Valuing companies with a fixed book-value leverage ratio

Pablo Fernandez*
IESE Business School

Abstract

We develop valuation formulae for a company that maintains a fixed book-value leverage ratio and claim that it is more realistic than to assume, as Miles-Ezzell (1980), a fixed market-value leverage ratio. We show that the appropriate discount rates for the expected equity cash flows and for the expected values of the equity are different. Modigliani-Miller and Miles-Ezzell do not make any assumption about the appropriate discount rate for the increases of the book value of assets, but this assumption is needed to calculate the value of the taxes paid by the levered and the unlevered company

November 16, 2005

JEL classification: G12; G31; G32

Keywords: Company valuation, value of tax shields, present value of the net increases of debt, required return to equity

I thank my colleagues José Manuel Campa and Charles Porter for their wonderful help revising earlier manuscripts of this paper, and Enrique Arzac and Ian Cooper for very helpful comments. I also thank Rafael Termes and my colleagues at IESE for their sharp questions that encouraged me to explore valuation problems.

*Contact information:
IESE Business School, University of Navarra. Camino del Cerro del Aguila 3. 28023 Madrid, Spain.
E-mail: fernandezpa@iese.edu

The value of tax shields defines the increase in the company's value as a result of the tax saving obtained by the payment of interest. However, there is no consensus in the existing literature regarding the correct way to compute the value of tax shields. Most authors think of calculating the value of the tax shield in terms of the appropriate present value of the tax savings due to interest payments on debt, but Modigliani-Miller (1963) proposes to discount the tax savings at the risk-free rate (R_F)[1], whereas Harris and Pringle (1985) and Ruback (1995, 2002) propose discounting these tax savings at the cost of capital for the unlevered firm (Ku). Miles and Ezzell (1985) propose discounting these tax savings the first year at the cost of debt and the following years at the cost of capital for the unlevered firm. Reflecting this lack of consensus, Copeland et al. (2000, p. 482) claim that "the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct."

We show that the value of tax shields depends only upon the nature of the stochastic process of the net increase of debt. More specifically, we prove that the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the value of the future net increases of debt.

We provide an alternative to Modigliani-Miller (1963) and to Miles-Ezzell (1980): we develop valuation formulae for companies that maintain a fixed book value leverage ratio. Modigliani-Miller (1963) formula should be used when the company has a preset amount of debt; Miles-Ezzell (1980) should be used only if debt will be always a multiple of the equity market value.

While two theories assume a constant discount rate for the increases of debt (the risk-free rate in Modigliani-Miller, and the appropriate discount rate for the increases of assets if the company maintains a constant book value leverage ratio), Miles-Ezzell assume one rate for $t = 1$ and Ku for $t>1$. The appropriate discount rate for the increase of debt in $t = 1$ is negative, according to Miles-Ezzell, if the expected growth (g) is smaller than $(Ku- R_F)/(1+ R_F)$.

Although Miles and Ezzell provide a computationally elegant solution (as shown in Arzac-Glosten, 2005), it is not a realistic one. We claim that it makes much more sense to characterize the debt policy of a company with expected constant leverage ratio as a fixed book value leverage ratio instead of as a fixed market value leverage ratio because:

[1] Myers (1974) propose to discount it at the cost of debt (Kd).

1. The amount of debt does not depend on the movements of the stock market,
2. It is easier to follow for non quoted companies, and
3. Managers should prefer so because the value of tax shields is more valuable.

On top of that, the Miles-Ezzell framework provides some results with dubious economic meaning:

1. The present value of the debt increases is negative under many scenarios (see sections 2D and 6).
2. The appropriate discount rate for the expected increase of debt of the next period is very small: -177.6% in the example of this paper (see section 6).
3. The appropriate discount rate for the expected equity cash flow of the next period is very big: 119% in the example of this paper (see section 4A).
4. The appropriate discount rate for the expected taxes of the levered firm is equal or smaller than the appropriate discount rate for the expected taxes of the unlevered firm under many scenarios (see section 9).

The Miles-Ezzell setup is equivalent to assume that the increase of debt is proportional to the increase of the free cash flow in every period, whereas we propose the increase of debt being proportional to the free cash flow.

The paper is organized as follows. In section 1 we derive the general formula for the value of tax shields. In section 2 we apply this formula to specific situations including a company that maintains a constant book-value leverage ratio. Section 3 is a numerical example.

In section 4 we show that the appropriate discount rate for the expected equity cash flows is different than the appropriate discount rate for the expected value of the equity. The appropriate discount rate for the expected equity cash flows is not constant in every period. Although the equity value of a growing perpetuity can be computed by discounting the expected value of the equity cash flow with a unique average rate (Ke), the appropriate discount rates for the expected values of the equity cash flows are not constant.

In sections 5, 6, 7 and 8 we calculate, respectively, the appropriate discount rates for the tax shields, for the increases of debt, for the value of debt and for the value of tax shields.

In section 9 we calculate the present value of taxes for the levered and the unlevered firm. Modigliani-Miller and Miles-Ezzell do not make any assumption about the appropriate discount rate for the increases of the book value of assets, but this assumption is needed to calculate the value of the taxes paid by the levered and the unlevered company.

Section 10 presents the appropriate discount rates for capital gains. Section 11 discusses the influence of growth on the risk of the cash flows. Section 12 concludes. Table 1 is a map to locate the different formulae in the paper. In the Appendix we derive additional formulae for the three theories discussed in this paper applied to growing perpetuities.

1. General expression of the value of tax shields

The value of the debt today (D_0) is the value today of the future stream of interest minus the value today of the future stream of the increases of debt (ΔD_t):

$$D_0 = \sum_1^{\infty} E[M_t \cdot \text{Interest}_t] - \sum_1^{\infty} E[M_t \cdot \Delta D_t] \tag{1}$$

As the value of tax shields is the value of the interest times the tax rate,

$$VTS_0 = T\sum_1^{\infty} E[M_t \cdot \text{Interest}_t] = T \cdot D_0 + T\sum_1^{\infty} E[M_t \cdot \Delta D_t] \tag{2}$$

Equation (2), valid for perpetuities and for companies with any pattern of growth, shows that the value of tax shields depends only upon the nature of the stochastic process of the net increase of debt. The problem of equation (2) is how to calculate the value today of the increases of debt that depends of the financing strategy.

2. Value of tax shields and value of the increases of debt in specific situations

We apply the result in (2) to specific situations and show how this formula is consistent with previous formulae under restrictive scenarios.

The value today of the levered company (V_{L0}) is equal to the value of debt (D_0) plus that of the equity (S_0). It is also equal to the value of the unlevered company (Vu_0)[2] plus the value of tax shields due to interest payments (VTS_0):

$$V_{L0} = S_0 + D_0 = Vu_0 + VTS_0 \tag{3}$$

[2] According to our notation, $Vu_0 = \sum_1^{\infty} E[M_t \cdot FCF_t]$ and $S_0 = \sum_1^{\infty} E[M_t \cdot ECF_t]$, being FCF_t the free cash flow of period t, and ECF_t the equity cash flow of period t.

2A. Debt is proportional to the Equity book value

If D_t = $K \cdot Ebv_t$, being Ebv the book value of equity, then ΔD_t = $K \cdot \Delta Ebv_t$. The increase of the book value of equity is equal to the profit after tax (PAT) minus the equity cash flow (ECF). The relationship between the profit after tax of the levered company (PAT_L) and the equity cash flow (ECF) is:

$$ECF_t = PAT_{Lt} - \Delta A_t + \Delta D_t \tag{4}$$

Notation being, ΔA_t = Increase of net assets in period t (Increase of Working Capital Requirements plus Increase of Net Fixed Assets); $\Delta D_t = D_t - D_{t-1}$ = Increase of Debt in period t.

Similarly, the relationship between the profit after tax of the unlevered company (PATu) and the free cash flow (FCF) is:

$$FCF_t = PATu_t - \Delta A_t \tag{5}$$

According to equation (4), as $\Delta Ebv_t = \Delta D_t/K$,

$$\Delta Ebv_t = PAT_{Lt} - ECF_t = \Delta A_t - \Delta D_t = \Delta D_t / K \tag{6}$$

In this situation, the increase of debt is proportional to the increases of net assets and the risk of the increases of debt is equal to the risk of the increases of assets:

$$\Delta D_t = \Delta A_t \cdot K / (1+K) \tag{7}$$

The value today of the increases of debt is:

$$E_0[M_{0,t} \cdot \Delta D_t] = \left(\frac{K}{1+K}\right) E_0[M_{0,t} \cdot \Delta A_t] \tag{8}$$

We will assume that the increase of net assets follows the stochastic process defined by $\Delta A_{t+1} = \Delta A_t (1+g)(1+\phi_{t+1})$. ϕ_{t+1} is a random variable with expected value equal to zero, but with a value today smaller than zero:

$$E_t[M_{t,t+1} \cdot \phi_{t+1}] = -\frac{f}{1+R_F} \tag{9}$$

Then, in the case of a growing perpetuity:

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta A_0 \left(\frac{K}{1+K}\right) \frac{(1+g)^t (1-f)^t}{(1+R_F)^t} \tag{10}$$

If we call $(1+\alpha) = (1+R_F) / (1-f)$, then

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta D_0 \frac{(1+g)^t}{(1+\alpha)^t} \tag{11}$$

α is the appropriate discount rate for the expected increases of the book value of assets[3] and, in this case, the appropriate discount rate for the expected increases of debt. $\sum_{1}^{\infty} E[M_t \Delta D_t]$ is the sum of a geometric progression with growth rate = $(1+g)/(1+\alpha)$. Then:

$$\sum_{1}^{\infty} E[M_t \Delta D_t] = \frac{\Delta D_0 (1+g)}{\alpha - g} = \frac{g D_0}{\alpha - g} \tag{12}$$

Substituting (12) in (2), we get:

$$VTS_0 = \frac{D_0 \alpha T}{(\alpha - g)} \tag{13}$$

As we show in section 5, equation (13) is <u>not</u> the present value of $D_0 \alpha T$ discounted at α, but the sum of the present values of the expected tax shields (D_{t-1} T R_F) discounted at different rates in each period.

2B. Debt is proportional to the Equity book value and the increase of assets is proportional to the free cash flow

If the increase of assets (ΔA_t) is proportional to the free cash flow (FCF_t), α = Ku and equation (12) is:

$$\sum_{1}^{\infty} E[M_t \Delta D_t] = \frac{g \cdot D_0}{Ku - g} \tag{14}$$

Substituting (14) in (2), we get:

$$VTS_0 = \frac{D_0 Ku\, T}{(Ku - g)} \tag{15}$$

As we assume that the increases of debt and assets are as risky as the free cash flows (α = Ku), the correct discount rate for the expected increases of debt is Ku, the required return to the unlevered company. (15) is equal to equation (28) in Fernandez (2004).[4] Cooper and Nyborg (2006) affirm that equation (15) violates value-additivity. It does not because equation (3) holds.

