The Valuation of Modular Projects: a Real Options Approach to the Value of Splitting*

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Abstract

This paper presents a real options approach to the valuation of modular projects, focusing on the value of splitting. Building upon the Baldwin and Clark (2000) approach to modularity, it proposes a more general model, which includes the possibility of delaying the option to split, the effect of the correlations, between the system and the modules, on the value of modularity. It also assumes a multi-staged product development. We study the impact of some of the variables which influence the optimal modular strategy, showing that value can be increased by modularisation depending on the relative values, costs and risk of each modular configuration. When the modules are perfectly correlated with each other and with the system and have an identical risk, there is no incentive to modularise, unless there is a "size" advantage, related to lower costs or higher values of the modules. The effects of the correlations on the value of modularity, depends on the relative risks of the system and the portfolio of modules. A significant part of the value added by modularity is related to the option to choose the best of an interconnected system and a portfolio of modules: when they are all required to be implemented, this is similar to the option to choose the best of two alternative systems. In our view, this cannot be attributed to modularity and, thus, we analyse the "net value" of splitting, which is the difference between the value of a modular project and the value of the project with the option mentioned above.

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1 Introduction

Complex investment projects might be decomposed into smaller units - modules - which may have a certain degree of independence: "A module is a unit whose structural elements are powerfully connected among themselves and relatively weakly connected to elements in other units... In other words, modules are units in a large system that are structurally independent of one another, but work together" (Baldwin and Clark, 2000, p. 63). *Information hiding* is one of the key issues of modularisation: once modularisation is achieved, the complexity of the module can be hidden, provided that an *interface* establishs the parameters of interdependence with other modules of the system.

Modularity can be achieved in three stages of the product development: design, production and consumption. The modularisation of designs, by the creation of "design rules" has been one of the main driving forces of innovation and growth of some industries, namelly the computer industry. Although it is difficult to distinguish between the design and the production phases of a product, there are some differences: (1) the output of a design process is a description of a product, whereas the output of a production process is the product itself; (2) the design and production phases are sometimes simultaneous, but when they are separated, design precedes production (Baldwin and Clark, 2000). The separation of the two phases is clearer when R&D is required to develop a product. Modularity is used also to simplify complex production processes, dividing the product into different manufacturing modules or "cells". One example of modularisation in production is the automobile industry: the car is divided into components/modules that can be manufactured in different sites or in "cells" within the same factory and then assembled. Finally, consumers can "mix and match" modules that work together to satisfy their desires/needs. For example, each consumer can "build" a computer with different components to meet his individual needs. There are numerous other examples of modularity in consumption like furniture and houses.

The valuation of modularity with a real options approach was addressed by Baldwin and Clark (1994, 1997, 2000) who developed a detailed model to value design modularity. Modularisation creates a portfolio of options which is more valuable than an option on a portfolio, provided that the aggregate distribution of value remains the same (Merton, 1973). That model relies on simple and convenient assumptions, which allows closed form solutions. Although they claim that the valuation of modularity should be market valued, the options valuation is made without any assumption about the market price of risk: it is a statistical approach rather than a classical real options approach. The main contribution of the model is, however, the detailed description of the potential options created by modularisation. A similar approach has been used by Gaynor and Bradner (2001) to value technology standardisation. They suggested modularity facilitates the staged development of the standards.

Sullivan, Griswold, Cai and Hallen (2001), adapted the Baldwin and Clark (2000) model to value modularity in software development. Their valuation approach is also very similar and the main difference relates to the methodology of partitioning a system into independent modules. The uncertainty creates an incentive to delay modularisation, while the main difficulty is precisely related to the estimation of the technical uncertainty associated with each module.

Sullivan, Chalasani, Jha and Sazawal (1999) argue that the Baldwin and Clark (1997) model is constructed from the perspective of the modularity, task structure and information hiding theory and not from a real options investment approach. The assumptions about the random walk and additivity of the modules' values does not enable the model to deal with more general modular systems. In their real options approach to the valuation of software design, the possibility of phased projects, which creates compound options, is addressed but a valuation model is not provided.

