QUADRATIC TERM STRUCTURE MODELS IN DISCRETE TIME

Marco Realdon*

15/3/06

Abstract

This paper extends the results on quadratic term structure models in continuous time to the discrete time setting. The continuous time setting can be seen as a special case of the discrete time one. Discrete time quadratic models have advantages over their continuous time counterparts as well as over discrete time affine models. Recursive closed form solutions for zero coupon bonds are provided even in the presence of multiple correlated underlying factors, time-dependent parameters, regime changes and "jumps" in the underlying factors. In particular regime changes and "jumps" cannot so easily be accommodated in continuous time quadratic models. Pricing bond options requires simple integration and model estimation does not require a restrictive choice of the market price of risk.

*Department of Economics, University of York, Alcuin College, University Rd, YO10 5DD, UK; tel: +44/(0)1904/433750; email: mr15@york.ac.uk.

Key words: quadratic term structure model, discrete time, bond valuation, regime change, jumps, bond option.

JEL classification: G12; G13.

1 Introduction

This paper presents a general class of quadratic term structure models (hereafter QTSM) in discrete time rather than in continuous time. The motivation for looking at QTSM in discrete time can be summarized as follows. After Ahn-Dittmar-Gallant (2002) we know that in continuous time QTSM have advantages over affine term structure models (hereafter ATSM). We also know, especially after Dai-Le-Singleton (2005), that ATSM in discrete have advantages over ATSM in continuous time. Then, since QSTM offer advantages and since the discrete time setting offers advantages, this paper explores QTSM in discrete time and finds that indeed these models offer advantages over QSTM in continuous time and also over ATSM in discrete time. This same argument is now articulated more precisely.

In continuous time QTSM overcome some of the limitations of ASTM. In fact Dai-Singleton (2000) find that term structure data suggest negative correlation between the state variables of ATSM as well as heteroscedasticity of the yields. But the admissibility conditions for ATSM entail a trade-off between matching yield heteroscedasticity and accommodating negative correlation of state variables at the same time. On the other hand Ahn-Dittmar-Gallant (2002) show that continuous time QTSM can reproduce yield heteroscedasticity and also accommodate any correlation between the state variables, thus providing an empirical performance superior to ATSM models. Yet QTSM still cannot fully capture the dynamics of the term structure: more flexibility is needed. The negative correlation between state variables is also important for credit risk pricing if one variable drives the default intensity and the other the short interest rate. In this respect Duffie-Liu (2001) had already shown how QSTM could accommodate this feature. Anyway in continuous time regime changes and "jumps" in the state variables compromise the tractability of QTSM, whereas ATSM in some cases admit (quasi) closed form solutions despite multiple regimes and "jumps", as shown in Duffie-Filipovic-Schachermayer (2003) and Dai-Singleton (2003). This seems an important advantage of ATSM since regime changes and "jumps" are supported by the empirical evidence on interest rates as explained in Johannes (2004), Sun (2005), Bansal-Zhou (2002), Ang-Bekaert (2002), Dai-Singleton-Yang (2005) and others. Virtually all the literature that has considered regimes changes or "jumps" in a no-arbitrage framework, has done so by employing just affine models.

On the other hand Dai-Le-Singleton (2005) show that ATSM in discrete time offer advantages over continuous time ATSM in that the discrete time setting provides much flexibility in specifying the market price of risk while the likelihood functions for the observed yields remains Gaussian. These are advantages in estimation. But, as in continuous time, also in discrete time the correlation between factors in an ATSM still cannot be negative, so that the problem highlighted by Dai-Singleton (2000) remains.

Given the above premise, this paper shows that discrete time QTSM offer advantages over both continuous time QTSM and discrete time ATSM. In discrete time QSTM provide closed form solutions for zero coupon bonds (and for moments of bond returns) even in the presence of regime changes and "jumps" in the underlying factors, whereas in continuous time only some ATSM seem to retain closed form solutions under such conditions. Moreover, as in Dai-Le-Singleton (2005), factors transition densities are Gaussian and the market price of risk can be freely specified, unlike in continuous time.

The advantages of discrete time QTSM over discrete time ATSM are that yields can be constrained to be non-negative and that the correlation between factors is unconstrained, unlike in Dai-Le-Singleton (2005). Moreover QTSM can accommodate regime changes and positive as well as negative "jumps" without any admissibility problem, while predicted yields still remain non-negative. Instead continuous or discrete time affine models either constrain jumps to be positive or allow yields to turn negative. Of course the results of this paper are applicable also to discrete time affine Gaussian models, which are special cases of discrete time QTSM, although the positivity of yields is not longer guaranteed in this setting. These results are of interest also to price credit risky bonds, since regime switches and "jumps" in the factors can model changes in ratings and the occasional sudden widening or narrowing of credit spreads.

The paper also makes various other points. Continuous time QTSM can be seen as a special case of discrete time QTSM as the discrete time steps converge to zero. Closed form solutions for zero coupon bond prices are available even in the presence of multiple correlated factors. This result differs from the continuous time setting, which requires the numerical solution of a system of ordinary differential equations, and is valid even if the model parameters are time dependent. This is of interest since time dependent parameters can significantly increase the fitting capability of the model. The conditions for the econometric identification of parameters are similar to those in continuous time. In a one factor setting simple integration gives the price of European bond options.

2 Literature

The literature most directly relevant to this paper is that on term structure models in discrete time and that on QTSM. Noteworthy discrete time models are that of Sun (1992), who proposes a discrete time version of the Cox-Ingersoll-Ross model, that of Gourieroux-Monfort-Polimenis (2002), who derive exact discrete time versions of continuous time affine models, and that of Ang-Piazzesi (2003), who propose a Gaussian model driven by macroeconomic and latent factors. More recently Dai-Le-Singleton (2005) study the discrete time counterparts of the continuous time term structure models of Duffie-Kan (1996) and Dai-Singleton (2000). Also the present paper considers the discrete time setting, but it analyses quadratic rather than affine term structure models. QTSM in discrete time are well defined, which is not the case for the discrete time version of the Cox-Ingersoll-Ross model appeared in Sun (1992), since in

Sun the short interest rate can be negative. Also in Ang-Piazzesi (2003) the short term interest rate can turn negative, although this does not mean that their Gaussian model is not well defined. QTSM overcome these problems since yields are always positive. To ensure positivity we just need to impose mild parameter constraints similar to the ones imposed by Ahn-Dittmar-Gallant (2002) in continuous time. Then, unlike in Ang-Piazzesi (2003), discrete time QTSM can capture the possible non-linear relationship between inflation or output gap and the level of the short interest rate or of longer term yields.

QTSM have been studied in continuous time. The first QTSM of Beaglehole-Tenney (1991) and that of Constantinides (1992) have been recently extended in Lieppold-Wu (2001, 2002), who price various contingent claims in the quadratic set up, in Ahn-Dittmar-Gallant (2002), who provide the maximally flexible QTSM, and in Chen-Filipovic-Poor (2005), who highlight the applicability of QTSM to credit risk pricing. The only discrete time QTSM to date seems that of Gourieroux-Sufana (2003, 2005), who set QTSM in the framework of affine models by introducing the Wishart matrix process for the underlying factors. In this way Gourieroux-Sufana show that the general class of affine models formulated by Duffie-Kan (1996) can be extended. This paper differs from Gourieroux-Sufana's in that it does not assume a Wishart process, rather it extends the results of the above continuous time QTSM to the discrete time setting and shows that the continuous time setting can be seen as a special case of the discrete time one. For example, unlike in Gourieroux-Sufana, in this paper the factors are not constrained to revert to zero and bond yields are affine-quadratic functions of the underlying factors, rather than simply quadratic functions thereof. More fundamentally this paper shows that discrete time QTSM still retain pricing closed form solutions even in the presence of regime changes and "jumps" in the underlying factors.

