Warrant Valuation and Strategic Exercise in Continuous Time and Imperfect Competition [*]

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Abstract

This paper focuses on the valuation of warrants using a game-theoretic approach, incorporating strategic exercise of the warrants. Unlike previous work on strategic exercise of warrants, this paper employs a continuous time framework for the case of an oligopolistic holding structure of the warrants and also allows the firm to apply a combination of uses for the exercise proceeds. The results suggest that American warrants might be worth less than European warrants, in much broader sets of parameters than previously shown in the literature.

A warrant is a right to buy a share of the firm at a certain price during a given period. While a call option is issued by an individual as a side bet on the stock value, the warrant is issued by the firm; when a warrant is exercised, the proceeds from exercising the warrant become a part of the firm's equity, and the firm issues a new stock to the warrant holder.

Strategic exercise of warrants and convertible securities was analyzed by Emanuel (1983), Constantinides (1984), Constantinides and Rosenthal (1984), and by Spatt and Sterbenz (1988). These papers incorporate the firm's policy regarding the use of the exercise proceeds and its dividends, into the warrant holders' considerations for the timing of the exercise, and consequently into the valuation models. The above papers focus on the case of a monopoly or perfect competition, i.e. the warrants are either held by a single holder or by a very large number of holders, each holding an infinitesimal number of warrants. Spatt and Sterbenz (1988) have also analyzed a special case of a two period discrete time model with an oligopolistic holding structure. However, the theoretical research on the field of strategic exercise of warrants has not yet been concluded. It is the purpose of this paper to contribute to the discussion by analyzing a continuous time model of an oligopolistic holding structure of warrants, and show that the results are very different from that of an oligopolistic holding structure in discrete time, as was presented by Spatt and Sterbenz (1988). The model discussed here also enables the firm to decide on a combination of uses for the exercise proceeds, and thus provides a better insight of the sensitivity of these models to the firm's use of the exercise proceeds.

I. **The Setup and Assumptions**

The model discussed in this paper is a continuous-time version of the two period discrete time model of warrants' strategic exercise in an oligopolistic holding structure, which was introduced by Spatt and Sterbenz (1988). The type of equilibrium which is discussed here is sub-game perfect. [1]

The motivation for using a continuous time model is as follows: In the two period model introduced by Spatt and Sterbenz (1988), when a warrant is exercised the firm uses the proceeds in order to pay an immediate extraordinary dividends to the stockholders. The model basically analyzes the equilibrium between two opposite incentives. On the one hand, there is an incentive to postpone the exercise of the warrants to the second period, in order to obtain the interest rate on the exercise proceeds over the lifespan of the warrants. On the other hand, the warrant holders who exercise in the first period receive their share of the dividend paid in this period, consists of the exercise proceeds of that time. Alternatively, the use of a continuous time model might result in a stronger incentive for the warrant holders to exercise before the expiration. An example might be that one warrant holder chooses to exercise shortly before the other. She might be better off that way, rather than to exercise at the exact time her fellow warrant holder chooses to do so, because she loses only a small fraction of the interest, and still receive a considerable share of the dividends, not having to share the dividends with her fellow warrant holder.

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[1] The notion of the sub-game perfect equilibrium was first introduced by Selten (1965) for sequential games. This kind of equilibrium imposes rationality in each sub-game of the main game.

The literature mentioned in the introduction employs a number of alternative assumptions about the use of warrants exercise proceeds:

1) The firm uses the proceeds in order to pay an extraordinary dividend to stockholders.

2) The firm uses the proceeds in order to reinvest in its regular projects.

3) The firm uses the proceeds in order to buy a risk free bond.

4) The firm uses the proceeds for stock repurchase.

Each model, however, uses only one of the assumptions, not allowing the firm to mix between these choices. In the henceforth presented model the firm will be allowed some flexibility in its choice of the use of the exercise proceeds. Explicitly, the firm will be able to use some combination of assumptions 1-3, mentioned above.

A. Assumptions

The firm's equity consists of n regular shares and m warrants (with an exercise ratio of 1:1). The warrants are held by K warrant holders, who are the players in the game. The warrants are infinitely divisible and any portion can be exercised between now ($t=0$) and time T, in which the firm pays a liquidation dividend.

