition to some of the previous findings, the table shows that the equity value, and the leveraged firm value are insensitive in relation to the percentage increase on the coupon rate. The same result does not hold for the partial refund rating trigger covenant case. Indeed, in this latter

[18] Nonetheless, the bond with no covenants would still have the highest par yields.

case, the greater the amortization of principal, the lower the equity and the corresponding leveraged firm value.

	Par coupon rate	Equity Value			Leveraged Firm value			Bond value		
		t = 0	τ_1 = 6	τ_1 = 8	t = 0	τ_1 = 6	τ_1 = 8	t = 0	τ_1 = 6	τ_1 =8
Δ = 1	12,44 %	68,41	28,55	22,83	168,40	104,31	102,9	100	76,76	80,07
Δ = 1,2	11,63 %	68,41	26,32	21,33	168,40	105,51	103,7	100	79,19	82,37
Δ = 1,3753	10,99 %	68,41	24,59	20,17	168,40	106,44	104,33	100	81,85	84,16

		Par coupon rate	Equity Value			Leveraged Firm value			Bond value		
			t = 0	τ_1 = 6	τ_1^+	t = 0	τ_1 = 6	τ_1^+	t = 0	τ_1 = 6	τ_1^+
θ = 0	Δ = 0,8	10,99 %	64.38	17,98	37,98	164,38	101,41	101,41	100	83,43	63,43
	Δ = 0,7	10,21 %	62.41	12,75	42,75	162,41	100,1	100,1	100	87,35	57,35
θ = 0,5	Δ = 0,8	11,39 %	64,05	19,34	29,34	164,05	100,26	90,26	100	80,92	60,92
	Δ = 0,7	10,69 %	61,80	14,06	29,06	161,80	98,36	83,36	100	84,3	54,3
θ = 1	Δ = 0,8	11,91 %	63,63	20,98	20,98	163,63	98,79	78,79	100	77,81	57,81
	Δ = 0,7	11,40 %	60,95	15,83	15,83	160,95	95,86	65,86	100	80,03	50,03

Table 1 – Equity, bond and leverage firm values at the issue date (t=0) and at the rating downgrade date (τ_1 = 6 and 8 for the no covenant and accrued coupon rate cases -first panel; and τ_1 = 6 for the partial amortization case - second panel). The bonds have a face value of 100 issued at par with a maturity of 10 years, and V_0=150; V_{B1} = 100; V_{B2} = 50; α_1 = 0,07; α_2 = 0,10; σ_1= 0,30; σ_2=0,45; r = 0,075, ρ_1 =0,8 and ρ_2 = 1.

7. Conclusion

Using a framework similar to that used by Bhanot (2003), we developed a model to price finite maturity coupon bonds with rating trigger based covenants, which resolved an inconsistency inherent in that model. namely, the absence of default at maturity. We showed that, this limitation in Bhanot's model could lead to a significant bond overvaluation . In addition, comparisons were made considering bonds with different types of rating trigger covenants. Although, the bond with a partial refund rating trigger covenant, when fully financed by equity infusion, offers the greatest bondholder "protection" and thus the lowest credit spread, the corresponding equity and leveraged firm values are lower when compared with those corresponding to the accrued coupon rating trigger covenant.

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Appendix

Preliminaries

- The dynamics of the logarithm of the assets value, $Y = \ln V$ are:

$$dY = \mu_i^X dt + \sigma_i dW^X$$

where W^X is a Wiener process under the probability measure Q_X and μ_i^X is the correspondent drift (X = B, V or m).

Q_B – is the probability measure when the assets value process is normalized by the saving account;

Q_V – is the probability measure when the assets value is used as numeraire;

Q_m – is the probability measure when assets paying one unit at default are used as numeraire.