3 Single factor discrete time QTSM

This section presents the basic one factor QTSM in discrete time. We define with $P_{n,t}$ the price of a zero coupon bond at time t and with n time periods to maturity. Each time period is of length $\Delta = 1$, thus the bond expires at time $t + n$. We define with r_t the risk-free interest rate at time t for the maturity equal to one period. Equivalently r_t can be viewed as the one period yield of the default-free zero coupon bond $P_{1,t}$, i.e. $r_t = -\ln P_{1,t}$. Then we invoke the basic risk-neutral valuation equation

$$P_{n,t} = E_t \left[e^{-\Sigma_{i=t}^{t+n-1} r_i} \right] = E_t \left[e^{-r_t} \cdot P_{n-1,t+1} \right] \tag{1}$$

where $E_t [..]$ denotes conditional expectation at time t under the risk-neutral measure. We state the following equations, which summarize the assumptions underlying the pricing model of this section, and then we explain the assumptions in turn:

$$r_t = \alpha + \beta x_t + \Psi x_t^2 \tag{2}$$

$$x_{t+1} = x_t \left(1 - \phi\right) + \phi\mu + \xi_{t+1} \tag{3}$$

$$\xi_t \backsim N\left(0, \sigma^2\right) \tag{4}$$

$$P_{n,t} = e^{A_n + B_n x_t + C_n x_t^2} \tag{5}$$

where α, β, Ψ, ϕ, μ, σ^2 are constant and A_n, B_n and C_n only depend on n. Equation 2 states that the one period interest rate r_t is a quadratic function of the underlying factor x_t. Equations 3 and 4 state that the factor x follows a Gaussian auto-regressive process, where the noise term ξ_{t+1} is normally distributed with mean of 0 and variance σ^2. This auto-regressive process is meant to be the process under the pricing measure or risk-neutral measure, rather than the process under the physical measure. The process under the physical measure will be considered later on. Equation 5 states our conjectured solution for $P_{n,t}$, which we are now going to verify. In fact we can restate the pricing equation as

$$P_{n,t} = E_t \left[e^{-\alpha - \beta x_t - \Psi x_t^2} e^{C_{n-1} x_{t+1}^2 + B_{n-1} x_{t+1} + A_{n-1}} \right] \tag{6}$$

which we can also re-write as

$$
\begin{aligned}
A_n + B_n x_t + C_n x_t^2 &= -\alpha - \beta x_t - \Psi x_t^2 + A_{n-1} + x_t (1-\phi) B_{n-1} \\
&\quad + \phi\mu B_{n-1} + C_{n-1} (x_t (1-\phi) + \phi\mu)^2 \\
&\quad + \ln E_t \left[e^{(B_{n-1} + 2C_{n-1} x_t (1-\phi) + 2C_{n-1}\phi\mu)\xi_{t+1} + C_{n-1}\xi_{t+1}^2} \right]
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
&\ln E_t \left[e^{(B_{n-1} + 2C_{n-1} x_t (1-\phi) + 2C_{n-1}\phi\mu)\xi_{t+1} + C_{n-1}\xi_{t+1}^2} \right] \\
&= -\ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2} - 2C_{n-1}\right) + \frac{\sigma^2 (B_{n-1} + 2C_{n-1} x_t (1-\phi) + 2C_{n-1}\phi\mu)^2}{(2 - 4C_{n-1}\sigma^2)}.
\end{aligned} \tag{8}
$$

In our setting $\frac{1}{\sigma^2} - 2C_{n-1} \geq 0$, since $C_{n-1} \leq 0$ and $\sigma \geq 0$, so the corresponding logarithm in the last equation is always well defined. Appendix 1 shows how equation 8 is derived and also shows that the above implies that

$$
\begin{aligned}
A_n &= -\alpha + A_{n-1} + \phi\mu B_{n-1} + C_{n-1}\phi^2\mu^2 \\
&\quad - \ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2} - 2C_{n-1}\right) + \frac{\sigma^2 (B_{n-1} + 2C_{n-1}\phi\mu)^2}{(2 - 4C_{n-1}\sigma^2)}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
B_n &= -\beta + (1-\phi) B_{n-1} + 2(1-\phi)\phi\mu C_{n-1} \\
&\quad + \frac{2C_{n-1}(1-\phi)\sigma^2 (B_{n-1} + 2C_{n-1}\phi\mu)}{(1 - 2C_{n-1}\sigma^2)}
\end{aligned} \tag{10}
$$

$$
C_n = -\Psi + (1-\phi)^2 C_{n-1} + \frac{\sigma^2 (2C_{n-1}(1-\phi))^2}{(2 - 4C_{n-1}\sigma^2)}. \tag{11}
$$

These recursive difference equations are subject to the initial conditions $A_0 = B_0 = C_0 = 0$ in the case of a zero coupon bond of unit face value. We notice that technically these equations, together with equation 5, provide a closed form solution for zero coupon bonds. At this point we can verify that at time t the one-period yield $y_{1,t}$ is

$$\begin{aligned} y_{1,t} &= -\ln P_{1,t} = -A_1 - B_1 x_t - C_1 x_t^2 \\ &= \alpha + \beta x_t + \Psi x_t^2 = r_t \end{aligned} \tag{12}$$

since $A_1 = -\alpha$, $B_1 = -\beta$ and $C_1 = -\Psi$.

Following Ahn-Dittmar-Gallant (2002) we note that, even in this discrete time setting, $r_t \geq 0$ as long as $\alpha \geq \frac{\beta^2}{4\Psi}$ and $\Psi > 0$. In fact $\frac{\partial r_t}{\partial x_t} = \beta + 2\Psi x_t = 0$, giving $x_t = -\frac{\beta}{2\Psi}$ and the corresponding lower bound for r_t, which is $r_t^* = \alpha - \beta\frac{\beta}{2\Psi} + \Psi\frac{\beta^2}{4\Psi^2} = \alpha - \frac{\beta^2}{4\Psi}$. This implies that, if $\alpha \geq \frac{\beta^2}{4\Psi}$, then the lower bound of r_t is $r_t^* \geq 0$. Hereafter we simply assume that this condition is met. Thus, whereas the discrete time version of Cox-Ingersoll-Ross type affine models, as in Sun (1992), poses the problem of possible negative values of r_t, the present discrete time QTSM does not pose such a problem. It is worth highlighting that ξ_{t+1} needs to have a Gaussian distribution in order for the above results to hold.

4 Multiple factors

Now we extend the previous single factor analysis to a setting of multiple factors. We redefine x_t, β, μ, ξ_{t+1}, B_n as $N \times 1$ vectors, and Ψ, ϕ, C_n, Σ as $N \times N$ matrixes. r_t, A_n and α are still scalars. In this multifactor setting we reformulate the model assumptions as

$$r_t = \alpha + \beta' x_t + x_t' \Psi x_t \tag{13}$$

$$x_{t+1} = (I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1} \tag{14}$$

$$\xi_{t+1} \backsim N(0, I) \tag{15}$$

$$P_{n,t} = e^{A_n + B_n' x_t + x_t' C_n x_t} \tag{16}$$

where I is the $N \times N$ identity matrix. These assumptions imply that the covariance matrix of $(x_{t+1} - x_t)$ is $\Sigma\Sigma'$. Again the auto-regressive process is specified under the risk-neutral measure. Without loss in generality we assume that Ψ and C_n are symmetric, which are conditions for the econometric identification of Ψ and C_n, just as Ahn-Dittmar-Gallant have pointed out for the continuous time case. We can derive closed form solutions for A_n, B_n and C_n also in this multifactor setting. To see how, first we restate the pricing equation for $P_{n,t}$ as

$$P_{n,t} = E_t\left[e^{-\alpha - \beta' x_t - x_t' \Psi x_t} e^{A_{n-1} + B_{n-1}' x_{t+1} + x_{t+1}' C_{n-1} x_{t+1}}\right]. \tag{17}$$