---

[3] A_t is the book value of assets, not the value of the assets which is the value of the unlevered equity (Vu).

[4] Fernandez (2004) wrongly considered as being zero the present value of a variable with expected value equal to zero. And he neglected to include in equations (5) to (14) terms with expected value equal to zero. Due to these errors, Equations (5) to (17), Tables 3 and 4, and Figure 1 of Fernandez (2004) are correct only if $PV_0[\Delta A_t] = PV_0[\Delta D_t] = 0$.

2C. The company has a preset amount of debt

In this situation, ΔD_t is known with certainty today and Modigliani-Miller (1963) applies: the appropriate discount rate for the ΔD_t is R_F, the risk-free rate.

$$E_0\left[M_{0,t} \cdot \Delta D_t\right] = \Delta D_0 \frac{(1+g)^t}{(1+R_F)^t} \tag{16}$$

Equation (16) is the sum of a geometric progression with growth rate $(1+g)/(1+R_F)$. Then:

$$\sum_1^{\infty} E[M_t \Delta D_t] = \frac{gD_0}{R_F - g} \tag{17}$$

Substituting (17) in (2), we get:

$$VTS_0 = \frac{D_0 R_F T}{(R_F - g)} \tag{18}$$

Modigliani-Miller may be viewed as just one extreme case of section 2A, in which $\alpha = R_F$. Fieten et al. (2005) argue that the Modigliani-Miller formula may be applied to all situations. However, it is valid only when the company has a preset amount of debt.

2D. Debt is proportional to the Equity market value

This is the assumption made by Miles and Ezzell (1980) and Arzac and Glosten (2005). If $D_t = L \cdot S_t$, the value today of the increase of debt in period 1 is:

$$E_0\left[M_{0,1} \cdot \Delta D_1\right] = \frac{D_0(1+g)}{1+Ku} - \frac{D_0}{1+R_F} \tag{19}$$

We prove in the Appendix (equation A.14) that:

$$\sum_1^{\infty} E[M_t \Delta D_t] = \frac{D_0}{Ku - g}\left(g - \frac{Ku - R_F}{1+R_F}\right) \tag{20}$$

Substituting (20) in (2), we get the well known Miles-Ezzell formula:

$$VTS_0 = \frac{D_0 R_F T}{(Ku - g)} \frac{(1+Ku)}{(1+R_F)} \tag{21}$$

We claim that it makes more sense to characterize the debt policy of a growing company with expected constant leverage ratio as a fixed book-value leverage ratio instead of as a fixed market-value leverage ratio because:

1. the debt does not depend on the movements of the stock market,
2. it is easier to follow for non quoted companies, and
3. managers should prefer so because the value of tax shields is more valuable: (21) is smaller than (15) and than (13).

The Miles-Ezzell setup works as if the company pays all the debt (D_{t-1}) at the end of every period t and simultaneously raises all new debt D_t. The risk of raising the new debt is equal to the risk of the free cash flow and, hence, the appropriate discount rate for the expected value of the new debt (the whole debt, not just the increase of debt) is Ku.

To assume $D_t = L \cdot S_t$ is not a good description of the debt policy of any company because if a company has only two possible states of nature in the following period, it is clear that under the worst state (low share price) the leveraged company will have to raise new equity and repay debt, and this is not the moment companies prefer to raise equity. Under the good state, the company will have to take a lot of debt and pay big dividends.

The Miles-Ezzell setup is equivalent to assume that the increase of debt is proportional to the increase of the free cash flow in every period, whereas in section 2B the increase of debt is proportional to the free cash flow.

Table 1 is a map of the formulae in this paper. Table 2 summarizes the implications of several approaches for the value of tax shields and for the value of the future increases of debt.

3. A numerical example

Table 3 contains the main valuation results for a constant growing company. It is interesting to note that according to Miles-Ezzell, the present value of the increases of debt is negative.

Table 4 contains the value of the tax shields (VTS) according to the different theories as a function of g and α. The VTS grows dramatically when g increases and decreases with α. It may be seen that Modigliani-Miller is equivalent to a constant book-value leverage ratio ($D_t = L \cdot Ebv_t$), when $\alpha = R_F = 4\%$. The VTS according to M-M is infinite when $g > R_F$.

4. Appropriate discount rates for the expected equity cash flows and for the expected value of the equity

The value of equity today (S_0) is equal to the present value of the equity cash flow in period 1 (ECF_1) plus the present value of the equity in period 1 (S_1). For perpetuities with a constant growth rate (g):

$$S_0 = \frac{ECF_0(1+g)}{(1+Ke_1)} + \frac{S_0(1+g)}{(1+K_{S1})} \tag{22}$$

Ke_1 is the appropriate discount rate for the expected equity cash flow in period 1 and K_{S1} is the appropriate discount rate for the expected value of the equity in period 1. We will see that both rates are different under all assumptions. The present value of the equity value in t = 1 is

$$\frac{S_0(1+g)}{(1+K_{S1})} = S_0 - \frac{ECF_0(1+g)}{(1+Ke_1)} \tag{23}$$

The general expression for the present value in t=0 of the equity value in t = t is:

$$\frac{S_0(1+g)^t}{(1+K_{S1})...(1+K_{St})} = S_0 - \frac{ECF_0(1+g)}{(1+Ke_1)} - ... - \frac{ECF_0(1+g)^t}{(1+Ke_1)...(1+Ke_t)}$$

To calculate the present value of the equity, we need to calculate the present value of the equity cash flows. The relationship between expected values in t=1 of the free cash flow (FCF), the equity cash flow and the debt cash flow is:

$$ECF_0(1+g) = FCF_0(1+g) - D_0 R_F (1-T) + g D_0 \tag{24}$$

Ke is the average appropriate discount rate for the expected equity cash flows, such that $S_0 = ECF_0(1+g)/(Ke-g)$. Ku is the appropriate discount rate for the expected free cash flows, such that $Vu_0 = FCF_0(1+g) / (Ku-g)$. Equation (24) is equivalent to:

$$S_0(Ke-g) = Vu_0(Ku-g) - D_0(R_F - g) + D_0 R_F T \tag{25}$$

As, according to equation (3), $S_0 = Vu_0 - D_0 + VTS_0$, we may rewrite (25) as:

$$S_0 Ke = Vu_0 Ku - D_0 R_F + VTS_0 g + D_0 R_F T \tag{26}$$

And the general equation for Ke is:

$$Ke = Ku + \frac{D_0}{S_0}\left[Ku - R_F(1-T)\right] - \frac{VTS_0}{S_0}(Ku - g) \tag{27}$$

This expression is the average Ke: it is not the required return to the equity cash flows (Ke_t) for all periods.

4A. Debt is proportional to the Equity market value

According to Miles and Ezzell (1980) and Arzac and Glosten (2005), substituting (21) in (27), we get:

$$Ke = Ku + \frac{D_0}{S_0}(Ku - R_F)\frac{(1+R_F - R_F T)}{(1+R_F)} \tag{28}$$

If $D_t = L \cdot S_t$, the appropriate discount rate for S_t (K_S) is also equal to the required return to the value of debt (K_D). We prove in the Appendix (equation A.10) that the appropriate discount rate for Vu_t is Ku. As according to (21) the VTS is proportional to D, following equation (3), D_t, S_t, Vu_t and VTS_t have the same risk and the appropriate discount rate for all of them is Ku. Then, the value of the equity value today is, according to equation (22):

$$S_0 = \frac{ECF_0(1+g)}{1+Ke_1} + \frac{S_0(1+g)}{1+Ku} \tag{29}$$

The appropriate discount rate for the expected equity cash flow in period 1 (Ke_1) is:

$$1+Ke_1 = \frac{ECF_0(1+g)(1+Ku)}{S_0(Ku-g)} = (1+Ku)\frac{(Ke-g)}{(Ku-g)} \tag{30}$$

The value of the equity today is also:

$$S_0 = \frac{ECF_0(1+g)}{(1+Ke_1)} + \frac{ECF_0(1+g)^2}{(1+Ke_1)(1+Ke_2)} + \frac{S_0(1+g)^2}{(1+Ku)^2} \tag{31}$$

Substituting (30) in (31), it is clear that Ke_2 = Ku. Following the same procedure, it may be shown that for t >1, Ke_t = Ku. In the example of table 3, Ke =16.07%, Ke_1=119.03% and K_{S1}= Ku = 9%.

4B. Debt is proportional to the Equity book value

Substituting (13) in (27), we get:

$$Ke = Ku + \frac{D_0}{S_0}\left[Ku - R_F(1-T) - \frac{\alpha T(Ku-g)}{\alpha - g}\right] \tag{32}$$

Calculating the expected value in t=0 of equation (24):

$$\frac{(1+g)ECF_0}{(1+Ke_1)} = \frac{(1+g)FCF_0}{(1+Ku)} - \frac{D_0 R_F(1-T)}{(1+R_F)} + \frac{gD_0}{(1+\alpha)} \tag{33}$$

As $(1+g)ECF_0=S_0$ (Ke-g) and $(1+g)FCF_0=Vu_0$ (Ku-g), the appropriate discount rate for the expected equity cash flow in period 1 is:

$$(1+Ke_1) = \frac{S_0(Ke-g)}{Vu_0\frac{(Ku-g)}{(1+Ku)} + D_0\left[\frac{g}{(1+\alpha)} - \frac{R_F(1-T)}{(1+R_F)}\right]} \tag{34}$$

And substituting (34) in (23):

$$\frac{S_0(1+g)}{(1+K_{S1})} = S_0 - Vu_0\frac{(Ku-g)}{(1+Ku)} - D_0\left[\frac{g}{(1+\alpha)} - \frac{R_F(1-T)}{(1+R_F)}\right] \tag{35}$$

In the Appendix we find the present value of the equity value in t (A.27) and the discount rate for the expected equity cash flow in t (A.30):

$$(1+Ke_1)...(1+Ke_t) = \frac{S_0(Ke-g)}{\frac{Vu_0(Ku-g)}{(1+Ku)^t} - \frac{D_0 R_F(1-T)}{(1+g)^{t-1}(1+R_F)^t}\left[\frac{R_F-\alpha+g(1+R_F)X^{t-1}}{g(1+R_F)+R_F-\alpha}\right] + \frac{gD_0}{(1+\alpha)^t}} \quad (36)$$

$$PV_0[S_t] = \frac{Vu_0(1+g)^t}{(1+Ku)^t} + \frac{D_0 g(1+g)^t}{(\alpha-g)(1+\alpha)^t} - \frac{(1-T)D_0}{(1+R_F)^t(\alpha-g)}\left[\alpha + \frac{gR_F(X^t-1)}{(X-1)}\right] \quad (37)$$

$X = (1+g)(1+ R_F)/(1+\alpha)$

In the example of table 3, if α=7%, Ke =11.63%, Ke_1=9.98% and K_{S1}=11.80%. In the example $PV_0[S_t]<0$ for t>25 and $PV_0[ECF_t]<0$ for t>46. $PV_0[S_t]<0$ only means that $PV_0[D_t] > PV_0[Vu_t] + PV_0[VTS_t]$. $PV_0[ECF_t]<0$ only means that $PV_0[D_{t-1} R_F(1-T)] > PV_0[FCF_t] + PV_0[\Delta D_t]$.