The corresponding drift terms are defined as follows:

$$\mu_i^B = (r - \alpha_i - \sigma_i^2/2)$$

$$\mu_i^V = \mu_i^B + \sigma_i^2 = (r - \alpha_i + \sigma_i^2/2)$$

$$\mu_i^m = \sqrt{\left(\mu_i^B\right)^2 + 2r\sigma_i^2}$$

- $E^{Q_X}\left[1_A \middle| \mathcal{F}_t\right] = Q_X\left(A \middle| \mathcal{F}_t\right)$ = probability, under measure Q_X, of event A occurring.

- The first passage time density at T (T > t) of the assets value from V_t to a barrier level V_B ($V_t > V_B$), under probability measure Q_X is given by:

$$g_X(T, V_t, V_B) = \frac{\ln\left(\frac{V_t}{V_B}\right)}{\sqrt{2\pi\sigma^2(T-t)^3}} e^{-\frac{1}{2}\left(\frac{\ln(V_t/V_B)+\mu^X(T-t)}{\sigma\sqrt{T-t}}\right)^2}$$

and the corresponding cumulative distribution function, is:

$$G_X(T, V_t, V_B) = \int_t^T g_X(u, V_t, V_B)du =$$

$$= N\left(\frac{-\ln\left(\frac{V_t}{V_B}\right) - \mu^X(T-t)}{\sigma\sqrt{(T-t)}}\right) + \left(\frac{V_t}{V_B}\right)^{-\frac{2\mu^X}{\sigma^2}} N\left(\frac{-\ln\left(\frac{V_t}{V_B}\right) + \mu^X(T-t)}{\sigma\sqrt{(T-t)}}\right) \quad \text{(P1)}$$

Where N[.] stands for the cumulative standard normal density function.

- Considering the definitions of τ_1 and τ_2 in the body of the text:

$Q_X(\tau_1 \leq T| \mathcal{F}_t) = G_X(T, V_t, V_{B1})$ - using μ_1^X and σ_1 in expression P1.

$Q_X(\tau_2 \leq T| \mathcal{F}_{\tau_1}) = G_X(T, V_{\tau_1}, V_{B2})$ and $Q_X(\tau_2 \leq T| \mathcal{F}_t) = G_X(T, V_t, V_{B2})$ – using μ_2^X and σ_2 in expression P1.

- $$E^{Q_B}\left[e^{-r(\tau_i - t)} 1_{\tau_i \leq T} \middle| \mathcal{F}_t\right] = \int_t^T e^{-r(u-t)} g_B(u, V_t, V_{Bi}) du = \left(\frac{V_t}{V_{Bi}}\right)^{\frac{\mu_i^m - \mu_i^B}{\sigma_i^2}} \int_t^T g_m(u, V_t, V_{Bi}) du =$$

$$= \left(\frac{V_t}{V_{Bi}}\right)^{\frac{\mu_i^m - \mu_i^B}{\sigma_i^2}} Q_m\left(\tau_i \leq T \middle| \mathcal{F}_t\right), \text{ where i = 1, 2.}$$

- $$E^{Q_B}\left[e^{-r(\tau_2 - \tau_1)} 1_{\tau_2 \leq T} \middle| \mathcal{F}_{\tau_1}\right] = \left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \leq T \middle| \mathcal{F}_{\tau_1}\right)$$

- $$E^{Q_V}\left[e^{-\alpha_i(\tau_i - t)} 1_{\tau_i \leq T} \middle| \mathcal{F}_t\right] = \int_t^T e^{-\alpha_i(u-t)} g_V(u, V_t, V_{Bi}) du = \left(\frac{V_t}{V_{Bi}}\right)^{\left(\frac{\mu_i^m - \mu_i^B}{\sigma_i^2} - 1\right)} \int_t^T g_m(u, V_t, V_{Bi}) du =$$

$$= \left(\frac{V_t}{V_{Bi}}\right)^{\left(\frac{\mu_i^m - \mu_i^B}{\sigma_i^2} - 1\right)} Q_m\left(\tau_i \leq T \middle| \mathcal{F}_t\right), \text{ where i = 1, 2.}$$