Noting that

$$
\begin{aligned}
x_{t+1}' C_{n-1} x_{t+1} &= \left((I-\phi)\, x_t + \phi\mu + \Sigma\xi_{t+1}\right)' C_{n-1} \left((I-\phi)\, x_t + \phi\mu + \Sigma\xi_{t+1}\right) \\
&= x_t' (I-\phi)' C_{n-1} (I-\phi)\, x_t + (\phi\mu)' C_{n-1}\phi\mu + \left(\Sigma\xi_{t+1}\right)' C_{n-1}\Sigma\xi_{t+1} \\
&\quad +2x_t' (I-\phi)' C_{n-1}\phi\mu + 2x_t' (I-\phi)' C_{n-1}\Sigma\xi_{t+1} + 2\,(\phi\mu)' C_{n-1}\Sigma\xi_{t+1}
\end{aligned}
$$

we define

$$
\begin{aligned}
Q &= x_t' (I-\phi)' C_{n-1} (I-\phi)\, x_t + (\phi\mu)' C_{n-1}\phi\mu + 2x_t' (I-\phi)' C_{n-1}\phi\mu \\
F' &= 2x_t' (I-\phi)' C_{n-1} + 2\,(\phi\mu)' C_{n-1}.
\end{aligned}
$$

Then we can rewrite equation 17 as

$$
\begin{aligned}
A_n + B_n' x_t + x_t' C_n x_t &= -\alpha - \beta' x_t - x_t'\Psi x_t + A_{n-1} \qquad (18) \\
&\quad +B_{n-1}' \left((1-\phi)\, x_t + \phi\mu\right) \\
&\quad + \left((1-\phi)\, x_t + \phi\mu\right)' C_{n-1} \left((1-\phi)\, x_t + \phi\mu\right) \\
&\quad + \ln E_t \left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1} + \xi_{t+1}'\Sigma' C_{n-1}\Sigma\xi_{t+1}}\right]
\end{aligned}
$$

where

$$\ln E_t \left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1}+\xi'_{t+1}\Sigma' C_{n-1}\Sigma\xi_{t+1}} \right] = \ln \frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N} \left((B_{n-1}+F)'\,\gamma_i \right)^2 \tag{19}$$

with γ_i being the i-th column of the $N \times N$ matrix $\gamma = \left((\Sigma\Sigma')^{-1} - 2C_{n-1} \right)^{-1/2}$. Appendix 2 derives equation 19 and shows that equations 18 and 19 imply three recursive equations for A_n, B_n and C_n, which are

$$\begin{aligned} A_n &= -\alpha + A_{n-1} + B'_{n-1}\phi\mu + (\phi\mu)'\,C_{n-1}\phi\mu + \ln \frac{|\gamma|}{abs\,|\Sigma|} \\ &\quad + \frac{1}{2}\sum_{i=1}^{N} \left(\begin{array}{c} B'_{n-1}\gamma_i B'_{n-1}\gamma_i + B'_{n-1}\gamma_i 2\,(\phi\mu)'\,C_{n-1}\gamma_i \\ +2\,(\phi\mu)'\,C_{n-1}\gamma_i B'_{n-1}\gamma_i + 2\,(\phi\mu)'\,C_{n-1}\gamma_i 2\,(\phi\mu)'\,C_{n-1}\gamma_i \end{array} \right) \end{aligned} \tag{20}$$

$$\begin{aligned} B'_n &= -\beta' + B'_{n-1}\,(1-\phi) + 2\,(\phi\mu)'\,C_{n-1}\,(I-\phi) \\ &\quad + \sum_{i=1}^{N} \left(\begin{array}{c} B'_{n-1}\gamma_i\,(C_{n-1}\gamma_i)'\,(I-\phi) + B'_{n-1}\gamma_i\gamma'_i C_{n-1}\,(I-\phi) \\ +2\,(\phi\mu)'\,C_{n-1}\gamma_i\gamma'_i C_{n-1}\,(I-\phi) + 2\,(\phi\mu)'\,C_{n-1}\gamma_i\gamma'_i C_{n-1}\,(I-\phi) \end{array} \right) \end{aligned} \tag{21}$$

$$C_n = -\Psi + (I-\phi)'\,C_{n-1}\,(I-\phi) + 2\sum_{i=1}^{N} (I-\phi)'\,C_{n-1}\gamma_i\gamma'_i C'_{n-1}\,(I-\phi) \tag{22}$$

subject to the terminal conditions $A_n = 0$, $B'_n = \mathbf{1}' \cdot \mathbf{0}$ and $C_n = \mathbf{0}$, where $\mathbf{0}$ is an $N \times N$ matrix of zeros and $\mathbf{1}$ is a $N \times 1$ vector of ones. Of course these equations are a generalization of the corresponding ones derived above in the single factor setting. We notice that technically these equations are again closed form solutions and imply a closed form solution for zero coupon bonds even in this multifactor setting. On the other hand, in continuous time closed form solutions are not known for QTSM in the presence of multiple correlated factors. Rather in continuous time a system of ODE's needs to be solved numerically. Moreover the above closed form solutions are ideal to accommodate parameters whose values change deterministically from one time period to the next. This seems important as time dependent parameters can greatly improve the capability of the model to fit the cross section of the term structure. We also notice that in this multifactor setting $r_t \geq 0$ as long as $\alpha \geq \frac{1}{4}\beta' \Psi^{-1'} \beta$ and Ψ is positive semi-definite. In fact $\frac{\partial r_t}{\partial x_t} = \beta + 2\Psi x_t = 0$, gives $x_t = -\frac{1}{2}\Psi^{-1}\beta$ and the corresponding lower bound for r_t is $r_t^* = \alpha - \frac{1}{4}\beta' \Psi^{-1}\beta$. Hence, if $\alpha \geq \frac{1}{4}\beta' \Psi^{-1}\beta$, then the lower bound is $r_t^* \geq 0$.

Finally we notice that the above term structure model can be reinterpreted as a reduced form model of credit risk. For example r can be reinterpreted as a risk-neutral default intensity, $P_{n,t}$ as the survival probability between t and $t + \Delta n$, $P_{n,t} - P_{n+1,t}$ as the probability of default in the time period $]t+n, t+n+1]$, and so on.

4.1 Convergence to the continuous time counterpart

We can consider the above model as the discrete time counterpart of QTSM in continuous time such those in Ahn-Dittmar-Gallant (2002) or Lieppold Wu

(2002). In continuous time the risk-neutral process of the state vector x is described by the stochastic differential equation $dx = k(\mu - x)\,dt + \sigma dz$, where k and σ are $N \times N$ square matrixes of constants, μ and x are $N \times 1$ column vectors and dz is an $N \times 1$ column vector of differentials of independent Wiener processes. But the above discrete time auto-regressive Markov process can be re-expressed as $x_{t+\Delta} - x_t = \phi(\mu - x_t) + \Sigma\xi_{t+1}$, where Δ is the length of the time step. Above we set $\Delta = 1$. But if $\Delta \to 0$, then $x_{t+\Delta} - x_t$ converges to dx if only we set $\phi = k\Delta$ and $\Sigma\Sigma' = \sigma\sigma'\Delta$. This is why we can think of the continuous time QTSM as special cases of the above discrete time model as $\Delta \to 0$.

4.2 Conditions for parameter identification

If the state variables x are not observable, we need to add some restrictions to the above QTSM in order to be able to uniquely identify the model parameters. As already shown by Ahn-Dittmar-Gallant (2002) in the continuous time setting, also in the present discrete time setting parameter identification requires that:

- Ψ be symmetric; we normalize Ψ by requiring that its diagonal be made up ones;

- $\mu \geq 0$, $\alpha \geq 0$, $\beta = 0$ in order for μ to be identifiable;

- Σ be diagonal (triangular) and ϕ be triangular (diagonal).

These restrictions are explained in Appendix 3. In other words the restrictions for the econometric identification of the discrete time model are similar to the corresponding restrictions in continuous time.