Keppo and Samila (2004) have valued the option to substitute and upgrade modular products from the perspective of the consumer, assuming that each upgrade is a Poisson jump in the value of the product.

In the real options theory literature the valuation of product development, R&D projects and staged projects, give some insights to the valuation of modular projects. Childs, Ott and Triantis (1998) compared parallel and sequential development of correlated projects. Childs and Triantis (1999) valued R&D projects with multiple products, incorporating learning-by-doing, collateral learning between different projects and interactions between project values. The technical and market uncertainty associated with R&D projects have been modelled with different approaches and remains a crucial parameter of the valuation of the options embedded in such projects. Lint and Pennings (2001) studied the impact of uncertainty and "moneyness" in a two-staged product development project of a single product. Uncertainty about the project value creates an incentive to delay investment, which is less valuable when the opportunity is shared by other competitors in the market. The value of waiting was first introduced by McDonald and Siegel (1986).

Modularity creates a portfolio of options contingent on several assets, whose valuation is not an easy task, unless simplified assumptions are made about the stochastic behaviour of the state variables. Furthermore those options can be exercised several times before maturity, i.e. they have Bermudan/American features. The value of a portfolio of options with interacting features can be significantly different from the sum of the individual options. A general model to price multiple interacting options was proposed by Kulatilaka and Trigeorgis (Kulatilaka, 1995; Kulatilaka and Trigeorgis, 1994) . An investment project with multiple embedded options can be decomposed into different "interacting modes". The options are the transitions between the modes with an exercise price equal to the cost of switching. This general real options model is only practicable in low dimensions.

A portfolio of options can also be valued using lattices[1]. However, if those options are contingent on multiple state variables, the valuation with lattices becomes impractical due to the curse of dimensionality.

Simulation has been a promising alternative in the last decade to value path-dependent options, American options and options with multiple state variables and under general stochastic processes. Unlike in the lattice methods, the computational effort increases only linearly in the number of stochastic factors. The main advantage of the Monte Carlo simulation methods is that the convergence depends only on the number of simulations and is independent of the dimension of the problems.

Monte Carlo simulation was first used to value options by Boyle (1977). Traditionally simulation was presented as a forward-looking technique, so it was seen as inadequate to deal with American options. In recent years, several authors proposed different methods to match simulation and dynamic programming which is a backward-looking technique. Among those methods, some use an estimation of the continuation value obtained by the projection of discounted payoffs onto a set of basis functions (Carrière, 1996; Tsitsiklis and Van Roy, 1999; Longstaff and Schwartz, 2001; Tsitsiklis and Van Roy, 2001; Carrière, 2001). Maybe due to its simplicity Longstaff and Schwartz (2001) Least-Squares Method (LSM) gained an increasing attention.

Gamba (2003) proposed a model that decomposes complex multiple real options problems (with interacting options) into simple hierarchical sets of individual options. Extending the LSM approach this model deals also with American and Bermudan real options, which are frequent in capital budgeting projects. The decomposition principle can be used in combination with any kind of methodology based on dynamic programming and the Bellman equation. Gamba presents the following types of possible interactions between real options: independent options, compound options, mutually exclusive options and switching problems.

Rodrigues and Armada (2005) have assessed empirically the accuracy of the estended LSM method, showing that, with a carefull choice of parameters, it is a very accurate method to value real options.

[1]Gamba and Trigeorgis (2001) proposed an extension to the Trigeorgis (1991) log-transformed binomial lattice to value a portfolio of interacting options.

This paper focuses on the value of splitting a system into a modular product, extending the Baldwin and Clark (2000) model, in several ways, using a real options perspective. In Section 2 we present the Baldwin and Clark (2000) model and point out some of its limitations. In Section 3, the model to value splitting of modular projects is presented, using a Monte Carlo simulation approach, as proposed by Longstaff and Schwartz (2001) and Gamba (2003). Section 4 presents some numerical results and sensitivity analysis. An analysis of the sources of the value of modularity and the proposed "net value of splitting" are presented in section 5. Section 6 concludes.

2 The Baldwin and Clark (2000) model to value modularity

Baldwin and Clark (1994, 1997, 2000) have developed a model to value modularity in design, with some insights from real options theory. The process of design modularisation is assumed to have three stages and the final design is created by six modular *operators*.

