Dry Markets and Statistical Arbitrage Bounds for European Derivatives

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Abstract

We derive statistical arbitrage bounds for the buying and selling price of European derivatives under incomplete markets. In this paper, incompleteness is generated due to the fact that the market is dry, i.e., the underlying asset cannot be transacted at certain points in time. In particular, we re…ne the notion of statistical arbitrage in order to extend the procedure for the case where dryness is random, i.e., at each point in time the asset can be transacted with a given probability. We analytically characterize several properties of the statistical arbitrage-free interval, show that it is narrower than the super-replication interval and dominates somehow alternative intervals provided in the literature. Moreover, we show that, for su¢ ciently incomplete markets, the statistical arbitrage interval contains the reservation price of the derivative.

1 Introduction

In complete markets and under the absence of arbitrage opportunities, the value of a European derivative must be the same as the cheapest portfolio that replicates exactly its value at any given point in time. However, in the presence of some market imperfections, markets may become incomplete, and it is not possible to exactly replicate the value of the European derivative at all times anymore. Nevertheless, it is possible to derive an arbitrage-free range of variation for the value of the derivative. This interval depends on two different factors. First, on the nature of market incompleteness; second, on the notion of arbitrage opportunities.

In what follows we consider that market incompleteness is generated by the fact that agents cannot trade the underlying asset on which the derivative is written whenever they please. In fact, and as opposed to the traditional asset pricing assumptions, markets are very rarely liquid and immediacy is not always available. As Longstaff (1995, 2001, 2004) recalls, the relevance of this fact is pervasive through many financial markets. The markets for many assets such as human capital, business partnerships, pension plans, saving bonds, annuities, trusts, inheritances and residential real estate, among others, are generally very illiquid and long periods of time (months, sometimes years) may be required to sell an asset. This point becomes extremely relevant for the case of option pricing when we consider that it is an increasingly common phenomenon even in well-established securities markets, as illustrated by the 1998 Russian default crisis, leading many traders to be trapped in risky positions they could not unwind.

To address the impact of this issue on derivatives' pricing, we consider a discrete-time setting such that transactions are not possible within a subset of points in time. Although clearly very stylized, the advantage of this setting is that it incorporates in a very simple way the notion of market illiquidity as the absence of immediacy. Under such illiquidity we say that markets are dry. In this framework, dryness changes what is otherwise a complete market into a dynamically incomplete market. This was also the approach in Amaro de Matos and Antão (2001) when characterizing the specific superreplication bounds for options in such markets and its implications. We further extend this setting by assuming that transactions occur at each point in time with a given probability, reflecting a more realistic ex-ante uncertainty about the market.

As stressed above, there is not a unique arbitrage-free value for a derivative under market incompleteness. However, for any given derivative, portfolios can be found that have the same payoff as the derivative in some states of nature, and higher payoffs in the other states. Such portfolios are said to be superreplicating. Holding such a portfolio should be worth more than the option itself and therefore, the value of the cheapest of such portfolios should be seen as an upper bound on the selling value of the option. Similarly, a lower bound for the buying price can be constructed. The nature of the superreplicating bounds is well characterized in the context of incomplete markets in the papers by El Karoui and Quenez (1991,1995), Edirisinghe, Naik and Uppal (1993) and Karatzas and Kou (1996). The superreplicating bounds establish the limits of the interval for arbitrage-free value of the option. If the price is outside this range, then a positive profit is attainable with probability one. Therefore, the equilibrium prices at which the derivative is transacted should lie between those bounds.

Most of the times, however, these superreplicating bounds are trivial, in the sense that they are too broad, not allowing a useful characterization of equilibrium prices' vicinity. As an alternative, Bernardo and Ledoit (2000) propose a utility-based approach, restricting the no-arbitrage condition to rule out investment opportunities offering high gain-loss ratios, where gain (loss) is the expected positive (negative) part of excess payoff. In this way, narrower bounds are obtained. Analogously, Cochrane and Saá-Requejo (2000) also restrict the no-arbitrage condition by not allowing transactions of "good deals", i.e. assets with very high Sharpe ratio. Following Hansen and Jagannathan (1991), they show that this restriction imposes an upper bound on the pricing kernel volatility and leads to narrower pricing implications when markets are incomplete.

Given a set of pricing kernels compatible with the absence of arbitrage opportunities, Cochrane and Sáa-Requejo exclude pricing kernels implying very high Sharpe ratios, whereas Bernardo and Ledoit exclude pricing kernels implying very high gain-loss ratios for a benchmark utility. Notice that, for a different utility, Bernardo and Ledoit would exclude a different subset of pricing kernels, for the same levels of acceptable gain-loss ratios. Also notice that the interval designed by Cochrane and Sáa-Requejo is not necessarily arbitrage free, and therefore does not necessarily contain the equilibrium price.

In order to avoid ad-hoc thresholds in either Sharpe or gain-loss ratios, or to make some parametric assumptions about a benchmark pricing kernel, as in Bernardo and Ledoit (2000), the work of Bondarenko (2003) introduces the notion of statistical arbitrage opportunity, by imposing a weak assumption on a functional form of admissible pricing kernels, yielding narrower pricing implications as compared to the superreplication bounds. A statistical arbitrage opportunity is characterized as a zero-cost trading strategy for which (i) the expected payoff is positive, and (ii) the conditional expected payoff in each final state of the economy is nonnegative. Unlike a pure arbitrage opportunity, a statistical arbitrage opportunity may allow for negative payoffs, provided that the average payoff in each final state is nonnegative. In particular, ruling out statistical arbitrage opportunities imposes a novel martingale-type restriction on the dynamics of securities prices. The important properties of the restriction are that it is model-free, in the sense that it requires no parametric assumptions about the true equilibrium model, and continues to hold when investors' beliefs are mistaken. Although Bondarenko's interval can be shown to be in the arbitrage-free region, it does not necessarily contain the equilibrium value of the derivative.

In this paper we extend the notion of statistical arbitrage opportunity to the case where the underlying asset can be transacted at each point in time with a given probability, and compare the statistical arbitrage-free bounds with the superreplication bounds. We show that the statistical arbitrage-free interval is narrower than the pure arbitrage bounds, and show also that, for sufficiently incomplete markets (probability not too close to 1), the statistical arbitrage interval contains the reservation price of the derivative. We also provide examples that allow comparison with the results of Cochrane and Saá-Requejo (2000) and discuss the comparison with Bernardo and Ledoit (2000).

This paper is organized as follows. In section 2, the model is presented and the pure arbitrage results are derived. In section 3 the notion of statistical arbitrage in the spirit of Bondarenko (2003) is defined. In section 4, the main results are presented. In Section 5 we first characterize the reservation prices and then show that, in a sufficiently dry market, they are contained in the statistical arbitrage interval. In Section 6 we illustrate how the statistical arbitrage-free interval somehow dominates alternative intervals provided in the literature. In section 7 some numerical examples are presented in order to illustrate some important properties of the bounds. In the last section several conclusions are presented.

2 The Model

Consider a discrete-time economy with T periods, with a risky asset and a riskless asset. At each point in time the price of the risky asset can be multiplied either by U or by D to get the price of the next point in time. Equilibrium requires that $U > R > D$, where R denotes one plus the risk-free interest rate. At time $t = 0$ and $t = T$ transactions are certainly possible. However, at $t = 1, \dots, T - 1$ there is uncertainty about the possibility of transaction of the risky asset. Transactions will occur with probability p at each of these points in time. A European Derivative with maturity T is considered.

Consider the Binomial tree process followed by the price of the risky asset. Let the set of nodes at date t be denoted by I_t; and let each of the $t + 1$ elements of I_t be denoted by $i_t = 1, \dots, t + 1$. For any $t' < t$, let $I_{t'}^{i_t}$ denote the set of all the nodes at time t' that are predecessors of a given node i_t. A path on the event tree is a set of nodes $w = \cup_{t \in \{0,1,\dots,T\}} i_t$ such that each element in the union satisfies $i_{t-1} \in I_{t-1}^{i_t}$. Let Ω denote the set of all paths on the event tree.

The payoffs of a European derivative, at each terminal node, will be denoted $G_T^{i_T}$. At each node i_t, the stock price is given by $S_t^{i_t} = U^{t+1-i_t} D^{i_t-1} S_0$. Moreover, at each node i_t, there is a number $\Delta_t^{i_t}$ representing the number of shares bought (or sold, if negative), and a number $B_t^{i_t}$ denoting the amount invested (or borrowed, if negative) in the risk-free asset. Hence, at t there are $t+1$ values of $\Delta_t^{i_t}$, composing a vector $\Delta_t \equiv (\Delta_t^1, \ldots, \Delta_t^{t+1}) \in \mathbb{R}^{t+1}$. Similarly, we construct the vector $B_t \equiv (B_t^1, \ldots, B_t^{t+1}) \in \mathbb{R}^{t+1}$.

Definition 1 A trading strategy is a portfolio process $\theta_t = (\Delta_t, B_t)$, composed of Δ_t units of the risky asset and an amount B_t invested in the riskless asset, such that the portfolio's cost is $\Delta_t S_t + B_t$ for $t = 0, 1, \ldots T-1$.

In order to find the upper (lower) bound of the arbitrage-free range of variation for the value of a European derivative we consider a financial institution that wishes to be fully hedged when selling (buying) that derivative. The objective of the institution is to minimize (maximize) the cost of replicating the exercise value of the derivative at maturity. The value determined under such optimization procedure avoids what is known as arbitrage opportunities, reflecting the possibility of certain profits at zero cost.

This section is organized as follows. We first characterize the upper bound, and then the lower bound for the interval of no-arbitrage admissible prices. For each bound, we first deal with the complete market case, and then with the fully incomplete market case, finally introducing random incompleteness.

2.1 The upper bound in the case $p = 0$ and $p = 1$.

First, we present the well-known case where $p = 1$. The usual definition of an arbitrage opportunity in our economy is as follows.

Definition 2 (Pure Arbitrage in the case $p = 1$) In this economy, an arbitrage opportunity is a zero cost trading strategy θ_t such that

1. the value of the portfolio is positive at any final node, i.e., $\Delta_{T-1}^{i_{T-1}} S_T^{i_T} + R B_{T-1}^{i_{T-1}} \geq 0$, for any $i_{T-1} \in I_{T-1}^{i_T}$ and all $i_T \in I_T$; and

2. the portfolio is self-financing, i.e., $\Delta_{t-1}^{i_{t-1}} S_t^{i_t} + R B_{t-1}^{i_{t-1}} \geq \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t}$, for any $i_{t-1} \in I_{t-1}^{i_t}$, all $i_t \in I_t$ and all $t \in \{0, \ldots, T-1\}$.

The upper bound for the value of the European option is the maximum value for which the derivative can be transacted, without allowing for arbitrage opportunities. This is the value of the cheapest portfolio that the seller of the derivative can buy in order to completely hedge his position against the exercise at maturity, without the need of additional financing at any rebalancing dates. Hence, for $p = 1$; the upper bound is C_u^1; given by

$$C_u^1 = \min_{\{\Delta_t, B_t\}_{t=0,\ldots,T-1}} \Delta_0 S_0 + B_0$$

subject to $\Delta_{T-1}^{i_{T-1}} S_T^{i_T} + R B_{T-1}^{i_{T-1}} \geq G_T^{i_T}$; with $i_{T-1} \in I_{T-1}^{i_T}$ and all $i_T \in I_T$; and the self-financing constraints $\Delta_{t-1}^{i_{t-1}} S_t^{i_t} + R B_{t-1}^{i_{t-1}} \geq \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t}$; for all $i_{t-1} \in I_{t-1}^{i_t}$; all $i_t \in I_t$ and all $t \in \{0, \ldots, T-1\}$; where the constraints reflect the absence of arbitrage opportunities. This problem leads to the familiar result

$$C_u^1 = \frac{1}{R^T} \sum_{j=0}^{T} \binom{T}{j} \left(\frac{R-D}{U-D} \right)^j \left(\frac{U-R}{U-D} \right)^{T-j} G_T^{T+1-j}.$$

Consider now the case where $p = 0$. In this case, the notion of a trading strategy satisfying the self-financing constraint is innocuous, since the portfolio θ_t cannot be rebalanced during the life of the option. Under the absence of arbitrage opportunities, the upper bound for the value is C_u^0 satisfying

$$C_u^0 = \min_{\{\Delta_0, B_0\}} \Delta_0 S_0 + B_0$$

subject to $\Delta_0 S_T^{i_T} + R^T \Delta_0 \geq G_T^{i_T}$; for all $i_T \in I_T$: In this case, the bound C_u^0 can be shown to solve the maximization problem on a set of positive constants $\{\theta_{i_T}\}$; with $\sum_{i_T=1}^{T+1} \theta_{i_T} = 1$;

$$C_u^0 = \max_{\theta_{i_T}} \frac{1}{R^T} \sum_{i_T=1}^{T+1} \theta_{i_T} G_T^{i_T}$$

subject to

$$S_0 = \frac{1}{R^T} \sum_{i_T=1}^{T+1} \theta_{i_T} S_T^{i_T};$$

For instance, if a call option with exercise K is considered, we have[1]

$$C_u^0 = \frac{1}{R^T} \left[\left(\frac{R^T - D^T}{U^T - D^T} \right) \left(U^T S_0 - K \right)^+ + \left(\frac{U^T - R^T}{U^T - D^T} \right) \left(D^T S_0 - K \right)^+ \right].$$

[1] See Amaro de Matos and Antão (2001).

2.2 The lower bound in the case $p = 0$ and $p = 1$:

The lower bound for the value of an American derivative is the minimum value for which the derivative can be transacted without allowing for arbitrage opportunities. This is the value of the most expensive portfolio that the buyer of the option can sell in order to be fully hedged, and without the need of additional financing at rebalancing dates.

For $p = 1$, the lower bound for the value of the derivative under the absence of arbitrage opportunities is thus C_l^1; given by

$$C_l^1 = \max_{\{\Delta_t; B_t\}_{t=0;\dots;T-1}} \Delta_0 S_0 + B_0$$

subject to $\Delta_{T-1}^{i_T} S_T^{i_T} + R B_{T-1}^{i_T} \le G_T^{i_T}$, with $i_{T-1} \in I_{T-1}^{i_T}$ and all $i_T \in I_T$; and the self-financing constraints $\Delta_{t-1}^{i_t} S_t^{i_t} + R B_{t-1}^{i_t} \le \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t}$; for all $i_{t-1} \in I_{t-1}^{i_t}$; all $i_t \in I_t$ and all $t \in \{0; \dots; T-1\}$; where the constraints reflect the absence of arbitrage opportunities This problem leads to the familiar result

$$C_l^1 = \frac{1}{R^T} \sum_{j=0}^{T} \binom{T}{j} \left(\frac{R-D}{U-D}\right)^j \left(\frac{U-R}{U-D}\right)^{T-j} G_T^{T+1-j}; \tag{1}$$

that coincides with the solution obtained for C_u^1:

In the case where $p = 0$; the lower bound for the value of the derivative is C_l^0; satisfying

$$C_l^0 = \max_{\{\Delta_0; B_0\}} \Delta_0 S_0 + B_0$$

subject to $\Delta_0 S_T + R^T B_0 \le G_T^{i_T}$: As above, it follows that, for a set of positive constants $\{\theta_{i_T}\}$; with $\sum_{i_T=1}^{T+1} \theta_{i_T} = 1$; this bound is given by

$$C_u^0 = \min_{\theta_{i_T}} \frac{1}{R^T} \sum_{i_T=1}^{T+1} \theta_{i_T} G_T^{i_T}$$

subject to

$$S_0 = \frac{1}{R^T} \sum_{i_T=1}^{T+1} \theta_{i_T} S_T^{i_T}:$$

In the case of a call option with exercise K, we have[2]

$$C_l^0 = \frac{1}{R^T}\left(\frac{R^T - U^{T-(i+1)}D^{i+1}}{U^{T-i}D^i - U^{T-(i+1)}D^{i+1}}\right)\left(U^{T-i}D^iS_0 - K\right)^+ \qquad (2)$$
$$+\frac{1}{R^T}\left(\frac{U^{T-i}D^i - R^T}{U^{T-i}D^i - U^{T-(i+1)}D^{i+1}}\right)\left(U^{T-i-1}D^{i+1}S_0 - K\right)^+ :$$

where i is defined as the unique integer satisfying $U^{n-(i+1)}D^{i+1} < R^n < U^{n-i}D^i$, and $0 \le i \le n-1$:

2.3 The Bounds on Probabilistic Markets

In the aforementioned cases we considered the cases where either $p = 0$ or $p = 1$. However, if p is not equal to neither 0 nor 1, the formulation has to be adjusted. If the risky asset can be transacted with a given probability $p \in (0;1)$, then the usual definition of arbitrage opportunity reads as follows.

Definition 3 (Pure Arbitrage for $p \in (0;1)$) In this economy, an arbitrage opportunity is a zero cost trading strategy such that

1. the value of the portfolio is positive at any final node, i.e.,

$$\Delta_t^{i_t}S_T^{i_T} + R^{T-t}B_t^{i_t} \ge 0;$$

$i_t \in I_t^{i_T}$ and all $i_T \in I_T$; and the self-financing constraints

2. the portfolio is self-financing, i.e.,

$$\Delta_{t-j}^{i_{t-j}}S_t^{i_t} + R^{j}B_{t-j}^{i_{t-j}} \ge \Delta_t^{i_t}S_t^{i_t} + B_t^{i_t};$$

for all $i_{t-j} \in I_{t-j}^{i_t}$; all $i_t \in I_t$ and all $t \in \{0;:::;T-1\}$:

The upper bound C_u^p is the solution of the following problem:

$$C_u^p = \min_{\{\Delta_t;B_t\}_{t=0;:::;T-1}} \Delta_0S_0 + B_0$$

where $\Delta_t; B_t \in R^{t+1}; t = 0;:::;T-1$; subject to the superreplicating conditions $\Delta_t^{i_t}S_T^{i_T} + R^{T-t}B_t^{i_t} \ge G_T^{i_T}$; with $i_t \in I_t^{i_T}$ and all $i_T \in I_T$; and the self-financing constraints $\Delta_{t-j}^{i_{t-j}}S_t^{i_t} + R^{j}B_{t-j}^{i_{t-j}} \ge \Delta_t^{i_t}S_t^{i_t} + B_t^{i_t}$ for all $i_{t-j} \in I_{t-j}^{i_t}$; all $i_t \in I_t$ and all $t \in \{0;:::;T-1\}$:

On the other hand, the lower bound C_l^p solves the following problem:

[2] See Amaro de Matos and Antão (2001).

$$C_l^p = \max_{\{\Delta_t; B_t\}_{t=0,\ldots,T-1}} \Delta_0 S_0 + B_0$$

where $\Delta_t, B_t \in \mathbb{R}^{t+1}, t = 0, \ldots, T-1$, subject to the conditions $\Delta_t^{i_t} S_T^{i_T} + R^{T-t} B_t^{i_t} \le G_T^{i_T}$, with $i_t \in I_t^{i_T}$ and all $i_T \in I_T$, and the self-financing constraints $\Delta_{t-1}^{i_{t-1}} S_t^{i_t} + R B_{t-1}^{i_{t-1}} \le \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t}$ for all $i_{t-1} \in I_{t-1}^{i_t}$, all $i_t \in I_t$ and all $t \in \{0, \ldots, T-1\}$.