A.1

- $$B(V, t, T; c, F, \Delta) = E^{Q_B}\left[\int_t^T cFe^{-r(s-t)} 1_{s<\tau_1} ds \middle| \mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(T-t)} F 1_{\tau_1 > T, V_T \geq F} \middle| \mathcal{F}_t\right] +$$

$$E^{Q_B}\left[e^{-r(T-t)} \rho_2 V_T 1_{\tau_1 > T, V_T < F} \middle| \mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(\tau_1 - t)} 1_{\tau_1 < T} B\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right) \middle| \mathcal{F}_t\right]$$

First term: $E^{Q_B}\left[\int_t^T cFe^{-r(s-t)} 1_{s<\tau_1} ds \middle| \mathcal{F}_t\right] = E^{Q_B}\left[\int_t^{T \wedge \tau_1} cFe^{-r(s-t)} ds \middle| \mathcal{F}_t\right] =$

$$= -\frac{cF}{r} E^{Q_B}\left[e^{-r(T-t)} 1_{\tau_1 > T} + e^{-r(\tau_1 - t)} 1_{\tau_1 \leq T} - 1 \middle| \mathcal{F}_t\right] = \frac{cF}{r}\left\{1 - e^{-r(T-t)} E^{Q_B}\left[1_{\tau_1 > T} \middle| \mathcal{F}_t\right] - E^{Q_B}\left[e^{-r(\tau_1 - t)} 1_{\tau_1 \leq T} \middle| \mathcal{F}_t\right]\right\} =$$

$$= \frac{cF}{r}\left[1 - e^{-r(T-t)} Q_B\left(\tau_1 > T \middle| \mathcal{F}_t\right) - \left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m - \mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \leq T \middle| \mathcal{F}_t\right)\right]$$

Second term: $E^{Q_B}\left[e^{-r(T-t)}F1_{\tau_1>T,V_T\geq F}\middle|\mathcal{F}_t\right] = Fe^{-r(T-t)}Q\left(\tau_1>T,V_T\geq F\middle|\mathcal{F}_t\right)$

Third term: $E^{Q_B}\left[e^{-r(T-t)}\rho_2 V_T 1_{\tau_1>T,V_T<F}\middle|\mathcal{F}_t\right] = \rho_2 V_t e^{-\alpha_1(T-t)}E^{Q_B}\left[\frac{e^{-(r-\alpha_1)(T-t)}V_T}{V_t}1_{\tau_1>T,V_T<F}\middle|\mathcal{F}_t\right] =$

$$\rho_2 V_t e^{-\alpha_1(T-t)}E^{Q_B}\left[\frac{e^{-(r-\alpha_1)(T-t)}V_T}{V_t}\middle|\mathcal{F}_t\right]E^{Q_V}\left[1_{\tau_1>T,V_T<F}\middle|\mathcal{F}_t\right] = \rho_2 V_t e^{-\alpha_1(T-t)}Q_V\left(\tau_1>T,V_T<F\middle|\mathcal{F}_t\right)$$

(note: $e^{-(r-\alpha_1)(T-t)}V_T$ is a martingale under probability measure Q_B)

Fourth term: $E^{Q_B}\left[e^{-r(\tau_1-t)}1_{\tau_1<T}B\left(V_{\tau_1},\tau_1,T,c,F,\Delta\right)\middle|\mathcal{F}_t\right] = \int_t^T e^{-r(\tau_1-t)}B\left(V_{B1},\tau_1,T,c,F,\Delta\right)g_B\left(\tau_1,V_t,V_{B1}\right)d\tau_1$

Collecting terms and noting that $Q_X\left(\tau_i>T\middle|\cdot\right)=1-Q_X\left(\tau_i\leq T\middle|\cdot\right)$, yields the expression (4) in the body of the text.