The oligopolistic holding structure of the warrants formally implies that $0 < K << \infty$ and that the number of warrants initially held by each of the warrant holders is not infinitesimal.

It is assumed that at the early exercise point ($t=0$) it is known that the warrants will subsequently expire in the money, making it optimal to exercise all remaining warrants at the expiration (time T).[2] Moreover, the exact value of the firm at T is known at $t=0$.[3]

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[2] This assumption has also been used by Spatt and Sterbenz (1988).

When a warrant is exercised, the firm uses the exercise proceeds as follows:

1) It invests a fraction $a \in [0,1)$ of the proceeds at the risk free rate r. Since the firm value is deterministic, assuming constant return to scale and no arbitrage opportunities, any sum of money invested in the firm's project should yield the risk free rate. Therefore, this fraction can also be interpreted as the fraction used to expand the firm's scale.

2) It uses the remaining proceeds (1- a of the proceeds) to pay an immediate extraordinary dividend to the current stockholders, including the stocks issued as a result of the warrants just exercised.

The choice of a is assumed to be exogenous to the model.[4]

No dividends are paid other than the extraordinary dividends from the exercise proceeds and the liquidation dividend.

Subject to these assumptions each warrant holder has to choose the number of warrants she wishes to exercise at each given time, until the expiration.

The sub-game perfect equilibrium is found by dividing the continuous time between $t=0$ and $t=T$ into short time periods, each with the duration of Δt, where $\Delta t \rightarrow 0$. It is also assumed that within each time period Δt, the players cannot exercise any warrants.

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[3] The use of certainty in this model is convenient because it makes it possible to focus on the specific interactions between the warrant holders, without having to deal with hedging issues. However, it should be noted that the equilibrium is independent of the actual value of the firm at the expiration point, namely FV. Thus, ceteris paribus, the equilibrium derived in this paper is also true in cases of stochastic behavior of the firm's value.

[4] A question that may arise from this model is what is at the best interest of the stockholders? Seemingly, their interest is to minimize the value of the warrants, and by that maximizing the portion of the stock value out of the firm's value. This, however, is usually not the case when the warrants are held by the firm's employees. Core and Guay (2001) support this hypothesis by showing that firms make additional options grants to employees, in order to reinstate incentive effects lost through options exercise.

B. *Definition of Symbols*

The following is defined for a certain time $t \in [0, T]$.

- Let k_i denote the number of warrants which player i exercises at time t.

- Let $\underline{k}_i$ denote the number of warrants which player i has exercised before time t.

- Let L_i denote the number of warrants which all the other players exercise at time t

  $(L_i = \sum_{-i} k_j)$.

- Let $\underline{L}_i$ denote the number of warrants which all the other players have exercised

  before time t ($\underline{L}_i = \sum_{-i} \underline{k}_j$).

- Let E denote the exercise price of a single warrant.

- Let r denote the continuous risk free interest rate.

- Let m_i denote the number of warrant initially held by player i.

- Let FV denote the value of the firm at time T (the "net" liquidation value, without the exercise proceeds).

- Let RF denote the value of the exercise proceeds invested by the firm at the risk free rate until time T, namely the deposits plus the interest accumulated.

II. **The Model's Equilibria**

In this section the model's equilibria is discussed.

Theorem: *If the condition* $\max_i(m_i) \le m - n\left(\dfrac{e^{rT}-1}{1-\boldsymbol{a}e^{rT}}\right)$ *is satisfied, then a unique sub-game perfect equilibrium exists where all warrant holders immediately exercise all of their warrants. If the previous condition is not satisfied, then the game has no sub-game perfect equilibrium.*