Notice that the constraints in the above optimization problems are implied by the absence of arbitrage opportunities and do not depend on the probability p.[3] Therefore, neither C_u^p nor C_l^p will depend on p. We are now in conditions to relate these values to C_u^0 and C_l^0 as follows.

Theorem 4 For $p \in (0, 1)$ the upper and lower bound for the prices above do not depend on p. The optimization problems above lead to the same solutions as when $p = 0$.

Proof. Consider first the case of the upper bound. The constraints characterizing C_u^p include all the constraints characterizing C_u^0. Thus, $C_u^p \ge C_u^0$. Now, let Δ_0^0 and B_0^0 denote the optimal values invested, at time $t = 0$, when $p = 0$. The trading strategy $\Delta_t^p = \Delta_0^0$ and $B_t^p = R^t B_0^0$, for all $t = 1, \ldots, T-1$, is an admissible strategy for any given p, hence $C_u^p = C_u^0$. The case of the lower bound is analogous. ■

The intuition for this result is straightforward. The upper (lower) bound of the European derivative remains the same as when $p = 0$, because with probability $1 - p$ it would not be possible to transact the stock at each point in time. In order to be fully hedged, as required by the absence of arbitrage opportunities, the worse scenario will be restrictive in spite of its possibly low probability. The fact that no intermediate transactions may occur dominates all other possibilities.

The above result is strongly driven by the definition of arbitrage opportunities. Nevertheless, if this notion is relaxed in an economic sensible way, a narrower arbitrage-free range of variation for the value of the European derivative may be obtained, possibly depending now on p. This is the subject of the rest of the paper.

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[3] This happens since, in order to have an arbitrage opportunity, we must ensure that, whether market exists or not at each time $t \in \{1, \ldots, T-1\}$, the agent will never lose wealth. Therefore, the optimization problem cannot depend on p.

3 Statistical Arbitrage Opportunity

Consider the economy described in the previous section. Let $T_p = \{1, ..., T - 1\}$ denote the set of points in time. At each of these points there is market with probability p; and there is no market with probability $1 - p$. The existence (or not) of the market at time t corresponds to the realization of a random variable y_t that assumes the value 0 (when there is no market) and 1 (when there is market). This random variable is defined for all $t \in T_p$ and it is assumed to be independent of the ordinary source of uncertainty that generates the price process. We can therefore talk about a market existence process. In order to construct one such process, let us start with the state space. Let $\#(T_p)$ denote the number of points in T_p. At each of these points, market may either exist or not exist, leading to $2^{\#(T_p)}$ possible states of nature. We then have the collection of possible states of nature denoted by $\hat{\Omega} = \{\omega_i\}_{i=1,...,2^{\#(T_p)}}$; each ω_i corresponding to a distinct state. Moreover, let $\hat{F} = \hat{F}_1, ..., \hat{F}_{T-1}$; where $\hat{F}_t$ is the σ-algebra generated by the random variable y_t. Let p_y be the probability associated with the random variable y_t. For all $t \in T_p$; we have $p_y(y_t = 1) = p$ and $p_y(y_t = 0) = 1 - p$.

3.1 The expected value of a portfolio

We now construct a random variable that allows to construct the expected future value of a portfolio in this setting. For $t < t'$; let $x_{t;t'}$ be a random variable identifying the last time that transactions take place before date t'; given that we are at time t; and transactions are currently possible. Let $\hat{\Omega}_t$ be the subset of $\hat{\Omega}$ such that $\hat{\Omega}_t = \left\{\omega_i \in \hat{\Omega} : y_t(\omega_i) = 1\right\}$. Then,

$$x_{t;t'} : \hat{\Omega}_t \to \left\{t, ..., t' - 1\right\}$$

Let $p_{x_{t;t'}}$ be the probability associated with $x_{t;t'}$. Then,

$$p_{x_{t;t'}}\left(x_{t;t'} = t\right) = (1 - p)^{t' - t - 1}.$$

Moreover, for a given $s \in (t; t')$;

$$p_{x_{t;t'}}\left(x_{t;t'} = s\right) = p(1 - p)^{t' - s - 1}.$$

Also note that

$$\sum_{s=t}^{t'-1} p_{x_{t;t'}}\left(x_{t;t'} = s\right) = (1 - p)^{t' - t - 1} + \sum_{s=t}^{t'-1} p(1 - p)^{t' - s - 1} = 1;$$

as it should.

Consider a given trading strategy $(\phi_t; B_t)_{t=0,\ldots,T}$; where $(\phi_t; B_t) \equiv (\phi_t^{i_t}; B_t^{i_t})_{i_t \in I_t}$ is a $(t+1)-$dimensional vector. Consider a given path w and $(i_s)_{s \in \{t,\ldots,T\}} \subset$ w. Suppose that the agent is at a given node i_t; where rebalancing is possible. As there is uncertainty about the existence of market at the future points in time, there is also uncertainty about the portfolio that the agent will be holding at a future node $i_{t'}$. In fact, the portfolio at $i_{t'}$ may be any of $\left(\phi_t^{i_t}; B_t^{i_t}\right)$; $\left(\phi_{t+1}^{i_{t+1}}; B_{t+1}^{i_{t+1}}\right)$; $\ldots$; or $\left(\phi_{t'-1}^{i_{t'-1}}; B_{t'-1}^{i_{t'-1}}\right)$; where $(i_s)_{s \in \{t,\ldots,t'-1\}} \subset$ w.

Clearly, the expected value of a given trading strategy at node $i_{t'}$, given that we are at node i_t; is

$$E_{i_t}^{p_{x_{t;t'}}}\left[\phi_x^{i_x} S_{t'}^{i} + R^{t'-x} B_x^{i_x}\right] = \sum_{s=t,\ldots,t'-1} p_{x_{t;t'}}\left(x_{t;t'} = s\right)\left[\phi_s^{i_s} S_{t'}^{i} + R^{t'-s} B_s^{i_s}\right];$$

where we use x to short notation for $x_{t;t'}$.

3.2 Statistical versus pure arbitrage

A pure arbitrage opportunity is a zero-cost portfolio at time t, such that the value of each possible portfolio at node i_T is positive, i.e.,

$$\phi_{t+j}^{i_{t+j}} S_T^{i_T} + R^{T-t-j} B_{t+j}^{i_{t+j}} \geq 0$$

for all i_{t+j} such that i_t is a predecessor, $j = 0, 1, \ldots, T - t - 1$ and

$$E_{i_t}^{p_{x_{t;T}}}\left[\phi_x^{i_x} S_T^{i_T} + R^{T-x} B_x^{i_x}\right] > 0;$$

together with the self-financing constraints

$$\phi_t^{i_t} S_{t+j}^{i_{t+j}} + R^{j-t} B_t^{i_t} \geq \phi_{t+j}^{i_{t+j}} S_{t+j}^{i_{t+j}} + B_{t+j}^{i_{t+j}};$$

for all i_{t+j} such that i_t is a predecessor, and $j = 1, \ldots, T - t - 1$.

If statistical arbitrage is considered, however, an arbitrage opportunity requires only that, at node i_T; the expected value of the portfolio at T is positive,

$$E_{i_t}^{p_{x_{t;T}}}\left[\phi_x^{i_x} S_T^{i_T} + R^{T-x} B_x^{i_x}\right] \geq 0;$$

together with weaker self-financing conditions. Let us regard these latter conditions in some detail.

Suppose that we are at a given node i_t. If there is market at the next point in time we then have, for sure, the portfolio $\left(\theta_t^{i_t}, B_t^{i_t}\right)$ at time $t+1$. Hence, if node i_{t+1} is reached, the self-financing condition is

$$\theta_t^{i_t} S_{t+1}^{i_{t+1}} + R B_t^{i_t} - \left(\theta_{t+1}^{i_{t+1}} S_{t+1}^{i_{t+1}} + B_{t+1}^{i_{t+1}}\right) \geq 0$$

Consider now that $t+2$ is reached. At time t there is uncertainty about the existence of the market at time $t+1$. Hence, at time $t+2$ we can either have the portfolio $\left(\theta_t^{i_t}, B_t^{i_t}\right)$ or the portfolio $\left(\theta_{t+1}^{i_{t+1}}, B_{t+1}^{i_{t+1}}\right)$. Under the concept of statistical arbitrage, we want to ensure that, in expected value, we are not going to lose at node i_{t+2}. Hence, the self-financing condition becomes

$$\sum_{s=t,t+1} p_{x_{t,t+2}}\left(x_{t,t+2}=s\right)\left(\theta_s^{i_s} S_{t+2}^{i_{t+2}} + R^{t+2-s} B_s^{i_s}\right) \geq \left(\theta_{t+2}^{i_{t+2}} S_{t+2}^{t+2} + B_{t+2}^{i_{t+2}}\right)$$

More generally, for any t at which transaction occurs and $t < t' < T$, the statistical self-financing condition becomes

$$E_{i_t}^{p_{x_{t,t'}}}\left[\theta_x^{i_x} S_{t'}^{i_{t'}} + R^{t'-x} B_x^{i_x}\right] \geq \left(\theta_{t'}^{i_{t'}} S_{t'}^{i_{t'}} + B_{t'}^{i_{t'}}\right)$$

Definition 5 [4] A statistical arbitrage opportunity is a zero-cost trading strategy for which

1. At any node i_t, the expected value of the portfolio at any final node is positive, i.e.,

$$E_{i_t}^{p_{x_{t,T}}}\left[\theta_{x_{t,T}}^{i_x} S_T^{i} + R^{T-x} B_{x_{t,T}}^{i_x}\right] \geq 0$$

[4] This notion of Arbitrage Opportunity is in the spirit of Bondarenko (2003). In his definition 2, a Statistical Arbitrage Opportunity (SAO) is defined as a zero-cost trading strategy with a payoff $Z_T = Z(F_T)$, such that

(i) $E[Z_T | F_0] > 0$; and
(ii) $E[Z_T | F_0; \xi_T] \geq 0$; for all ξ_T;

where ξ_t denotes the state of the Nature at time t; and $F_t = (\xi_1, \ldots, \xi_t)$ is the market information set, with $F_0 = \emptyset$. Also, the second expectation is taken at time $t = 0$ and is conditional to the terminal state ξ_T. However, notice that eliminating SAO's at time $t = 0$ does not imply the absence of SAO's at future times $t \in [1, T-1]$. Hence, in order to incorporate a dynamically consistent absence of SAO's, we refine the definition of a SAO as a zero-cost trading strategy with a payoff $Z_T = Z(F_T)$, such that

(i) $E[Z_T | F_0] > 0$; and
(ii) $E[Z_T | F_t; \xi_T] \geq 0$; for all ξ_T and all $t \in [0, T-1]$.

for any $i_t \in I_t$ and $t \in \{0, 1, \ldots, T-1\}$; and

2. The portfolio is statistically self-financing, i.e:,

$$E_{i_t}^{p_{x_{t,t'}}}\left[\delta_{x_{t,t'}}^{i_x} S_{t'}^{i} + R^{t'-x} B_{x_{t,t'}}^{i_x} - \left(\delta_{t'}^{i_{t'}} S_{t'}^{i_{t'}} + B_{t'}^{i_{t'}}\right)\right] \geq 0$$

for any $i_t \in I_t$; $t' > t$, $t \in \{0, 1, \ldots, T-2\}$ and $t' \in \{1, \ldots, T-1\}$.

The two definitions of arbitrage are related in the following.

Theorem 6 If there are no statistical arbitrage opportunities, then there are no pure arbitrage opportunities.

Proof. If there is a pure arbitrage opportunity then the inequalities present in the definition of arbitrage opportunity, definition 3, are respected. Hence, as these expressions are the terms under expectation in the definition of Statistical Arbitrage opportunity, presented in definition 5, there is also a statistical arbitrage opportunity. ■

The set of portfolios that represent a pure arbitrage opportunity is a subset of the portfolios that represent a statistical arbitrage opportunity, i.e., there are portfolios that, in spite of not being a pure arbitrage opportunity, represent a statistical arbitrage opportunity.

In order to have a statistical arbitrage opportunity it is not necessary (although it is sufficient) that the value of the portfolio at the final date is positive. It is only necessary that, for all t, the expected value of the portfolio at the final date is positive.

Consider now the self-financing conditions under statistical arbitrage. When rebalancing the portfolio it is not necessary (although it is sufficient) that the value of the new portfolio is smaller than the value of the old one. This happens because future rebalancing is uncertain, leading to uncertainty about the portfolio that the agent will be holding in any future moment. In order to avoid a statistical arbitrage opportunity it is only necessary that the expected value of the portfolio at a given point in time is larger than the value of the rebalancing portfolio.

Finally, notice that the concept of statistical arbitrage opportunity reduces to the usual concept of arbitrage opportunity in the limiting case $p = 0$.

3.3 Augmented measures

For technical reasons, we now define an augmented probability space Q on Ω. In order to do that, we define a semipath m from i_t to $i_{t'}$, which is a set of nodes $m = \cup_{k\in\{t,\ldots,t'\}} i_k$ such that $i_k \in I_k^{i_{k+1}}$. Let $\Omega^{+}_{i_t;i_{t'}}$ denote the set of semipaths from i_t to $i_{t'}$.

Definition 7 An augmented probability space in Ω is a set of probabilities $q^{(i_T;T);m}_{(i_t;t)}$ such that $i_t \in I_t$; $m \in \Omega^{+}_{i_t;i_T}$, $t = 0;\ldots;T$ and

$$\sum_{i_T}\sum_{t=0}^{T-1}\sum_{i_t}\sum_{m} q^{(i_T;T);m}_{(i_t;t)} = 1;$$

Definition 8 A modified martingale probability measure is an augmented probability measure $Q \in \mathcal{Q}$ which satisfies

(i)

$$S_0 = \frac{1}{R^T}\sum_{\{i_T \in I_T\}} q^{i_T} S_T^{i_T}$$

where

$$q^{i_T} = \sum_{t=0}^{T-1}\sum_{\left\{i_t : i_t \in I_t^{i_T}\right\}}\sum_{\left\{m \in \Omega^{+}_{i_t;i_T}\right\}} q^{(i_T;T);m}_{(i_t;t)} S_T^{i_T};$$

(ii)

$$S_{T-1}^{i_{T-1}} = \frac{1}{R}\sum_{\left\{i_T : i_{T-1} \in I_{T-1}^{i_T}\right\}} \pi^{(i_T;T);m}_{(i_t;t)} S_T^{i_T}$$

with

$$\sum_{\left\{i_T : i_{T-1} \in I_{T-1}^{i_T}\right\}} \pi^{(i_T;T);m}_{(i_t;t)} = 1$$

and

$$\pi^{(i_T;T);m}_{(i_t;t)} = \frac{1}{\Psi}\sum_{t=0}^{T-1} p_{x_{t;T}}(x_{t;T} = T-1)\sum_{\left\{i_t : i_t \in I_t^{i_{T-1}}\right\}}\sum_{\left\{m \in \Omega^{+}_{i_t;i_T} : i_{T-1} \in m\right\}} q^{(i_T;T);m}_{(i_t;t)}$$

where

$$\Psi = \sum_{t=0}^{T-1} p_{x_{t;T}}(x_{t;T} = T-1)\sum_{\left\{i_T : i_{T-1} \in I_{T-1}^{i_T}\right\}}\sum_{\left\{i_t : i_t \in I_t^{i_{T-1}}\right\}}\sum_{\left\{m \in \Omega^{+}_{i_t;i_T} : i_{T-1} \in m\right\}} q^{(i_T;T);m}_{(i_t;t)};$$

(iii) there exists $\left\{\alpha^{(i_{t'},t'),m}_{(i_t,t)}\right\}$ for all $i_{t'} \in I_{t'}$; $i_t \in I_t$; $m \in \Omega^+_{i_t,i_T}$ and $t' > t$ for all $t = 0, \ldots, T-1$ such that, for all $0 < k < T$,

$$S_k^{i_k} = \frac{1}{R^{T-k}} \sum_{\left\{i_T : i_k \in I_k^{i_T}\right\}} \theta^{(i_T,T),m}_{(i_t,t)} S_T^{i_T} + \sum_{t'>k} \frac{1}{R^{t'-k}} \varepsilon^{(i_{t'},t'),m}_{(i_t,t)} S_{t'}^{i_{t'}},$$

where

$$\sum_{\left\{i_T : i_k \in I_k^{i_T}\right\}} \theta^{(i_T,T),m}_{(i_t,t)} + \sum_{t'>k} \varepsilon^{(i_{t'},t'),m}_{(i_t,t)} = 1$$

and

$$\theta^{(i_T,T),m}_{(i_t,t)} = \frac{1}{\Theta} \sum_{t=0}^{k} p_{x_{t,T}}(x_{t,T} = k) \sum_{\left\{i_t : i_t \in I_t^{i_k}\right\}} \sum_{\left\{m \in \Omega^+_{i_t,i_T} : i_k \in m\right\}} q^{(i_T,T),m}_{(i_t,t)}$$

$$\varepsilon^{(i_{t'},t'),m}_{(i_t,t)} = \frac{1}{\Theta} \sum_{t<k} p_{x_{t,t'}} \left(x_{t,t'} = k\right) \sum_{\left\{i_t : i_t \in I_t^{i_k}\right\}} \sum_{\left\{m \in \Omega^+_{i_t,i_{t'}} : i_k \in m\right\}} \alpha^{(i_{t'},t'),m}_{(i_t,t)}$$

with

$$\Theta = \sum_{\left\{i_T : i_k \in I_k^{i_T}\right\}} \sum_{t=0}^{k} p_{x_{t,T}}(x_{t,T} = k) \sum_{\left\{i_t : i_t \in I_t^{i_k}\right\}} \sum_{\left\{m \in \Omega^+_{i_t,i_T} : i_k \in m\right\}} q^{(i_T,T),m}_{(i_t,t)} + \sum_{t'>k} \sum_{t<k} p_{x_{t,t'}} \left(x_{t,t'} = k\right) \sum_{\left\{i_t : i_t \in I_t^{i_k}\right\}} \sum_{\left\{m \in \Omega^+_{i_t,i_{t'}} : i_k \in m\right\}} \alpha^{(i_{t'},t'),m}_{(i_t,t)}.$$

We denote by Q_S the set of modified martingale probability measure. Such measures will help writing down the upper and lower bounds for the value of European derivatives under the absence of statistical arbitrage opportunities.