4C. Debt is proportional to the Equity book value and the increase of assets is proportional to the free cash flow

In this situation, as the increases of assets are proportional to the free cash flows ($\Delta A_{t+1} = Z \cdot FCF_t$), α = Ku, and equation (32) is:

$$Ke = Ku + \frac{D_0}{S_0}(Ku - R_F)(1-T) \quad (38)$$

If α = Ku, as $Vu_0 = S_0 + D_0 - VTS_0$, equations (34) and (35) are:

$$(1+Ke_1) = \frac{(Ke-g)(1+R_F)(1+Ku)}{(Ku-g)(1+R_F)+(Ke-Ku)} \quad (39)$$

$$\frac{S_0(1+g)}{(1+K_{S1})} = \frac{S_0[(1+R_F)(1+g)-Ke+Ku]}{(1+Ku)(1+R_F)} \quad (40)$$

And equations (36) and (37) are:

$$(1+Ke_1)...(1+Ke_t) = \frac{S_0(Ke-g)}{\frac{S_0(Ku-g)}{(1+Ku)^t} + \frac{D_0(1-T)(Ku-R_F)}{(Ku-R_F)-g(1+R_F)}\left[\frac{Ku-g}{(1+Ku)^t} - \frac{R_F(1+g)}{(1+g)^t(1+R_F)^t}\right]} \quad (41)$$

$$\frac{Ke-g}{(1+Ke_1)...(1+Ke_t)} = \frac{(Ku-g)}{(1+Ku)^t} + \frac{(Ke-Ku)}{Ku-g-R_F(1+g)}\left[\frac{Ku-g}{(1+Ku)^t} - \frac{R_F(1+g)}{(1+g)^t(1+R_F)^t}\right] \quad (41bis)$$

$$\frac{Ke-g}{(1+Ke_1)...(1+Ke_t)} = \frac{(Ku-g)}{(1+Ku)^t}\left(1+\frac{(Ke-Ku)}{Ku-g-R_F(1+g)}\right) - \frac{(Ke-Ku)}{Ku-g-R_F(1+g)}\left[\frac{R_F(1+g)}{(1+g)^t(1+R_F)^t}\right] =$$

$$= \frac{(Ku-g)}{(1+Ku)^t}\left(\frac{Ke-g-R_F(1+g)}{Ku-g-R_F(1+g)}\right) - \frac{(Ke-Ku)}{Ku-g-R_F(1+g)}\left[\frac{R_F(1+g)}{(1+g)^t(1+R_F)^t}\right]$$

(41bis)

$$\frac{(Ke-g)(Ku-g-R_F(1+g))}{(1+Ke_1)...(1+Ke_t)} = (Ku-g)\left(\frac{Ke-g-R_F(1+g)}{(1+Ku)^t}\right)-(Ke-Ku)\left[\frac{R_F(1+g)}{(1+g)^t(1+R_F)^t}\right]$$

$$PV_0[S_t] = \frac{S_0(1+g)^t}{(1+Ku)^t} + \frac{D_0(1-T)(Ku-R_F)}{(Ku-R_F)-g(1+R_F)}\left[\frac{(1+g)^t}{(1+Ku)^t} - \frac{1}{(1+R_F)^t}\right] \tag{42}$$

$$PV_0[S_t] = S_0\left(\frac{(1+g)^t}{(1+Ku)^t} + \frac{(Ke-Ku)}{(Ku-R_F)-g(1+R_F)}\left[\frac{(1+g)^t}{(1+Ku)^t} - \frac{1}{(1+R_F)^t}\right]\right) \tag{42bis}$$

$$PV_0[S_t] = \frac{S_0}{Ku-R_F-g(1+R_F)}\left(\frac{(1+g)^t}{(1+Ku)^t}(Ke-R_F-g(1+R_F)) - \frac{(Ke-Ku)}{(1+R_F)^t}\right)$$

When t tends to infinity, $Ke_t = K_{St} = (1+g)(1+R_F)-1$ if $(1+g)(1+R_F) < (1+Ku)$ and $Ke_t = K_{St} = Ku$ if $(1+g)(1+R_F) > (1+Ku)$.

In the example of table 3, if α=Ku=9%, Ke =12.09%, Ke_1=10.30% and K_{S1}=12.27%. In the example $PV_0[S_t]<0$ for t>24 and $PV_0[ECF_t]<0$ for t>44.

4D. The company has a preset amount of debt

Modigliani-Miller may be viewed as just one extreme case of section 4B, in which $\alpha = R_F$. Substituting (18) in (27) (or substituting α by R_F in (32)), we get:

$$Ke = Ku + \frac{D_0}{S_0}\left[Ku - R_F - R_F T\frac{Ku-R_F}{R_F-g}\right] \tag{43}$$

But this expression is the average Ke. It is not the required return to equity (Ke_t) for all the periods. Substituting α by R_F in (34) and (35):

$$(1+Ke_1) = \frac{(Ke-g)(1+Ku)(1+R_F)}{(Ku-g)(1+R_F)+(Ke-Ku)(1+g)} \tag{44}$$

$$\frac{S_0(1+g)}{(1+K_{S1})} = S_0 - Vu_0\frac{(Ku-g)}{(1+Ku)} + D_0\frac{R_F(1-T)-g}{(1+R_F)} = \frac{S_0(1+g)(1+R_F-Ke+Ku)}{(1+R_F)(1+Ku)} \tag{45}$$

In this situation, the appropriate discount rate for the expected value of tax shields (VTS) and for the expected debt is the risk-free rate. Substituting α by R_F in (36) and (37), and having into account that $Vu_0 = S_0 + D_0 - VTS_0$, we get:

$$(1+Ke_1)...(1+Ke_t) = \frac{S_0(Ke-g)}{\frac{Vu_0(Ku-g)}{(1+Ku)^t} - D_0\frac{R_F(1-T)-g}{(1+R_F)^t}} \tag{46}$$

$$E_0[M_{0,t}\cdot S_t] = Vu_0\left(\frac{1+g}{1+Ku}\right)^t - D_0\frac{R_F(1-T)-g}{R_F-g}\left(\frac{1+g}{1+R_F}\right)^t \tag{47}$$

In the appendix (A.33) we show that

$$\prod_{1}^{t}(1+K_{St}) = \frac{S_0(1+R_F)^t(1+Ku)^t}{(S_0-Vu_0)(1+Ku)^t+Vu_0(1+R_F)^t} \tag{48}$$

Comparing (46) and (47) it is clear that the appropriate discount rate for the equity cash flow is different that the appropriate discount rate for the expected value of the equity. When t tends to infinity, $Ke_t = K_{St} = R_F$.

From (47) we see that the present value of the equity is negative if

$$t > \log\left(\frac{Vu_0(R_F-g)}{D_0[R_F(1-T)-g]}\right) / \log\left(\frac{1+Ku}{1+R_F}\right)$$

In the example of table 3, Ke =9.80%, Ke_1=9.21% and K_{S1}=9.84%. $PV_0[S_t]<0$ for t>42 and $PV_0[ECF_t]<0$ for t>68.

Although the equity value of a growing perpetuity can be computed by discounting the expected value of the equity cash flow with a unique average rate Ke, the appropriate discount rates for the expected values of the equity cash flows are not constant. Table 5 presents the appropriate discount rates for the expected values of the equity cash flows of our example. According to Miles-Ezzell, Ke_t is 119.03% for t = 1 and 9% for the rest of the periods. According to Modigliani-Miller, Ke < Ku if g> R_F(1-T).

For all cases, the expected total return for the shareholder (K_{SHAR1}) is Ke for all periods because:

$$K_{SHAR1} = \frac{gS_0 + ECF_0(1+g)}{S_0} = \frac{gS_0 + S_0(Ke-g)}{S_0} = Ke$$

5. Appropriate discount rates for the tax shields (K_{TSt})

The tax shield of the next period is known with certainty (D_0 R_F T) under all methods and the appropriate discount rate is R_F.

If the company maintains a constant book-value leverage, the appropriate discount rate for the expected increases of debt is α; and the appropriate discount rate for the expected tax shield of t = 2 (K_{TS2}), is such that:

$$\frac{D_0(1+g)R_F T}{(1+R_F)(1+K_{TS2})} = R_F T\left[\frac{D_0}{(1+R_F)^2} + \frac{gD_0}{(1+R_F)(1+\alpha)}\right] \tag{49}$$

$$1+K_{TS2} = \frac{(1+R_F)(1+\alpha)(1+g)}{1+\alpha+g(1+R_F)} \tag{50}$$

In the appendix (A.18), we show that the present value of the tax shield in t is:

$$PV_0[TS_t] = \frac{R_F T D_0}{(1+R_F)^t}\left[1+\frac{g(1+R_F)(X^{t-1}-1)}{(1+\alpha)(X-1)}\right] \qquad (51)$$

We also prove that the appropriate discount rate for the expected tax shield of period t, for t>1, is:

$$(1+K_{TSt}) = \frac{PV_0[TS_{t-1}]}{PV_0[TS_t]}(1+g) = X(1+\alpha)\frac{(1+\alpha)(X-1)+g(1+R_F)(X^{t-2}-1)}{(1+\alpha)(X-1)+g(1+R_F)(X^{t-1}-1)} \qquad (52)$$

$$(1+K_{TS2})...(1+K_{TSt}) = \frac{X^{t-1}(1+\alpha)^t(X-1)}{(1+\alpha)(X-1)+g(1+R_F)(X^{t-1}-1)}$$

In the example of table 3, if α =7%, K_{TS2}=4.057%.