5 Physical process

The above multifactor model was derived while assuming that, under the risk-neutral measure, which we denote as Q^*, the process of the state variables was $x_{t+1} = (I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1}$. Now we specify the process for x under the physical measure, which is of interest for econometric estimation and risk management. To do so we need to specify a market price of risk. As highlighted by Dai-Le-Singleton (2005), the discrete time setting allows very flexible specifications of the market price of risk while still retaining tractable transition densities for the time series of the underlying factors or of the observed yields. To switch to the physical measure, which we denote with P^*, we assume that the Radon-Nykodim derivative is

$$\frac{dP^*}{dQ^*} = e^{\frac{1}{2}\left(2\xi'_{t+1} f(x_t) - f(x_t)' f(x_t)\right)} \tag{23}$$

where $f(x_t)$ is an $N \times 1$ vector of functions of x_t that do not depend on ξ_{t+1}. Then the conditional probability density of ξ_{t+1} under the physical measure, which we denote with $P^*\left(\xi_{t+1}\right)$, is

$$\begin{aligned} P^*\left(\xi_{t+1}\right) &= Q^*\left(\xi_{t+1}\right)\frac{dP^*}{dQ^*} = \frac{1}{\sqrt{(2\pi)^N}} e^{-\frac{1}{2}\xi'_{t+1}\xi_{t+1} + \frac{1}{2}\left(2\xi'_{t+1} f(x_t) - f(x_t)' f(x_t)\right)} \qquad (24) \\ &= \frac{1}{\sqrt{(2\pi)^N}} e^{-\frac{1}{2}\sum_{i=1}^{N}\left(\xi_{t+1,i} - f(x_t)_i\right)^2}. \end{aligned}$$

It follows that under the physical measure the process of x becomes

$$x_{t+1} = x_t(1-\phi) + \phi\mu + \Sigma\left(f(x_t) + \xi_{t+1}\right) \tag{25}$$

$$= x_t(1-\phi) + \phi\mu + \Sigma f(x_t) + \Sigma\xi_{t+1} \tag{26}$$

where again $\xi_{t+1} \backsim N(0, I)$. Here the point to note is that $f(x_t)$ is a constant at time t, so the choice of the function $f(x_t)$ can be very wide. On the other hand x_{t+1} will still have a conditional Gaussian distribution, irrespective of $f(x_t)$. This fact, already noted by Dai-Le-Singleton (2005), guarantees tractability in econometric testing as well as much freedom in the choice of $f(x_t)$, provided that such choice is consistent with the absence of arbitrage. For example $f(x_t)$ can be a set of polynomial functions such that

$$f(x_t) = \begin{bmatrix} f_{1,1} \cdot x_{1,t}^{q_1} + .. + f_{1,N} \cdot x_{N,t}^{q_N} \\ .. \\ f_{N,1} \cdot x_{1,t}^{q_1} + .. + f_{N,N} \cdot x_{N,t}^{q_N} \end{bmatrix} \tag{27}$$

where all $f_{i,j}$ and q_i (with $1 \leq i \leq N$ and $1 \leq j \leq N$) are constants to be estimated and where $[x_{1,t}, .., x_{N,t}]' = x_t$. $f(x_t)$ will determine the risk-premia demanded by the market as revealed by the level of excess expected bond returns over and above the risk-free one period yield r_t. To see this we can calculate the one period excess expected log return as

$$\ln \frac{E_t^{P^*}\left[P_{n-1,t+1}\right]}{P_{n,t}} - r_t = \ln E_t^{P^*}\left[P_{n-1,t+1}\right] - \left(A_n + B_n' x_t + x_t' C_n x_t\right) - r_t \quad (28)$$

where $E_t^{P^*}[..]$ denotes conditional expectation with respect to the physical measure. Invoking again equation 19 we obtain

$$\begin{aligned}
\ln E_t^{P^*}\left[P_{n-1,t+1}\right] &= A_{n-1} + B_{n-1}'\left(\left(1-\phi\right)x_t + \phi\mu + \Sigma f\left(x_t\right)\right) \quad (29) \\
&\quad + \left(\left(1-\phi\right)x_t + \phi\mu + \Sigma f\left(x_t\right)\right)' C_{n-1}\left(\left(1-\phi\right)x_t + \phi\mu + \Sigma f\left(x_t\right)\right) \\
&\quad + \ln \frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(B_{n-1} + F_{P^*}\right)'\gamma_i\right)^2
\end{aligned}$$

with

$$F_{P^*}' = 2x_t'\left(I-\phi\right)' C_{n-1} + 2\left(\phi\mu + \Sigma f\left(x_t\right)\right)' C_{n-1}. \quad (30)$$

Although unreported, we can also find closed form expressions for the expected value and variance of zero coupon bond yields under the physical measure. These tractable expressions for expected bond returns, expected future yields and variance of future yields under the physical measure are of interest for the econometric testing of the model. For example, they can be used to provide moment conditions to be used in GMM estimation along the lines of Lieppold and Wu (2003).

6 Multiple regimes and "jumps"

This section extends the results of the previous section to a setting whereby multiple regimes are possible. All other things being equal, we now assume that the risk-neutral process of x_t is such that

$$x_{t+1} = (I - \phi_i)\, x_t + \phi_i \mu_i + \Sigma_i \xi_{t+1} \tag{31}$$

where the integer i (with $i = 1, 2, .., m$) is the index that indicates the current regime. Thus the vector x_t still follows a first order auto-regressive process, but now the parameters ϕ_i, μ_i and Σ_i depend on the regime i. Regime transitions are governed by a Markov chain as explained below. For simplicity we assume that the market is risk-neutral toward the risk of a regime change. This assumption could be relaxed at no cost. Our objective is to price a zero coupon bond at time t and with n periods to maturity in this setting. To this end we first define an $m \times m$ matrix

$$\mathbf{\Lambda}_{n,t} = \left[.., (\mathbf{\Lambda}_{n,t})_i\, , ..\right] \tag{32}$$

which we are going to use to express zero coupon bond prices at time t under the m different regimes and with n periods to maturity. $(\mathbf{\Lambda}_{n,t})_i$ is the i-th column of $\Lambda_{n,t}$. We impose the conditions

$$\mathbf{1}' \cdot (\mathbf{\Lambda}_{n,t})_i = 1 \tag{33}$$

for $i = 1, 2, .., m$, where $\mathbf{1}' = [1, .., 1]$ is a $1 \times m$ row vector of ones. We suppose that, in regime i, the price at time t of a zero coupon bond with n periods to maturity is

$$\mathbf{p}'_{n,t} \left(\mathbf{\Lambda}_{n,t}\right)_i \tag{34}$$

where $\mathbf{p}'_{n,t}$ is an $1 \times m$ row vector such that

$$\mathbf{p}'_{n,t} = \left[e^{A_{n,1}+B'_{n,1}x_t+x'_tC_{n,1}x_t}, .., e^{A_{n,i}+B'_{n,i}x_t+x'_tC_{n,i}x_t}, .., e^{A_{n,m}+B'_{n,m}x_t+x'_tC_{n,m}x_t}\right]. \tag{35}$$

The functions $A_{n,i}$, $B_{n,i}$ and $C_{n,i}$ are given by equations 20, 21 and 22 if only we substitute ϕ, μ and Σ with ϕ_i, μ_i and Σ_i. It follows that the $1 \times m$ row vector of zero coupon bond prices corresponding to all regimes at time t and with n periods to maturity is $\mathbf{p}'_{n,t}\Lambda_{n,t}$. We define with $\mathbf{T}_t$ an $m \times m$ matrix describing the real world regime transition probabilities over one time step. In particular $\left(\mathbf{T}_t\right)_i$ is the the i-th column of $\mathbf{T}_t$ and it denotes the transition probabilities from regime i at time t to any regime at time $t+1$. Consistency requires that we impose the conditions $\mathbf{1}' \cdot \left(\mathbf{T}_t\right)_i = 1$, for $i = 1, 2, .., m$. Transition probabilities are independent of x_t. Under our assumption of risk-neutrality toward regime changes, risk-neutral valuation implies that zero coupon bond prices in all regimes satisfy the following system of pricing equations

$$\mathbf{p}'_{n,t}\mathbf{\Lambda}_{n,t} = e^{-\alpha-\beta'x_t-x'_t\Psi x_t} \cdot E_t\left[\mathbf{p}'_{n-1,t+1}\mathbf{\Lambda}_{n-1,t+1}\mathbf{T}_t\right] \tag{36}$$