The complete proof can be found in the appendix. The sketch of the proof is as follows: It is known that all remaining warrants are exercised at time T, therefore the last non-trivial decision point of how many warrants to exercise is $t = T - \Delta t$. The sub-game perfect equilibrium is found by analyzing the game starting from this last decision point ($t = T - \Delta t$). The present value of the payoff function of player i relative to time $T - \Delta t$ is:

$$\Pi^i(k_i, L_i) = \underbrace{\frac{k_i + \underline{k}_i}{k_i + \underline{k}_i + L_i + \underline{L}_i + n}(1-\boldsymbol{a})(L_i + k_i)E}_{\text{Player } i\text{'s share in the extraordinary dividend paid at T-}\Delta t} - \underbrace{E \cdot k_i}_{\substack{\text{The cost of the} \\ \text{exercise at T-}\Delta t}} - \underbrace{\frac{E(m_i - k_i - \underline{k}_i)}{e^{r\Delta t}}}_{\substack{\text{The cost of the} \\ \text{exercise at T}}} + \tag{1}$$

$$+ \underbrace{\frac{m_i}{(m+n)e^{r\Delta t}}\left\{FV + \left[\boldsymbol{a}\cdot E(L_i + k_i) + RF\right]e^{r\Delta t} + (m - k_i - \underline{k}_i - L_i - \underline{L}_i)E\right\}}_{\text{Player } i\text{'s share in the extraordinary dividend to be paid at T}}$$

In equilibrium each of the warrant holders maximizes the present value of his payoffs, while regarding the other warrant holders' actions as given (namely L_i). A Nash equilibrium is calculated for the last decision point. The equilibrium at each of the

preceding decision point is then calculated, while using the results derived in the subsequent decision points.

The *Theorem* implies that if the largest warrant holder is not "too large", the model has a unique sub-game perfect equilibrium, in which all the warrants are exercised immediately. Under the assumptions of the model, a monopolistic warrant holder would have exercised all of his warrants at the expiration, and this kind of behavior would thus maximize the total value of the warrants. The equilibrium derived in the *Theorem* is interesting because it actually minimizes the total value of the warrants, hence implying on a total coordination failure amongst the warrant holders. In the next section, this equilibrium is compared to that found by Spatt and Sterbenz (1988) for the discrete two period model. When the largest warrant holder is "large enough", the game has no sub-game perfect equilibrium.

The *Theorem* also reveals the exact effect that the firm's policy regarding the use of the exercise proceeds has on the equilibrium. The equilibrium where all the warrant holders exercise at the earliest possible time tends to hold where a large fraction of the exercise proceeds is paid as a dividend, i.e. when a is small.

III. Full Dividend Policy

We now discuss the specific case where $a = 0$. That is, when a warrant is exercised, the firm uses all of the exercise proceeds in order to pay an immediate extraordinary dividend to its shareholders. Based on the *Theorem*, and letting a be 0, it is easy to see that a sub-game perfect equilibrium exists if and only if the following condition holds:

$$\max_i(m_i) \leq m - n\left(e^{rT} - 1\right) \tag{2}$$

In this equilibrium all the warrants are exercised immediately (at $t=0$).

The conditions for early exercise of the warrants found in this paper, for the case where $a = 0$, are very different from those found by Spatt and Sterbenz (1988). This can be explicitly shown by assuming that the initial holding of warrants is equal amongst the warrant holders, similar to the assumption used by Spatt and Sterbenz (1988). Let r now denote the risk free interest rate on the entire period between time 0 and time T.

Spatt and Sterbenz (1988) argue that the two period game has a Nash equilibrium where everyone exercises at the early exercise point only if:

$$r \leq \frac{m - m_i}{n} \cdot \frac{m_i}{n + m} \tag{3}$$

In contrast, in the continuous time model, condition (2) implies that early exercise of all warrants will take place if the following condition holds:

$$r \leq \frac{m - m_i}{n} \tag{4}$$

Hence, there is an immense difference in the predicted equilibrium between the two models. The early exercise equilibrium holds for broader sets of parameters in the continuous time model than in the discrete two period model In general, each warrant holder wishes to precede the other warrant holders in exercising, in order to have a greater portion of the dividends. The continuous time assumption allows one warrant holder to precede another by a very small fraction of time, with a negligible loss of interest income. The warrant holders' motivation to do so intensifies the competition between them considerably, forcing them to an early exercise, and reducing the value of their warrants.