In the first stage, the *design rules* are formulated and the six operators can be used to create a design structure and the corresponding task structure, defining the number, boundaries and interfaces of the modules:

1. If the initial design is interdependent, the *splitting* operator can be used to create a modular design;
2. *Substituting* one module for another;
3. *Augmenting* the number of modules, adding new modules to the system;
4. *Excluding* a module from the system;
5. *Inverting* to create new design rules;
6. *Porting* a module to another system.

The choice of a design, and its task structure, corresponds to the choice of a random payoff function. This means that the outcome (value) of the design is not deterministic: it has a probability distribution. The formulation of the design rules is not free: to create a modular task structure some costs are incurred and they increase with the complexity of the design.

In the second stage, the task structure created in the previous stage is implemented and experimented. After this stage the design is ready to be tested and integrated with the system. In the final stage the value of the design is known and the options associated with the design can be exercised.

The advantages of a modular design are essentially related to its greater flexibility, and have to be weighted against the costs that it implies: they are associated with the formulation of the design rules and also with the possibility of underperformance of the new modular design compared with an interconnected system.

Splitting creates a portfolio of options that must be valued to find the optimal strategy, i.e. the optimal number of modules. If the module values are not perfectly correlated[2], modularisation increases value, and higher degrees of modularisation enhance this result. This is also the case when modularisation makes the aggregate distribution better. When the opposite occurs, the final result is unknown, and can be a design with lower value. This could be the case where splitting is done without sufficient knowledge of modules interdependencies. However, the decision of splitting a design should not be delayed until it produces the same aggregate distribution. Some deterioration of aggregate value will be compensated by the options created by modularisation.

The model can deal with both symmetric an asymmetric modules (i.e. modules with different size, costs and uncertainties) and explains different rates of investment across the asymmetric modules.

The model makes some convenient assumptions about the uncertainty of the value of modules:

- The value of each module has a normal distribution with a variance proportional to its complexity, measured by the number of tasks, and they are uncorrelated with each other. This makes possible to add the values of the modules, since the sum of normally distributed random variables has also a normal distribution.

- The risk (variance) of the module value depends on the complexity associated with its size, i.e. , with the number of tasks. It is assumed that larger designs have a higher risk of failure/success.

- Modularisation does not change the probability distribution of the value of the design, if the number of tasks is held constant.

These assumptions are designed to make the model tractable and simple, but some of them are not realistic.

The assumption that the module values are uncorrelated and normally distributed is a convenient choice, which allows the value of the portfolio of modules to be just simply the sum of the individual values. However, in reality, the values of the modules are correlated with each other and with the system value. The effect of such correlations is not neglegible.

The uncertainty of the modules value is modeled as being independent from time. The length of each stage is, on the other hand, not considered.

[2]The model assumes that the modules are uncorrelated.

Although the authors admit that the design process is a sequence of exercises of the options associated with each modular operator, the model seems to suggest that the exercise of those options is confined to the moment of the valuation. The number of modules is chosen, at that moment, to maximise the net option value. However, they suggest that the compoundeness and recursiveness of the design process can only be valued with a more complex valuation framework.

3 A real options approach to the value of splitting

We propose a model which extends the Baldwin and Clark (2000) valuation model in several ways:

- The options embedded in the modular operators can be exercised at different moments in time. The option to split, the option develop and the option to implement can be delayed (the options are American or, at least, Bermudan).
- The optimal modular configuration is endogenous and optimally given by the model.
- The design process is a staged process and this is addressed in the valuation model.
- The variables on which the decision to split (modularise) is contingent are not independent. Therefore the correlation between the stochastic variables is addressed explicitly.

We assume that the options are finite lived: the value of the business erodes with time and, eventually, disappears as a result of either the entrance of competitors or the market value of the assets becoming worthless.

Let us assume that the modular investment project under valuation has the following stages and options embedded:

Stage 1 - Design formulation: at this stage the *option to split* can be exercised. The optimal number of modules - including the possibility of an one-module (interconnected) system - is the outcome of this stage. The costs incurred in this stage are dependent on the number of modules. We assume that the option to split can be delayed until the beginning of the next stage.