4 Main Results

4.1 The upper bound

4.1.1 The Problem

The problem of determining the upper bound of the statistical arbitrage-free range of variation for the value of a European derivative, can be stated as

$$C_u = \min_{\{\Delta_t, B_t\}_{t=0,\ldots,T-1}} \Delta_0 S_0 + B_0$$

where

$$\Delta_t, B_t \in \mathbb{R}^{t+1}, t = 0, \ldots, T-1$$

subject to the conditions of a positive expected payoff

$$E^{p_{x_t,T}}_{i_t}\left[\Delta_x^{i_x} S_T^{i} + R^{T-x} B_x^{i_x}\right] \geq G_T^{i_T};$$

for any $i_t \in I_t$ and $t \in \{0, 1, \ldots, T-1\}$[5]; and self-financing conditions

$$E^{p_{x_t,t'}}_{i_t}\left[\Delta_x^{i_x} S_{t'}^{i} + R^{t'-x} B_x^{i_x}\right] - \left(\Delta_{t'}^{i_{t'}} S_{t'}^{i_{t'}} + B_{t'}^{i_{t'}}\right) \geq 0$$

for any $i_t \in I_t$; $t' > t$, $t \in \{0, 1, \ldots, T-2\}$ and $t' \in \{1, \ldots, T-1\}$[6].

Example 9 Illustration of the optimization problem with $T = 3$. The evolution of the price underlying asset can be represented by the tree in figure 1.

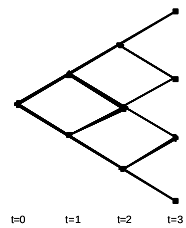

Figure 1: Evolution of the undelying asset' price.

In what concerns the evolution of the price process there are eight different states, i.e., $\Omega = \{w_i\}_{i=1,\ldots,8}$. The problem that must be solved in order to find the upper bound is the following.

$$C_u = \min_{\{\Delta_t, B_t\}_{t=0,\ldots,2}} \Delta_0 S_0 + B_0$$

where

$$\begin{aligned} \{\Delta_0; B_0\} &= \{(\Delta_0; B_0)\} \\ \{\Delta_1; B_1\} &= \left\{\left(\Delta_1^1; B_1^1\right); \left(\Delta_1^2; B_1^2\right)\right\} \\ \{\Delta_2; B_2\} &= \left\{\left(\Delta_2^1; B_2^1\right); \left(\Delta_2^2; B_2^2\right); \left(\Delta_2^3; B_2^3\right)\right\} \end{aligned}$$

[5] For each i_t there are $2^{(T-t)}$ paths, and as a result, $2^{(T-t)}(t+1)$ restrictions at time t. The total number of restrictions is $\sum_{t=0}^{T-1} 2^{(T-t)}(t+1)$.

[6] For each i_t there are $\sum_{t'=t+1}^{T-1} 2^{t'-t}$. Hence, for each t there are $(t+1)\sum_{t'=t+1}^{T-1} 2^{t'-t}$. Hence, there are $\sum_{t=0}^{T-2}(t+1)\sum_{t'=t+1}^{T-1} 2^{t'-t}$ restrictions.

subject to the conditions of a positive expected payoff

$$\left[\Delta_2^{i_2} S_3^{i_3} + R B_2^{i_2}\right] \geq G_3^{i_3};$$

for all $i_3 \in I_3$ and $i_2 \in I_2$ such that $i_2 \in I_2^{i_3}$: (these are 6 constraints);

$$p\left[\Delta_2^{i_2} S_3^{i_3} + R B_2^{i_2}\right] + (1-p)\left[\Delta_1^{i_1} S_3^{i_3} + R^2 B_1^{i_1}\right] \geq G_3^{i_3}$$

for all $i_3 \in I_3$; $i_2 \in I_2$ and $i_1 \in I_1$ such that $i_2 \in I_2^{i_3}$: and $i_1 \in I_1^{i_2}$; i.e., i_1; i_2 and i_3 belong to the same path (these are 8 constraints) and

$$\left[p^2 + p(1-p)\right]\left[\Delta_2^{i_2} S_3^{i_3} + R B_2^{i_2}\right] + p(1-p)\left[\Delta_1^{i_1} S_3^{i_3} + R^2 B_1^{i_1}\right] + \\ + (1-p)^2\left[\Delta_0 S_3^{i_3} + R^3 B_2^{i_2}\right] \geq G_3^{i_3}$$

for all $i_3 \in I_3$; $i_2 \in I_2$ and $i_1 \in I_1$ such that $i_2 \in I_2^{i_3}$: and $i_1 \in I_1^{i_2}$; i.e., i_1; i_2 and i_3 belong to the same path (these are 8 constraints). Moreover, the self-financing constraints must also be considered

$$\Delta_0 S_1^{i_1} + R B_0 \geq \Delta_1^{i_1} S_1^{i_1} + B_1^{i_1}$$

for any $i_1 \in I_1$ (2 constraints),

$$(1-p)\left[\Delta_0 S_2^{i_2} + R^2 B_2^{i_2}\right] + p\left[\Delta_1^{i_1} S_2^{i_2} + R B_1^{i_1}\right] \geq \Delta_2^{i_2} S_2^{i_2} + B_2^{i_2}$$

for any $i_2 \in I_2$ and $i_1 \in I_1$ such that $i_1 \in I_1^{i_2}$; i.e., i_1;and i_2 belong to the same path (these are 4 constraints) and, finally,

$$\Delta_1^{i_1} S_2^{i_2} + R B_1^{i_1} \geq \Delta_2^{i_2} S_2^{i_2} + B_2^{i_2}$$

for any $i_2 \in I_2$ and $i_1 \in I_1$ such that $i_1 \in I_1^{i_2}$; i.e., i_1;and i_2 belong to the same path (these are 4 constraints).

4.1.2 Solution

Theorem 10 There exists a modified martingale probability measure, $q^{i_T} \in Q_S$; such that the upper bound for arbitrage-free value of a European option can be written as

$$C_u = \max_{q^{i_T} \in Q_S} \frac{1}{R^T} \sum_{\{i_T \in I_T\}} q^{i_T} G_T^{i_T}: \tag{3}$$

Proof. See proof in appendix A.1. ■

Remark 11 If a Call Option is considered, the values for q^{i_T}, in a model with two periods are explicitly calculated in appendix A.3. In that case it can be shown that for a strictly positive p, the q^1, q^2 and q^3 are also strictly positive.

In what follows we characterize some relevant properties of C_u:

1. $C_u \leq C_u^0$

   Proof. Let Δ_0^0 and B_0^0 denote the optimal values invested, at time $t = 0$, in the stock and in the risk-free asset respectively, when $p = 0$. The trading strategy $\hat{\Delta}_t = \Delta_0^{p=0}$ and $\hat{B}_t = R^t B_0^{p=0}$, for $t = 1, \ldots, T-1$, is an admissible strategy for any given p. As a result, the solution of the problem for any p cannot be larger that the value of this portfolio at $t = 0$ (which is C_u^0). ■

2. $C_u \geq C_u^1$.

   Proof. Consider the trading strategy $\left(\hat{\Delta}_t^{*}, \hat{B}_t^{*}\right)_{t=0,\ldots,T-1}$ that solves the maximization problem that characterizes the upper bound for a $p \in (0,1)$. This is an admissible strategy for the case $p = 1$, because it is self-financing, i.e.,

   $$\Delta_{t-1}^{i_{t-1}} S_t^{i_t} + R B_{t-1}^{i_{t-1}} \geq \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t},$$

   and superreplicates the payoff of the European derivative at maturity, i.e.,

   $$\Delta_{T-1}^{i_{T-1}} S_T^{i_T} + R B_{T-1}^{i_{T-1}} \geq G_T^{i_T}.$$

   Hence, the solution of the problem for $p = 1$ cannot be higher than the value of this portfolio at $t = 0$ (which is C_u). ■

3. $\lim_{p \to 0} C_u = C_u^0$ and $\lim_{p \to 1} C_u = C_u^1$.

   Proof. See Appendix A.4 ■

   An example for a Call Option and T=2 is also show in appendix A.4.

4. C_u is a decreasing function of p.

   Proof. See Appendix A.4 ■

5. For a Call Option and $T = 2$, we can prove that

$$C_u \le pC_u^1 + (1 - p)C_u^0$$

meaning that the probabilistic upper bound is a convex linear combination of the perfectly liquid upper bound and the perfectly illiquid upper bound.

Proof. See appendix A.4. ■

4.2 The Lower Bound

The organization of this section is analogous to the section for the upper bound.

4.2.1 The Problem

The problem of determining the lower bound of the statistical arbitrage-free range of variation for the value of an European derivative, can be stated as

$$C_l = \max_{\{\Delta_t; B_t\}_{t=0;:::;T-1}} \Delta_0 S_0 + B_0$$

where

$$\Delta_t; B_t \in R^{t+1}; t = 0; :::; T - 1$$

subject to the conditions of a positive expected payoff

$$E_{i_t}^{p_{x_{t,T}}}\left[\Delta_x^{i_x} S_T^{i} + R^{T-x} B_x^{i_x}\right] \le G_T^{i_T};$$

for any $i_t \in I_t$ and $t \in \{0; 1; ::: ; T - 1\}$[7];and self-financing conditions

$$E_{i_t}^{p_{x_{t,t'}}}\left[\Delta_x^{i_x} S_{t'}^{i} + R^{t'-x} B_x^{i_x}\right] - \left(\Delta_{t'}^{i_{t'}} S_{t'}^{i_{t'}} + B_{t'}^{i_{t'}}\right) \le 0$$

for any $i_t \in I_t$; $t' > t$, $t \in \{0; 1; ::: ; T - 2\}$ and $t' \in \{1; ::: ; T - 1\}$[8]:

[7] As in the upper bound case, for each i_t there are $2^{(T-t)}$ paths, and as a result, $2^{(T-t)}(t + 1)$ restrictions at time t. The total number of restrictions is $\sum_{t=0}^{T-1} 2^{(T-t)}(t + 1)$:

[8] As in the upper bound case, for each i_t there are $\sum_{t'=t+1}^{T-1} 2^{t'-t}$: Hence, for each t there are $(t + 1)\sum_{t'=t+1}^{T-1} 2^{t'-t}$: Hence, there are $\sum_{t=0}^{T-2}(t + 1)\sum_{t'=t+1}^{T-1} 2^{t'-t}$ restrictions.

4.2.2 Solution

Theorem 12 There exists an modified martingale probability measure, $q^{i_T} \in Q_S$, such that the upper bound for arbitrage-free value of an European option can be written as

$$C_l = \min_{q^{i_T} \in Q_S} \frac{1}{R^T} \sum_{\{i_T \in I_T\}} q^{i_T} G_T^{i_T}.$$

Proof. The proof is analogous to the upper bound. ∎

Remark 13 If a call option is considered, the values for q^{i_T}, in a model with two periods are explicitly calculated in appendix B.2. In that case it can be shown that for a strictly positive p, the q^1, q^2 and q^3 are also strictly positive.

In what follows we characterize some relevant properties of C_l:

1. $C_l \geq C_l^0$.
2. $C_l \leq C_l^1$.
3. $\lim_{p \to 0} C_l = C_l^0$ and $\lim_{p \to 0} C_l = C_l^1$.

   An example for a call option and $T = 2$ is shown in appendix ??
4. C_l is a increasing function of p.

   The proofs of these properties are analogous to those presented for the upper bound.

5 Utility Functions and Reservation Prices

In this section we show that the price for which any agent is indifferent between transacting or not transacting the derivative, to be called the reservation price of the derivative, is contained within the statistical arbitrage bounds derived above.

Let $u_t(.)$ denote a utility function representing the preferences of an agent at time t. The argument of the utility function is assumed to be the consumption at time t. Let y be the initial endowment of the agent, and Z_t denote the vector of consumption at time t, i.e., $Z_t = \left(Z_t^{i_t}\right)_{i_t \in I_t}$. Let ρ be a discount factor. If an agent sells a European derivative by the amount C, and that derivative has a payoff at maturity given $G_T^{i_T}$, the maximum utility that he or she can attain is

$$u^*_{sell}(C;p) = \sup_{\{\Delta_t, B_t\}_{t=0,\ldots,T-1}} E_0^{G,P}\left[\sum_{t=0}^{T} \rho^t u_t(Z_t)\right]$$

subject to

$$\begin{aligned} Z_0 + \Delta_0 S_0 + B_0 &\leq C + y \\ Z_t^{i_t} + \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t} &\leq \Delta_{t-j}^{i_{t-j}} S_t^{i_t} + R^j B_{t-j}^{i_{t-j}} \\ Z_T^{i_T} &\leq \Delta_{T-j}^{i_{T-j}} S_T^{i_T} + R^j B_{T-j}^{i_{T-j}} - G_T^{i_T} \end{aligned}$$

for all $i_T \in I_T$; $i_t \in I_t$, $j \leq t$ and $t = 1, \ldots, T-1$ where $E_0^{G,P}$ denotes a bivariate expected value, at $t = 0$, with respect to the probability P induced by the market existence and the probability G underlying the stochastic evolution of the price process.

Similarly, if the agent decides not to include derivatives in his or her portfolio, the maximum utility that he or she can attain is given by

$$u^*(p) = \sup_{\{\Delta_t, B_t\}_{t=0,\ldots,T-1}} E_0^{G,P}\left[\sum_{t=0}^{T} \rho^t u_t(Z_t)\right]$$

subject to

$$\begin{aligned} Z_0 + \Delta_0 S_0 + B_0 &\leq y \\ Z_t^{i_t} + \Delta_t^{i_t} S_t^{i_t} + B_t^{i_t} &\leq \Delta_{t-j}^{i_{t-j}} S_t^{i_t} + R^j B_{t-j}^{i_{t-j}} \\ Z_T^{i_T} &\leq \Delta_{T-j}^{i_{T-j}} S_T^{i_T} + R^j B_{T-j}^{i_{T-j}} \end{aligned}$$

Lemma 14 In the case of random dryness, there is $p^* > 0$ such that, for all $p < p^*$; the utility attained selling the derivative by C_u, is larger than the utility attained if the derivative is not included in the portfolio.

$$u^*_{sell}(C_u;p) \geq u^*(p):$$

Proof. Let, for a given p; $\{\Delta_t; B_t\}_{t=0;\ldots;T-1}$ denote the solution of the utility maximization problem with no derivative and $\{\Delta_t^u; B_t^u\}_{t=0;\ldots;T-1}$ denote the solution of the minimization problem that must be solved to find the upper bound if statistical arbitrage opportunities are considered (see section 4.1). Moreover, let $\left(\Delta_t^{sell}; B_t^{sell}\right)_{t=0;\ldots;T-1}$ denote an admissible solution of the utility maximization problem when the agent sells one unit of the derivative. Now, consider the limit case, when p approaches zero. In that case, the portfolio

$$\left\{\Delta_t^{sell}; B_t^{sell}\right\}_{t=0;\ldots;T-1} \equiv \{\Delta_t + \Delta_t^u; B_t + B_t^u\}_{t=0;\ldots;T-1}$$

is an admissible solution of the utility maximization problem when the agent sells one unit of derivative by C_u. The reason is as follows. The constraint set of the problem that must be solved to find the upper bound is continuous in p. Hence, when $p \to 0$, the solution of the problem is $\{\Delta_t^u; B_t^u\}_{t=1;\ldots;T-1} = \left(\Delta_0^u; R^t B_0^u\right)$ where $\{\Delta_0^u; B_0^u\}$ is the solution of following problem

$$\min_{\Delta;B} \left\{\Delta S_0 + B \text{ s.a. } \Delta S_T^{i_T} + R^T B \geq G_T^{i_T}; \forall i_T\right\}:$$

As $C = C_u = \Delta_0^u S_0 + B_0$; the portfolio $\{\Delta_t^u; B_t^u\}_{t=1;\ldots;T-1}$ is an admissible solution of the utility maximization problem when one unit of the derivative is being sold. Moreover, it guarantees a positive expected utility.[9] Hence, the portfolio $\left\{\Delta_t^{sell}; B_t^{sell}\right\}_{t=0;\ldots;T-1}$ is also admissible solution for the optimization problem, when one unit of the derivative is being sold, which guarantees a higher payoff than the portfolio $\{\Delta_t; B_t\}_{t=0;\ldots;T-1}$. Hence,

$$u_{sell}^*(C_u; 0) \geq u^*(0):$$

Continuity on p of both u_{sell}^* and u^* ensure the result. ■

Remark 15 Notice that the existence of p^* follows from the continuity of the utilities in p. Furthermore, it is possible to have $p^* = 1$. Examples with different values of p^* are given in the end of this paper. The range of values $p < p^*$ characterizes what was vaguely described as "sufficiently incomplete markets" in the introduction.

The reservation price for an agent that is selling the option is defined as the value of C that makes $u_{sell}^*(C; p) = u^*(p)$. Let R_u denote such reservation price.

Theorem 16 For all $p < p^*$ we have $R_u \leq C_u$.

Proof. The optimal utility value;

$$u_{sell}^*(C; p) = \sup_{\{\Delta_t; B_t\}_{t=0;\ldots;T-1}} C + y - (\Delta_0 S_0 + B_0) + E_0^{G;P}\left[\sum_{t=1}^{T} \rho^t u_t(Z_t)\right];$$

---

[9] It suffices to consider $Z_0 = y$; $Z_{i_t} = 0$ and $Z_{i_T} = \Delta S_T^{i_T} + R^T B - G_T^{i_T} \geq 0$.

is increasing in C: This, together with lemma 14 leads to the result. ■

The same applies for the case when the agent is buying a derivative. In that case, if an agent is buying the derivative by C, the maximum utility that he or she can attain is

$$u^*_{buy}(C;p) = \sup_{\{\Delta_t, B_t\}_{t=0,\dots,T-1}} E_0^{G,P} \sum_{t=0}^{T} \rho^t u_t(Z_t)$$

subject to

$$Z_0 + \Delta_0 S_0 + B_0 \le -C + y$$
$$Z_t^{\omega^t} + \Delta_t^{\omega^t} S_t^{\omega^t} + B_t^{\omega^t} \le \Delta_{t-1}^{\omega^{t-1}} S_t^{\omega^t} + R B_{t-1}^{\omega^{t-1}}$$
$$Z_T^{\omega^T} \le \Delta_{T-1}^{\omega^{T-1}} S_T^{\omega^T} + R B_{T-1}^{\omega^{T-1}} + G_T^{\omega^T}$$

Lemma 17 In the case of random dryness, there is $p^* > 0$ such that, for all $p < p^*$; the utility attained buying the derivative by C_l, is larger than the utility attained if the derivative is not included in the portfolio.

$$u^*_{buy}(C_l;p) \geq u^*(p):$$

Proof. The proof is analogous to the one in proposition 14 ■

Let R_l denote the reservation selling price, i.e., the price such that $u^*(p) = u^*_{buy}(R_l;p)$.

Theorem 18 For all $p < p^*$ we have $R_l \geq C_l^p$:

Proof. The proof is analogous to the one presented in theorem (16). However, in this case the utility is a decreasing function of C; and we obtain

$$u^*_{buy}(C_l;p) \geq u^*(p) \Rightarrow R_l \geq C_l:$$

■

Several illustrations are presented in section 7.