The standard assumption in most theoretical models of warrant valuation is that the exercise proceeds would be immediately distributed as a cash dividend. This policy, however, may cause American warrants to be exercised very early. The valuation of American warrants (which can be exercised at any time until the expiration), in this case, is very different from that of European warrants (which can be exercised only at the expiration time). Hence, taking into account the long lives of warrants, the common way to value warrants by adjusting the Black & Scholes formula by the dilution factor, might result in considerable overpricing of these warrants.[5]

IV. Summary and Discussion

This paper focuses on the valuation of warrants using a game-theoretic approach, taking into account strategic exercise of the warrants. A continuous time model is discussed for the case of an oligopolistic holding structure of the warrants.

The results suggest that if, when warrants are exercised, the firm uses some of the exercise proceeds to pay an immediate extraordinary dividend, and uses the rest of the proceeds to buy a riskless bond, then the only possible sub-game perfect equilibrium, if one exists, is immediate exercise of all warrants. This equilibrium is the worst possible solution, from the collective point of view of the warrant holders, since it minimizes the total value of the warrants.

For the specific case in which the firm uses all of the exercise proceeds in order to pay an immediate extraordinary dividend ($a = 0$), this equilibrium holds in a much broader set of parameters than previously found in the literature for the discrete case.

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[5] Galai and Schneller (1978) were the first to introduce the dilution factor for the pricing of European warrants.

It is implied from the paper that the firm's policy regarding the use of the exercise proceeds should be carefully analyzed when valuating American warrants. Moreover, the equilibrium suggests that the value of European warrants might prove to be considerably higher than that of American warrants.

Spatt and Sterbenz (1988) show that if the firm invests the proceeds from any warrant exercise in a riskless zero-coupon bond that matures at the warrants' expiration (meaning, explicitly, that $a = 1$), then all warrants would be exercised at maturity. On the other hand, as shown in section *III*, in a continuous time framework the only sub-game perfect equilibrium where the exercise proceeds are used for the payment of an immediate extraordinary dividend ($a = 0$), is that in which all the warrants are exercised at the earliest time possible. These two policies create opposite incentives for the warrant holders, regarding the timing of exercise. In this paper, the balance between these two opposite forces is directly analyzed.

Monitoring the firm's use of the exercise proceeds might prove to be very difficult. In any case, it is reasonable to assume that most firms use some kind of mixed policy. Thus, if such data should become available, it might be possible to estimate a, and find the implied equilibrium.

In practice, however, the exercise of warrants doesn't necessarily occur in a rational, competitive manner. For example, Core and Guay (2001) report that the exercise of warrants held by employees is greater (less) when the firm's stock price hit 52-week highs (lows). These findings confirm the existence of some kind of psychological bias in the timing of the exercise. Still, much more theoretical research should be conducted in the field of strategic exercise of warrants and its effect on the warrants' valuation.

Appendix

In this Appendix, we show the detailed proof of the *Theorem* shown in the paper. The sub-game perfect equilibrium is found by analyzing the game starting from the last decision point ($t = T - \Delta t$). The present value of the payoff function of player i relative to time $T - \Delta t$ is:

$$\Pi^i(k_i, L_i) = \underbrace{\frac{k_i + \underline{k}_i}{k_i + \underline{k}_i + L_i + \underline{L}_i + n}(1-a)(L_i + k_i)E}_{\text{Player } i\text{'s share in the extraordinary dividend paid at T-}\Delta t} - \underbrace{E \cdot k_i}_{\substack{\text{The cost of the} \\ \text{exercise at T-}\Delta t}} - \underbrace{\frac{E(m_i - k_i - \underline{k}_i)}{e^{r\Delta t}}}_{\substack{\text{The cost of the} \\ \text{exercise at T}}} + \quad \textbf{(A1)}$$

$$+ \underbrace{\frac{m_i}{(m+n)e^{r\Delta t}}\left\{FV + \left[a \cdot E(L_i + k_i) + RF\right]e^{r\Delta t} + (m - k_i - \underline{k}_i - L_i - \underline{L}_i)E\right\}}_{\substack{\text{Player } i\text{'s share in the extraordinary} \\ \text{dividend to be paid at T}}}$$