When t tends to infinity, K_{TSt}= MIN[α, (1+ R_F)(1+g)-1]

It is also easy to calculate that, using (51), $VTS_0 = \sum_{t=0}^{\infty} PV_0[TS_t] = \frac{D_0 T\alpha}{\alpha - g}$

According to Miles-Ezzell, the appropriate discount rate for the expected tax shields is R_F for t =1 and Ku for t>1.

According to Modigliani-Miller, as the debt in any period is known today, the appropriate discount rate for the expected tax shields of any period (K_{TS}) is R_F.[5]

6. Appropriate discount rates for the increases of debt (K_{DDt})

If the company maintains a constant book-value leverage, the appropriate discount rate for the expected increases of debt ($K_{\Delta Dt}$) is α. According to Modigliani-Miller, as the debt in any period is known today, the appropriate discount rate for the expected increases of debt is R_F. According to Miles-Ezzell, the equivalent discount rate for the expected increase of debt in period 1 ($K_{\Delta D1}$) is such that:

$$\frac{E_0[\Delta D_1]}{1+K_{\Delta D1}} = \frac{gD_0}{1+K_{\Delta D1}} = \frac{D_0(1+g)}{1+Ku} - \frac{D_0}{1+R_F} \qquad (53)$$

Some algebra permits to express $1+K_{\Delta D1} = \frac{g(1+Ku)(1+R_F)}{g(1+R_F)+R_F-Ku}$ (54)[6]

In our example, $K_{\Delta D1}$ = -177.6%.

[5] This result may be obtained also calculating (52) when α=R_F

[6] If g=0, then $K_{\Delta D1}$ according to (54) is -100%, which does not make any economic sense. In this situation the expected value of the increase of debt is 0, but $E_0[M_{0,1}\cdot\Delta D_1] = \frac{D_0}{1+Ku} - \frac{D_0}{1+R_F}$.

For t=2: $E_0\left[M_{0,2}\cdot\Delta D_2\right] = D_0\,\frac{(1+g)}{(1+Ku)}\left(\frac{(1+g)}{(1+Ku)} - \frac{1}{(1+R_F)}\right) = \frac{g\,D_0\,(1+g)}{(1+K_{\Delta D1})(1+K_{\Delta D2})}$

After equation (53) it is obvious that $K_{\Delta D2}$ = Ku. Repeating this exercise, we find that $K_{\Delta Dt}$ = Ku. Under Miles-Ezzell the appropriate discount rate for Vu_t, D_t, VTS_t and Vu_t is Ku, and as all of them are multiples of the free cash flow, also ΔD_t is a multiple of the ΔFCF_t: $\Delta D_t = [D_0 / FCF_0]\ \Delta FCF_0$.

Table 6 contains the value today of the increases of debt in different periods and the sum of all of them. According to Miles-Ezzell the value today of the increases of debt in every period is negative. It is interesting to note that while two theories assume a constant discount rate for the increases of debt (Modigliani-Miller assume R_F and constant book value leverage assumes α), Miles-Ezzell assume one rate for t = 1 and Ku for t>1. The appropriate discount rate for the increase of debt in t = 1 is, according to Miles-Ezzell, equation (53), which is negative if $g < (Ku - R_F)/(1+R_F)$.

.

7. Appropriate discount rates for the value of debt (K_{Dt})

The expected value of debt in t=1 ($D_0(1+g)$) and the value of the debt today (D_0) must accomplish equation (55):

$$D_0 = \frac{D_0\ R_F}{1+R_F} - \frac{gD_0}{1+K_{\Delta D1}} + \frac{D_0(1+g)}{1+K_{D1}} \tag{55}$$

Substituting the expressions for $K_{\Delta D2}$ (appropriate discount rate for the expected increases of debt) from the previous section, we find that:

a) according to Miles-Ezzel, K_{D1} = Ku

b) according to Modigliani-Miller, $K_{D1} = R_F$

c) with constant book-value leverage, $1+K_{D1} = \frac{(1+g)(1+\alpha)(1+R_F)}{1+\alpha+g(1+R_F)}$ (56)

As $K_{Dt} = K_{TSt+1}$?we prove in the appendix (A.20) that:

$$(1+K_{D1})...(1+K_{Dt}) = \frac{(1+g)^t(1+R_F)^t}{1+\dfrac{g(1+R_F)\left[X^t-1\right]}{(1+\alpha)\left[X-1\right]}} \tag{57}$$

if $\alpha = R_F$ $\quad (1+K_{D1})...(1+K_{Dt}) = (1+R_F)^t$. For t = 1, (57) is equal to (56)

In the example of table 3, K_{D1} is 4.06% and K_{D2} is 4.11%. When tends to infinity, $1+K_{Dt} = (1+g)(1+R_F)$ if X < 1, and $1+K_{Dt} = (1+\alpha)$ if X >1.

d) with constant book-value leverage and ΔD_t= M FCF_t :

$$1+K_{D1}=\frac{(1+g)(1+Ku)(1+R_F)}{1+Ku+g(1+R_F)}$$

In the example of table 3, K_{D1} is 4.09% and K_{D2} is 4.18%.

8. Appropriate discount rates for the value of tax shields (K_{VTSt})

The expected value of the tax shield in t=1 ($VTS_0(1+g)$) and the value of tax shields today (VTS_0) must accomplish equation (58):

$$VTS_0=\frac{D_0\,R_F\,T}{1+R_F}+\frac{VTS_0(1+g)}{1+K_{VTS1}} \tag{58}$$

Substituting the expressions for the value of the tax shields (equations (13), (15), (18) and (21)), we find that:

a) according to Miles-Ezzel, K_{VTS1} = Ku

b) according to Modigliani-Miller, $K_{VTS1} = R_F$

c) with constant book-value leverage, $1+K_{VTS1}=\frac{\alpha(1+R_F)(1+g)}{\alpha+gR_F}$ (59)

In the example of table 3, K_{VTS1} is 4.88% and K_{VTS2} is 4.91%.

d) with constant book-value leverage and ΔD_t= M FCF_t : $1+K_{VTS1}=\frac{Ku(1+R_F)(1+g)}{Ku+gR_F}$

In the example of table 3, K_{VTS1} is 5.15% and K_{VTS2} is 5.18%.

With constant book-value leverage (D_t=K Ebv_t), K_{D1} and K_{VTS1} are not equal:

$$K_{VTS1}-K_{TS1}=\frac{g(1+R_F)(1+g)}{Ku+gR_F}$$

In the case of constant book-value leverage, we prove in the Appendix (A.22) that:

$$PV_0[VTS_t]=\frac{TD_0}{(1+R_F)^t}\left[1+\frac{gR_F X^t}{[X-1](\alpha-g)}-\frac{g(1+R_F)}{(1+\alpha)[X-1]}\right] \tag{60}$$

From (60), we get:

$$(1+K_{VTS1})...(1+K_{VTSt})=\frac{\alpha(X-1)(1+R_F)^t(1+g)^t}{gR_F X^t-\frac{(\alpha-R_F)(\alpha-g)}{(1+\alpha)}} \tag{61}$$

9. Value today of the expected taxes

We also derive the appropriate discount rates for the expected values of the taxes. If we assume that the appropriate discount rate for the increases of assets is Ku, then the appropriate discount rate for the expected value of the taxes of the unlevered company is also Ku. But the appropriate discount rate for the expected value of the taxes of the levered company (K_{TAXL}) is different according to the three theories. According to Modigliani-Miller and according to Fernandez, the taxes of the levered company are riskier than the taxes of the unlevered company. However, according to Miles-Ezzell, both taxes are equally risky for t > 1.[7]

If leverage costs do not exist, that is, if the expected free cash flows are independent of leverage,[8] the value of tax shields (VTS) may be stated as follows

$$VTS_0 = Gu_0 - G_{L0} \tag{62}$$

where Gu_0 is the present value of the taxes paid by the unlevered company and G_{L0} is the present value of the taxes paid by the levered company.

Taking into consideration Eq. (4) and (5), the taxes paid every year by the unlevered company ($Taxes_U$) and by the levered company ($Taxes_L$) are:

$$Taxes_{Ut} = [T/(1\text{-}T)]\ PATu = [T/(1\text{-}T)]\ (FCF_t + \Delta A_t) \tag{63}$$

$$Taxes_{Lt} = [T/(1\text{-}T)]\ (ECF_t + \Delta A_t - \Delta D_t) \tag{64}$$

The present values in t=0 of equations (63) and (64) are:

$$Gu_0 = \left(\frac{T}{1-T}\right)\left(Vu_0 + \sum_1^{\infty} E[M_t \cdot \Delta A_t]\right) \tag{65}$$

$$G_{L0} = \left(\frac{T}{1-T}\right)\left(S_0 + \sum_1^{\infty} E[M_t \cdot \Delta A_t] - \sum_1^{\infty} E[M_t \cdot \Delta D_t]\right) \tag{66}$$

The value of tax shields is the difference between Gu (65) and G_L (66).

In section 2A we defined α as the appropriate discount rate for the expected increases of the book value of assets. Modigliani-Miller and Miles-Ezzell do not make

[7] It the risk of the increase of assets is smaller than the risk of the free cash flows, then Miles-Ezzell provides a surprising result: the taxes of the levered company are less risky than the taxes of the unlevered company.

[8] When leverage costs do exist, the total value of the levered company is lower than the total value of the unlevered company. A world with leverage cost is characterized by the following relation:

$$Vu + Gu = S + D + G_L + \text{Leverage Cost} > S + D + G_L$$

Leverage cost is the reduction in the company's value due to the use of debt.

any assumption about the appropriate discount rate for the increases of the book value of assets, but this assumption is needed to calculate the value of the taxes paid by the levered and the unlevered company. The appropriate discount rate for the expected taxes of the unlevered company (K_{TAXU1}) is such that:

$$\frac{E_0[Taxes_{U1}]}{(1+K_{TAXU1})}=\left[\frac{T}{1-T}\right]\left(\frac{(1+g)FCF_0}{(1+Ku)}+\frac{gA_0}{(1+\alpha)}\right)$$

As $E[Taxes_{U1}]=[T/(1-T)]\ [FCF_0(1+g) + gA_0]$, we can calculate K_{TAXU1}.

$$(1+K_{TAXU1})=\frac{(1+g)FCF_0+gA_0}{(1+g)FCF_0(1+\alpha)+gA_0(1+Ku)}(1+Ku)(1+\alpha) \tag{67}$$

If α = Ku, then K_{TAXUt}= Ku.