6 Comparisons with the Literature

In what follows we compare our methodology with others in the literature, namely Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000). Cochrane and Saá-Requejo (2000) introduce the notion of "good deals", or investment opportunities with high Sharpe ratios. They show that ruling out investment opportunities with high Sharpe ratios, they can obtain narrower bounds on securities prices. However, as stressed in Bondarenko (2003), not all pure arbitrage opportunities qualify as "good deals". Moreover, for a given set of parameters we found out that in order to contain the reservation prices of a risk neutral agent the interval is more broad than the one that was obtained in our formulation.

We first provide a simple example to compare our bounds with pure arbitrage bounds.

Example 19 Consider a simple two periods example, where transactions are certainly possible at times $t = 0$ and $t = 2$. At time $t = 1$ there are transactions with a given probability $p = 0.65$. The initial stock price is $S_0 = 100$ and it may either increase in each period with a probability 0.55, or decrease with a probability 0.45. We take and $U = 1.2$, $D = 0.8$ and $R = 1.1$. A call option that matures at time $T = 2$ with exercise price $K = 80$ is considered. Using pure arbitrage arguments we find the following range of variation for the value of the call option

$$[33.88; 37.69]$$

Using the notion of statistical arbitrage opportunity, the above range gets narrower and is given by

$$[34.31; 35.17];$$

clearly narrower that the above interval.

If markets were complete ($p = 1$), the value of the option would be 34.71. Also, the reservation price[10] for a risk neutral agent is equal to 35.09. Notice that both intervals include the complete market value and the reservation price.

We now use the same example to compare our methodology with the one presented by Cochrane and Saá-Requejo (2000). We show that either our interval is contained in theirs, or else, their interval do not contain the above mentioned reservation price.

With the Sharpe ratio methodology the lower bound is given by

$$\underline{C} = \min_{\{m\}} E\left\{ m\,[S_2 - K; 0]^+ \right\}$$

[10] In this example, the reservation price for an agent who is buying the derivative coincides with that of an agent who is selling it.

subject to

$$S_0 = E[mS_2];\ m \geq 0;\ \sigma(m) \leq \frac{h}{R^2};$$

where S_0 is the initial price of the risky asset, and S_2 is the price of the risky asset at time $t = 2$.[11] The upper bound is

$$\overline{C} = \max_{\{m\}} E\left\{m[S_2 - K; 0]^+\right\}$$

subject to

$$p = E[mS_2];\ m \geq 0;\ \sigma(m) \leq \frac{h}{R^2}.$$

Example 20 In order to compare the statistical arbitrage interval with the Sharpe ratio bounds, we must choose the ad-hoc factor h so as to make one of the limiting bounds to coincide. If we want the upper bound of the Sharpe Ratio methodology to coincide with the upper bound obtained with statistical arbitrage, we must take $h = 0.3173$. In that case, the lower bound will be 33.88 and the range of variation will be

$$[33.88; 35.17];$$

worse than the statistical arbitrage interval.

Alternatively, if we want the lower bound of the Sharpe Ratio methodology to coincide with the lower bound obtained with statistical arbitrage, we must take $h = 0.28359$.[12] In that case, the upper bound will be 34.49 and the range of variation will be

$$[34.31; 34.49].$$

Although this interval is tighter than the statistical arbitrage interval, it does not contain the reservation price for a risk neutral agent.

In a different paper Bernardo and Ledoit (2000) preclude investments offering high gain-loss ratios to a benchmark investor, somehow analogous

[11] As stressed by Cochrane and Saá-Requejo, in a former paper Hansen and Jagannathan (1991) have shown that a constraint on the discount factor volatility is equivalent to impose an upper limit on the Sharpe ratio of mean excess return to standard deviation.

[12] In order to get a lower bound higher than 33.88 it is necessary to impose aditionally that $m > 0$. If that were not the case, then the lower bound would only be defined for $h \geq 0.2980$ and would be equal to 33.88.

to the “good deals” of Cochrane and Sáa-Requejo. The criterion, however, is different since Bernardo and Ledoit (2000) propose a utility-based approach, as stressed in the Introduction. In this way, the arbitrage-free range of variation for the value of the European derivative is narrower than in the case of pure arbitrage. Let $\tilde{z}^+$ denote the (random) gain and $\tilde{z}^-$ denote the (random) loss of a given investment opportunity. The utility of a benchmark agent characterizes a pricing kernel that induces a probability measure, according to which the expected gain-loss ratio is bounded from above

$$\frac{E(\tilde{z}^+)}{E(\tilde{z}^-)} \le \bar{L}.$$

The fair price is the one that makes the net result of the investment to be null. In other words, for a benchmark investor, it would correspond to the pricing kernel that would make

$$E\left(\tilde{z}^+ - \tilde{z}^-\right) = 0 \iff \frac{E(\tilde{z}^+)}{E(\tilde{z}^-)} = 1.$$

This last equality characterizes the benchmark pricing kernel for a given utility.

Notice that the fair price constructed in this way coincides with our definition of the reservation price. Therefore, by choosing $\bar{L}$ larger than one, the interval built by Bernardo and Ledoit contains by construction the reservation price of the benchmark agent.

On the other hand, the arbitrary threshold $\bar{L}$ can be chosen such that their interval is contained in the statistical arbitrage-free interval.

The disadvantages, however, are clear. First, the threshold is ad-hoc, just as in the case of Cochrane and Sáa-Requejo; second, the constructed interval depends on the benchmark investor; and finally, the only reservation price that is contained for sure in that interval, is the reservation price of the benchmark investor. In other words, we cannot guarantee that the reservation price of an arbitrary agent, different from the benchmark, is contained in that interval.

7 Numerical Examples

7.1 Upper and Lower Bounds

In this section several numerical examples are provided in order to illustrate the properties of the upper and lower bounds presented in the previous sections.

Using numerical examples we can conclude that, for a call option,

$$C_u \leq pC_u^1 + (1 - p)C_u^0.$$

If the Call Option is sold by the expected value of the call, regarding the existence (or not) of market, there will be an arbitrage opportunity in statistical terms. The reason is that the agent that sells the call option can buy a hedging portfolio (in a statistical sense) by an amount smaller than the expected value of the call option. As a result, there is an arbitrage opportunity, because he is receiving more for the call option than is paying for the hedging portfolio.

However, in what concerns the lower bound, it is not possible to conclude whether $C_l \leq pC_l^1 + (1 - p)C_l^0$ or $C_l \geq pC_l^1 + (1 - p)C_l^0$. That depends on the value of the parameters. Although in the two-period simulation in Figure 2 we seem to have the former case, the three period example in Figure ?? seems to suggest the latter, since the lower bound behaves as a concave function of p for most of its domain.

Finally, we may use Figure 4 to illustrate several features.

The Upper and Lower bounds of an European Call Option for different values of p and K ($K = 80$, $K = 100$, $K = 120$, $K = 140$ e $K = 160$) in a three period model, with $U = 1.2$; $R = 1.1$; $D = 0.8$ and $S = 100$.

First, let us regard the situations in this Figure that are related to pure arbitrage. This includes the value of the derivative under perfectly liquid ($p = 1$) and perfectly illiquid ($p = 0$) markets. In the former case, the unique value of the derivative clearly decreases with the exercise price K, as it should. In the latter, both the upper bound C_u^0 and the lower bound C_l^0 also decrease with K. More curiously, however, the spread $C_u^0 - C_l^0$ has a non-monotonic behaviour. Obviously this difference is null for K less than the infimum value of the stock at maturity and must go to zero as the strike approaches the supremum of the stock's possible values at maturity. In the middle of this range it attains a maximum. In our numerical example we observe that the maximum value of the spread is attained for K close to 120.

Regarding the statistical arbitrage domain when $p \in (0, 1)$, we notice that all the above remarks remain true. The Figure also suggests that, for any given p, the spread attains its maximum for the same value of K as before. Notice that the spread $C_u - C_l$ decreases with p for fixed strike K converging to zero as $p \to 1$. Hence, although somehow different from the traditional definition of arbitrage, the notion of statistical arbitrage seems

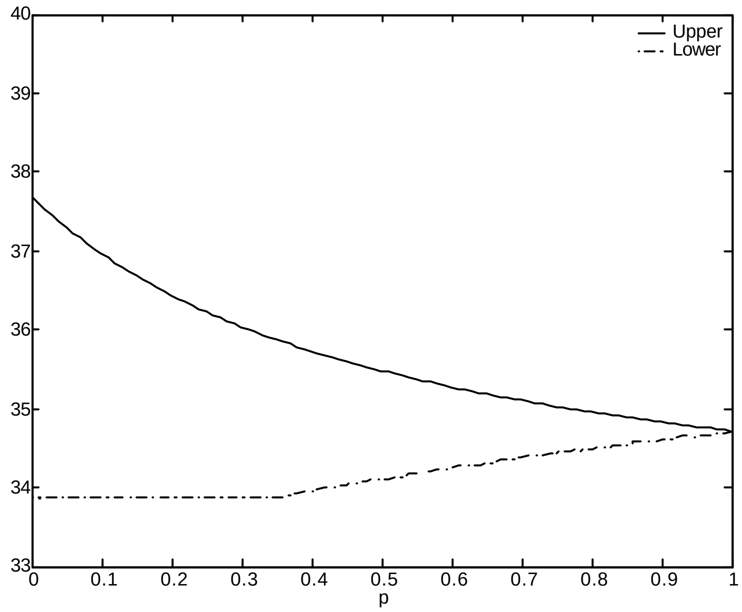

Figure 2: The Upper and Lower Bounds for a European Call Option with di¤erent value of p in a two period model, with $U = 1:2$, $R = 1:1$, $D = 0:8$; $S_0 = 100$ and $K = 80$:

to provide a very nice bridge, for $0 < p < 1$, between the two extreme cases above ($p = 0$ and $p = 1$), where the original concept of arbitrage makes sense. This can be seen in ...gure 5.

A third issue driven by the ...gure is the remark that, for intermediate values of K; there is an overlapping of the di¤erent spreads C_u^0 ¡ C_l^0: Take $K = 120$ and $K = 140$; for instance, when $p = 0$. The upper bound for $K = 140$ is above the lower bound of the $K = 120$ derivative. The value 15 is in-between. The spread is constructed in a way such that if the $K = 120$ derivative is transacted by 15; there are no arbitrage opportunities. But, if the $K = 140$ derivative is transacted by 15; there are also no arbitrage opportunities. Hence, we may sell the $K = 140$ derivative by 15 and use the proceeds to buy the $K = 120$ derivative by 15. Since the payo¤ of the $K = 120$ derivative is always larger than the payo¤ of the $K = 140$ derivative, we would have an arbitrage opportunity... This paradox has a simple explanation. Our bounds are constructed under the assumption that there is a single derivative. In fact, the presence of more derivatives may

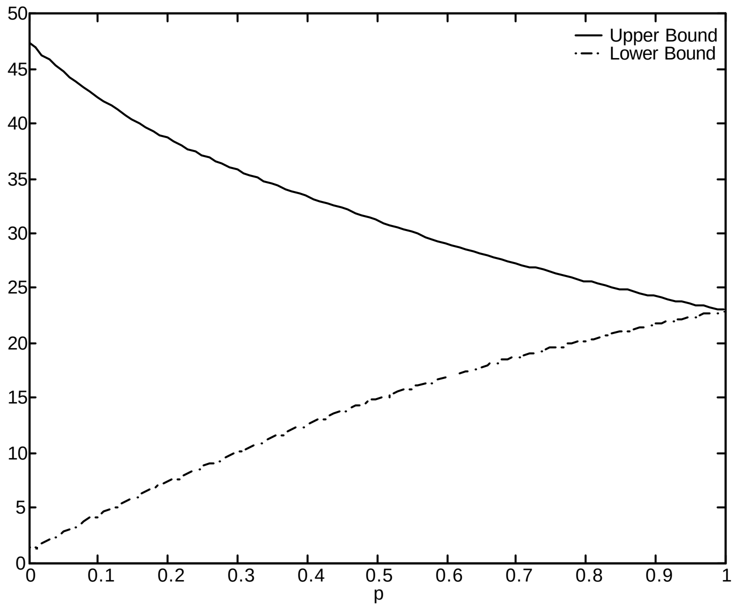

Figure 3: The Upper and Lower Bounds for a European Call Option with di¤erent value of p in a three period model, with $U = 1{:}3$, $R = 1$, $D = 0{:}6$; $S_0 = 100$ and $K = 100$:

help to complete the market, making the overlapping arbitrage-free regions not viable. As markets become complete, the arbitrage-free regions shrink to a point, corresponding to the unique value of the derivatives under complete markets.

7.2 Utility and Reservation Prices

In this section we illustrate several aspects related to the determination of the reservation price. In Figure 6 we represent the utility of an agent in three di¤erent situations. Without the derivative; a short position on the derivative, when the instrument is sold by the statistical arbitrage upper bound; and a long position on the derivative when the instrument is bought by the statistical arbitrage lower bound. Notice that for $p = 0$ the best situation is the short position on the derivative and the worst is without trading the derivative. This is consistent with lemma 14 and lemma 17. Notice that there is a value of p such that, for larger probabilities, the utility without trading the derivative is no longer the worst. That critical

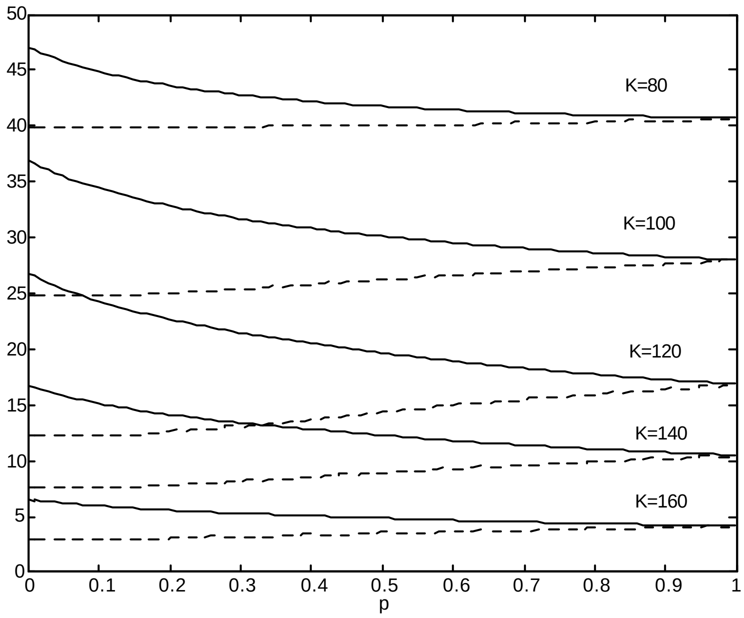

Figure 4: The Upper and Lower bounds of an European Call Option for di¤erent values of p and K (K = 80, K = 100, K = 120, K = 140 e K = 160) in a three period model, with U = 1:2; R = 1:1; D = 0:8 and S = 100.

value of p is what we called p¤:

Figure 7 represents the statistical arbitrage-free interval together with the reservation price for a risk neutral agent. By construction, the probability associated to the point where the reservation price coincides with the upper bound, corresponds to the critical probability p¤: Notice from Figure 6 that the utility of the position associated to a long position on the derivative is always above the utility without trading the derivative. This implies that the reservation price is always above the lower bound. Likewise, the fact that the utility of the short position on the derivative goes below the utility without trading the derivative, implies that the reservation price goes above the upper bound.

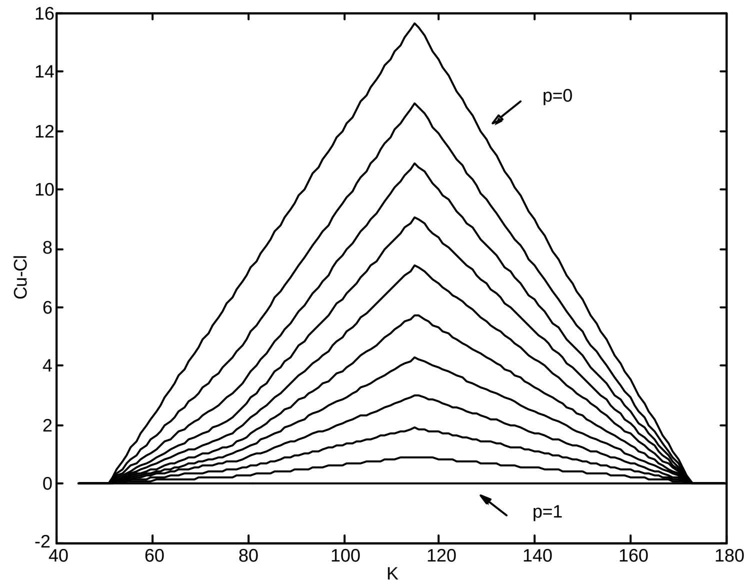

Figure 5: The Spread $(C_u - C_l)$ of an European Call Option for different values of K and p ($p = 0; \ldots; 1$ with increments of 0.1) in a three period model, with $U = 1.2$; $R = 1.1$; $D = 0.8$ and $S = 100$.

8 Conclusion

In this paper we have characterized the statistical arbitrage-free bounds for the value of an option written on an asset that may not be transacted. This statistical arbitrage-free interval is by construction tighter than the usual arbitrage-free interval, obtained under the superreplication strategy. In that sense, our result is close to the results of Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000). By using a concept of statistical arbitrage, in the spirit of Bondarenko's (2003), we were able to avoid the arbitrary threshold that led the former approaches to constrain the arbitrage-free interval.

In a framework characterized by the fact that transactions of the underlying asset are possible with a given probability, we derived the range of variation for the statistical arbitrage-free value of an European derivative. If transactions were possible at all points in time there would be a unique arbitrage-free value for the European derivative that is contained in the statistical arbitrage-free range. Moreover, the statistical arbitrage-free range is

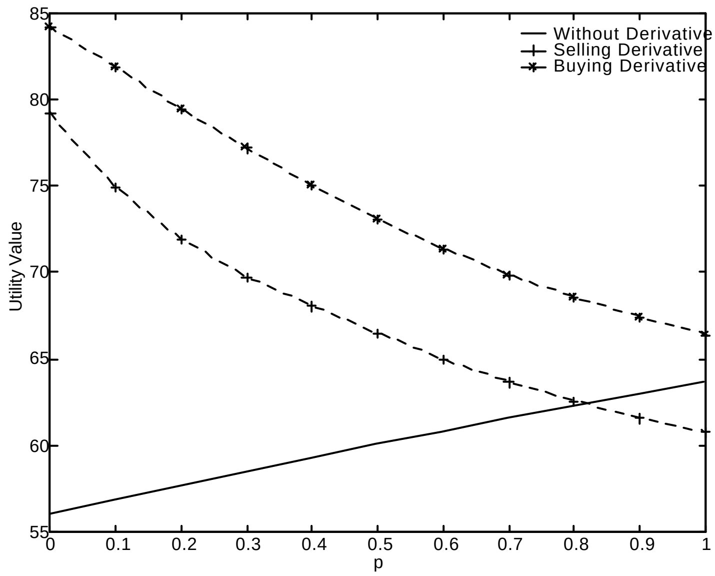

Figure 6: Utility value for the following parameters: U = 1:3, D = 0:6, R = 1, S_0 = 100, K = 100, ½ = 1=R, q = 0:5, y = 50 and T = 3.

contained in the arbitrage-free range of variation if the market is perfectly illiquid. The upper bound is a decreasing function in the probability of existence of the market and the lower bound is a increasing function. They are asymptotically well behaved both when p ! 0 and when p ! 1.