The second partial derivative, relative to k_i (taking the other players actions, L_i, as given) is:

$$\frac{\partial^2 \Pi^i}{\partial k_i^2} = \frac{(k_i + \underline{k}_i + L_i + \underline{L}_i + n) - (k_i + \underline{k}_i)}{(k_i + \underline{k}_i + L_i + \underline{L}_i + n)^2}(1-a)E + \quad \textbf{(A2)}$$

$$+ E(1-a)\frac{(k_i + \underline{k}_i + L_i + \underline{L}_i + n) - (k_i + \underline{k}_i)}{(k_i + \underline{k}_i + L_i + \underline{L}_i + n)^2} - (1-a)E(L_i + k_i)\frac{L_i + \underline{L}_i + n}{2(k_i + \underline{k}_i + L_i + \underline{L}_i + n)^3} > 0$$

The strict convexity of the payoff function implies that the best reply of player i to any vector of rival strategies is to exercise all or none of her remaining warrants at time $T - \Delta t$. The payoffs in any one of these actions are as follows:

When the player i exercises all her remaining warrants at time T (*i.e.* k_i=0) her payoff is:

$$\Pi^i(0, L_i) = \frac{k_i \cdot L_i \cdot E(1-a)}{\underline{k}_i + L_i + \underline{L}_i + n} - \frac{E(m_i - \underline{k}_i)}{e^{r\Delta t}} + \quad \textbf{(A3)}$$

$$+ \frac{m_i}{(m+n)e^{r\Delta t}}\left[FV + (a \cdot E \cdot L_i + RF)e^{r\Delta t} + (m - \underline{k}_i - L_i - \underline{L}_i)E\right]$$

When player i exercises all her remaining warrants at time $T\text{-}\Delta t$ (i.e. $k_i = m_i - \underline{k}_i$) her payoff is:

$$\Pi^i\left(m_i - \underline{k}_i, L_i\right) = \frac{m_i\left(L_i + m_i - \underline{k}_i\right)\left(1-\boldsymbol{a}\right)E}{m_i + L_i + \underline{L}_i + n} - E\left(m_i - \underline{k}_i\right) + \qquad \textbf{(A4)}$$

$$+\frac{m_i}{\left(m+n\right)e^{r\Delta t}}\left\{FV + \left[\boldsymbol{a}\cdot E\left(L_i + m_i - \underline{k}_i\right) + RF\right]e^{r\Delta t} + \left(m - m_i - L_i - \underline{L}_i\right)E\right\}$$

Hence, exercising the remaining warrants of player i at time T would only take place if $\Pi^i\left(0, L_i\right) \geq \Pi^i\left(m_i - \underline{k}_i, L_i\right)$:

$$\frac{\underline{k}_i L_i\left(1-\boldsymbol{a}\right)}{\underline{k}_i + L_i + \underline{L}_i + n} + \frac{m_i\left[1 - \boldsymbol{a}e^{r\Delta t}\right]\left(m_i - \underline{k}_i\right)}{\left(m+n\right)e^{r\Delta t}} + \left(1 - \frac{1}{e^{r\Delta t}}\right)\left(m_i - \underline{k}_i\right) \geq \qquad \textbf{(A5)}$$

$$\geq \frac{m_i\left(L_i + m_i - \underline{k}_i\right)\left(1-\boldsymbol{a}\right)}{m_i + L_i + \underline{L}_i + n}$$

Taking the limit of the above condition where $\Delta t \to 0$ gives:

$$m_i \underline{k}_i\left(m_i + \underline{L}_i + L_i - m\right) + \underbrace{\left[m_i^2 + m_i\underline{L}_i + m_iL_i - m_im - L_im - L_in\right]}_{\leq 0}\underbrace{\left(\underline{L}_i + L_i + n\right)}_{} \geq 0 \qquad \textbf{(A6)}$$

$$\underbrace{\phantom{m_i \underline{k}_i\left(m_i + \underline{L}_i + L_i - m\right)}}_{\leq 0}$$

When $\Delta t \to 0$, condition (A6) never holds, thus making it optimal for all players to exercise all their remaining warrants at $T\text{-}\Delta t$, and not to wait for the expiration (i.e. time T). The only case when it is optimal for player i to wait until time T is when she is the only remaining warrant holder left at $T\text{-}\Delta t$. When this happens she is a monopoly warrant holder and thus under the above assumptions should always wait until the expiration.

The second part of the proof reveals what constitutes the optimal strategy for player i at time t when she knows that in time $t + \Delta t$ all the remaining warrants in the market will be exercised.