- $B(V_{\tau_1}, \tau_1, T; c, F, \Delta) = E^{Q_B}\left[\int_{\tau_1}^{T}\Delta cFe^{-r(s-\tau_1)}1_{s<\tau_2}ds\middle|\mathcal{F}_{\tau_1}\right] + E^{Q_B}\left[e^{-r(T-\tau_1)}F1_{\tau_2>T,V_T\geq F}\middle|\mathcal{F}_{\tau_1}\right] +$

$$+\ E^{Q_B}\left[e^{-r(T-\tau_1)}\rho_2 V_T 1_{\tau_2>T,V_T<F}\middle|\mathcal{F}_{\tau_1}\right] + E^{Q_B}\left[e^{-r(\tau_2-\tau_1)}\rho_1 V_{\tau_2}1_{\tau_2<T}\middle|\mathcal{F}_{\tau_1}\right]$$

Notice that the first three terms in the above expression are similar to those relating B(V, t, T; c, F, Δ), the difference being the filtration considered (τ_1 instead of t) and the coupon (ΔcF instead of cF). Thus, applying the same derivation, the solution will be the same as that obtained previously after replacing V_t by $V_{\tau_1} = V_{B1}$, t by τ_1 and cF by ΔcF.

For the fourth term: $E^{Q_B}\left[e^{-r(\tau_2-\tau_1)}\rho_1 V_{\tau_2}1_{\tau_2<T}\middle|\mathcal{F}_{\tau_1}\right] = \rho_1 V_{B2}E^{Q_B}\left[e^{-r(\tau_2-\tau_1)}1_{\tau_2<T}\middle|\mathcal{F}_{\tau_1}\right] =$

$$= \rho_1 V_{B2}\left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m-\mu_2^B}{\sigma_2^2}}Q_m\left(\tau_2\leq T\middle|\mathcal{F}_{\tau_1}\right)$$

Collecting terms and noting that $Q_X\left(\tau_i>T\middle|\cdot\right)=1-Q_X\left(\tau_i\leq T\middle|\cdot\right)$, yields the expression (5) in the body of the text.

- After the downgrade, the bond value is defined as B(V_{τ_1} , τ_1, T; c, F, Δ), but since t > τ_1 the relevant filtration is $\mathcal{F}_t$, thus:

$$B(V_t, t, T; c, F, \Delta) = E^{Q_B}\left[\int_t^T \Delta cFe^{-r(s-t)}1_{s<\tau_2}\,ds\middle|\mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(T-t)}F1_{\tau_2>T,V_T\geq F}\middle|\mathcal{F}_t\right] +$$

$$+ E^{Q_B}\left[e^{-r(T-t)}\rho_2 V_T 1_{\tau_2>T,V_T<F}\middle|\mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(\tau_2-t)}\rho_1 V_{\tau_2} 1_{\tau_2<T}\middle|\mathcal{F}_t\right]$$

Whose solution will be the same of B(V_{τ_1} , τ_1, T; c, F, Δ) (expression 5) after replacing, V_{B1} by V_t and τ_1 by t yielding expression (5').

A.2

- Equity value at t < τ_1:

$$E(V_t, t, T, c, F, \Delta) = E^{Q_B}\left[\int_t^T (\alpha_1 V_s - cF(1-\iota))e^{-r(s-t)}1_{s<\tau_1}\,ds\middle|\mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(T-t)}(V_T - F)1_{\tau_1>T,V_T\geq F}\middle|\mathcal{F}_t\right] +$$

$$+ E^{Q_B}\left[e^{-r(\tau_1-t)}1_{\tau_1<T}E(V_{\tau_1}, \tau_1, T, c, F, \Delta)\middle|\mathcal{F}_t\right]$$