Finally, we could also prove that, in the case of random illiquidity, the reservation prices (both for selling and buying positions) are contained in the statistical arbitrage-free range of variation for the value of the European Option.

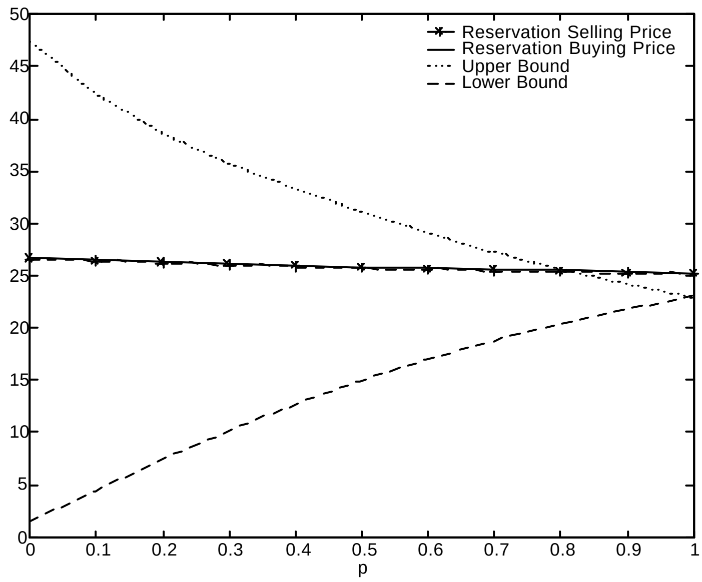

Figure 7: Statistical Arbitrage free bounds and reservation prices for the following parameters: U = 1:3, D = 0:6, R = 1, S_0 = 100, K = 100, ½ = 1=r, q = 0:5, y = 40 and T = 3.

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A Some Proofs on the Solution of the Upper Bound for Statistical Arbitrage Opportunities

A.1 Proof of theorem 10

Proof. For any given path $m \in \Omega^{+}_{i_t;i_T}$ let $\lambda^{(i_T;T);m}_{(i_t;t)}$ be the dual variable associated with the superreplication constraint

$$E_{i_t}^{p_{x_{t,T}}}\left[\sum_{s=t;\dots;t'-1} p_{x_{t;t'}}\left[x_{t;t'} = s\right]\left(\Delta_s^{i_s} S_{t'}^{i} + R^{t'-s} B_s^{i_s}\right)\right] \geq G_T^{i_T}$$

with $i_k \in I_k^{i_{k+1}}$ and $k = t;\dots;T-1$. Let $n_t^{i_T}$ be the number of nodes that are predecessors of node i_T at time t; where $n_t^{i_T}$ is given by

$$n_t^{i_T} = \min\{T-(i_T-1); i_T-1; T-t\} + 1$$

At each node i_t; that is a predecessor of i_T; there are $\#\left(\Omega^{+}_{i_t;i_T}\right)$ paths to reach i_T. For any given path $m \in \Omega^{+}_{i_t;i_{t'}}$ let $\alpha^{(i_{t'};t');m}_{(i_t;t)}$ be the dual variable associated with the self-financing constraints

$$E_{i_t}^{p_{x_{t,t'}}}\left[\Delta^{i_x}_{x_{t;t'}} S_{t'}^{i} + R^{t'-x} B^{i_x}_{x_{t;t'}}\right] - \left(\Delta_{t'}^{i_{t'}} S_{t'}^{i_{t'}} + B_{t'}^{i_{t'}}\right) \geq 0.$$

Considering $n_t^{i_{t'}}$ be the number of nodes that are predecessors of node $i_{t'}$ at time t we have

$$n_t^{i_{t'}} = \min\left\{t'-(i_{t'}-1); i_{t'}-1; t'-t\right\} + 1$$

At each node i_t that is a predecessor of $i_{t'}$ there are $\#\left(\Omega^{+}_{i_t;i_{t'}}\right)$.

The problem that must be solved in order to find the upper bound of the range of variation of the arbitrage-free value of an European derivative is a linear programming problem. Its dual problem is

$$\min_{\lambda^{i_T}} \sum_{j=1}^{T} \lambda^{i_T} G_T^{i_T}$$

where λ^{i_T} is the sum of the dual variables associated with the positive expected payoff constraints that have the right member equal to $G_T^{i_T}$; i.e.,

$$\lambda^{i_T} = \sum_{t=0}^{T-1} \sum_{\{i_t: i_t \in I_t\}} \sum_{\left\{m \in \Omega^{+}_{i_t;i_T}\right\}} \lambda^{(i_T;T);m}_{(i_t;t)} S_T^{i_T}$$

The first set of constraints is of nonnegativity of each dual variable, i.e, $\lambda^{(i_T,T);m}_{(i_t,t)}, \alpha^{(i_{t'},t');m}_{(i_t,t)} \geq 0$. The other set of constraints consists of equality constraints, one constraint associated with each variable of the primal problem. As there are

$$2\sum_{t=0}^{T-1}(t+1) = 2\frac{1+(T-1+1)}{2}T = T(T+1)$$

primal variables there are also $T(T+1)$ constraints of the dual problem, which are equality constraints because the variables of the primal problem are free.

The constraint for Δ_0 is:

$$p_{x_{0,T}}(x_{0,T}=0)\left[\sum_{i_T\in I_T}\sum_{m\in\mathcal{M}^{+}_{i_0,i_T}}\lambda^{(i_T,T);m}_{(i_0,0)}S_T^{i_T}\right]+\alpha^{(1,1)}_{(i_0,0)}S_1^1+\alpha^{(2,1)}_{(i_0,0)}S_1^2$$

$$+\sum_{t=2}^{T-1}\left\{\prod_{j=1}^{t-1}p_y(y_j=0)\sum_{\left\{i_t: i_0\in I_0^{i_t}\right\}}\sum_{m\in\mathcal{M}^{+}_{i_0,i_t}}\alpha^{(i_t,t);m}_{(i_0,0)}S_t^{i_t}\right\}=S_0$$

The constraint for B_0 is:

$$p_{x_{0,T}}(x_{0,T}=0)R^T\sum_{i_T\in I_T}\sum_{m\in\mathcal{M}^{+}_{i_0,i_T}}\lambda^{i_T;m}_{i_0}+R\left(\alpha^{(1,1)}_{(i_0,0)}+\alpha^{(2,1)}_{(i_0,0)}\right)$$

$$+\sum_{t=2}^{T-1}\left\{\prod_{j=1}^{t-1}p_y(y_j=0)R^t\sum_{\left\{i_t: i_0\in I_0^{i_t}\right\}}\sum_{m\in\mathcal{M}^{+}_{i_0,i_t}}\alpha^{(i_t,t);m}_{(i_0,0)}\right\}=1$$

For the constraint that concerns Δ_{i_k} the term in λ is

$$p_{x_{0,T}}(x_{0,T}=k)\sum_{m\in\mathcal{M}^{+}_{i_{0k},i_T}}\sum_{\left\{i_T: i_0\in I_0^{i_T}\right\}}\lambda^{(i_T,T);m}_{(i_0,0)}S_T^{i_T}+$$

$$p_{x_{1,T}}(x_{1,T}=k)\sum_{\left\{i_1: i_1\in I_1^{i_k}\right\}}\sum_{\left\{m\in\mathcal{M}^{+}_{i_1,i_T}: i_k\in m\right\}}\sum_{\left\{i_T: i_k\in I_k^{i_T}\right\}}\lambda^{(i_T,T);m}_{(i_1,1)}S_T^{i_T}$$

$$\vdots$$

$$p_{x_{t,T}}(x_{t,T}=k)\sum_{\left\{i_t: i_t\in I_t^{i_k}\right\}}\sum_{\left\{m\in\mathcal{M}^{+}_{i_t,i_T}: i_k\in m\right\}}\sum_{\left\{i_T: i_k\in I_k^{i_T}\right\}}\lambda^{(i_T,T);m}_{(i_t,t)}S_T^{i_T}=$$

$$\left[\sum_{t=0}^{k}p_{x_{t,T}}(x_{t,T}=k)\sum_{\left\{m\in\mathcal{M}^{+}_{i_t,i_T}: i_k\in m\right\}}\sum_{\left\{i_t: i_t\in I_t^{i_k}\right\}}\sum_{\left\{i_T: i_k\in I_k^{i_T}\right\}}\lambda^{(i_T,T);m}_{(i_t,t)}S_T^{i_T}\right]$$

The terms that involve α are

$$\left[\sum_{t<k}\sum_{t'>k}p_{x_{t,t'}}\left(x_{t,t'}=k\right)\sum_{\left\{i_t: i_t\in I_t^{i_k}\right\}}\sum_{\left\{m\in\mathcal{M}^{+}_{i_t,i_{t'}}: i_k\in m\right\}}\alpha^{(i_{t'},t');m}_{(i_t,t)}S_{t'}^{i_{t'}}\right]-$$

$$-\sum_{t<k}\sum_{\left\{i_t: i_t\in I_t^{i_k}\right\}}\alpha^{(i_k,k);m}_{(i_t,t)}S_k^{i_k}=0.$$

Hence, the constraint for β_{i_k} is

$$\sum_{t=0}^{k} \left[p_{x_{t,T}}(x_{t,T}=k) \sum_{\{i_T: i_k \in I_k^{i_T}\}} \sum_{\{i_t: i_t \in I_t^{i_k}\}} \sum_{\{m \in \Omega_{i_t,i_T}: i_k \in m\}} \lambda_{(i_t,t)}^{(i_T,T),m} S_T^{i_T} \right] + \sum_{t<k} \sum_{t'>k} \left[p_{x_{t,t'}} \left(x_{t,t'} = k \right) \sum_{\{i_t: i_t \in I_t^{i_k}\}} \sum_{\{m \in \Omega_{i_t,i_{t'}}: i_k \in m\}} \alpha_{(i_t,t)}^{(i_{t'},t'),m} S_{t'}^{i_{t'}} \right] + \\ - \sum_{t<k} \sum_{\{i_t: i_t \in I_t^{i_k}\}} \alpha_{(i_t,t)}^{(i_k,k),m} S_k^{i_k} = 0$$

The constraint for B_{i_k} is:

$$\sum_{t=0}^{k} \left[p_{x_{t,T}}(x_{t,T}=k) \sum_{\{i_t: i_t \in I_t^{i_k}\}} \sum_{\{m \in \Omega_{i_t,i_T}: i_k \in m\}} \sum_{\{i_T: i_k \in I_k^{i_T}\}} \lambda_{(i_t,t)}^{(i_T,T),m} R^{T-k} \right] + \sum_{t<k} \sum_{t'>k} \left[p_{x_{t,t'}} \left(x_{t,t'} = k \right) \sum_{\{i_t: i_t \in I_t^{i_k}\}} \sum_{\{m \in \Omega_{i_t,i_{t'}}: i_k \in m\}} \alpha_{(i_t,t)}^{(i_{t'},t'),m} R^{t'-t} \right] \\ - \sum_{t<k} \sum_{\{i_t: i_t \in I_t^{i_k}\}} \alpha_{(i_t,t)}^{(i_k,k),m} = 0$$

Note that if $k = T-1$, the constraint for $\beta_{i_{T-1}}$ the constraint for β_{i_k} is

$$\sum_{t=0}^{T-1} \left[p_{x_{t,T}}(x_{t,T}=T-1) \sum_{\{i_T: i_{T-1} \in I_{T-1}^{i_T}\}} \sum_{\{i_t: i_t \in I_t^{i_{T-1}}\}} \sum_{\{m \in \Omega_{i_t,i_T}: i_{T-1} \in m\}} \lambda_{(i_t,t)}^{(i_T,T),m} S_T^{i_T} \right] + \\ - \sum_{t<T-1} \sum_{\{i_t: i_t \in I_t^{i_{T-1}}\}} \alpha_{(i_t,t)}^{(i_{T-1},T-1),m} S_{T-1}^{i_{T-1}} = 0$$

The constraint for $B_{i_{T-1}}$ is

$$\sum_{t=0}^{T-1} \left[p_{x_{t,T}}(x_{t,T}=T-1) \sum_{\{i_T: i_{T-1} \in I_{T-1}^{i_T}\}} \sum_{\{i_t: i_t \in I_t^{i_{T-1}}\}} \sum_{\{m \in \Omega_{i_t,i_T}: i_{T-1} \in m\}} \lambda_{(i_t,t)}^{(i_T,T),m} R \right] + \\ - \sum_{t<T-1} \sum_{\{i_t: i_t \in I_t^{i_{T-1}}\}} \alpha_{(i_t,t)}^{(i_{T-1},T-1),m} = 0$$

The left member of each constraint is a linear combination of the variables of the dual problem. The right member is equal to S_0 and 1 for the dual constraints associated with the variables β_0 and B_0, respectively. For the remaining constraints the right member is equal to zero. First, let us consider only the constraints associated with primal variables β's. For a given i_T the terms involving λ in the dual constraints regarding β_{i_k} is

$$p_{x_{t,T}}(x_{t,T}=k) \sum_{\{i_t: i_t \in I_t^{i_k}\}} \sum_{\{m \in \Omega_{i_t,i_T}: i_k \in m\}} \lambda_{(i_t,t)}^{(i_T,T),m} S_T^{i_T}$$

Summing up all the constraints that concern Δ_{i_k} with $i_k \in I_k$ the term associated with and the term associated with S_T^{i} is

$$p_{x_{t,T}}(x_{t,T}=k)\sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Gamma^{+}_{i_t,i_T}\right\}} \lambda^{(i_T,T),m}_{(i_t,t)} S_T^{i_T}$$

As,

$$\sum_{k=t}^{T-1} p_{x_{t,T}}(x_{t,T}=k) = 1;$$

summing for all $k \geq t$, the term associated with $S_T^{i_T}$ that is multiplying by $p_{x_{t,T}}(x_{t,T}=\cdot)$ is

$$\sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Gamma^{+}_{i_t,i_T}\right\}} \lambda^{(i_T,T),m}_{(i_t,t)} S_T^{i_T}$$

Hence, summing up over all constraints associated with primal variables Δ's we have that the terms in λ associated with $S_T^{i_T}$ are

$$\sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Gamma^{+}_{i_t,i_T}\right\}} \lambda^{(i_T,T),m}_{(i_t,t)} S_T^{i_T}$$

Therefore, the sum over all $S_T^{i_T}$ is

$$\sum_{\{i_T : i_T \in I_T\}} \sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Gamma^{+}_{i_t,i_T}\right\}} \lambda^{(i_T,T),m}_{(i_t,t)} S_T^{i_T} = S_0$$

Still considering only the constraints associated with primal variables Δ's; in what follows we describe the terms in α. For a given $S_t^{i_t}$; the terms in α are

$$\underbrace{i \sum_{s<t} \sum_{\left\{i_s : i_s \in I_s^{i_t}\right\}} \alpha^{(i_t,t),m}_{(i_s,s)} S_t^{i_t}}_{\text{from the constraint } \Delta_{i_t}}$$

$$+ \underbrace{\sum_{k=0}^{t-1} \sum_{s=k}^{t-1} \left[p_{x_{k,t}}(x_{k,t}=s) \sum_{\left\{i_k : i_k \in I_k^{i_t}\right\}} \sum_{\left\{m \in \Gamma^{+}_{i_k,i_t} : i_s \in m\right\}} \alpha^{(i_t,t),m}_{(i_k,k)} S_t^{i_t} \right]}_{\text{summing up all the constraints } \Delta_{i_k};\ \text{with } i_k \in I_k^{i_t}}$$

As,

$$\sum_{s=k}^{t-1} p_{x_{k,t}}(x_{k,t} = s) = 1$$

the above equations sum up to zero. Summing up over all $S_t^{i_t}$ a zero will also be obtained. Hence, if all dual constraints that concern $S_t^{i_t}$ are summed up, the following relation is obtained:

$$S_0 = \sum_{\{i_T : i_T \in I_T\}} \sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Omega^+_{i_t, i_T}\right\}} \lambda^{(i_T,T),m}_{(i_t,t)} S_T^{i_T} \tag{4}$$

Now, proceeding in a similar way but considering the dual constraints associated with B's: Because the right member of the constraints is equal to 0, excepting the one associated B_0, we multiply each constraint by a constant. The constraint associated with the variable B_{i_k} is multiplied by R^k. Then, all the constraints associated with B's are summed up, and

$$\sum_{\{i_T : i_T \in I_T\}} \sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Omega^+_{i_t, i_T}\right\}} q^{(i_T,T),m}_{(i_t,t)} = 1$$

where

$$q^{(i_T,T),m}_{(i_t,t)} = R^{T-t} \lambda^{(i_T,T),m}_{(i_t,t)}.$$

Denoting,

$$q^{i_T} = \sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{m \in \Omega^+_{i_t, i_T}\right\}} q^{(i_T,T),m}_{(i_t,t)} S_T^{i_T}$$

equation (4) can be written as

$$S_0 = \frac{1}{R^T} \sum_{\{i_T : i_T \in I_T\}} q^{i_T} S_T^{i_T}$$

with

$$\sum_{\{i_T : i_T \in I_T\}} q^{i_T} = 1$$

■

A.2 Proof of theorem 10 with $T = 3$

Proof. As the problem sketched in the example of section to obtain the upper bound is a linear programming problem, considering $S_t^{i_t} = U^{t-(i_t-1)}D^{i_t-1}$; its dual is written as follows:

$$\min_{\left\{\lambda^{(i_T;T);m}_{(i_t;t)};\ \alpha^{(i_{t'};t');m}_{(i_t;t)}\right\}} \left(\lambda^{(1;3)}_{(0;0)}+\lambda^{(1;3)}_{(1;1)}+\lambda^{(1;3)}_{(1;2)}\right)G_3^1$$

$$+\left(\lambda^{(2;3);1}_{(0;0)}+\lambda^{(2;3);3}_{(0;0)}+\lambda^{(2;3);3}_{(0;0)}+\lambda^{(2;3);1}_{(1;1)}+\lambda^{(2;3);2}_{(1;1)}+\lambda^{(2;3)}_{(2;1)}+\lambda^{(2;3)}_{(1;2)}+\lambda^{(2;3)}_{(2;2)}\right)G_3^2$$

$$+\left(\lambda^{(3;3);1}_{(0;0)}+\lambda^{(3;3);2}_{(0;0)}+\lambda^{(3;3);3}_{(0;0)}+\lambda^{(3;3)}_{(1;1)}+\lambda^{(3;3);1}_{(2;1)}+\lambda^{(3;3);2}_{(2;1)}+\lambda^{(3;3)}_{(2;2)}+\lambda^{(3;3)}_{(3;2)}\right)G_3^3$$

$$+\left(\lambda^{(4;3)}_{(0;0)}+\lambda^{(4;3)}_{(2;1)}+\lambda^{(4;3)}_{(3;2)}\right)G_3^4$$

subject to the non-negativity constraints of the dual variables

$$\lambda^{(i_T;T);m}_{(i_t;t)} \geq 0;$$

for all $i_t \in I_t^{i_T}$; $i_T \in I_T$ and $t = 0; 1$ and 2;

$$\alpha^{(i_{t'};t');m}_{(i_t;t)} \geq 0;$$

for all $i_t \in I_t^{i_{t'}}$; $i_{t'} \in I_{t'}$ and $t' = 0; 1$ and 2; and subject to twelve equality constraints, each one associated with a variable of the primal problem. The constraint associated with Δ_0 is given by

$$(1-p)^2\left[\lambda^{(1;3)}_{(0;0)}S_3^1+\sum_{m=1;2;3}\lambda^{(2;3);m}_{(0;0)}S_3^2+\sum_{m=1;2;3}\lambda^{(3;3);m}_{(0;0)}S_3^3+\lambda^{(4;3)}_{(0;0)}S_3^4\right]$$

$$+\alpha^{(1;1)}_{(0;0)}S_1^1+\alpha^{(2;1)}_{(0;0)}S_1^2+ \tag{5}$$

$$+(1-p)\left[\alpha^{(1;2)}_{(0;0)}S_2^1+\left(\alpha^{(2;2);1}_{(0;0)}+\alpha^{(2;2);2}_{(0;0)}\right)S_3^2+\alpha^{(3;2)}_{(0;0)}S_2^3\right]=S_0$$