subject to the terminal conditions

$$\mathbf{p}'_{0,t+n}\mathbf{\Lambda}_{0,t+n} = \mathbf{1}'. \tag{37}$$

Since transition probabilities are independent of x_t, the solution to equation 36 is such that

$$\mathbf{p}'_{n,t} = e^{-\alpha-\beta' x_t - x_t'\Psi x_t} \cdot E_t\left[\mathbf{p}'_{n-1,t+1}\right] \tag{38}$$

$$\mathbf{\Lambda}_{n,t} = \mathbf{\Lambda}_{n-1,t+1}\mathbf{T}_t. \tag{39}$$

The solution to equation 38 is indeed given by equation 35 and Appendix A.4 shows that, if $\mathbf{T}_t$ is constant over time an equal to $\mathbf{T}$, then $\Lambda_{n,t} = \mathbf{T}^{n-1}$, so that

$$\mathbf{p}'_{n,t}\mathbf{\Lambda}_{n,t} = \left[e^{A_{n,1}+B'_{n,1}x_t+x_t'C_{n,1}x_t}, .., e^{A_{n,i}+B'_{n,i}x_t+x_t'C_{n,i}x_t}, .., e^{A_{n,m}+B'_{n,m}x_t+x_t'C_{n,m}x_t}\right]\cdot\mathbf{T}^{n-1}. \tag{40}$$

It is worth highlighting that technically this is a closed form solution for zero coupon bonds in the presence of regime changes. Regime changes lend much flexibility to the model and represent an important advantage over continuous time QTSM, since we know of no closed form solution for continuous time QTSM in the presence of regime changes. Moreover estimation in this setting can make use of closed form solutions for moments of bond returns of all orders. For example, under the real measure and under regime i, the first three moments of one period gross returns on zero coupon bonds are respectively

$$\frac{E_t^{P^*}\left[\mathbf{p}'_{n-1,t+1}\right]\boldsymbol{\Lambda}_{n-1,t+1}\left(\mathbf{T}_t\right)_i}{\mathbf{p}'_{n,t}\left(\boldsymbol{\Lambda}_{n,t}\right)_i}, E_t^{P^*}\left[\left(\frac{\mathbf{p}'_{n-1,t+1}\boldsymbol{\Lambda}_{n-1,t+1}\left(\mathbf{T}_t\right)_i}{\mathbf{p}'_{n,t}\left(\boldsymbol{\Lambda}_{n,t}\right)_i}\right)^2\right], E_t^{P^*}\left[\left(\frac{\mathbf{p}'_{n-1,t+1}\boldsymbol{\Lambda}_{n-1,t+1}\left(\mathbf{T}_t\right)_i}{\mathbf{p}'_{n,t}\left(\boldsymbol{\Lambda}_{n,t}\right)_i}\right)^3\right] \tag{41}$$

where $E_t^{P^*}\left[\mathbf{p}'_{n-1,t+1}\right]$, $E_t^{P^*}\left[\left(\mathbf{p}'_{n-1,t+1}\boldsymbol{\Lambda}_{n-1,t+1}\left(\mathbf{T}_t\right)_i\right)^2\right]$ and $E_t^{P^*}\left[\left(\mathbf{p}'_{n-1,t+1}\boldsymbol{\Lambda}_{n-1,t+1}\left(\mathbf{T}_t\right)_i\right)^3\right]$ have closed form solutions, which we can find by using equation 29. Although this section has concentrated on quadratic models, similar results are applicable to affine models. In particular, if $\Psi = \mathbf{0}$, where $\mathbf{0}$ is the $N \times N$ matrix of zeros, then the model of this section becomes a multi-factor Vasicek-type model with multiple regimes.

The model in this section is again applicable to price bonds subject to credit risk. In particular regime changes may correspond to rating transitions. Then the regime index i can be viewed as a credit rating index. This seems of interest since Duffee (1999) finds evidence suggesting that the processes of risk-neutral default intensities change as the credit rating worsens.

6.1 Modelling "jumps" through mixtures of Gaussian densities

A special case of the above model with multiple regimes is one whereby the conditional density of the state variables x is a mixture of Gaussian densities. Such mixture allows us to model jumps in the factors x. For example, we can assume that the error terms ξ_t driving x_t at any time t be distributed according to a mixture of Gaussian densities, each density having a different variance. For example we can assume that x_t, ϕ_i, μ_i and Σ_i are scalars, and that $i = [1, 2]$, $\phi_1 = \phi_2$, $\mu_1 = \mu_2$. Then we can impose that

$$\xi_t \backsim \begin{cases} N(0, \Sigma_1) \text{ with probability } p_1, \\ N(0, \Sigma_2) \text{ with probability } p_2 = (1 - p_1). \end{cases} \tag{42}$$

In other words, with probability p_1, ξ_t is distributed according to a normal density with mean 0 and variance Σ_1 and, with probability p_2, ξ_t is distributed according to a normal density with mean 0 and variance Σ_2. If Σ_2 is much greater than Σ_1 and if p_1 is close to 1, then this model can describe infrequent sizable jumps in x_t. In this case we would set

$$(\mathbf{T}_t)_1 = (\mathbf{T}_t)_2 = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. \tag{43}$$

Of course this model can be immediately generalized: x_t, ϕ_i, μ_i and Σ_i may not be scalars and ξ_t may be distributed according to a mixture of an arbitrary number of Gaussian densities, each with different mean and variance. The weights of the mixture would be the elements of $(\mathbf{T}_t)_i$, with $\mathbf{1}' \cdot (\mathbf{T}_t)_i = 1$. Thus we can accommodate positive as well as negative "jumps" in discrete time QTSM while still retaining closed form pricing formulae. Again we know of no closed form solutions for continuous time QTSM in the presence of factor "jumps".

7 Bond options

In this section we provide a semi-closed form solution also for bond options in discrete time and in a single factor setting. The generalization to a multi-factor setting is straightforward, but then option valuation requires the numerical solution of multiple integrals. We denote with $O_{n,t}$ the price of a European call option at time t that expires at time $t+n$. The call gives the right to buy a zero coupon bond which expires at time $t+m$ and whose value at t is denoted as $P_{m,t}$. We set $m>n$. At the option expiry date the bond is worth $P_{m-n,t+n}=e^{A_*+B_*x_{t+n}+C_*x_{t+n}^2}$, where A_*, B_* and C_* can be found as shown above in the single factor setting. Invoking again risk-neutral valuation we can write

$$O_{n,t}=E_t\left[e^{-\Sigma_{i=t}^{t+n-1}r_i}\max\left(e^{A_*+B_*x_{t+n}+C_*x_{t+n}^2}-K,0\right)\right]. \tag{44}$$

We notice that the the option expires at the money when $A_*+B_*x_{t+n}+C_*x_{t+n}^2=\ln K$, which implies that the call will be exercise as long as the following two conditions are simultaneously met

$$\frac{-B_*-\sqrt{\left(B_*\right)^2-4C_*\left(A^*-\ln K\right)}}{2C_*}=x_{t+1}^*\leq x_{t+1} \tag{45}$$

$$x_{t+1}\leq x_{t+1}^{**}=\frac{-B_*+\sqrt{\left(B_*\right)^2-4C_*\left(A_*-\ln K\right)}}{2C_*}. \tag{46}$$