First term: $E^{Q_B}\left[\int_t^T (\alpha_1 V_s - cF(1-\iota))e^{-r(s-t)}1_{s<\tau_1}\,ds\middle|\mathcal{F}_t\right] =$

$$= \left[\int_t^T \alpha_1 E^{Q_B}\left[V_s e^{-r(s-t)}1_{s<\tau_1}\middle|\mathcal{F}_t\right]ds\right] - E^{Q_B}\left[\int_t^T cF(1-\iota)e^{-r(s-t)}1_{s<\tau_1}\,ds\middle|\mathcal{F}_t\right] \qquad \text{(by Fubini's theorem)}$$

$$= \left[\int_t^T \alpha_1 V_t e^{-\alpha_1(s-t)}E^{Q_V}\left[1_{s<\tau_1}\middle|\mathcal{F}_t\right]ds\right] - E^{Q_B}\left[\int_t^{T\wedge\tau_1} cF(1-\iota)e^{-r(s-t)}ds\middle|\mathcal{F}_t\right] =$$

$$= V_t E^{Q_V}\left[\int_t^{T\wedge\tau_1}\alpha_1 e^{-\alpha_1(s-t)}ds\middle|\mathcal{F}_t\right] + \frac{cF(1-\iota)}{r}E^{Q_B}\left[e^{-r(T-t)}1_{\tau_1>T} + e^{-r(\tau_1-t)}1_{\tau_1\leq T} - 1\middle|\mathcal{F}_t\right] =$$

$$= V_t E^{Q_V}\left[1 - e^{-\alpha_1(T-t)}1_{\tau_1>T} - e^{-\alpha_1(\tau_1-t)}1_{\tau_1\leq T}\middle|\mathcal{F}_t\right] - \frac{cF(1-\iota)}{r}\left\{1 - e^{-r(T-t)}E^{Q_B}\left[1_{\tau_1>T}\middle|\mathcal{F}_t\right] - E^{Q_B}\left[e^{-r(\tau_1-t)}1_{\tau_1\leq T}\middle|\mathcal{F}_t\right]\right\}$$

$$= V_t\left\{1 - e^{-\alpha_1(T-t)}Q_V\left(\tau_1 > T\middle|\mathcal{F}_t\right) - E^{Q_V}\left[e^{-\alpha_1(\tau_1-t)}1_{\tau_1\leq T}\middle|\mathcal{F}_t\right]\right\} -$$

$$- \frac{cF(1-\iota)}{r}\left[1 - e^{-r(T-t)}Q_B\left(\tau_1 > T\middle|\mathcal{F}_t\right) - \left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m - \mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \leq T\middle|\mathcal{F}_t\right)\right]$$

$$= V_t\left[1-e^{-\alpha_1(T-t)}Q_V\left(\tau_1>T\middle|\mathcal{F}_t\right)-\left(\frac{V_t}{V_{B1}}\right)^{\left(\frac{\mu_1^m-\mu_1^B}{\sigma_1^2}-1\right)}Q_m\left(\tau_1\le T\middle|\mathcal{F}_t\right)\right]-$$

$$-\frac{cF(1-\iota)}{r}\left[1-e^{-r(T-t)}Q_B\left(\tau_1>T\middle|\mathcal{F}_t\right)-\left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m-\mu_1^B}{\sigma_1^2}}Q_m\left(\tau_1\le T\middle|\mathcal{F}_t\right)\right]$$

Second term:

$$E^{Q_B}\left[e^{-r(T-t)}(V_T-F)1_{\tau_1>T,V_T\ge F}\middle|\mathcal{F}_t\right]=V_te^{-\alpha_1(T-t)}E^{Q_V}\left[1_{\tau_1>T,V_T\ge F}\middle|\mathcal{F}_t\right]-Fe^{-r(T-t)}E^{Q_B}\left[1_{\tau_1>T,V_T\ge F}\middle|\mathcal{F}_t\right]=$$

$$=V_te^{-\alpha_1(T-t)}Q_V\left(\tau_1>T,V_T\ge F\middle|\mathcal{F}_t\right)-Fe^{-r(T-t)}Q_B\left(\tau_1>T,V_T\ge F\middle|\mathcal{F}_t\right)=$$

$$=V_te^{-\alpha_1(T-t)}\left[Q_V\left(\tau_1>T\middle|\mathcal{F}_t\right)-Q_V\left(\tau_1>T,V_T<F\middle|\mathcal{F}_t\right)\right]-Fe^{-r(T-t)}Q_B\left(\tau_1>T,V_T\ge F\middle|\mathcal{F}_t\right)$$