The constraint associated with B_0 is given by

$$(1-p)^2R^3\left[\lambda^{(1;3)}_{(0;0)}+\sum_{m=1;2;3}\lambda^{(2;3);m}_{(0;0)}+\sum_{m=1;2;3}\lambda^{(3;3);m}_{(0;0)}+\lambda^{(4;3)}_{(0;0)}\right]$$

$$+R\left(\alpha^{(1;1)}_{(0;0)}+\alpha^{(2;1)}_{(0;0)}\right)+(1-p)R^2\left[\alpha^{(1;2)}_{(0;0)}+\alpha^{(2;2);1}_{(0;0)}+\alpha^{(1;2);2}_{(0;0)}+\alpha^{(3;2)}_{(0;0)}\right]=1 \tag{6}$$

The constraint associated with Δ_1^1 is given by

$$
\begin{aligned}
&(1-p)\left[\lambda^{(1;3)}_{(1;1)}S_3^1+\sum_{m=1;2}\lambda^{(2;3);m}_{(1;1)}S_3^2+\lambda^{(3;3)}_{(1;1)}S_3^3\right]+\\
&+p(1-p)\left[\lambda^{(1;3)}_{(0;0)}S_3^1+\sum_{m=1;2}\lambda^{(2;3);m}_{(0;0)}S_3^2+\lambda^{(3;3);1}_{(0;0)}S_3^3\right]+\\
&-\alpha^{(1;1)}_{(0;0)}S_1^1+p\left[\alpha^{(1;2)}_{(0;0)}S_2^1+\alpha^{(2;2);1}_{(0;0)}S_2^2\right]+\alpha^{(1;2)}_{(1;1)}S_2^1+\alpha^{(2;2)}_{(1;1)}S_2^2=0
\end{aligned}
\tag{7}
$$

The constraint associated with B_1^1 is given by

$$
\begin{aligned}
&(1-p)R^2\left[\lambda^{(1;3)}_{(1;1)}+\sum_{m=1;2}\lambda^{(2;3);m}_{(1;1)}+\lambda^{(3;3)}_{(1;1)}\right]+\\
&+p(1-p)R^2\left[\lambda^{(1;3)}_{(0;0)}+\sum_{m=1;2}\lambda^{(2;3);m}_{(0;0)}+\lambda^{(3;3);1}_{(0;0)}\right]+\\
&-\alpha^{(1;1)}_{(0;0)}+pR\left[\alpha^{(1;2)}_{(0;0)}+\alpha^{(2;2);1}_{(0;0)}\right]+R\left[\alpha^{(1;2)}_{(1;1)}+\alpha^{(2;2)}_{(1;1)}\right]=0
\end{aligned}
\tag{8}
$$

The constraint associated with Δ_1^2 is given by

$$
\begin{aligned}
&(1-p)\left[\lambda^{(2;3);3}_{(1;1)}S_3^2+\sum_{m=1;2}\lambda^{(3;3);m}_{(1;1)}S_3^3+\lambda^{(4;3)}_{(1;1)}S_3^4\right]+\\
&+p(1-p)\left[\lambda^{(2;3);3}_{(0;0)}S_3^2+\sum_{m=2;3}\lambda^{(3;3);m}_{(0;0)}S_3^3+\lambda^{(4;3)}_{(0;0)}S_3^4\right]+\\
&-\alpha^{(2;1)}_{(0;0)}S_1^2+p\left[\alpha^{(2;2);2}_{(0;0)}S_2^2+\alpha^{(3;2)}_{(0;0)}S_2^3\right]+\alpha^{(2;2)}_{(2;1)}S_2^2+\alpha^{(3;2)}_{(2;1)}S_2^3=0
\end{aligned}
\tag{9}
$$

The constraint associated with B_1^2 is given by

$$\begin{aligned}
&(1-p)R^2\left[\lambda^{(2;3);3}_{(1;1)} + \sum_{m=1;2}\lambda^{(3;3);m}_{(1;1)} + \lambda^{(4;3)}_{(1;1)}\right] + \\
&+p(1-p)R^2\left[\lambda^{(2;3);3}_{(0;0)} + \sum_{m=2;3}\lambda^{(3;3);m}_{(0;0)} + \lambda^{(4;3)}_{(0;0)}\right] + \\
&-\alpha^{(2;1)}_{(0;0)} + pR\left[\alpha^{(2;2);2}_{(0;0)} + \alpha^{(3;2)}_{(0;0)}\right] + R\left[\alpha^{(2;2)}_{(2;1)} + \alpha^{(3;2)}_{(2;1)}\right] = 0
\end{aligned} \tag{10}$$

The constraint associated with Δ_2^1 is given by

$$\begin{aligned}
&\lambda^{(1;3)}_{(1;2)}U^3S_0 + \lambda^{(2;3)}_{(1;2)}U^2DS_0 + p\left(\lambda^{(1;3)}_{(1;1)}U^3S_0 + \lambda^{(2;3);1}_{(1;1)}U^2DS_0\right) \\
&+p\left(\lambda^{(1;3)}_{(0;0)}U^3S_0 + \lambda^{(2;3);1}_{(0;0)}U^2DS_0\right) - \alpha^{(1;2)}_{(0;0)}U^2S_0 - \alpha^{(1;2);1}_{(1;1)}U^2S_0 = 0
\end{aligned} \tag{11}$$

The constraint associated with B_2^1 is given by

$$\begin{aligned}
&R\left[\lambda^{(1;3)}_{(1;2)} + \lambda^{(2;3)}_{(1;2)}\right] + pR\left[\lambda^{(1;3)}_{(1;1)} + \lambda^{(2;3);1}_{(1;1)}\right] \\
&+pR\left[\lambda^{(1;3)}_{(0;0)} + \lambda^{(2;3);1}_{(0;0)}\right] - \alpha^{(1;2);1}_{(0;0)} - \alpha^{(1;2)}_{(1;1)} = 0
\end{aligned} \tag{12}$$

The constraint associated with Δ_2^2 is given by

$$\begin{aligned}
&\lambda^{(2;3)}_{(2;2)}S_3^2 + \lambda^{(3;3)}_{(2;2)}S_3^3 + p\left[\left(\lambda^{(2;3);2}_{(1;1)} + \lambda^{(2;3)}_{(2;1)}\right)S_3^2 + \left(\lambda^{(3;3)}_{(1;1)} + \lambda^{(3;3);1}_{(2;1)}\right)S_3^3\right] \\
&+p\left[\left(\lambda^{(2;3);2}_{(0;0)} + \lambda^{(2;3);3}_{(0;0)}\right)S_3^2 + \left(\lambda^{(3;3);1}_{(0;0)} + \lambda^{(3;3);2}_{(0;0)}\right)S_3^3\right] \\
&-\left[\alpha^{(2;2);1}_{(0;0)} + \alpha^{(2;2);2}_{(0;0)}\right]S_2^2 - \left[\alpha^{(2;2)}_{(1;1)} + \alpha^{(2;2)}_{(2;1)}\right]S_2^2 = 0
\end{aligned} \tag{13}$$

The constraint associated with B_2^2 is given by

$$\begin{aligned}
&R\left[\lambda^{(2;3)}_{(2;2)} + \lambda^{(3;3)}_{(2;2)}\right] + pR\left[\lambda^{(2;3);2}_{(1;1)} + \lambda^{(2;3)}_{(2;1)} + \lambda^{(3;3)}_{(1;1)} + \lambda^{(3;3);1}_{(2;1)}\right] \\
&+pR\left[\lambda^{(2;3);2}_{(0;0)} + \lambda^{(2;3);3}_{(0;0)} + \lambda^{(3;3);1}_{(0;0)} + \lambda^{(3;3);2}_{(0;0)}\right] \\
&-\left[\alpha^{(2;2);1}_{(0;0)} + \alpha^{(2;2);2}_{(0;0)}\right] - \left[\alpha^{(2;2)}_{(1;1)} + \alpha^{(2;2)}_{(2;1)}\right] = 0
\end{aligned} \tag{14}$$

The constraint associated with Δ_2^3 is given by

$$\begin{gathered}\lambda^{(3,3)}_{(3,2)} U D^2 S_0 + \lambda^{(4,3)}_{(3,2)} D^3 S_0 + p\left[\lambda^{(3,3),2}_{(2,1)} U D^2 S_0 + \lambda^{(4,3)}_{(2,1)} D^3 S_0\right] \\ + p\left[\lambda^{(3,3),3}_{(0,0)} U D^2 S_0 + \lambda^{(4,3)}_{(0,0)} D^3 S_0\right] - \alpha^{(3,2)}_{(0,0)} D^2 S_0 - \alpha^{(3,2)}_{(1,1)} D^2 S_0 = 0\end{gathered} \tag{15}$$

The constraint associated with B_2^3 is given by

$$\begin{gathered}R\left[\lambda^{(3,3)}_{(3,2)} + \lambda^{(4,3)}_{(3,2)}\right] + pR\left[\lambda^{(3,3),2}_{(2,1)} + \lambda^{(4,3)}_{(2,1)}\right] + \\ + pR\left[\lambda^{(3,3),3}_{(0,0)} + \lambda^{(4,3)}_{(0,0)}\right] - \alpha^{(3,2)}_{(0,0)} - \alpha^{(3,2)}_{(1,1)} = 0\end{gathered} \tag{16}$$

Summing up equations (5), (7), (9), (11),(13) and (15) we obtain

$$\begin{gathered}S_0 = \left(\lambda^{(1,3)}_{(0,0)} + \lambda^{(1,3)}_{(1,1)} + \lambda^{(1,3)}_{(1,2)}\right) S_3^1 + \\ + \left(\sum_{m=1,2,3} \lambda^{(2,3),m}_{(0,0)} + \sum_{m=1,2} \lambda^{(2,3),m}_{(1,1)} + \lambda^{(2,3)}_{(2,1)} + \lambda^{(2,3)}_{(1,2)} + \lambda^{(2,3)}_{(2,2)}\right) S_3^2 \\ + \left(\sum_{m=1,2,3} \lambda^{(3,3),m}_{(0,0)} + \lambda^{(3,3)}_{(1,1)} + \sum_{m=1,2} \lambda^{(3,3),m}_{(2,1)} + \lambda^{(3,3)}_{(2,2)} + \lambda^{(3,3)}_{(3,2)}\right) S_3^3 \\ + \left(\lambda^{(4,3)}_{(0,0)} + \lambda^{(4,3)}_{(2,1)} + \lambda^{(4,3)}_{(3,2)}\right) S_3^4\end{gathered} \tag{17}$$

Multiplying equations (8) and (10) by R and equations (12), (14) and (15) by R^2 and then summing up with equation (6) we obtain

$$\begin{gathered}\lambda^{(1,3)}_{(0,0)} + \sum_{m=1,2,3} \lambda^{(2,3),m}_{(0,0)} + \sum_{m=1,2,3} \lambda^{(3,3),m}_{(0,0)} + \lambda^{(4,3)}_{(0,0)} + \\ \lambda^{(1,3)}_{(1,1)} + \sum_{m=1,2} \lambda^{(2,3),m}_{(1,1)} + \lambda^{(3,3)}_{(1,1)} + \lambda^{(2,3)}_{(2,1)} + \sum_{m=1,2} \lambda^{(3,3),m}_{(2,1)} + \lambda^{(4,3)}_{(2,1)} + \\ + \lambda^{(1,3)}_{(1,2)} + \lambda^{(2,3)}_{(1,2)} + \lambda^{(2,3)}_{(2,2)} + \lambda^{(3,3)}_{(2,2)} + \lambda^{(3,3)}_{(3,2)} + \lambda^{(4,3)}_{(3,2)} = \frac{1}{R^3}\end{gathered}$$

Hence, denoting

$$q^{(i_{t'},t')}_{(i_t,t)} = R^3 \lambda^{(i_{t'},t')}_{(i_t,t)}$$

and

$$q^1 = q^{(1,3)}_{(0,0)} + q^{(1,3)}_{(1,1)} + q^{(1,3)}_{(1,2)}$$
$$q^2 = \sum_{m=1,2,3} q^{(2,3);m}_{(0,0)} + \sum_{m=1,2} q^{(2,3);m}_{(1,1)} + q^{(2,3)}_{(2,1)} + q^{(2,3)}_{(1,2)} + q^{(2,3)}_{(2,2)}$$
$$q^3 = \sum_{m=1,2,3} q^{(3,3);m}_{(0,0)} + q^{(3,3)}_{(1,1)} + \sum_{m=1,2} q^{(3,3);m}_{(2,1)} + q^{(3,3)}_{(2,2)} + q^{(3,3)}_{(3,2)}$$
$$q^4 = q^{(4,3)}_{(0,0)} + q^{(4,3)}_{(2,1)} + q^{(4,3)}_{(3,2)}$$

we can rewrite (17) as

$$S_0 = \frac{1}{R^3}\left[q^1 S_3^1 + q^2 S_3^2 + q^3 S_3^3 + q^4 S_3^4 \right]$$

with q^i are defined above and $q^1 + q^2 + q^3 + q^4 = 1$.

Taking into consideration equation (10), equation (9) can be rewritten as

$$S_2^1 = \frac{1}{R}\left[\pi^{(1,3)}_{(1,2)} S_3^1 + \pi^{(2,3)}_{(1,2)} S_3^2 \right]$$

where

$$\pi^{(1,3)}_{(1,2)} = \frac{\lambda^{(1,3)}_{(1,2)} + \sqrt{\lambda^{(1,3)}_{(1,1)}} + \sqrt{\lambda^{(1,3)}_{(0,0)}}}{\lambda^{(1,3)}_{(1,2)} + \sqrt{\lambda^{(1,3)}_{(1,1)}} + \sqrt{\lambda^{(1,3)}_{(0,0)}} + \lambda^{(2,3)}_{(1,2)} + \sqrt{\lambda^{(2,3);1}_{(1,1)}} + \sqrt{\lambda^{(2,3);1}_{(0,0)}}}$$
$$\pi^{(2,3)}_{(1,2)} = \frac{\lambda^{(2,3)}_{(1,2)} + \sqrt{\lambda^{(2,3);1}_{(1,1)}} + \sqrt{\lambda^{(2,3);1}_{(0,0)}}}{\lambda^{(1,3)}_{(1,2)} + \sqrt{\lambda^{(1,3)}_{(1,1)}} + \sqrt{\lambda^{(1,3)}_{(0,0)}} + \lambda^{(2,3)}_{(1,2)} + \sqrt{\lambda^{(2,3);1}_{(1,1)}} + \sqrt{\lambda^{(2,3);1}_{(0,0)}}}$$

and $\pi^{(1,3)}_{(1,2)} + \pi^{(2,3)}_{(1,2)} = 1$. Proceeding in an analogous way with constraints (13), (14), (15) and (16) we obtain

$$S_2^2 = \frac{1}{R}\left[\pi^{(2,3)}_{(2,2)} S_3^2 + \pi^{(3,3)}_{(2,2)} S_3^3 \right]$$

where

$$\pi^{(2,3)}_{(2,2)} = \frac{\lambda^{(2,3)}_{(2,2)} + \sqrt{\left(\lambda^{(2,3);2}_{(1,1)} + \lambda^{(2,3)}_{(2,1)} + \lambda^{(2,3);2}_{(0,0)}\right) + \lambda^{(2,3);3}_{(0,0)}}}{\lambda^{(2,3)}_{(2,2)} + \sqrt{\left(\lambda^{(2,3);2}_{(1,1)} + \lambda^{(2,3)}_{(2,1)} + \lambda^{(2,3);2}_{(0,0)}\right) + \lambda^{(2,3);3}_{(0,0)}} + \lambda^{(3,3)}_{(2,2)} + \sqrt{\lambda^{(3,3)}_{(1,1)} + \lambda^{(3,3);1}_{(2,1)} + \lambda^{(3,3);1}_{(0,0)} + \lambda^{(3,3);2}_{(0,0)}}}$$
$$\pi^{(3,3)}_{(2,2)} = \frac{\lambda^{(3,3)}_{(2,2)} + \sqrt{\left(\lambda^{(3,3)}_{(1,1)} + \lambda^{(3,3);1}_{(2,1)} + \lambda^{(3,3);1}_{(0,0)}\right) + \lambda^{(3,3);2}_{(0,0)}}}{\lambda^{(2,3)}_{(2,2)} + \sqrt{\left(\lambda^{(2,3);2}_{(1,1)} + \lambda^{(2,3)}_{(2,1)} + \lambda^{(2,3);2}_{(0,0)}\right) + \lambda^{(2,3);3}_{(0,0)}} + \lambda^{(3,3)}_{(2,2)} + \sqrt{\lambda^{(3,3)}_{(1,1)} + \lambda^{(3,3);1}_{(2,1)} + \lambda^{(3,3);1}_{(0,0)} + \lambda^{(3,3);2}_{(0,0)}}}$$