Her payoff function is (from the decision point at t):

$$\Pi^i(k_i, L_i) = \underbrace{\frac{k_i + \underline{k}_i}{k_i + \underline{k}_i + L_i + \underline{L}_i + n}(1-\boldsymbol{a})(L_i + k_i)E}_{\text{Player } i\text{'s share in the extraordinary dividend paid at t}} \underbrace{- E \cdot k_i}_{\text{The cost of the exercise at t}} + \tag{A7}$$

$$+ \underbrace{\frac{m_i(1-\boldsymbol{a})(m - k_i - \underline{k}_i - L_i - \underline{L}_i)E}{(m+n)e^{r\Delta t}}}_{\text{Player } i\text{'s share in the extraordinary dividend to be paid at t+}\Delta t} \underbrace{- \frac{E(m_i - k_i - \underline{k}_i)}{e^{r\Delta t}}}_{\text{The cost of the exercise at t+}\Delta t} +$$

$$\underbrace{+ \frac{m_i \cdot \left[FV + RFe^{r(T-t)} + \boldsymbol{a}(L_i + k_i)Ee^{r(T-t)} + \boldsymbol{a}(m - k_i - \underline{k}_i - L_i - \underline{L}_i)Ee^{r(T-t-\Delta t)}\right]}{(m+n)e^{r(T-t)}}}$$

Player i's share in the liquidation dividend to be paid at T

The second partial derivative is the same as the second partial derivative of the payoff function calculated above in (A2), for the decision point at T-Δt. Once again, the strict convexity of the payoff function implies that the best reply of player i to any vector of rival strategies is to exercise all of her remaining warrants at time t or all at $t+\Delta t$. The payoffs in any one of these actions are as follows:

When player i exercises all her remaining warrants at time $t+\Delta t$ (i.e. $k_i=0$) her payoff is:

$$\Pi^i(0, L_i) = \frac{\underline{k}_i(1-\boldsymbol{a})L_i E}{\underline{k}_i + L_i + \underline{L}_i + n} + \frac{m_i(1-\boldsymbol{a})(m - \underline{k}_i - L_i - \underline{L}_i)E}{(m+n)e^{r\Delta t}} + \tag{A8}$$

$$+ \frac{m_i \cdot \left[FV + RFe^{r(T-t)} + \boldsymbol{a}L_i Ee^{r(T-t)} + \boldsymbol{a}(m - \underline{k}_i - L_i - \underline{L}_i)Ee^{r(T-t-\Delta t)}\right]}{(m+n)e^{r(T-t)}} - \frac{E(m_i - \underline{k}_i)}{e^{r\Delta t}}$$

When player i exercises all her remaining warrants at time t (i.e. $k_i = m_i - \underline{k}_i$) her payoff is:

$$\Pi^i\left(m_i - \underline{k}_i, L_i\right) = \frac{m_i(1-\boldsymbol{a})(L_i + m_i - \underline{k}_i)E}{m_i + L_i + \underline{L}_i + n} - \tag{A9}$$

$$- E(m_i - \underline{k}_i) + \frac{m_i(1-\boldsymbol{a})(m - m_i - L_i - \underline{L}_i)E}{(m+n)e^{r\Delta t}} +$$

$$+ \frac{m_i \cdot \left[FV + RFe^{r(T-t)} + \boldsymbol{a}(L_i + m_i - \underline{k}_i)Ee^{r(T-t)} + \boldsymbol{a}(m - m_i - L_i - \underline{L}_i)Ee^{r(T-t-\Delta t)}\right]}{(m+n)e^{r(T-t)}}$$

Exercising the remaining warrants of player i at time $t+\Delta t$ would take place if and only if $\Pi^i\left(0, L_i\right) \geq \Pi^i\left(m_i - \underline{k}_i, L_i\right)$. The condition is similar to condition (A5):

$$\frac{\underline{k}_i L_i(1-\boldsymbol{a})}{\underline{k}_i + L_i + \underline{L}_i + n} + \frac{m_i\left[1 - \boldsymbol{a}(1+r)^{\Delta t}\right](m_i - \underline{k}_i)}{(m+n)e^{r\Delta t}} + \left(1 - e^{-r\Delta t}\right)(m_i - \underline{k}_i) \geq \tag{A10}$$

$$\geq \frac{m_i(L_i + m_i - \underline{k}_i)(1-\boldsymbol{a})}{m_i + L_i + \underline{L}_i + n}$$

Hence, it can be seen that, unless player i is the only remaining warrant holder in the market, she will exercise all her remaining warrants at t rather than at $t+\Delta t$. Thus, by induction, a sub-game perfect equilibrium is such that all the warrant holders exercise immediately at t=0. (Looking from the warrant holders collective point of view, this equilibrium is the worst possible solution.)