The appropriate discount rate for the expected taxes of the levered company is:

$$\frac{E_0[Taxes_{L1}]}{(1+K_{TAXL1})}=\left[\frac{T}{1-T}\right]\left(\frac{(1+g)FCF_0}{(1+Ku)}+\frac{gA_0}{(1+\alpha)}-\frac{D_0R_F(1-T)}{(1+R_F)}\right)$$

As $E[Taxes_{L1}] = [T/(1-T)]\ [FCF_0(1+g) + gA_0 - D_0\ R_F\ (1-T)]$

$$1+K_{TAXL1}=\frac{(1+g)FCF_0+gA_0-D_0R_F(1-T)}{\frac{(1+g)FCF_0}{(1+Ku)}+\frac{gA_0}{(1+\alpha)}-\frac{D_0R_F(1-T)}{(1+R_F)}} \tag{68}$$

For t > 1, (for example, for t=2), the present value is:

$$PV_0[Taxes_{L2}]=\frac{E_0[Taxes_{L1}](1+g)}{(1+K_{TAXL1})(1+K_{TAXL2})}$$

According to Miles-Ezzell, K_{TAXL2}= Ku if α=Ku

From equation (62) we van calculate the present value of the levered taxes also as:

$$G_{L0}=Gu_0-VTS_0=\frac{\mathbf{T}}{1-T}\left[Vu_0+\frac{g\,A_0}{(\alpha-g)}\right]-VTS_0 \tag{69}$$

Although K_{TAXUt} and K_{TAXLt} are not constant, we can calculate K_{TAXU} and K_{TAXL} such that G_{U0} = $Taxes_{U0}$ (1+g) / (K_{TAXU} - g) and G_{L0} = $Taxes_{L0}$ (1+g) / (K_{TAXL} - g). Some algebra permits to find, for all theories:

$$K_{TAXU}=\frac{Vu_0(\alpha-g)Ku+g\alpha A_0}{Vu_0(\alpha-g)+gA_0} \tag{70}$$

$$K_{TAXL}=g+\frac{S_0(Ke-g)+g(A_0-D_0)}{Vu_0+\frac{gA_0}{(\alpha-g)}-\frac{VTS_0(1-T)}{T}} \tag{71}$$

In our example (Table 3), if α = 7%, Gu = 870.48, and K_{TAXU} = 8.437%, but K_{TAXU1} is 8.556% and tends to 7% when t tends to infinity. If α = 9% = Ku, Gu = 946.67, and K_{TAXU} = K_{TAXUt} is 9%. According to Miles-Ezzell, K_{TAXL} < K_{TAXU}.

Table 7 presents the appropriate discount rates for the expected values of the taxes in the initial periods for our example and their average. According to Miles-Ezzell, if α = Ku = 9%, K_{TAXLt} is 10.19% for t = 1 and 9% (equal to K_{TAXUt}) for the rest of the periods. According to Miles-Ezzell, if α = 7%, K_{TAXL1} is 9.64% and K_{TAXL2} is 8.44% (smaller than K_{TAXU2}). According to the other theories, K_{TAXLt} is higher than Ku (9%) and grows with t.

10. Appropriate discount rate for capital gains

In the Appendix, we deduct the appropriate discount rate for the expected capital gains in formulae (A.37) to (A.41). It may be seen that for our example the appropriate discount rate for the capital gains in the first periods are negative according to all theories. This result contradicts Cooper and Nyborg (2006) who affirm that "since capital gains are known with certainty, the appropriate discount rate for them is the risk free rate."

11. Is Ku independent of growth?

Up to now we have assumed that Ku is constant, independent of growth. From equation (6) we know that $FCF_t = PATu_t - \Delta A_t$.

If we consider that the risk of the unlevered profit after tax (PATu) is independent of growth, and that K_{PATu} is the required return to the expected PATu, the present value of equation (6) is:

$$Vu_0 = \frac{(1+g)FCF_0}{(Ku-g)} = \frac{(1+g)PATu_0}{(K_{PATu}-g)} - \frac{gA_0}{(\alpha-g)}$$

$$Ku = g + \frac{(1+g)FCF_0}{\dfrac{(1+g)PATu_0}{(K_{PATu}-g)} - \dfrac{gA_0}{(\alpha-g)}}$$

Table 8 contains the required return to the free cash flows (Ku) as a function of α (required return to the increase of assets) and g (expected growth). It may be seen that Ku is increasing in g[9] if $\alpha < K_{PATu}$, and decreasing in g if $\alpha > K_{PATu}$

[9] This result contradicts Cooper and Nyborg (2006) that maintain that "Ku is decreasing in g".

12. Conclusions

The value of tax shields depends only upon the nature of the stochastic process of the net increase of debt. More specifically, the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the value today of the net increases of debt. This expression is equivalent to the difference between the present values of two different cash flows, each with its own risk: the value today of taxes for the unlevered company and the value today of taxes for the levered company. The critical parameter for calculating the value of tax shields is the value today of the net increases of debt.

When the debt level is fixed, Modigliani-Miller (1963) applies, and the tax shields should be discounted at the required return to debt. If the leverage ratio (D/E) is fixed at market value, then Miles-Ezzell (1980) applies with the caveats discussed. If the leverage ratio is fixed at book values and the increases of assets are as risky as the free cash flows (the increases of debt are as risky as the free cash flows), then Fernandez (2004) applies. We have developed new formulas for the situation in which the leverage ratio is fixed at book values but the increases of assets have a different risk than the free cash flows.

We argue that it is more realistic to assume that a company maintains a fixed book-value leverage ratio than to assume, as Miles-Ezzell (1980) do, that the company maintains a fixed market-value leverage ratio because:

1. The amount of debt does not depend on the movements of the stock market,
2. It is easier to follow for non quoted companies, and
3. Managers should prefer so because the value of tax shields is more valuable.

On top of that, the Miles-Ezzell framework provides some results with dubious economic meaning:

1. The present value of the debt increases is negative under many scenarios
2. The appropriate discount rate for the expected increase of debt of the next period is too big: -177.6% in the example of this paper.
3. The appropriate discount rate for the expected equity cash flow of the next period is too big: 119% in the example of this paper.
4. The appropriate discount rate for the expected taxes of the levered firm is equal or smaller than the appropriate discount rate for the expected taxes of the unlevered firm under many scenarios.

Table 1

Index to the formulae in this paper

	General	Miles-Ezzell D_t=L S_t	Modigliani-Miller D_t fixed	Debt proportional to equity book value (D_t = K Ebv_t)	
					ΔD_t=L FCF_t
VTS_t	(2)	(21)	(18)	(13)	(15)
$K_{\Delta D1}$, $K_{\Delta Dt}$		(54), Ku	R_F, R_F	α, α	Ku, Ku
$PV_0(\Delta D_t)$		(19), (20)	(16), (17)	(8), (12)	(14)
Ke	(27)	(28)	(43)	(32)	(38)
Ke_1, Ke_t		(30), Ku	(44)	(34)	(39)
$\Pi(1+Ke_t)$		(31)	(46)	(36)	(41)
K_{S1}, K_{St}	(23)	Ku, Ku	(45), (48)	(35), (37)	(40), (42)
K_{D1}, K_{Dt}		Ku, Ku	R_F, R_F	(56), (57)	(56), (57)
K_{VTSt}	(58)	Ku	R_F	(59), (61)	(59), (61)
K_{Vut}	(A.10) Ku	Ku	Ku	Ku	Ku
$K_{\Delta Vut}$ =$K_{\Delta FCFt}$	(A.12) Ku	Ku	Ku	Ku	Ku
K_{TS1}, K_{TSt}		R_F, Ku	R_F, R_F	R_F, (52)	R_F, (52)
$PV_0(S_t)$		(31)	(47)	(37)	(42)
$Taxes_{Ut}$	(63)				
$Taxes_{Lt}$	(64)				
Gu	(65)				
G_L	(66)				
K_{TAXUt}	(67), (70)				
K_{TAXLt}	(68), (71)				
$PV_0(\Delta Ebv_t)$		(A.34)	(A.35)	(A.36)	
K_{CG1}, K_{CGt}	(A.37), (A.41)	(A.38), (A.41)	(A.39), (A.41)	(A.41)	(A.40), (A.41)
Vu	(A.10)				

VTS = value of tax shields
$K_{\Delta D}$ = required return to the expected increases of debt
Ke = required return to the expected equity cash flows
K_S, K_D, K_{VTS}, K_{Vu} = required return to the equity value (S), to the debt value (D), to the value of tax shields (VTS) and to the unlevered equity value (Vu).
$K_{\Delta Vu}$ =$K_{\Delta FCF}$ = required return to the increases of the unlevered equity value (Vu) and to the increases of the free cash flow (FCF)
K_{TS} = required return to the tax shields (TS)
$PV_0(S_t)$ = present value in t = 0 of the equity value in t (S_t)
$Taxes_U$, $Taxes_L$ = Taxes paid by the unlevered company ($Taxes_U$) and by the unlevered company ($Taxes_L$)
Gu, G_L = Present value of taxes paid by the unlevered (Gu) and by the unlevered company (G_L)
K_{TAXU}, K_{TAXL} = required return to the expected taxes paid by the unlevered company (K_{TAXU}) and by the unlevered company (K_{TAXL})
$PV_0(\Delta Ebv_t)$ = present value in t= 0 of the increase of the expected increase of the equity book-value in t.
K_{CG} = required return to the expected capital gains (CG).

Table 2

Value today of the increases of debt implicit in the most popular formulae for calculating the value of tax shields.

Perpetuities growing at a constant rate g

Authors	**VTS_0**	**$\sum PV_0[\Delta D_t]$**
Miles-Ezzell (1980) Arzac-Glosten (2005)	$\frac{D_0\ R_F\ T}{(Ku-g)}\frac{(1+Ku)}{(1+R_F)}$	$\frac{D_0}{Ku-g}\left(g-\frac{Ku-R_F}{1+R_F}\right)$
Modigliani-Miller (1963)	$\frac{D_0\ R_F\ T}{(R_F-g)}$	$\frac{g \cdot D_0}{R_F-g}$
Constant book-value leverage	$\frac{D_0\ a\ T}{(a-g)}$	$\frac{g \cdot D_0}{\alpha-g}$
Constant book-value leverage Debt as risky as assets	$\frac{D_0\ Ku\,T}{(Ku-g)}$	$\frac{g \cdot D_0}{Ku-g}$

Ku = unlevered cost of equity

T = corporate tax rate

D_0 = debt value today

R_F = risk-free rate

α = required return to the increases of assets

Table 3

Example. Valuation of a constant growing company

FCF_0 = 70; A_0 = 1,000; D_0 = 700;

R_F = 4%; Ku = 9%; a = 7%; T = 40%; g = 2%; Vu_0 = 1,020.