with $\pi^{(2,3)}_{(2,2)} + \pi^{(3,3)}_{(2,2)} = 1$; and

$$S_2^3 = \frac{1}{R}\left[\pi^{(3,3)}_{(3,2)} S_3^3 + \pi^{(4,3)}_{(3,2)} S_3^4 \right]$$

where

$$\pi^{(3,3)}_{(3,2)} = \frac{\lambda^{(3,3)}_{(3,2)} + p\left(\lambda^{(3,3);2}_{(2,1)} + \lambda^{(3,3);3}_{(0,0)}\right)}{\lambda^{(3,3)}_{(3,2)} + p\left(\lambda^{(3,3);2}_{(2,1)} + \lambda^{(3,3);3}_{(0,0)}\right) + \lambda^{(4,3)}_{(3,2)} + p\left(\lambda^{(4,3)}_{(2,1)} + \lambda^{(4,3)}_{(0,0)}\right)}$$

$$\pi^{(4,3)}_{(3,2)} = \frac{\lambda^{(4,3)}_{(3,2)} + p\left(\lambda^{(4,3)}_{(2,1)} + \lambda^{(4,3)}_{(0,0)}\right)}{\lambda^{(3,3)}_{(3,2)} + p\left(\lambda^{(3,3);2}_{(2,1)} + \lambda^{(3,3);3}_{(0,0)}\right) + \lambda^{(4,3)}_{(3,2)} + p\left(\lambda^{(4,3)}_{(2,1)} + \lambda^{(4,3)}_{(0,0)}\right)}$$

with $\pi^{(3,3)}_{(3,2)} + \pi^{(4,3)}_{(3,2)} = 1.$

Moreover, using equations (7), (8), (9) and (10) we have

$$S_1^1 = \frac{1}{R^2}\left[\theta^{(1,3)}_{(1,1)} S_3^1 + \theta^{(2,3)}_{(1,1)} S_3^2 + \theta^{(3,3)}_{(1,1)} S_3^3\right] + \frac{1}{R}\left[\varepsilon^{(1,2)}_{(1,1)} S_2^1 + \varepsilon^{(2,2)}_{(1,1)} S_2^2\right]$$

where

$$\theta^{(1,3)}_{(1,1)} = \frac{R^2\left[(1-p)\lambda^{(1,3)}_{(1,1)} + p(1-p)\lambda^{(1,3)}_{(0,0)}\right]}{\Theta}$$

$$\theta^{(2,3)}_{(1,1)} = \frac{R^2\left[(1-p)\sum_{m=1,2}\lambda^{(2,3);m}_{(1,1)} + p(1-p)\sum_{m=1,2}\lambda^{(2,3);m}_{(0,0)}\right]}{\Theta}$$

$$\theta^{(3,3)}_{(1,1)} = \frac{R^2\left[(1-p)\lambda^{(3,3)}_{(1,1)} + p(1-p)\lambda^{(3,3);1}_{(0,0)}\right]}{\Theta}$$

$$\varepsilon^{(1,2)}_{(1,1)} = \frac{R\left[p\alpha^{(1,2)}_{(0,0)} + \alpha^{(1,2)}_{(1,1)}\right]}{\Theta}; \quad \varepsilon^{(2,2)}_{(1,1)} = \frac{R\left[p\alpha^{(2,2);1}_{(0,0)} + \alpha^{(2,2)}_{(1,1)}\right]}{\Theta}$$

$$\begin{aligned}\Theta = {} & R^2\left[(1-p)\lambda^{(1,3)}_{(1,1)} + p(1-p)\lambda^{(1,3)}_{(0,0)}\right] \\ & + R^2\left[(1-p)\sum_{m=1,2}\lambda^{(2,3);m}_{(1,1)} + p(1-p)\sum_{m=1,2}\lambda^{(2,3);m}_{(0,0)}\right] \\ & + R^2\left[(1-p)\lambda^{(3,3)}_{(1,1)} + p(1-p)\lambda^{(3,3);1}_{(0,0)}\right] + R\left[p\alpha^{(1,2)}_{(0,0)} + \alpha^{(1,2)}_{(1,1)} + p\alpha^{(2,2);1}_{(0,0)} + \alpha^{(2,2)}_{(1,1)}\right]\end{aligned}$$

and

$$S_1^2 = \frac{1}{R^2}\left[\theta^{(1,3)}_{(2,1)} S_3^2 + \theta^{(2,3)}_{(2,1)} S_3^3 + \theta^{(3,3)}_{(2,1)} S_3^4\right] + \frac{1}{R}\left[\varepsilon^{(1,2)}_{(2,1)} S_2^2 + \varepsilon^{(2,2)}_{(2,1)} S_2^3\right], \text{ with}$$

$$\theta^{(1,3)}_{(2,1)} = \frac{R^2\left[(1-p)\lambda^{(2,3);3}_{(1,1)} + p(1-p)\lambda^{(2,3);3}_{(0,0)}\right]}{\Upsilon}$$

$$\theta^{(2,3)}_{(2,1)} = \frac{R^2\left[(1-p)\sum_{m=1,2}\lambda^{(3,3);m}_{(1,1)} + p(1-p)\sum_{m=2,3}\lambda^{(3,3);m}_{(0,0)}\right]}{\Upsilon}$$

$$\theta^{(3,3)}_{(2,1)} = \frac{R^2\left[(1-p)\lambda^{(4,3)}_{(1,1)} + p\lambda^{(4,3)}_{(0,0)}(1-p)\right]}{\Upsilon}$$

$$\varepsilon^{(1,2)}_{(2,1)} = \frac{R\left[p\alpha^{(2,2);2}_{(0,0)} + \alpha^{(2,2)}_{(2,1)}\right]}{\Upsilon}, \quad \varepsilon^{(2,2)}_{(2,1)} = \frac{R\left[p\alpha^{(3,2)}_{(0,0)} + \alpha^{(3,2)}_{(2,1)}\right]}{\Upsilon}$$

$$
\begin{aligned}
\Upsilon \quad = \quad & R^2\left[(1-p)\,\lambda_{(1;1)}^{(2;3);3} + p(1-p)\,\lambda_{(0;0)}^{(2;3);3}\right] \\
& + R^2\left[(1-p)\sum_{m=1;2}\lambda_{(1;1)}^{(3;3);m} + p(1-p)\sum_{m=2;3}\lambda_{(0;0)}^{(3;3);m}\right] + \\
& + R^2\left[(1-p)\,\lambda_{(1;1)}^{(4;3)} + p\,\lambda_{(0;0)}^{(4;3)}(1-p)\right] + R\left[p\alpha_{(0;0)}^{(2;2);2} + \alpha_{(2;1)}^{(2;2)} + p\alpha_{(0;0)}^{(3;2)} + \alpha_{(2;1)}^{(3;2)}\right].
\end{aligned}
$$

■

A.3 Upper Bound for $T = 2$

For $T = 2$ the problem that must be solved to find the upper bound of the arbitrage free range of variation is the following

$$
C_u = \min_{\{\Delta_0;B_0;\Delta_1^1;B_1^1;\Delta_1^2;B_1^2\}} \Delta_0 S_0 + B_0
$$

subject to the conditions of positive expected payoff at time $t = 0$;

$$
\begin{aligned}
p\left(\Delta_1^1 U^2 S_0 + RB_1^1\right) + (1-p)\left(\Delta_0 U^2 S_0 + R^2 B_0\right) &\geq \left(U^2 S_0 - K\right)^+ &(18)\\
p\left(\Delta_1^1 UDS_0 + RB_1^1\right) + (1-p)\left(\Delta_0 UDS_0 + R^2 B_0\right) &\geq (UDS_0 - K)^+ \\
p\left(\Delta_1^2 UDS_0 + RB_1^2\right) + (1-p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right) &\geq (UDS_0 - K)^+ \\
p\left(\Delta_1^2 D^2 S_0 + RB_1^2\right) + (1-p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right) &\geq \left(D^2 S_0 - K\right)^+ &(19)
\end{aligned}
$$

positive expected payoff at time $t = 1$;

$$
\begin{aligned}
\left(\Delta_1^1 U^2 S_0 + RB_1^1\right) &\geq \left(U^2 S_0 - K\right)^+ \\
\left(\Delta_1^1 UDS_0 + RB_1^1\right) &\geq (UDS_0 - K)^+ &(20)\\
\left(\Delta_1^2 UDS_0 + RB_1^2\right) &\geq (UDS_0 - K)^+ &(21)\\
\left(\Delta_1^2 D^2 S_0 + RB_1^2\right) &\geq \left(D^2 S_0 - K\right)^+
\end{aligned}
$$

and self-financing,

$$
\begin{aligned}
\Delta_1^1 US_0 + B_1^1 &\leq \Delta_0 US_0 + RB_0 &(22)\\
\Delta_1^2 DS_0 + B_1^2 &\leq \Delta_0 DS_0 + RB_0. &(23)
\end{aligned}
$$

Construct the Lagrangean

$$
\begin{aligned}
L \;=\;& \Delta_0 S_0 + B_0 + \\
&+\lambda^{(1,2)}_{(0,0)}\left[\left(U^2S_0 - K\right)^+ - p\left(\Delta_1^1 U^2 S_0 + RB_1^1\right) - (1-p)\left(\Delta_0 U^2 S_0 + R^2 B_0\right)\right] + \\
&+\lambda^{(2,2),1}_{(0,0)}\left\{(UDS_0 - K)^+ - p\left(\Delta_1^1 UDS_0 + RB_1^1\right) - (1-p)\left(\Delta_0 UDS_0 + R^2 B_0\right)\right\} \\
&+\lambda^{(2,2),2}_{(0,0)}\left\{(UDS_0 - K)^+ - p\left(\Delta_1^2 UDS_0 + RB_1^2\right) - (1-p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right)\right\} \\
&+\lambda^{(3,2)}_{(0,0)}\left[\left(D^2S_0 - K\right)^+ - p\left(\Delta_1^2 D^2 S_0 + RB_1^2\right) - (1-p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right)\right] \\
&+\alpha^{(2,1)}_{(0,0)}\left\{\Delta_1^1 US_0 + B_1^1 - \Delta_0 US_0 - RB_0\right\} \\
&+\alpha^{(1,1)}_{(0,0)}\left\{\Delta_1^2 DS_0 + B_1^2 - (\Delta_0 DS_0 + RB_0)\right\} \\
&+\lambda^{(1,2)}_{(1,1)}\left[\left(U^2S_0 - K\right)^+ - \left(\Delta_1^1 U^2 S_0 + RB_1^1\right)\right] \\
&+\lambda^{(2,2)}_{(1,1)}\left\{(UDS_0 - K)^+ - \left(\Delta_1^1 UDS_0 + RB_1^1\right)\right\} \\
&+\lambda^{(2,2)}_{(2,1)}\left\{(UDS_0 - K)^+ - \left(\Delta_1^2 UDS_0 + RB_1^2\right)\right\} \\
&+\lambda^{(3,2)}_{(2,1)}\left[\left(D^2S_0 - K\right)^+ - \left(\Delta_1^2 D^2 S_0 + RB_1^2\right)\right]
\end{aligned}
$$

The solution is characterized by

$$\lambda^{(2,2),1}_{(0,0)} = \lambda^{(2,2),2}_{(0,0)} = \lambda^{(1,2)}_{(1,1)} = \lambda^{(3,2)}_{(2,1)} = 0$$

and

$$\lambda^{(1,2)}_{(0,0)}, \lambda^{(3,2)}_{(0,0)}, \alpha^{(2,1)}_{(0,0)}, \alpha^{(1,1)}_{(0,0)}, \lambda^{(2,2)}_{(1,1)}, \lambda^{(2,2)}_{(2,1)} \geq 0$$

Using the fact that equations (18), (19), (22), (23), (20) and (21) are binding the optimal values Δ_0^*; B_0^*; ; Δ_1^{1*}; B_1^{1*}; Δ_1^{2*}; B_1^{2*}can be obtained. In particular, Δ_0^* e B_0^* are given by

$$
\begin{aligned}
\Delta_0^* &= \frac{\left(U^2S_0 - K\right)^+ \left[(U - R)(R - D) + p(R - D)^2\right]}{S_0\Lambda} \\
&\quad - \frac{\left(D^2S_0 - K\right)^+ \left[(U - R)(R - D) + p(R - U)^2\right]}{S_0\Lambda} \\
&\quad - \frac{p(UDS_0 - K)^+ \left[-2DR + 2RU + D^2 - U^2\right]}{S_0\Lambda}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
B_0^* &= -\frac{\left(U^2S_0 - K\right)^+ \left[D^2(U - R)(R - D) + pUD(R - D)^2\right]}{R^2\Lambda} \\
&\quad + \frac{\left(D^2S_0 - K\right)^+ \left[U^2(U - R)(R - D) + pUD(R - U)^2\right]}{R^2\Lambda} \\
&\quad + \frac{pR(UDS_0 - K)^+ \left(-2DU^2 + RU^2 - RD^2 + 2D^2U\right)}{R^2\Lambda}
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda &= U^2\left[(U - R)(R - D) - p\left(-R^2 + 4RD - DU\right)\right] \\
&\quad - D^2\left[(U - R)(R - D) - p\left(-R^2 - UD + 4RU\right)\right]
\end{aligned}
$$

The remaining equations are also satisfied.

As a result, after some trivial algebra, we obtain

$$
C_u = \Delta_0^* S_0 + B_0^* = \frac{1}{R^2}\left[q_1\left(U^2S_0 - K\right)^+ + q_2(UDS_0 - K)^+ + q_3\left(D^2S_0 - K\right)^+\right]
$$

with

$$
\begin{aligned}
q_1 &= \frac{(U - R)(R - D)\left(R^2 - D^2\right) + p(R - D)^2\left(R^2 - UD\right)}{\Lambda} \\
q_2 &= p\frac{(R - D)^2\left(U^2 - R^2\right) + (R - U)^2\left(R^2 - D^2\right)}{\Lambda} \\
q_3 &= \frac{(U - R)(R - D)\left(U^2 - R^2\right) + p(R - U)^2\left(UD - R^2\right)}{\Lambda}
\end{aligned}
$$

It is easy to check that $q_1; q_2; q_3 \geq 0$ and $q_1 + q + q_3 = 1$:

A.4 Proofs of the Properties and Examples

A.4.1 Property 3

1. **Proof.** Let the set of admissible solutions that characterize the upper bound for the case $p \in (0,1)$ be denoted by $A(p)$, where $A(p)$ is a correspondence such that

$$A(p) : [0,1] \to R^{t(t+1)}.$$

The portfolio $(\Delta, B) = (\Delta_{i_t}, B_{i_t})_{i_t \in I_t,\, t=0,\dots,T-1} \in R^{t(t+1)}$ is said to be an admissible solution for the problem defined in section 4.1.1 if $(\Delta, B) \in A(p)$.

Moreover, let $A(p = 0)$ and $A(p = 1)$ denote, respectively, the admissible solutions for the problems characterizing the upper bound in the case $p = 0$ and $p = 1$, presented in section 2.1.

By the Theorem of the Maximum,[13] if the constraint correspondence $A(p)$ is continuous and if the objective function is continuous on p, then the value of the objective function in the optimum is also continuous on p.

First consider the case $p \to 1$. In this case, $\lim_{p \to 1} A(p) = A(p = 1)$. Hence,

$$\lim_{p \to 1} C_u^p = C_u^p(p = 1) = C_u^1.$$

However, the same does not apply when $p \to 0$. In this case

$$\lim_{p \to 0} A(p) \subset A(p = 0)$$

which implies that

$$C_u^p(p = 0) \geq C_u^0.$$

However, using property 1, we conclude $C_u^p(p = 0) = C_u^0$. ■

Example 21 (for $T=2$.) When $p \to 0$, and in the case of a Call Option, which means that the market is completely illiquidity, the optimal values of Δ_0 and B_0 in expression (24) converge to

$$\Delta_0^* = \frac{\left(U^2 S_0 - K\right)^+ - \left(D^2 S_0 - K\right)^+}{S_0\left[U^2 - D^2\right]}$$

$$B_0^* = \frac{U^2\left(D^2 S_0 - K\right)^+ - D^2\left(U^2 S_0 - K\right)^+}{R^2\left(U^2 - D^2\right)}$$

[13] See, for instance, Mas-Collel et al. (1995), page 963.

then,

$$\begin{aligned}
\lim_{p\to 0} C_u^p &= \Delta_0^* S_0 + B_0^* \\
&= \frac{1}{R^2}\left[\left(\frac{R^2-D^2}{U^2-D^2}\right)\left(U^2 S_0 - K\right)^+ + \left(\frac{U^2-R^2}{U^2-D^2}\right)\left(D^2 S_0 - K\right)^+\right];
\end{aligned}$$

i.e., $\lim_{p\to 0} C_u^p = C_u^0$. On the other hand, when $p \to 1$, which means that there is no illiquidity, the optimal values of Δ_0 and B_0 in expression (24) converge to

$$\begin{aligned}
\Delta_0^* &= \frac{\left(U^2 S_0 - K\right)^+ (R-D) - \left(D^2 S_0 - K\right)^+ (U-R) + (UDS_0 - K)^+ (U - 2R + D)}{S_0 R (U-D)^2} \\
B_0^* &= -\frac{D(R-D)}{R^2 (U-D)^2}\left(U^2 S_0 - K\right)^+ + \frac{U(U-R)}{R^2(U-D)^2}\left(D^2 S_0 - K\right)^+ \\
&\quad + \frac{[-D(U-R) + U(R-D)]}{R^2(U-D)^2}(UDS_0 - K)^+
\end{aligned}$$

then,

$$\begin{aligned}
\lim_{p\to 0} C_u^p &= \Delta_0^* S_0 + B_0^* \\
&= \frac{1}{R^2}\sum_{j=0}^{2}\binom{2}{j}\left(\frac{R-D}{U-D}\right)^j\left(\frac{U-R}{U-D}\right)^{2-j}\left(U^j D^{T-j} S_0 - K\right)^+;
\end{aligned}$$

i.e., $\lim_{p\to 0} C_u^p = C_u^1$.

A.4.2 Property 4

Proof. Consider the trading strategy $\left(\Delta_t^{1*}, B_t^{1*}\right)_{t=0,\ldots,T-1}$ that solves the maximization problem that characterizes the upper bound for a $p \in (0,1)$.