The only possible exception to this equilibrium is the possibility of a single warrant holder who exercises all her warrants at the expiration (time T), while the others exercise at t=0. This is potentially a possible equilibrium, since if there is just a single warrant holder left in the market, she will act as a monopoly warrant holder, and thus will wait for the expiration. We will now check the stability of the first type of sub-game perfect equilibrium.

In equilibrium, each warrant holder regards the other warrant holders' actions as given (namely L_i). In order for the first type of equilibrium to hold (i.e. all warrant holders exercise at $t=0$), no warrant holder should have incentive to deviate from the equilibrium by exercising at the expiration. In order to examine the equilibrium stability, the payoff to player i at each of the possible actions should be compared.

If i exercises her warrants at $t=0$ her payoff is (taking into account that $\underline{L}_i = 0$, $L_i = m - m_i$ and $k_i = m_i$):

$$\Pi^i(k_i, L_i) = \underbrace{\frac{m_i m(1-\boldsymbol{a})E}{m+n}}_{\substack{\text{Player } i\text{'s share in the} \\ \text{extraordinary} \\ \text{dividend paid at t=0}}} - \underbrace{Em_i}_{\substack{\text{The cost} \\ \text{of} \\ \text{exercise}}} + \underbrace{\frac{m_i \cdot \left[FV + m\boldsymbol{a}Ee^{rT}\right]}{(m+n)e^{rT}}}_{\substack{\text{Player } i\text{'s share in the} \\ \text{liquidation dividend} \\ \text{to be paid at T}}} \qquad \textbf{(A11)}$$

If i exercises her warrants at $t=T$ her payoff is (taking into account that $\underline{L}_i = 0$, $L_i = m - m_i$ and $k_i = 0$):

$$\Pi^i(k_i, L_i) = \underbrace{\frac{m_i\left[FV + Em_i + \boldsymbol{a}(m - m_i)Ee^{rT}\right]}{(m+n)e^{rT}}}_{\substack{\text{Player } i\text{'s share in the} \\ \text{dividend to be paid at T}}} - \underbrace{\frac{Em_i}{e^{rT}}}_{\substack{\text{The cost} \\ \text{of the} \\ \text{exercise at T}}} \qquad \textbf{(A12)}$$

In order for the first equilibrium to exist, her payoff in (A12) should not exceed her payoff in (A11), meaning that the following condition must hold:

$$m - n\left(\frac{e^{rT} - 1}{1 - \boldsymbol{a}e^{rT}}\right) \geq m_i \qquad \textbf{(A13)}$$

The above condition implies that the incentive to deviate from the equilibrium and wait for the expiration is greater for a player who initially holds a higher number of warrants.

Hence, in order that no warrant holder has an incentive to deviate, the following condition must hold:

$$m - n\left(\frac{e^{rT} - 1}{1 - \boldsymbol{a}\,e^{rT}}\right) \geq \max_i\left(m_i\right) \qquad \textbf{(A14)}$$

Condition (A14) guarantees that the second type of equilibrium (where all the warrant holders exercise at $t=0$ except one who exercises at the expiration) will not hold.

The possible exception to the first type of equilibrium is the possibility of a single warrant holder who exercises all her warrants at the expiration (time T), while the others exercise at $t=0$. Let i now be the player with the largest amount of warrants held initially in the game. The first condition for the existence of the second equilibrium is:

$$m - n\left(\frac{e^{rT} - 1}{1 - \boldsymbol{a}\,e^{rT}}\right) \leq m_i \qquad \textbf{(A15)}$$

Condition (A15) implies that player i should not deviate from her strategy of exercising all her warrants at $t=T$, knowing that all the other players exercise at $t=0$. The second condition should verify that all other players, knowing that player i waits for the expiration, will have incentive to exercise immediately. If one of these players (j) should deviate, she will exercise all her warrants at T-Δt, in order to postpone her exercise as much as she can, and get the alternative interest. Yet, by exercising before T she makes sure that she doesn't share the extraordinary dividend (paid by the firm because of her exercise) with player i.