	Modigliani-Miller	**Miles-Ezzell**	**Debt proportional to equity book value ($D_t = K \cdot Ebv_t$)**	
	D_t fixed	**$D_t = K \cdot E_t$**	a = 7%	a **= 9% = Ku**
	$\Delta D_t = K \cdot CFd_t$	$\Delta D_t = K \cdot \Delta FCF_t$	$\Delta D_t = \Delta A_t/(1+1/K)$	$\Delta D_t = K \cdot FCF_t$
VTS_0	560.00	167.69	392.00	360.00
Equity value (S_0)	880.00	487.69	712.00	680.00
$PV_0[\Delta D_t]$	700.00	-280.77	280.00	200.00
Gu = $PV_0[TAX_{Ut}]$	946.67	946.67	946.67	870.48
G_L = $PV_0[TAX_{Lt}]$	386.67	778.97	554.67	510.48
Ke average	9.80%	16.07%	11.63%	12.09%
K_{TAXU} average	8.44%	8.44%	8.44%	9%
K_{TAXL} average	14.86%	8.38%	10.97%	11.74%

	Modigliani-Miller		**Miles-Ezzell**		**Debt proportional to equity book value ($D_t = K \cdot Ebv_t$)**			
	D_t fixed		**$D_t = K \cdot E_t$**		a = 7%		a **= 9% = Ku**	
	t=1	**t=2**	**t=1**	**t=1**	**t=1**	**t=2**	**t=1**	**t=2**
Ke_t	9.21%	9.23%	119.03%	10.30%	9.87%	9.92%	10.30%	10.35%
K_{St}	9.84%	9.89%	9%	12.27%	11.80%	11.99%	12.27%	12.47%
$K_{\Delta Dt}$	4%	4%	-177.6%	9%	7%	7%	9%	9%
K_{Dt}	4%	4%	9%	4.09%	4.06%	4.11%	4.09%	4.18%
K_{Vut}	9%	9%	9%	9%	9%	9%	9%	9%
K_{VTSt}	4%	4%	9%	5.145%	4.88%	4.91%	5.15%	5.18%
K_{TSt}	4%	4%	4%	4%	4%	4.06%	4%	4.09%
K_{TAXUt}	8.56%	8.55%	8.56%	9%	8.56%	8.55%	9%	9%
K_{TAXLt}	9.64%	9.69%	9.64%	10.19%	9.64%	9.67%	10.19%	10.24%

Table 4

Value of the tax shields (VTS) according to the different theories as a function of g (expected growth) and a (required return to the increase of assets).

D_0 = 700; R_F = 4%; Ku = 9%; T = 40%

	g					
	0%	1%	2%	3%	4%	5%
Miles-Ezzell	130.43	146.73	167.69	195.64	234.77	293.46
Modigliani-Miller	280.00	373.33	560.00	1120.00	∞	∞
$D_t = L \cdot Ebv_t$; α=5%	280.00	350.00	466.67	700.00	1399.90	13266.67
$D_t = L \cdot Ebv_t$; α=7%	280.00	326.67	392.00	490.00	653.33	980.00
$D_t = L \cdot Ebv_t$; α=9%	280.00	315.00	360.00	420.00	504.00	630.00
$D_t = L \cdot Ebv_t$; α=11%	280.00	308.00	342.22	385.00	440.00	513.33
$D_t = L \cdot Ebv_t$; α=15%	280.00	300.00	323.08	350.00	381.82	420.00

Table 5

Appropriate discount rates for the expected values of the equity cash flows (Ke_t)

$FCF_0 = 70$; $D_0 = 700$; $R_F = 4\%$; $Ku = 9\%$; $T = 40\%$; $g = 2\%$.

$D_t = L{\cdot}Ebv_t$ means that the company maintains a constant book-value leverage ratio. α is the appropriate discount rate for the increases of assets.

	Ke average	Ke_t						
		t=1	t=2	t=5	t=10	t=20	t=30	t=40
Miles-Ezzell	16.07%	119.03%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
Modigliani-Miller	9.80%	9.21%	9.23%	9.26%	9.33%	9.56%	9.96%	10.73%
$D_t = L{\cdot}Ebv_t$; α=5%	10.72%	9.44%	9.46%	9.55%	9.73%	10.32%	11.50%	14.58%
$D_t = L{\cdot}Ebv_t$; α=7%	11.63%	9.87%	9.92%	10.07%	10.39%	11.44%	13.86%	24.19%
$D_t = L{\cdot}Ebv_t$; α=9%	12.09%	10.30%	10.35%	10.53%	10.89%	12.11%	15.11%	32.07%
$D_t = L{\cdot}Ebv_t$; α=11%	12.36%	10.71%	10.76%	10.91%	11.25%	12.43%	15.44%	33.17%

Table 6

Value today of the increases of debt in different periods and the sum of all of them

D_0 = 700; R_F = 4%; Ku = 9%; T = 40%; g = 2%.

$PV_0(\Delta D_t)$	t=1	t=2	t=3	t=4	t=5	t=10	t=20	t=30	t=40	t=50	Sum
Miles-Ezzell	-18.03	-16.87	-15.79	-14.78	-13.83	-9.92	-5.11	-2.63	-1.35	-0.70	-280.77
Modigliani-Miller	13.46	13.20	12.95	12.70	12.46	11.30	9.31	7.67	6.31	5.20	700.00
$D_t = L\cdot Ebv_t$; α=5%	13.33	12.95	12.58	12.22	11.87	10.27	7.69	5.75	4.30	3.22	466.67
$D_t = L\cdot Ebv_t$; α=7%	13.08	12.47	11.89	11.33	10.80	8.51	5.27	3.27	2.02	1.25	280.00
$D_t = L\cdot Ebv_t$; α=9%	12.84	12.02	11.25	10.53	9.85	7.07	3.64	1.87	0.96	0.50	200.00
$D_t = L\cdot Ebv_t$; α=11%	12.61	11.59	10.65	9.79	8.99	5.89	2.53	1.09	0.47	0.20	155.56

Table 7

Appropriate discount rates for the expected value of the taxes of the levered and unlevered company.

Comparison of the results under three financial policies: Miles-Ezzell (ME), Modigliani-Miller (MM) and the debt proportional to the book value of equity (D=K·Ebv).

Ku = 9%; FCF_0 = 70; D_0 = 700; R_F = 4%; T = 40%; g = 2%.

		K_{TAXU}	K_{TAXL}		
			D=K·Ebv	ME	MM
a	4%	6,52%	8,32%	6,22%	8,32%
	7%	8,44%	10,97%	8,38%	14,86%
	8%	8,75%	11,40%	8,77%	16,53%
	9%	9,00%	11,74%	9,08%	18,02%
	10%	9,20%	12,01%	9,32%	19,35%
	13%	9,61%	12,57%	9,85%	22,62%

		K_{TAXU1}	K_{TAXU2}	K_{TAXL1}			K_{TAXL2}		
				D=K·Ebv	ME	MM	D=K·Ebv	ME	MM
a	4%	7,87%	7,82%	8,78%	8,78%	8,78%	8,77%	7,55%	8,77%
	7%	8,56%	8,55%	9,64%	9,64%	9,64%	9,67%	8,44%	9,69%
	8%	8,78%	8,78%	9,92%	9,92%	9,92%	9,96%	8,73%	9,98%
	9%	9,00%	9,00%	10,19%	10,19%	10,19%	10,24%	9,00%	10,26%
	10%	9,22%	9,22%	10,47%	10,47%	10,47%	10,51%	9,27%	10,54%
	13%	9,85%	9,83%	11,26%	11,26%	11,26%	11,28%	10,03%	11,33%

Table 8

Ku as a function of g (growth) and a (required return to the increase of assets) if the required return to the profit after tax of the unlevered company (K_{PATu}) is fixed

K_{PATu}= 9%; FCF_0 = 70; D_0 = 700; R_F = 4%; T = 40%.

		g					
		0%	1%	2%	3%	4%	5%
a	7%	9.00%	9.40%	9.88%	10.58%	11.89%	17.51%
	8%	9.00%	9.16%	9.34%	9.54%	9.80%	10.17%
	9%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
	10%	9.00%	8.88%	8.76%	8.66%	8.58%	8.52%
	12%	9.00%	8.70%	8.46%	8.27%	8.15%	8.10%
	15%	9.00%	8.54%	8.20%	7.97%	7.85%	7.84%

Appendix

General set up and derivation of some valuation formulae

To avoid arguments about the appropriate discount rates, we will use pricing kernels. The price of an asset that pays a random amount x_t at time t is the sum of the expectation of the product of x_t and M_t, the pricing kernel for time t cash flows:

$$P_x = \sum_1^\infty E[M_t \cdot x_t]$$

We will assume that

$$FCF_{t+1} = FCF_t (1 + g)(1 + \varepsilon_{t+1}) \tag{A.1}$$

ε_{t+1} is a random variable with expected value equal to zero ($E_t[\varepsilon_{t+1}] = 0$), but with a value today smaller than zero:

$$E_t[M_{t,t+1} \cdot \varepsilon_{t+1}] = -\frac{d}{1+R_F} \tag{A.2}$$

The risk free rate corresponds to the following equation:

$$\frac{1}{1+R_F} = \sum_1^\infty E[M_t \cdot 1] \tag{A.3}$$

First, we deduct the value of the unlevered equity. If $M_{t,t+1}$ is the one period pricing kernel at time t for cash flows at time t+1,

$$Vu_t = E_t[M_{t,t+1} \cdot FCF_{t+1}] + E_t[M_{t,t+1} \cdot Vu_{t+1}] \tag{A.4}$$

A solution must be $Vu_t = a \cdot FCF_t$; then:

$$Vu_t = E_t[M_{t,t+1} \cdot FCF_{t+1}] + E_t[M_{t,t+1} \cdot aFCF_{t+1}] = (1+a)E_t[M_{t,t+1} \cdot FCF_{t+1}] \tag{A.5}$$

According to (A.1):

$$E_t[M_{t,t+1} \cdot FCF_{t+1}] = E_t[M_{t,t+1} \cdot FCF_t(1+g)] + E_t[M_{t,t+1} \cdot FCF_t(1+g)\varepsilon_{t+1}] \tag{A.6}$$

Using equation (A.6) and defining Ku = (R_F + d) / (1 - d):

$$E_t[M_{t,t+1} \cdot FCF_{t+1}] = \frac{FCF_t(1+g)}{1+R_F} - \frac{FCF_t(1+g)d}{1+R_F} = \frac{FCF_t(1+g)(1-d)}{1+R_F} = \frac{FCF_t(1+g)}{1+Ku} \tag{A.7}$$

$$Vu_t = aFCF_t = (1+a)\frac{FCF_t(1+g)}{1+Ku}; \qquad a = \frac{(1+g)}{Ku-g} \tag{A.8}$$

Then:

$$Vu_t = \sum_1^\infty E[M_t \cdot FCF_t] = \frac{(1+g)}{Ku-g} FCF_t \tag{A.9}$$

The appropriate discount rate for Vu_t is also Ku because:

$$Vu_0 = \frac{FCF_0(1+g)}{1+Ku} + \frac{Vu_0(1+g)}{1+Ku} = \frac{Vu_0(Ku-g)}{1+Ku} + \frac{Vu_0(1+g)}{1+Ku} = Vu_0 \tag{A.10}$$

If $D_t = L \cdot S_t$, the appropriate discount rate for the expected value of the unlevered equity (Vu_t), for the expected value of the debt (D_t), for the expected value of the tax shields (VTS_t), and for the expected value of the equity (S_t) is Ku in all periods.