To determine the option value $O_{n,t}$ we proceed as follows. We denote with $O_{t,n}\left(x_{t+1}^{**}\right)$ the value of the contingent claim that pays off the bond at time $t+1$

if and only if $x_{t+1} = x_{t+1}^{**}$, in which case the bond is worth $e^{A_* + B_* x_{t+1}^{**} + C_* x_{t+1}^{**2}} = H$. Then we can write the pricing equation for the one period option $O_{t,1}\left(x_{t+1}^{**}\right)$ as

$$O_{t,1}\left(x_{t+1}^{**}\right) = e^{-\alpha - \beta x_t - \Psi x_t^2} \cdot E_t\left[1_{x_{t+1}=x_{t+1}^{**}} \cdot e^{A_* + B_* x_{t+1} + C_* x_{t+1}^2}\right] \tag{47}$$

with

$$\begin{aligned} E_t\left[1_{x_{t+1}=x_{t+1}^{**}} \cdot e^{A_* + B_* x_{t+1} + C_* x_{t+1}^2}\right] &= e^{A_* + B_* x_t(1-\phi) + B_* \phi\mu + C_*(x_t(1-\phi)+\phi\mu)^2} \cdot \\ &\quad \cdot e^{(B_* + 2(\phi\mu + x_t(1-\phi))C_*)\xi^{**} + C_*(\xi^{**})^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2} \end{aligned} \tag{48}$$

$$\begin{aligned} \xi^{**} &= x_{t+1}^{**} - x_t(1-\phi) - \phi\mu \\ &= \frac{-B_* + \sqrt{(B_*)^2 - 4C_*(A_* - \ln H)}}{2C_*} - x_t(1-\phi) - \phi\mu. \end{aligned} \tag{49}$$

Then we assume that $O_{t,1}\left(x_{t+1}^{**}\right) = e^{A_1^o + B_1^o x_t + C_1^o x_t^2}$ so that we can write

$$\begin{aligned} e^{A_1^o + B_1^o x_t + C_1^o x_t^2} &= e^{-\alpha - \beta x_t - \Psi x_t^2} \cdot e^{A_* + B_* x_t(1-\phi) + B_* \phi\mu + C_*(x_t(1-\phi)+\phi\mu)^2} \cdot \\ &\quad \cdot e^{(B_* + 2(\phi\mu + x_t(1-\phi))C_*)\xi^{**} + C_* \xi^{**2}} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2}. \end{aligned} \tag{50}$$

A_1^o, B_1^o and C_1^o only depend on n, the time to the option expiry. The superscript o highlights that these functions refer to the contingent claim under consideration. Solving the last equation gives

$$A_1^o = -\alpha + \ln \frac{1}{\sigma\sqrt{2\pi}} - \frac{\left(x_{t+1}^{**} - \phi\mu\right)^2}{2\sigma^2} + A_* + B_* x_{t+1}^{**} + C_* x_{t+1}^{**2} \quad (51)$$

$$B_1^o = -\beta + \left(x_{t+1}^{**} - \phi\mu\right) \frac{(1-\phi)}{\sigma^2} \quad (52)$$

$$C_1^o = -\Psi - \frac{(1-\phi)^2}{2\sigma^2}. \quad (53)$$

Now we highlight the dependence of x_{t+1}^{**} on H explicitly by writing $x_{t+1}^{**}(H)$. Then in order to find $O_{t,n}\left(x_{t+1}^{**}(H)\right)$, we notice that, given A_1^o, B_1^o and C_1^o, we can find A_n^o, B_n^o, C_n^o for $n > 1$ as we found A_n, B_n, C_n in the single factor bond valuation setting. Similarly we can find $O_{t,n}\left(x_{t+1}^{*}(H)\right)$. Then integrating $O_{t,n}\left(x_{t+1}^{**}(H)\right)$ and $O_{t,n}\left(x_{t+1}^{*}(H)\right)$ over H we can find the solution for the present value of the option since

$$\begin{aligned} O_{t,n} = & \int_K^\infty \left[O_{t,n}\left(x_{t+1}^{**}(H)\right) + O_{t,n}\left(x_{t+1}^{*}(H)\right)\right] dH \\ & -K \int_K^\infty \left[C_{t,n}\left(x_{t+1}^{**}(H)\right) + C_{t,n}\left(x_{t+1}^{*}(H)\right)\right] dH \end{aligned} \quad (54)$$

where $C_{t,n}\left(x_{t+1}^{**}(H)\right)$ is the value of a claim that pays 1 at expiry if $x_{t+1} = x_{t+1}^{**}$. As above, it can be shown that risk-neutral valuation implies that

$$C_{t,1}\left(x_{t+1}^{**}\right)=e^{-\alpha-\beta x_t-\Psi x_t^2}E_t\left[1_{x_{t+1}=x_{t+1}^{**}}\right] \tag{55}$$

with

$$E_t\left[1_{x_{t+1}=x_{t+1}^{**}}\right]=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2} \tag{56}$$

giving

$$A_1^c+B_1^c x_t+C_1^c x_t^2=-\alpha-\beta x_t-\Psi x_t^2+\ln\frac{1}{\sigma\sqrt{2\pi}}-\frac{1}{2}\left(\frac{x_{t+1}^{**}-x_t\left(1-\phi\right)-\phi\mu}{\sigma}\right)^2 \tag{57}$$

and

$$A_1^c = -\alpha+\ln\frac{1}{\sigma\sqrt{2\pi}}-\frac{\left(x_{t+1}^{**}-\phi\mu\right)^2}{2\sigma^2} \tag{58}$$

$$B_1^c = B_1^o \tag{59}$$

$$C_1^c = C_1^o. \tag{60}$$

For $n>1$, A_n^c, B_n^c and C_n^c will be equal to A_n, B_n and C_n, which are employed for bond valuation in the single factor setting.

8 Conclusion

Overall this paper adds to recent research which shows the advantages of switching from continuous to discrete time pricing models. This paper has studied quadratic term structure models (QSTM) in discrete time, providing closed form solutions for zero coupon bonds even in the presence of multiple correlated factors, time-dependent parameters, regime changes and "jumps" in the underlying factors. Tractability even in the presence of regime changes and "jumps" highlights important advantages in switching from continuous QTSM to discrete time QSTM. The continuous time setting can be seen as a special case of the discrete time one. Closed forms are also available for state prices and the valuation of bond options in the presence of one stochastic factor requires simple numerical integration. As already noted by Dai-Le-Singleton (2005) in the context of affine term structure models (ATSM), also for QTSM the discrete time setting provides much flexibility in specifying the market price of risk while the factors transition density remains Gaussian, which are advantages in estimation. Overall quadratic models in discrete time seem to offer additional advantages over QTSM in continuous time as well as over ATSM in discrete time. A step for future research is the empirical testing of the model presented here.

A Appendixes

A.1 The one factor model

This Appendix considers the setting with one single factor. We can derive equation 8 as follows. We define $u \sim N\left(0, \sigma^2\right)$, and a and b arbitrary constants. Then we want to evaluate the expectation

$$E\left[e^{au+bu^2}\right] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left(-\frac{u^2}{2\sigma^2}+au+bu^2\right)}du. \tag{61}$$

Since $\sigma^2 \geq 0$ and $b \leq 0$ in our model, we can put $\gamma = \sqrt{\left(\frac{1}{\sigma^2}-2b\right)^{-1}}$ and write

$$\begin{aligned}
-\left(\frac{1}{2\sigma^2}-b\right)u^2+au &= -\frac{u^2}{2\gamma^2}+au = -\frac{u^2}{2\gamma^2}+au-\frac{\gamma^2a^2}{2}+\frac{\gamma^2a^2}{2} \qquad (62)\\
&= -\frac{1}{2}\left(\frac{u}{\gamma}-\gamma a\right)^2+\frac{\gamma^2a^2}{2}.
\end{aligned}$$

We need not consider the case $\gamma = -\sqrt{\left(\frac{1}{\sigma^2}-2b\right)^{-1}}$ in our setting, since it would lead to $E\left[e^{au+bu^2}\right]$ being negative, which has no economic meaning. It follows that

$$\begin{aligned}
\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left(-\frac{u^2}{2\sigma^2}+au+bu^2\right)}du &= \frac{\gamma}{\gamma}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{\gamma^2a^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{\left(u-\gamma^2a\right)^2}{2\gamma^2}}du \qquad (63)\\
&= \frac{\gamma}{\sigma}e^{\frac{\gamma^2a^2}{2}} = \frac{1}{\sigma\sqrt{\left(\frac{1}{\sigma^2}-2b\right)}}e^{\frac{\sigma^2a^2}{2\left(1-2b\sigma^2\right)}}.
\end{aligned}$$

and thus

$$\ln E\left[e^{au+bu^2}\right] = -\ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2} - 2b\right) + \frac{\sigma^2 a^2}{(2-4b\sigma^2)}. \tag{64}$$

Now if we substitute $u = \xi_{t+1}$, $a = B_{n-1} + 2C_{n-1}x_t(1-\phi) + 2C_{n-1}\phi\mu$ and $b = C_{n-1}$ into this last equation, we get equation 8 in the text.