Third Term: $E^{Q_B}\left[e^{-r(\tau_1-t)}1_{\tau_1<T}E\left(V_{\tau_1},\tau_1,T,c,F,\Delta\right)\middle|\mathcal{F}_t\right]=\int_t^T e^{-r(\tau_1-t)}E\left(V_{B1},\tau_1,T,c,F,\Delta\right)g_B\left(\tau_1,V_t,V_{B1}\right)d\tau_1$

Collecting terms yields the expression (9) in the body of the text.

- Equity value at t = τ_1, accrued coupon rate case

$$E\left(V_{\tau_1},\tau_1,T,c,F,\Delta\right)=$$

$$E^{Q_B}\left[\int_{\tau_1}^{T}(\alpha_2V_s-c\Delta F(1-\iota))e^{-r(s-\tau_1)}1_{s<\tau_2}\,ds\middle|\mathcal{F}_{\tau_1}\right]+E^{Q_B}\left[(V_T-F)e^{-r(T-\tau_1)}1_{\tau_2>T,V_T\ge F}\middle|\mathcal{F}_{\tau_1}\right]=$$

Applying the same reasoning as for the two first terms in the previous case but considering now the filtration at τ_1 yields the expression (10) in the body of the text.

A.3.
The value of the tax benefits is given by:

- $TB\left(V_t,t,T,c,F,\Delta,\iota\right)=E^{Q_B}\left[\int_t^T cF\iota e^{-r(s-t)}1_{s<\tau_1}\,ds\middle|\mathcal{F}_t\right]+E^{Q_B}\left[e^{-r(\tau_1-t)}1_{\tau_1<T}TB\left(V_{\tau_1},\tau_1,T,c,F,\Delta\right)\middle|\mathcal{F}_t\right]$

First term:

$$=E^{Q_B}\left[\int_t^{T\wedge\tau_1}cF\iota e^{-r(s-t)}ds\middle|\mathcal{F}_t\right]=-\frac{cF\iota}{r}E^{Q_B}\left[e^{-r(T-t)}1_{\tau_1>T}+e^{-r(\tau_1-t)}1_{\tau_1\le T}-1\middle|\mathcal{F}_t\right]=$$

$$= \frac{cF\iota}{r}\left\{1 - e^{-r(T-t)}E^{Q_B}\left[1_{\tau_1 > T}\middle|\mathcal{F}_t\right] - E^{Q_B}\left[e^{-r(\tau_1 - t)}1_{\tau_1 \le T}\middle|\mathcal{F}_t\right]\right\}$$

$$= \frac{cF\iota}{r}\left[1 - e^{-r(T-t)}Q_B\left(\tau_1 > T\middle|\mathcal{F}_t\right) - \left(\frac{V_t}{V_{B1}}\right)^{\frac{\mu_1^m - \mu_1^B}{\sigma_1^2}} Q_m\left(\tau_1 \le T\middle|\mathcal{F}_t\right)\right]$$

Second term:

$$E^{Q_B}\left[e^{-r(\tau_1 - t)}1_{\tau_1 < T}TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right)\middle|\mathcal{F}_t\right] = \int_t^T e^{-r(\tau_1 - t)}TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right)g_B\left(\tau_1, V_t, V_{B1}\right)d\tau_1$$

Collecting terms yields expression (11) in the body of the text.