Using (A.10), the appropriate discount rate for ΔVu_1 ($K_{\Delta Vu1}$) is:

$$Vu_0 = \frac{Vu_0(Ku-g)}{1+Ku} + \frac{Vu_0(1+g)}{1+Ku} = \frac{Vu_0(Ku-g)}{1+Ku} + \frac{Vu_0}{1+R_F} + \frac{gVu_0}{1+K_{\Delta Vu1}} \quad \text{(A.11)}$$

$$1+K_{\Delta Vu1} = \frac{g(1+Ku)(1+R_F)}{g(1+R_F)-(Ku-R_F)} \quad \text{(A.12)}$$

As $Vu_t = a \cdot FCF_t$; $K_{\Delta Vut} = K_{\Delta FCFt}$. Looking at (54), $K_{\Delta Vu1} = K_{\Delta Dt}$.

For t=2, as the expected value of ΔVu_2 is $gVu_0(1+g)$, the expected value of the difference difference between Vu_2,and Vu_1, known in t=1:

$$\frac{gVu_0(1+g)}{(1+K_{\Delta Vu1})(1+K_{\Delta Vu2})} = \frac{Vu_0(1+g)^2}{(1+Ku)^2} - \frac{Vu_0(1+g)}{(1+Ku)(1+R_F)}$$

It is clear that $K_{\Delta Vu2} = Ku = K_{\Delta FCF2}$. Using the same argument, it may be shown that for t>1, $K_{\Delta Vut} = Ku = K_{\Delta FCFt}$

Miles-Ezzell present value of the increases of debt

Equation (19) is the present value of the expected increase of debt in period 1. The present value of the expected increase of debt in period t (as D_{t-1} is known in period t-1) is:

$$E_0[M_{0,t} \cdot \Delta D_t] = \frac{D_0(1+g)^t}{(1+Ku)^t} - \frac{D_0 1+g)^{t-1}}{(1+R_F)(1+Ku)^{t-1}} \quad \text{(A.13)}$$

The sum of all the present values of the expected increases of debt is a geometric progression with growth rate = (1+g)/(1+Ku). The sum is:

$$\sum_{t=1}^{\infty} E_0[M_{0,t} \cdot \Delta D_t] = \frac{D_0}{(Ku-g)}\left(g - \frac{Ku-R_F}{1+R_F}\right) \quad \text{(A.14)}$$

Miles-Ezzell formulae with continuous adjustment of debt

If debt is adjusted continuously, not only at the end of the period, then the Miles-Ezzell formula (21) changes to

$$VTS_0 = \int_0^{\infty} T\rho D_t e^{(\gamma-\kappa)t} dt = \frac{D_0 ? T}{\kappa - \gamma} \quad \text{(A.15)}$$

where $\rho = \ln(1+R_F)$, $\gamma = \ln(1+g)$, and $\kappa = \ln(1+Ku)$.

Perhaps formula (A.15) induces to Cooper and Nyborg (2006) and Ruback (1995 and 2002) to use (A.16) as the expression for the value of tax shields when the company maintains a constant market value leverage ratio (D_t = L S_t):

$$VTS_0 = \frac{D_0\ R_F\ T}{Ku - g} \tag{A.16}$$

But (A.16) is incorrect for discrete time: (21) is the correct formula.

$D_t = L \cdot E_t$ is absolutely equivalent to $D_t = M \cdot Vu_t$. In both cases $\Delta D_t = X \cdot \Delta FCF_t$, being $X = D_0 / FCF_0$.

Derivation of formulas if debt is proportional to the book value of equity

The present value of the tax shield of period t is:

$$\frac{D_0(1+g)^{t-1} R_F T}{(1+R_F)(1+K_{TS2})...(1+K_{TSt})} = \frac{D_0 R_F T}{(1+R_F)^t} + \frac{g D_0 R_F T}{(1+R_F)^{t-1}(1+\alpha)} + ... + \frac{g D_0 (1+g)^{t-2} R_F T}{(1+R_F)(1+\alpha)^{t-1}} \tag{A.17}$$

K_{TS} is the appropriate discount rate for the tax shields (TS). (A.17) takes into consideration the fact that the appropriate discount rate for the increases of debt is α. (A.17) is the sum of a geometric progression with a factor $X = (1+g)(1+R_F)/(1+\alpha)$. The solution is:

$$PV_0[TS_t] = \frac{R_F T D_0}{(1+R_F)^t}\left[1 + \frac{g(1+R_F)(X^{t-1}-1)}{(1+\alpha)(X-1)}\right] \tag{A.18}$$

And the appropriate discount rate for the expected tax shield of period t is:

$$(1+K_{TSt}) = \frac{PV_0[TS_{t-1}]}{PV_0[TS_t]}(1+g) = X(1+\alpha)\frac{(1+\alpha)(X-1) + g(1+R_F)(X^{t-2}-1)}{(1+\alpha)(X-1) + g(1+R_F)(X^{t-1}-1)} \tag{A.19}$$

$$(1+K_{TSt}) = \frac{PV_0[TS_{t-1}]}{PV_0[TS_t]}(1+g) = X(1+\alpha)\frac{1+R_F-(1+\alpha)^2 + g(1+R_F)X^{t-2}}{1+R_F-(1+\alpha)^2 + g(1+R_F)X^{t-1}}$$

As $K_{Dt} = K_{TSt+1}$ using (A.18), we know that:

$$PV_0[TS_{t+1}] = \frac{R_F T D_0 (1+g)^t}{(1+R_F)(1+K_{D1})...(1+K_{Dt})} = \frac{R_F T D_0}{(1+R_F)^{t+1}}\left[1 + \frac{g(1+R_F)(X^t-1)}{(1+\alpha)(X-1)}\right]$$

$$(1+K_{D1})...(1+K_{Dt}) = \frac{(1+R_F)^t(1+g)^t}{1 + \frac{g(1+R_F)(X^t-1)}{(1+\alpha)(X-1)}} \tag{A.20}$$

And the present value of the debt in t is:

$$PV_0[D_t] = \frac{D_0(1+g)^t}{(1+K_{D1})...(1+K_{Dt})} = \frac{D_0}{(1+R_F)^t}\left[1+\frac{g(1+R_F)(X^t-1)}{(1+\alpha)(X-1)}\right] \quad (A.21)$$

We calculate the present value of the value of tax shields in t from the equation:

$$VTS_0 = PV_0[TS_1] + ... + PV_0[TS_t] + \frac{VTS_0(1+g)^t}{(1+K_{VTS1})(1+K_{VTS2})...(1+K_{VTSt})}$$

It is clear that:

$$PV_0[VTS_t] = \frac{VTS_0(1+g)^t}{(1+K_{VTS1})(1+K_{VTS2})...(1+K_{VTSt})} = PV_0[TS_{t+1}] + ... = \sum_{t+1}^{\infty} PV_0[TS_t]$$

$$PV_0[VTS_t] = R_F T D_0 \sum_{t=t+1}^{\infty} \frac{1}{(1+R_F)^t}\left[1+\frac{g(1+R_F)(X^{t-1}-1)}{(1+\alpha)(X-1)}\right]$$

We have three geometric progressions with different growth factors. The result is:

$$PV_0[VTS_t] = \frac{TD_0}{(1+R_F)^t(X-1)}\left[X-1+\frac{gR_F X^t}{(\alpha-g)} - \frac{g(1+R_F)}{(1+\alpha)}\right] \quad (A.22)$$

if $\alpha = R_F$; $X = (1+g)$ and $PV_0[VTS_t] = \frac{R_F T D_0}{R_F - g}\frac{(1+g)^t}{(1+R_F)^t}$; $(1+K_{VTSt}) = (1+R_F)$

To calculate the present value of the equity in t, we start with equation (A.23)

$$S_0 = \frac{ECF_0(1+g)}{(1+Ke_1)} + ... + \frac{ECF_0(1+g)^t}{(1+Ke_1)...(1+Ke_t)} + \frac{S_0(1+g)^t}{(1+K_{S1})...(1+K_{S1})} \quad (A.23)$$

It is clear that

$$PV_0[S_t] = \frac{S_0(1+g)^t}{(1+K_{S1})...(1+K_{St})} = \sum_{t+1}^{\infty} \frac{ECF_0(1+g)^t}{(1+Ke_1)...(1+Ke_t)}$$ **WRONG**

$$PV_0[S_t] = \frac{S_0(1+g)^t}{(1+K_{S1})...(1+K_{St})}$$ **and** $$E_0[S_t] = S_0(1+g)^t = \sum_{i=t+1}^{\infty} \frac{ECF_0(1+g)^i}{(1+Ke_i)(1+Ke_{i+1})...}$$

From equation (24), we know that:

new $$E_0[S_t] = \sum_{i=t+1}^{\infty} \frac{FCF_0(1+g)^i}{(1+Ku)^{i-t}} + \sum_{t+1}^{\infty} \frac{D_0 g(1+g)^{i-1}}{(1+\alpha)^{i-t}} - \sum_{t+1}^{\infty} \frac{D_0 R_F(1-T)(1+g)^{i-1}}{(1+K_{Di})(1+K_{Di+1})...} \quad (A.24)$$

(A.22) is also the sum of the present values of the tax shields from t+1 on, then, the present value of the last term of equation (A.24) is:

$$PV_0[VTS_t]\frac{1-T}{T} = \frac{(1-T)D_0}{(1+R_F)^t(X-1)}\left[X-1+\frac{gR_F X^t}{(\alpha-g)} - \frac{g(1+R_F)}{(1+\alpha)}\right] \quad (A.25)$$