Fix a given path w, such that $(i_t)_{t=0,\ldots,T} \in w$. For each path we have T superreplicating constraints, one for each $t \in \{0,\ldots,T-1\}$. For $T-1$;

$$\Delta^{1*}_{i_{T-1}} S_T^{i_T} + R B^{1*}_{i_{T-1}} \geq G_T^{i_T};$$

for $t = T-2$

$$(1-p)\left(\Delta^{1*}_{i_{T-2}} S_T^{i_T} + R^2 B^{1*}_{i_{T-2}}\right) + p\left(\Delta^{1*}_{i_{T-1}} S_T^{i_T} + R B^{1*}_{i_{T-1}}\right) \geq G_T^{i_T};$$

which can be rewritten as

$$(1-p)\left[\Delta^*_{i_{T-2}}S_T^{i_T}+R^2B^*_{i_{T-2}}-G_T^{i_T}\right]+p\left[\Delta^*_{i_{T-1}}S_T^{i_T}+RB^*_{i_{T-1}}-G_T^{i_T}\right]\geq 0;$$

for $t=T-3$

$$\begin{aligned}(1-p)^2\left(\Delta^*_{i_{T-3}}S_T^{i_T}+R^3B^*_{i_{T-3}}\right)+p(1-p)\left(\Delta^*_{i_{T-2}}S_T^{i_T}+R^2B^*_{i_{T-2}}\right)+\\+\left[p(1-p)+p^2\right]\left(\Delta^*_{i_{T-1}}S_T^{i_T}+RB^*_{i_{T-1}}\right)\geq G_T^{i_T};\end{aligned}$$

which can be rewritten as

$$\begin{aligned}(1-p)^2\left[\Delta^*_{i_{T-3}}S_T^{i_T}+R^3B^*_{i_{T-3}}-G_T^{i_T}\right]+p\left[\Delta^*_{i_{T-1}}S_T^{i_T}+RB^*_{i_{T-1}}-G_T^{i_T}\right]\\+p(1-p)\left[\Delta^*_{i_{T-2}}S_T^{i_T}+R^2B^*_{i_{T-2}}-G_T^{i_T}\right]\geq 0;\end{aligned}$$

More generally, for t

$$\begin{aligned}E_{i_t} &= E_{i_{t+1}}+(1-p)^{T-t-1}\left(\Delta^*_{i_t}S_T^{i_T}+R^{T-t}B^*_{i_t}\right)+\\&\quad-(1-p)^{T-t-1}\left(\Delta^*_{i_{t+1}}S_T^{i_T}+R^{T-t}B^*_{i_{t+1}}\right)\\&= E_{i_{t+1}}+(1-p)^{T-t-1}\left[\left(\Delta^*_{i_t}S_T^{i_T}+R^{T-t}B^*_{i_t}\right)-\left(\Delta^*_{i_{t+1}}S_T^{i_T}+R^{T-t}B^*_{i_{t+1}}\right)\right]\geq 0\end{aligned}$$

By backward induction since $T=t-1$; we can prove that $E_{i_{t+1}}$ is positive. Hence, if E_{i_t} is positive for a given p it will also be positive for another p; because when p increases $(1-p)^{T-t-1}$ decreases. Hence, whatever the sign of

$$\left(\Delta^*_{i_t}S_T^{i_T}+R^{T-t}B^*_{i_t}\right)-\left(\Delta^*_{i_{t+1}}S_T^{i_T}+R^{T-t}B^*_{i_{t+1}}\right)$$

E_{i_t} will remain positive.

The same applies for the self-financing conditions. ■

A.4.3 Property 5

The upper bound of the ask price in a complete market can be written as

$$C_u^1=\frac{1}{R^2}\left[\pi_1^{liq}\left(U^2S_0-K\right)^+ +\pi_2^{liq}(UDS_0-K)^+ +\pi_3^{liq}\left(D^2S_0-K\right)^+\right]$$

with

$$\pi_1^{liq} = \frac{(R-D)^2}{(U-D)^2}$$

$$\pi_1^{liq} = \frac{(R-D)(U-R)}{(U-D)^2}$$

$$\pi_1^{liq} = \frac{(U-R)^2}{(U-D)^2}$$

and the upper bound of the ask price in an illiquid market can be written as

$$C_u^0 = \frac{1}{R^2}\left[\pi_1^{illiq}\left(U^2S_0 - K\right)^+ + \pi_3^{illiq}\left(D^2S_0 - K\right)^+\right]$$

with

$$\pi_1^{illiq} = \frac{R^2 - D^2}{U^2 - D^2}$$

$$\pi_3^{illiq} = \frac{U^2 - R^2}{U^2 - D^2}$$

Hence, the upper bound for the ask price in a random dry market can be written

$$C_u^p = \alpha C_u^1 + (1-\alpha)\, C_u^0 \tag{25}$$

with

$$\alpha = \frac{p\frac{R(U-D)^2}{(U-R)(U+D)(R-D)}}{1 + p\left[\frac{R(U-D)^2}{(U-R)(U+D)(R-D)} - 1\right]}$$

$$1-\alpha = \frac{1-p}{1 + p\left[\frac{R(U-D)^2}{(U-R)(U+D)(R-D)} - 1\right]}$$

Taking the derivative of α with respect to p we obtain

$$\frac{\partial\alpha}{\partial p} = \frac{R(U-D)^2(R+D)(U-R)(U+D)\left(R^2-D^2\right)}{\left\{(U-R)(U+D)(R^2-D^2) + p\left[(R-D)^2U(R+D) + (U-R)^2D(R+D)\right]\right\}^2} \geq 0$$

Hence, taking the derivative of C_u^0 with respect to p we conclude that C_u^0 is a decreasing function of p;

$$\begin{aligned} \frac{\partial C_u^p}{\partial p} &= \frac{\partial \alpha}{\partial p} C_u^1 - \frac{\partial \alpha}{\partial p} C_u^0 \\ &= \frac{\partial \alpha}{\partial p} \left(C_u^1 - C_u^0 \right) \leq 0. \end{aligned}$$

Taking the second derivative we can also conclude that C_u^p is a convex function with respect to p;

$$\begin{aligned} \frac{\partial^2 C_u^p}{\partial p^2} &= \frac{\partial^2 \alpha}{\partial p^2} C_u^1 - \frac{\partial^2 \alpha}{\partial p^2} C_u^0 \\ &= \frac{\partial^2 \alpha}{\partial p^2} \left(C_u^1 - C_u^0 \right) \end{aligned}$$

as

$$\frac{\partial^2 \alpha}{\partial p^2} \leq 0$$

we have

$$\frac{\partial^2 C_u^p}{\partial p^2} \geq 0.$$

Alternative proof: After some algebra we obtain that following relation between α, in equation (25) and p:

$$\alpha > p$$

So, as

$$C_0^{ask\ liq} - C_0^{ask\ illiq} \leq 0$$

we obtain,

$$\begin{aligned} (p - \alpha) \left(C_0^{ask\ liq} - C_0^{ask\ illiq} \right) &\geq 0 \\ pC_0^{ask\ liq} + (1 - p) C_0^{ask\ illiq} - \left[\alpha C^{ask\ liq} + (1 - \alpha) C^{ask\ illiq} \right] &\geq 0 \end{aligned}$$

and follows that

$$C_0^{ask} \leq pC_0^{ask\ liq} + (1 - p) C_0^{ask\ illiq}.$$

B Some Proofs on the Solution of the Lower Bound for Statistical Arbitrage Opportunities

B.1 Proof of theorem 12

Proof. The problem that must be solved in order to find the upper bound of the range of variation of the arbitrage-free value of an European derivative is a linear programming problem. Its dual problem is

$$\max_{\lambda^{i_T}} \sum_{j=1}^{T-1} \lambda^{i_T} G_T^{i_T}$$

where λ^{i_T} is the sum of the dual variables associated with the positive expected payoff constraints that have the right member equal to $G_T^{i_T}$, i.e.,

$$\lambda^{i_T} = \sum_{t=0}^{T-1} \sum_{\{i_t : i_t \in I_t\}} \sum_{\left\{ m \in \Omega^{+}_{i_t;i_T} \right\}} \lambda^{(i_T;T);m}_{(i_t;t)} S_T^{i_T}$$

The first set of constraints is of nonnegativity of each dual variable, i.e, $\lambda^{(i_T;T);m}_{(i_t;t)}, \alpha^{(i_{t'};t');m}_{(i_t;t)} \geq 0$: The other set of constraints consists of equality constraints, one constraint associated with each variable of the primal problem. The other set are equality constraints which are equal to the ones obtained for the upper bound. Using the same argument as in the proof of theorem 1 the proof is straightforward. ■

B.2 Lower Bound for T=2

For $T = 2$ the problem that must be solved to find the lower bound of the arbitrage free range of variation is the following

$$C_l = \max_{\{\Delta_0; B_0; \Delta_1^1; B_1^1; \Delta_1^2; B_1^2\}} \Delta_0 S_0 + B_0$$

subject to the conditions of positive expected payoff at time $t = 0$;

$$\begin{aligned}
p\left[\Delta_1^1 U^2 S_0 + R B_1^1\right] + (1-p)\left[\Delta_0 U^2 S_0 + R^2 B_0\right] &\leq \left[U^2 S_0 - K\right]^+ \\
p\left[\Delta_1^1 U D S_0 + R B_1^1\right] + (1-p)\left[\Delta_0 U D S_0 + R^2 B_0\right] &\leq (U D S_0 - K)^+ \\
p\left[\Delta_1^2 U D S_0 + R B_1^2\right] + (1-p)\left[\Delta_0 D^2 S_0 + R^2 B_0\right] &\leq (U D S_0 - K)^+ \\
p\left[\Delta_1^2 D^2 S_0 + R B_1^2\right] + (1-p)\left[\Delta_0 D^2 S_0 + R^2 B_0\right] &\leq \left[D^2 S_0 - K\right]^+
\end{aligned}$$

positive expected payoff at time $t = 1$;

$$
\begin{aligned}
\left(\Delta_1^1 U^2 S_0 + R B_1^1\right) &\leq \left(U^2 S_0 - K\right)^+ \\
\left(\Delta_1^1 U D S_0 + R B_1^1\right) &\leq (U D S_0 - K)^+ \\
\left(\Delta_1^2 U D S_0 + R B_1^2\right) &\leq (U D S_0 - K)^+ \\
\left(\Delta_1^2 D^2 S_0 + R B_1^2\right) &\leq \left(D^2 S_0 - K\right)^+
\end{aligned}
$$

and self-financing,

$$
\begin{aligned}
\Delta_1^1 U S_0 + B_1^1 &\geq \Delta_0 U S_0 + R B_0 \\
\Delta_1^2 D S_0 + B_1^2 &\geq \Delta_0 D S_0 + R B_0 .
\end{aligned}
$$

The Lagrangean of the problem is

$$
\begin{aligned}
L \;=\;& \Delta_0 S_0 + B_0 + \\
&+ \mu_{(0,0)}^{(1,2)} \left[\left(U^2 S_0 - K\right)^+ - p\left(\Delta_1^1 U^2 S_0 + R B_1^1\right) - (1 - p)\left(\Delta_0 U^2 S_0 + R^2 B_0\right)\right] + \\
&+ \mu_{(0,0)}^{(2,2),1} \left[(U D S_0 - K)^+ - p\left(\Delta_1^1 U D S_0 + R B_1^1\right) - (1 - p)\left(\Delta_0 U D S_0 + R^2 B_0\right)\right] \\
&+ \mu_{(0,0)}^{(2,2),2} \left[(U D S_0 - K)^+ - p\left(\Delta_1^2 U D S_0 + R B_1^2\right) - (1 - p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right)\right] \\
&+ \mu_{(0,0)}^{(3,2)} \left[\left(D^2 S_0 - K\right)^+ - p\left(\Delta_1^2 D^2 S_0 + R B_1^2\right) - (1 - p)\left(\Delta_0 D^2 S_0 + R^2 B_0\right)\right] \\
&+ \alpha_{(0,0)}^{(2,1)} \left[\Delta_1^1 U S_0 + B_1^1 - \Delta_0 U S_0 - R B_0\right] \\
&+ \alpha_{(0,0)}^{(1,1)} \left[\Delta_1^2 D S_0 + B_1^2 - (\Delta_0 D S_0 + R B_0)\right] \\
&+ \mu_{(1,1)}^{(1,2)} \left[\left(U^2 S_0 - K\right)^+ - \left(\Delta_1^1 U^2 S_0 + R B_1^1\right)\right] \\
&+ \mu_{(1,1)}^{(2,2)} \left[(U D S_0 - K)^+ - \left(\Delta_1^1 U D S_0 + R B_1^1\right)\right] \\
&+ \mu_{(2,1)}^{(2,2)} \left[(U D S_0 - K)^+ - \left(\Delta_1^2 U D S_0 + R B_1^2\right)\right] \\
&+ \mu_{(2,1)}^{(3,2)} \left[\left(D^2 S_0 - K\right)^+ - \left(\Delta_1^2 D^2 S_0 + R B_1^2\right)\right]
\end{aligned}
$$

The constraints that are binding depend on the value of the parameters. First, if $R^2 - UD < 0$ and $p > \frac{UD - R^2}{UD - R^2 + R(U - D)}$ or $R^2 - UD > 0$ and $p > \frac{R^2 - UD}{R^2 - UD + R(U - D)}$ the optimal solution of the dual problem is characterized by

$$
\mu_{(0,0)}^{(1,2)} = \mu_{(0,0)}^{(3,2)} = \mu_{(1,1)}^{(2,2)} = \mu_{(2,1)}^{(2,2)} = 0
$$

and

$$\lambda^{(2;2);1}_{(0;0)}, \lambda^{(2;2);2}_{(0;0)}, \alpha^{(2;1)}_{(0;0)}, \alpha^{(1;1)}_{(0;0)}, \lambda^{(1;2)}_{(1;1)}, \lambda^{(3;2)}_{(2;1)} \geq 0.$$

Second, if $R^2 - UD < 0$ and $p < \frac{UD - R^2}{UD - R^2 + R(U - D)}$ the optimal solution of the dual problem is characterized by

$$\lambda^{(1;2)}_{(0;0)} = \lambda^{(2;2);1}_{(0;0)} = \lambda^{(1;2)}_{(1;1)} = \lambda^{(2;2)}_{(1;1)} = \alpha^{(1;1)}_{(0;0)} = 0$$

and

$$\lambda^{(2;2);2}_{(0;0)}, \lambda^{(3;2)}_{(0;0)}, \alpha^{(2;1)}_{(0;0)}, \lambda^{(2;2)}_{(2;1)}, \lambda^{(3;2)}_{(2;1)} \geq 0.$$

Finally, if $R^2 - UD > 0$ and $p < \frac{R^2 - UD}{R^2 - UD + R(U - D)}$ the optimal solution of the dual problem is characterized by

$$\lambda^{(2;2);2}_{(0;0)} = \lambda^{(3;2)}_{(0;0)} = \lambda^{(2;2)}_{(2;1)} = \lambda^{(3;2)}_{(2;1)} = \alpha^{(2;1)}_{(0;0)} = 0$$

and

$$\lambda^{(1;2)}_{(0;0)}, \lambda^{(2;2);1}_{(0;0)}, \alpha^{(1;1)}_{(0;0)}, \lambda^{(1;2)}_{(1;1)}, \lambda^{(2;2)}_{(1;1)} \geq 0.$$

Hence, after some algebra the lower bound can be written as:

$$C_l = q^1 \left[U^2 S_0 - K\right]^+ + q^2 \left[UDS_0 - K\right]^+ + q^3 \left[U^2 S_0 - K\right]^+ \tag{26}$$

with

$$q^1 = \frac{(R-D)^2 \left(UD - R^2\right) + p(U-R)(R-D)\left(D^2 - R^2\right)}{R^2 \left[(U-R)^2 (D^2 - UD) + (R-D)^2 (UD - D^2) + p(U-R)(R-D)(D^2 - U^2)\right]}$$

$$q^2 = \frac{(U-R)^2 \left(D^2 - R^2\right) + (R-D)^2 \left(R^2 - U^2\right)}{R^2 \left[(U-R)^2 (D^2 - UD) + (R-D)^2 (UD - D^2) + p(U-R)(R-D)(D^2 - U^2)\right]}$$

$$q^3 = \frac{(U-R)^2 \left(R^2 - UD\right) + p(U-R)(R-D)\left(R^2 - U^2\right)}{R^2 \left[(U-R)^2 (D^2 - UD) + (R-D)^2 (UD - D^2) + p(U-R)(R-D)(D^2 - U^2)\right]}$$

if $R^2 - UD < 0$ and $p > \frac{UD - R^2}{UD - R^2 + R(U - D)}$ or $R^2 - UD > 0$ and $p > \frac{R^2 - UD}{R^2 - UD + R(U - D)}$,

$$C_l = \frac{1}{R^2} \left\{ \frac{R^2 - D^2}{D(U-D)} \left[UDS_0 - K\right]^+ + \frac{UD - R^2}{D(U-D)} \left[D^2 S_0 - K\right]^+ \right\} \tag{27}$$

if $R^2 - UD < 0$ and $p < \frac{UD - R^2}{UD - R^2 + R(U - D)}$ and

$$C_l = \frac{1}{R^2}\left\{\frac{R^2 - UD}{U(U - D)}\left[U^2 S_0 - K\right]^+ + \frac{U^2 - R^2}{U(U - D)}\left[UDS_0 - K\right]^+\right\} \qquad (28)$$

if $R^2 - UD > 0$ and $p < \frac{R^2 - UD}{R^2 - UD + R(U - D)}$:

B.3 Example on Property 3

Example 22 (for $T=2$) When $p \to 0$, which means that the market is completely illiquidity, the values of the lower bound are given by (27) or (28) depending on $R^2 - UD \gtrless 0$. These values coincide with the ones presented in (2) for $T=2$. On the other hand, when $p \to 1$, which means that there is no illiquidity, the values of q_1; q_2 and q_3 presented in (26) tend to

$$\begin{aligned}
\lim_{p \to 1} q_1 &= \frac{(R - D)^2\left(UD - R^2\right) + (U - R)(R - D)\left(D^2 - R^2\right)}{R^2(U - R)^2(D^2 - UD) + R^2(R - D)^2(UD - U^2) + R^2(U - R)(R - D)(D^2 - U} \\
&= \frac{(R - D)^2}{R^2(U - D)^2} \\
\lim_{p \to 1} q_2 &= \frac{(U - R)^2\left(D^2 - R^2\right) + (R - D)^2\left(R^2 - U^2\right)}{R^2(U - R)^2(D^2 - UD) + R^2(R - D)^2(UD - D^2) + R^2(U - R)(R - D)(D^2 - U} \\
&= 2\frac{(R - D)(U - R)}{R^2(U - D)^2} \\
\lim_{p \to 1} q_3 &= \frac{(U - R)^2\left(R^2 - UD\right) + (U - R)(R - D)\left(R^2 - U^2\right)}{R^2(U - R)^2(D^2 - UD) + R^2(R - D)^2(UD - D^2) + R^2(U - R)(R - D)(D^2 - U} \\
&\quad \frac{(U - R)^2}{R^2(U - D)^2}
\end{aligned}$$

then,

$$\begin{aligned}
\lim_{p \to 0} C_l &= \frac{(R - D)^2}{R^2(U - D)^2}\left[U^2 S_0 - K\right]^+ + 2\frac{(R - D)(U - R)}{R^2(U - D)^2}\left[UDS_0 - K\right]^+ \\
&\quad + \frac{(U - R)^2}{R^2(U - D)^2}\left[U^2 S_0 - K\right]^+
\end{aligned}$$

which coincides with C_l; presented in equation (1).