- $TB\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) = E^{Q_B}\left[\int_{\tau_1}^{T} c\Delta F\iota e^{-r(s-\tau_1)}1_{s<\tau_2}\,ds\middle|\mathcal{F}_{\tau_1}\right] =$

$$= \frac{c\Delta F\iota}{r}\left\{1 - e^{-r(T-\tau_1)}E^{Q_B}\left[1_{\tau_2 > T}\middle|\mathcal{F}_{\tau_1}\right] - E^{Q_B}\left[e^{-r(\tau_2 - \tau_1)}1_{\tau_2 \le T}\middle|\mathcal{F}_{\tau_1}\right]\right\}$$

Whose value, for the accrued coupon rate case (Δ>1), will be given by:

$$\frac{c\Delta F\iota}{r}\left[1 - e^{-r(T-\tau_1)}Q_B\left(\tau_2 > T\middle|\mathcal{F}_{\tau_1}\right) - \left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \le T\middle|\mathcal{F}_{\tau_1}\right)\right] \qquad (12)$$

And, for the partial amortization case (Δ<1, and $V_{\tau_1} = V_{B1} - \theta(1-\Delta)F$) :

$$\frac{c\Delta F\iota}{r}\left[1 - e^{-r(T-\tau_1)}Q_B^*\left(\tau_2 > T\middle|\mathcal{F}_{\tau_1}\right) - \left(\frac{V_{B1} - \theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m^*\left(\tau_2 \le T\middle|\mathcal{F}_{\tau_1}\right)\right] \qquad (12')$$

A.4. In relation to bankruptcy costs:

$$BC\left(V_t, t, T, c, F, \Delta\right) = E^{Q_B}\left[e^{-r(T-t)}1_{\tau_1 > T, V_T < F}\left(1-\rho_2\right)V_T\middle|\mathcal{F}_t\right] + E^{Q_B}\left[e^{-r(\tau_1 - t)}1_{\tau_1 < T}BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right)\middle|\mathcal{F}_t\right]$$

$$= \left(1-\rho_2\right)V_t e^{-\alpha_1(T-t)}E^{Q_V}\left[1_{\tau_1 > T, V_T \ge F}\middle|\mathcal{F}_t\right] + \int_t^T e^{-r(\tau_1 - t)}BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right)g_B\left(\tau_1, V_t, V_{B1}\right)d\tau_1 =$$

$$= \left(1-\rho_2\right)V_t e^{-\alpha_1(T-t)}Q_V\left(\tau_1 > T, V_T < F\middle|\mathcal{F}_t\right) + \int_t^T e^{-r(\tau_1 - t)}BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta\right)g_B\left(\tau_1, V_t, V_{B1}\right)d\tau_1$$

- $BC\left(V_{\tau_1}, \tau_1, T, c, F, \Delta, \iota\right) =$

$$= E^{Q_B}\left[e^{-r(T-\tau_1)} 1_{\tau_2 > T, V_T < F}(1-\rho_2)V_T \middle| \mathcal{F}_{\tau_1}\right] + E^{Q_B}\left[e^{-r(\tau_2-\tau_1)} 1_{\tau_2 < T}(1-\rho_1)V_{B2} \middle| \mathcal{F}_{\tau_1}\right]$$

Whose value, for the accrued coupon rate case (Δ>1), will be given by:

$$(1-\rho_2)V_{B1}e^{-\alpha_2(T-\tau_1)}Q_V\left(\tau_2 > T, V_T < F \middle| \mathcal{F}_{\tau_1}\right) + (1-\rho_1)V_{B2}\left(\frac{V_{B1}}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right) \qquad (14)$$

And, for the partial amortization case (Δ<1, and $V_{\tau_1} = V_{B1} - \theta(1-\Delta)F$) :

$$(1-\rho_2)[V_{B1} - \theta(1-\Delta)F]e^{-\alpha_2(T-\tau_1)}Q_V^*\left(\tau_2 > T, V_T < \Delta F \middle| \mathcal{F}_{\tau_1}\right) +$$

$$+(1-\rho_1)V_{B2}\left(\frac{V_{B1} - \theta(1-\Delta)F}{V_{B2}}\right)^{\frac{\mu_2^m - \mu_2^B}{\sigma_2^2}} Q_m^*\left(\tau_2 \le T \middle| \mathcal{F}_{\tau_1}\right) \qquad (14')$$