A.2 The multi-factor model

This Appendix considers the setting with multiple factors. Equation 19 is derived as follows. We define $w = \Sigma\xi_{t+1}$ and notice that $w \sim N(\mathbf{0}, \Sigma\Sigma')$, where w and $\mathbf{0}$ are $N \times 1$ vectors. Then we set $a = B_{n-1} + F$ and notice that

$$\begin{aligned}
& E_t\left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1}+\xi'_{t+1}\Sigma' C_{n-1}\Sigma\xi_{t+1}}\right] && (65)\\
&= \frac{1}{\sqrt{(2\pi)^N}}\int e^{-\frac{1}{2}\xi'_{t+1}\xi_{t+1}+a'w+w'C_{n-1}w}d\xi_{t+1} \\
&= \frac{1}{\sqrt{(2\pi)^N}abs\left|\Sigma\right|}\int e^{-\frac{1}{2}w'\left(\Sigma\Sigma'\right)^{-1}w+a'w+w'C_{n-1}w}dw \\
&= E\left[e^{a'w+w'C_{n-1}w}\right] && (66)
\end{aligned}$$

where we have made the substitutions $\xi_{t+1} = \Sigma^{-1}w$ and $d\xi_{t+1} = abs\left|\Sigma^{-1}\right|dw$ and where $abs\left|\Sigma^{-1}\right|$ denotes the absolute value of the determinant of Σ^{-1}. Then $\left(\left(\Sigma\Sigma'\right)^{-1} - 2C_{n-1}\right)$ is positive semi-definite and symmetric. This is the case since $\Sigma\Sigma'$ is symmetric and positive semi-definite and so is $\left(\Sigma\Sigma'\right)^{-1}$. Then C_{n-1} can also be assumed to be symmetric and negative definite for our purposes without loss in generality. It follows that $\gamma = \left((\Sigma\Sigma')^{-1} - 2C_{n-1} \right)^{-1/2}$ exists and is symmetric. Then we can write the following

$$\begin{aligned} & -\frac{1}{2} w' \left(\Sigma\Sigma'\right)^{-1} w + a'w + w' C_{n-1} w \\ & = -\frac{1}{2} w' \left((\Sigma\Sigma')^{-1} - 2C_{n-1} \right) w + a'w = -\frac{1}{2} w' \gamma^{-2} w + a'w \\ & = -\frac{1}{2} \left(\gamma^{-1} w\right)' \gamma^{-1} w + a'w = -\frac{1}{2} v'v + a'\gamma v \end{aligned} \tag{67}$$

where $v = \gamma^{-1} w$. Hence, if γ is of full rank, it follows that the differential dw is such that

$$dw = abs\left|\gamma\right| dv = \left|\gamma\right| dv \tag{68}$$

where $abs\left|\gamma\right|$ is the absolute value of $\left|\gamma\right|$ and $abs\left|\gamma\right| = \left|\gamma\right|$ since γ is non-negative definite. At this point we can write

$$\begin{aligned} & \frac{1}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \int e^{-\frac{1}{2} w' \left(\Sigma\Sigma'\right)^{-1} w + a'w + w' C_{n-1} w} dw \\ & = \frac{1}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \int e^{-\frac{1}{2} v'v + a'\gamma v} \left|\gamma\right| dv = \frac{\left|\gamma\right|}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \Pi_{i=1}^{N} \int e^{-\frac{v_i^2}{2} + a'\gamma_i v_i} dv_i \\ & = \frac{\left|\gamma\right|}{abs\left|\Sigma\right|} \Pi_{i=1}^{N} e^{\frac{(a'\gamma_i)^2}{2}} \end{aligned} \tag{69}$$

where γ_i denotes the i-th column of γ, and substituting for $a = B_{n-1} + F$ into the last line we get equation 19. We notice that the last line makes use of the fact that

$$\frac{1}{\sqrt{2\pi}}\int e^{\left(-\frac{u^2}{2}+au\right)}du = \frac{1}{\sqrt{2\pi}}e^{\frac{a^2}{2}}\int e^{-\frac{(u-a)^2}{2}}du = e^{\frac{a^2}{2}}. \tag{70}$$

Then we can find the recursive solutions for A_n, B_n and C_n in the multifactor setting. Equations 18 and 19 imply

$$\begin{aligned}
&A_n + B_n' x_t + x_t' C_n x_t = -\alpha - \beta' x_t - x_t' \Psi x_t \qquad (71)\\
&+A_{n-1} + B_{n-1}'\left(\left(1-\phi\right)x_t + \phi\mu\right)\\
&+x_t'\left(I-\phi\right)' C_{n-1}\left(I-\phi\right)x_t + \left(\phi\mu\right)' C_{n-1}\phi\mu + 2x_t'\left(I-\phi\right)' C_{n-1}\phi\mu\\
&\ln\frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(B_{n-1} + \left(2x_t'\left(I-\phi\right)' C_{n-1} + 2\left(\phi\mu\right)' C_{n-1}\right)'\right)'\gamma_i\right)^2 \qquad (72)
\end{aligned}$$

Then we invoke the matching principle to separate the variables and find that equation 71 implies the following system of difference equations

$$\begin{aligned}
A_n &= -\alpha + A_{n-1} + B_{n-1}'\phi\mu + \left(\phi\mu\right)' C_{n-1}\phi\mu + \ln\frac{|\gamma|}{abs\,|\Sigma|}\\
&\quad+\frac{1}{2}\sum_{i=1}^{N}\begin{pmatrix} B_{n-1}'\gamma_i B_{n-1}'\gamma_i + B_{n-1}'\gamma_i 2\left(\phi\mu\right)' C_{n-1}\gamma_i \\ +2\left(\phi\mu\right)' C_{n-1}\gamma_i B_{n-1}'\gamma_i + 2\left(\phi\mu\right)' C_{n-1}\gamma_i 2\left(\phi\mu\right)' C_{n-1}\gamma_i \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
B_n' x_t &= -\beta' x_t + B_{n-1}'\left(1-\phi\right)x_t + 2\left(\phi\mu\right)' C_{n-1}\left(I-\phi\right)x_t\\
&\quad+\sum_{i=1}^{N}\begin{pmatrix} B_{n-1}'\gamma_i\left(C_{n-1}\gamma_i\right)'\left(I-\phi\right)x_t + B_{n-1}'\gamma_i\gamma_i' C_{n-1}\left(I-\phi\right)x_t \\ +2\left(\phi\mu\right)' C_{n-1}\gamma_i\gamma_i' C_{n-1}\left(I-\phi\right)x_t + 2\left(\phi\mu\right)' C_{n-1}\gamma_i\gamma_i' C_{n-1}\left(I-\phi\right)x_t \end{pmatrix}
\end{aligned}$$

$$x_t' C_n x_t = -x_t' \Psi x_t + x_t' (I-\phi)' C_{n-1} (I-\phi) x_t + 2 \sum_{i=1}^{N} x_t' (I-\phi)' C_{n-1} \gamma_i \gamma_i' C_{n-1}' (I-\phi) x_t.$$

The equations for B_n' and for C_n in the text follow immediately.