Calculating the present value of equation (A.24) (we need to calculate the sum of the two geometric progressions) and using (A.25), we get:

$$E_0[\Delta D_t] = \sum_{t+1}^{\infty} \frac{D_0 g(1+g)^{i-1}}{(1+\alpha)^{i-t}} = \frac{D_0 g(1+g)^t}{\alpha - g}$$

$$E_0[Vu_t] = \sum_{i=t+1}^{\infty} \frac{FCF_0(1+g)^i}{(1+Ku)^{i-t}} = \frac{FCF_0(1+g)^{t+1}}{Ku - g} = Vu_0(1+g)^t$$

$$PV_0[S_t] = \frac{Vu_0(1+g)^t}{(1+Ku)^t} + \frac{D_0 g(1+g)^t}{(\alpha - g)(1+\alpha)^t} - \frac{(1-T)D_0}{(1+R_F)^t(X-1)}\left[X - 1 + \frac{gR_F X^t}{(\alpha - g)} - \frac{g(1+R_F)}{(1+\alpha)}\right] \quad \text{(A.26)?}$$

$$PV_0\left[\frac{D_0 g(1+g)^t}{\alpha - g} \textbf{ in t}\right] = ?????$$

(A.26) may be simplified into:

$$PV_0[S_t] = \frac{Vu_0(1+g)^t}{(1+Ku)^t} + \frac{D_0 g(1+g)^t}{(\alpha - g)(1+\alpha)^t} - \frac{(1-T)D_0}{(1+R_F)^t(\alpha - g)}\left[\alpha + \frac{gR_F(X^t - 1)}{(X-1)}\right] \quad \text{(A.27)?}$$

If $\alpha = R_F$, (A.27) is:

$$PV_0[S_t] = \frac{Vu_0(1+g)^t}{(1+Ku)^t} - \frac{D_0(1+g)^t}{(1+R_F)^t} + \frac{D_0 T R_F(1+g)^t}{(R_F - g)(1+R_F)^t}$$

If t = 0, (A.27) is: $S_0 = Vu_0 - D_0 + \frac{D_0 \alpha T}{(\alpha - g)}$

$$PV_0[S_t] < 0 \quad \textbf{if} \quad \frac{(1-T)D_0}{(1+R_F)^t(\alpha - g)}\left[\alpha + \frac{gR_F(X^t - 1)}{(X-1)}\right] > \frac{Vu_0(1+g)^t}{(1+Ku)^t} + \frac{D_0 g(1+g)^t}{(\alpha - g)(1+\alpha)^t} -$$

The present value of the unlevered equity in t is

$$PV_0[Vu_t] = \frac{Vu_0(1+g)^t}{(1+Ku)^t} \quad \text{(A.28)}$$

(A.27), (A.22), (A.21) and (A.28) satisfy the condition:

$$PV_0[Vu_t] + PV_0[VTS_t] = PV_0[D_t] + PV_0[S_t]$$

To calculate the discount rate of the expected equity cash flow in t, we use equation (A.23):

$$\frac{ECF_0(1+g)^t}{(1+Ke_1)...(1+Ke_t)} = PV_0[S_{t-1}] - PV_0[S_t] \quad \text{(A.29)}$$

Using (A.27) and some algebra permits to find:

$$(1+Ke_1)...(1+Ke_t) = \frac{S_0(Ke - g)}{\frac{Vu_0(Ku - g)}{(1+Ku)^t} + \frac{D_0 R_F(1-T)}{(1+g)^{t-1}(1+R_F)^t}\left[\frac{R_F - \alpha + g(1+R_F)X^{t-1}}{g(1+R_F) + R_F - \alpha}\right] + \frac{gD_0}{(1+\alpha)^t}} \quad \text{(A.30)}$$

The appropriate discount rate for the expected value of equity implied by Modigliani-Miller

Calculating present value of equation (1) in t =1:

$$E_0[M_{0,1} \cdot VTS_1] = E_0[M_{0,1} \cdot S_1] + E_0[M_{0,1} \cdot D_1] - E_0[M_{0,1} \cdot Vu_1]$$

$$VTS_0 \frac{(1+g)}{(1+R_F)} = S_0 \frac{(1+g)}{(1+K_{S1})} + D_0 \frac{(1+g)}{(1+R_F)} - Vu_0 \frac{(1+g)}{(1+Ku)}$$

$$\frac{S_0}{(1+K_{S1})} = \frac{S_0 + (VTS_0 - D_0)Ku + Vu_0 R_F}{(1+R_F)(1+Ku)} \tag{A.31}$$

$$(1+K_{St}) = (1+R_F)(1+Ku)\frac{(S_0 - Vu_0)(1+Ku)^{t-1} + Vu_0(1+R_F)^{t-1}}{(S_0 - Vu_0)(1+Ku)^t + Vu_0(1+R_F)^t} \tag{A.32}$$

$$\prod_1^t (1+K_{St}) = \frac{S_0(1+R_F)^t(1+Ku)^t}{(S_0 - Vu_0)(1+Ku)^t + Vu_0(1+R_F)^t} \tag{A.33}$$

Present value of the expected increases of the book-value of equity

Using equation (4), the present value of the future increases of equity is equal to the present value of the future increases of assets minus the present value of the future increases of debt. Then, the present value of the future increases of equity, according to the different theories is:

ME: $$\sum_{t=1}^{\infty} E_0[M_{0,t} \cdot \Delta Ebv_t] = \frac{gA_0}{(\alpha - g)} - \frac{D_0}{(Ku - g)}\left(g - \frac{Ku - R_F}{1+R_F}\right) \tag{A.34}$$

MM: $$\sum_{t=1}^{\infty} E_0[M_{0,t} \cdot \Delta Ebv_t] = \frac{gA_0}{(\alpha - g)} - \frac{gD_0}{(R_F - g)} \tag{A.35}$$

$D_t = K \cdot Ebv_t$: $$\sum_{t=1}^{\infty} E_0[M_{0,t} \cdot \Delta Ebv_t] = \frac{g(A_0 - D_0)}{(\alpha - g)} = \frac{gEbv_0}{(\alpha - g)} \tag{A.36}$$

Appropriate discount rate for capital gains, K_{CGt}

$$PV_0[(S_1 - S_0)_{\text{in } \mathbf{t=1}}] = \frac{S_0(1+g)}{1+K_{S1}} - \frac{S_0}{1+R_F} = \frac{gS_0}{1+K_{CG1}}$$

$$\frac{1}{1+K_{CG1}} = \frac{(1+g)}{g(1+K_{S1})} - \frac{1}{g(1+R_F)} \tag{A.37}$$

According to Miles-Ezzell, as K_{St} = Ku, (A.37) is:

$$\frac{1}{1+K_{CG1}} = \frac{1+g}{g(1+Ku)} - \frac{1}{g(1+R_F)} \tag{A.38}$$

K_{CGt} = Ku if t>1. In our example, K_{CG1} = -177.6%.

According to Modigliani-Miller, using (45), (A.37) is:

$$\frac{1}{1+K_{CG1}} = \frac{1}{(1+R_F)} - \frac{(Ke-R_F)(1+g)}{g(1+R_F)(1+Ku)} \tag{A.39}$$

In our example, K_{CG1} = -160.8%.

If Debt is proportional to the Equity book value and the increase of assets is proportional to the free cash flow, using (40), (A.37) is:

$$\frac{1}{1+K_{CG1}} = \frac{1}{(1+Ku)} - \frac{Ke-R_F}{g(1+Ku)(1+R_F)} \tag{A.40}$$

In our example, K_{CG1} = -137.7%.

The present value of the expected capital gain in t is:

$$PV_0\left[(S_t - S_0)_{\text{in } \mathbf{t=t}}\right] = \frac{S_0(1+g)^t}{(1+K_{S1})...(1+K_{St})} - \frac{S_0}{(1+R_F)^t} = \frac{gS_0(1+g)^{t-1}}{(1+K_{CG1})...(1+K_{CGt})}$$

$$\frac{1}{(1+K_{CG1})...(1+K_{CGt})} = \frac{(1+g)}{g(1+K_{S1})...(1+K_{St})} - \frac{1}{g(1+g)^{t-1}(1+R_F)^t} \tag{A.41}$$

It is interesting to note that K_{CGt} (except for t = 1) are equal for Miles-Ezzell and under the constant book value leverage ratio.

<u>Total expected return for the shareholder</u>

The total expected return for the shareholder is Ke in every period because

$$K_{SHAR1} = \frac{gS_0 + ECF_0(1+g)}{S_0} = \frac{gS_0 + S_0(Ke-g)}{S_0} = Ke \tag{A.42}$$

REFERENCES

Arzac, E.R, Glosten, L.R., 2005, A Reconsideration of Tax Shield Valuation, European Financial Management 11/4, 453-461.

Cooper, I. A., and Nyborg, K. G., 2006, The Value of Tax Shields IS Equal to the Present Value of Tax Shields, Journal of Financial Economics, forthcoming.

Copeland, T. E., Koller T., Murrin J., 2000. Valuation: Measuring and Managing the Value of Companies (Third edition. Wiley, New York).

Damodaran, A., 1994. Damodaran on Valuation (John Wiley and Sons, New York).

Fernandez, P., 2004, The value of tax shields is NOT equal to the present value of tax shields, Journal of Financial Economics 73/1, 145-165.

Fieten, P., Kruschwitz, L., Laitenberger, J., Löffler, A., Tham, J.. Vélez-Pareja, I., and N. Wonder, 2005, Comment on 'The value of tax shields is NOT equal to the present value of tax shields', Quarterly Review of Economics and Finance 45, #1, 184-187.

Harris, R.S., Pringle, J.J., 1985, Risk–adjusted discount rates extensions from the average-risk case, Journal of Financial Research 8, 237-244.

Miles, J.A., Ezzell, J.R., 1980, The weighted average cost of capital, perfect capital markets and project life: a clarification, Journal of Financial and Quantitative Analysis 15, 719-730.

Modigliani, F., Miller, M., 1963, Corporate income taxes and the cost of capital: a correction, American Economic Review 53, 433-443.

Myers, S.C., 1974, Interactions of corporate financing and investment decisions – implications for capital budgeting, Journal of Finance 29, 1-25.

Ruback, R., 1995, A note on capital cash flow valuation, Harvard Business School Case No. 9-295-069.

Ruback, R., 2002, Capital cash flows: a simple approach to valuing risky cash flows, Financial Management 31, 85–103.