Player j's payoff function (relative to $t=0$) if she chooses to exercise in $t=0$ is:

$$\Pi^j(0) = \underbrace{\frac{m_j}{m-m_i+n}(1-a)(m-m_i)E}_{\substack{\text{Player } j\text{'s share in} \\ \text{the extraordinary} \\ \text{dividend paid at t=0}}} - \underbrace{E \cdot m_j}_{\substack{\text{The cost of} \\ \text{the exercise} \\ \text{at t=0}}} + \underbrace{\frac{m_j\left[m_i E + FV + (m-m_i)Eae^{rT}\right]}{(m+n)e^{rT}}}_{\substack{\text{Player } j\text{'s share in the liquidation} \\ \text{dividend and the extraordinary} \\ \text{dividend to be paid at T}}} \quad \textbf{(A16)}$$

Player j's payoff function (relative to $t=0$) if she chooses to exercise in $t=T-\Delta t$ is:

$$\Pi^j(T-\Delta t) = \underbrace{\frac{m_j^2 E(1-a)}{(m-m_i+n)e^{r(T-\Delta t)}}}_{\substack{\text{Player } j\text{'s share in the} \\ \text{extraordinary dividend} \\ \text{paid at t=T-}\Delta t}} - \underbrace{\frac{E \cdot m_j}{e^{r(T-\Delta t)}}}_{\substack{\text{The cost of} \\ \text{the exercise} \\ \text{at t= T-}\Delta t}} + \quad \textbf{(A17)}$$

$$+ \underbrace{\frac{m_j\left[m_i E + FV + (m-m_i-m_j)Eae^{rT} + m_j Eae^{r\Delta T}\right]}{(m+n)e^{rT}}}_{\substack{\text{His share in the liquidation dividend and the} \\ \text{extraordinary dividend to be paid at T}}}$$

The second equilibrium exists only if $\Pi^j(0) \geq \Pi^j(T-\Delta t)$ for all players other than i.

Thus, in addition to condition (A15), the following condition must also hold:

$$\frac{m_j}{m-m_i+n}(1-a)(m-m_i)E - E \cdot m_j + \frac{m_j\left[m_i E + FV + (m-m_i)aEe^{rT}\right]}{(m+n)e^{rT}} \geq \quad \textbf{(A18)}$$

$$\geq \frac{m_j^2 E(1-a)}{(m-m_i+n)e^{r(T-\Delta t)}} - \frac{E \cdot m_j}{e^{r(T-\Delta t)}} +$$

$$+ \frac{m_j\left[m_i E + FV + (m-m_i-m_j)aEe^{rT} + m_j aEe^{r\Delta T}\right]}{(m+n)e^{rT}}$$

Taking the lim of condition ($A18$) where $\Delta t \to 0$ will derive:

$$m - m_i - n\frac{e^{rT} - 1}{1 - \boldsymbol{a}e^{rT}} \geq m_j \left\{ 1 - \frac{m_i \boldsymbol{a}\left[1 - e^{rT}\right]}{(m+n)\left[1 - \boldsymbol{a}e^{rT}\right]} \right\} \qquad \textbf{(A19)}$$

From integrating condition (A15) and ($A19$), it follows that:

$$m_j \left\{ 1 - \frac{m_i \boldsymbol{a}\left[1 - e^{rT}\right]}{(m+n)\left[1 - \boldsymbol{a}e^{rT}\right]} \right\} \leq 0 \qquad \textbf{(A20)}$$

Taking into account that $m - m_i > 0$, a necessary condition in order for condition (A15) to hold is that $0 < \boldsymbol{a}e^{rT} < 1$. However, when $0 < \boldsymbol{a}e^{rT} < 1$, condition (A20) doesn't hold, meaning that there is no sub-game equilibrium.

This completes the proof. ‖

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