A.3 Conditions for econometric identification of parameters

This Appendix discusses the conditions for the econometric identification of the model parameters when the factors are not observable. We focus on the general setting with multiple factors. We consider linear invariant transformations of x, since only linear transformations will retain the Gaussian distribution given that x has Gaussian distribution. We denote the generic invariant transformation as $x = \Omega y + \Theta$, where Θ and y are $N \times 1$ vectors and Ω is an $N \times N$ matrix. Ω^{-1} is assumed to exist. Then, since we assumed that $r_t = \alpha + \beta' x_t + x_t' \Psi x_t$ and $x_{t+1} = (I-\phi) x_t + \phi\mu + \Sigma \xi_{t+1}$, we can re-express such assumptions as

$$r_t = \alpha + \beta'\Theta + \Theta'\Psi\Theta + \beta'\Omega y_t + y_t'\Omega'\Psi\Theta + \Theta'\Psi\Omega y_t + y_t'\Omega'\Psi\Omega y_t \quad (73)$$

$$y_{t+1} - y_t = \Omega^{-1}\phi(\mu - \Theta - \Omega y_t) + \Omega^{-1}\Sigma\xi_{t+1}. \quad (74)$$

Then we notice that only if $\beta = 0$ then $\Theta = 0$ in order for the transformation to be invariant. And only if $\Theta = 0$ can μ be uniquely identified in estimation. Then, since Ψ is symmetric, $\beta = 0$ and $\Theta = 0$, we can re-express r_t and $y_{t+1} - y_t$ under the invariant transformation as

$$r_t = \alpha + y_t' \Omega' \Psi \Omega y_t \tag{75}$$

$$y_{t+1} - y_t = \Omega^{-1} \phi \Omega \left(\Omega^{-1} \mu - y_t \right) + \Omega^{-1} \Sigma \xi_{t+1}. \tag{76}$$

Then, in order for the transformation to be invariant and for Ω to be constrained to be equal to the identity matrix I, either Σ is diagonal and ϕ is triangular or Σ is triangular and ϕ is diagonal. These conditions imply that Ω must be diagonal. But, since $\Omega' \Psi \Omega$ must have all diagonal terms equal to 1, since the transformation must be invariant and since Ψ has all diagonal terms equal to 1, then $\Omega = I$.

A.4 Multiple regimes

Given that $\Lambda_{n,t}$ and $\mathbf{T}_t$ are independent of x, we can re-write equation 36 as

$$\mathbf{p}_{n,t}' \mathbf{\Lambda}_{n,t} = e^{-\alpha - \beta x_t - x_t' \Psi x_t} \cdot E_t \left[\mathbf{p}_{n-1,t+1}' \right] \cdot \mathbf{\Lambda}_{n-1,t+1} \mathbf{T}_t. \tag{77}$$

If we impose that $\mathbf{p}_{0,t+n}' = \mathbf{1}'$, it follows that the terminal conditions $\mathbf{p}_{0,t+n}' \Lambda_{0,t+n} = \mathbf{1}'$ imply that

$$\begin{aligned} \mathbf{p}_{1,t+n-1}' \mathbf{\Lambda}_{1,t+n-1} &= e^{-\alpha - \beta x_{t+n-1} - x_{t+n-1}' \Psi x_{t+n-1}} \cdot E_{t+n-1} \left[\mathbf{p}_{0,t+n}' \right] \cdot \mathbf{\Lambda}_{0,t+n} \mathbf{T}_{t+n-1} \\ &= e^{-\alpha - \beta x_{t+n-1} - x_{t+n-1}' \Psi x_{t+n-1}} \cdot \mathbf{1}' \end{aligned} \tag{78}$$

giving

$$
\begin{aligned}
E_{t+n-1}\left[\mathbf{p}'_{0,t+n}\right] \cdot \mathbf{\Lambda}_{0,t+n}\mathbf{T}_{t+n-1} &= \mathbf{1}' \qquad (79)\\
\mathbf{1}' \cdot \mathbf{\Lambda}_{0,t+n}\mathbf{T}_{t+n-1} &= \mathbf{1}'
\end{aligned}
$$

which implies that $\Lambda_{0,t+n} = \mathbf{T}^{-1}_{t+n-1}$. Then, if $\mathbf{T}_t$ is constant and equal to $\mathbf{T}$, using equation 39 we get $\Lambda_{0,t+n} = \mathbf{T}^{-1}$, $\Lambda_{1,t+n-1} = \mathbf{I}$, $\Lambda_{2,t+n-2} = \mathbf{T}$, $\Lambda_{3,t+n-3} = \mathbf{T} \cdot \mathbf{T}$, or more generally $\Lambda_{n,t} = \mathbf{T}^{n-1}$.

References

[1] Ahn D., Dittmar R., Gallant R., 2002, "Quadratic term structure models: theory and evidence", The Review of financial studies 15, n.1, 243-288.

[2] Ang A. and Piazzesi M., 2003, "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variable", Journal of Monetary Economics 50, 745-787.

[3] Ang A. and Bekaert G., 2002, "Short rate nonlinearities and regime switches", Journal of Economic Dynamics and Control 26, 1243-1274.

[4] Bansal R. and Zhou H., 2002, "Term structure of interest rates with regime shifts", The Journal of Finance 57, n. 5, 1997-2043.

[5] Beaglehole D. and Tenney M., 1991, "General solutions of some contingent claim pricing equations, Journal of Fixed Income 1, 69-83.

[6] Chen L., Filipovic D. and Poor V., 2004, "Quadratic term structure models for risk-free and defaultable rates", Mathematical Finance, 14, n.4, 515-536.

[7] Constantinides G., 1992, "A theory of the nominal term structure of interest rates", The Review of Financial Studies 5, n.4, 531-552.

[8] Cox J. and Ingersoll J.E.jr and Ross S.A., 1985, "A theory of the term structure of interest rates", Econometrica 53, n.2, 384-408.

[9] Dai Q. and Singleton K., 2003, "Term structure dynamics in theory and reality", The Review of Financial Studies 16, n.3, 631-678.

[10] Dai Q., Le A. and Singleton K., 2005, "Discrete-time dynamic term structure models with generalised market prices of risk", working paper.

[11] Dai Q., Singleton K. and Yang W., 2005, "Regime shifts in a dynamic term structure model of US treasury bond yields", Working paper, Stanford University.

[12] Duffee G., 1999, "Estimating the price of default risk", The Review of Financial Studies 12, n.1, 197-226.

[13] Duffie D., Filipovic D. and Schachermayer W., 2003, "Affine processes and applications in finance", Annals of Applied Probability, vol. 13, 984-1053.

[14] Duffie D. and Kan R., 1996, "A yield factor model of interest rates", Mathematical Finance 6, 379-406.

[15] Duffie D. and Liu J., 2001, "Floating-fixed credit spreads", Financial Analysts Journal, May-June, 76-87.

[16] Duffie D. and Singleton K., 1999, "Modeling term structures of defaultable bonds", The Review of financial studies 12, n.4, 687-720.

[17] Evans M., 2000, "Regime shifts, risk and the term structure", Working paper, Georgetown University.

[18] Gourieroux C, Monfort A. and Polimenis V., 2002, "Affine term structure models", Working paper CREST.

[19] Gourieroux C. and Sufana R., 2003, "Wishart quadratic term structure models", Working paper CREF HEC Montreal.

[20] Gourieroux C. and Sufana R., 2005, "Wishart quadratic term structure models", Working paper University of Toronto.

[21] Johannes M., 2004, "The statistical and economic role of jumps in continuous time interest rate models", The Journal of Finance 59, n.1, 227-260.

[22] Leippold M. and Wu L., 2002, "Asset pricing under the quadratic class", Journal of Financial and Quantitative Analysis 37, n.2, 271-294.

[23] Leippold M. and Wu L., 2003, "Design and estimation of quadratic term structure models", European Finance Review 7, 47-73.

[24] Longstaff F., Mithal S. and Neis E., 2004, "Corporate yield spreads: default risk or liquidity? New evidence from the credit default swap market", forthcoming in Journal of finance.

[25] Sun T., 1992, "Real and nominal interest rates: a discrete time model and its continuous time limit", The Review of Financial Studies 5, n.4, 581-611.

[26] Sun L., 2005, "Regime shifts in interest rate volatility", Journal of Empirical Finance 12, 418-434.