Sage 2 - R&D: the *option to develop* the modules can be exercised with a capital outlay. This stage requires a certain amount of time, after which the module is ready to be launched in the market. The option to develop can be delayed until the maturity of the project less the required time to develop the module.

Stage 3 - Production and marketing: once the model is developed, the *option to implement* can be exercised, starting the production and launching the product in the market. We assume, fo the sake of simplicity, that this stage is instantaneous.

Let us also assume that the value of an interconnected design (a system with a single module) follows the following stochastic process:

$$dS\left(t\right)=\left(\mu-\delta\right)S\left(t\right)dt+\sigma S\left(t\right)dW \qquad S\left(0\right)=S \tag{1}$$

where $S\left(t\right)>0$, μ and σ are, respectively, the drift parameter and the instantaneous volatility, δ is the rate of lost cash flows. Finally, dW is the increment of a Wiener process.

Assuming market completeness, there is a unique risk-neutral probability measure under which the asset price stochastic process is:

$$dS\left(t\right)=\left(r-\delta\right)S\left(t\right)dt+\sigma S\left(t\right)dW \qquad S\left(0\right)=S \tag{2}$$

where r is the riskless interest rate.

Each of the individual m module values evolves also according to a geometric Brownian motion process:

$$dM_i^{(m)}\left(t\right)=\left(\mu_i^{(m)}-\delta_i^{(m)}\right)M_i^{(m)}\left(t\right)dt+\sigma_i M_i^{(m)}\left(t\right)dW_i^{(m)} \qquad M_i^{(m)}\left(0\right)=M_i^{(m)} \tag{3}$$

where $M_i^{(m)}$ denotes the value of the i-th module in a system with m modules, and $dW_i^{(m)}$ are increments of Wiener processes.

For notational convenience equation 1 can be expressed as:

$$dM_1^{(1)}\left(t\right)=\mu_1^{(1)}M_1^{(1)}\left(t\right)dt+\sigma M_1^{(1)}\left(t\right)dW_1^{(1)} \tag{4}$$

These processes are correlated: the module values within each modular configuration $\left(\mathbb{E}\left[dW_i^{(m)}dW_j^{(m)}\right]=\rho_{ij}^{(m)}dt\right)$; the module values in different configurations $\left(\mathbb{E}\left[dW_i^{(m)}dW_j^{(n)}\right]=\rho_{ij}^{(m)(n)}dt\right)$ and the module and system values $\left(\mathbb{E}\left[dW_i^{(m)}dW_1^{(1)}\right]=\rho_i^{(1)}dt\right)$.

If a given system can only be split into three modules, we have six state variables:

$$\begin{aligned}
dM_1^{(1)}(t) &= \left(\mu_1^{(1)} - \delta_1^{(1)}\right) M_1^{(1)}(t)\,dt + \sigma_1^{(1)} M_1^{(1)}(t)\,dW_1^{(1)} \\
dM_1^{(2)}(t) &= \left(\mu_1^{(2)} - \delta_1^{(2)}\right) M_1^{(2)}(t)\,dt + \sigma_1^{(2)} M_1^{(2)}(t)\,dW_1^{(2)} \\
dM_2^{(2)}(t) &= \left(\mu_2^{(2)} - \delta_2^{(2)}\right) M_2^{(2)}(t)\,dt + \sigma_2^{(2)} M_2^{(2)}(t)\,dW_2^{(2)} \\
dM_1^{(3)}(t) &= \left(\mu_1^{(3)} - \delta_1^{(3)}\right) M_1^{(3)}(t)\,dt + \sigma_1^{(3)} M_1^{(3)}(t)\,dW_1^{(3)} \\
dM_2^{(3)}(t) &= \left(\mu_2^{(3)} - \delta_2^{(3)}\right) M_2^{(3)}(t)\,dt + \sigma_2^{(3)} M_2^{(3)}(t)\,dW_2^{(3)} \\
dM_3^{(3)}(t) &= \left(\mu_3^{(3)} - \delta_3^{(3)}\right) M_3^{(3)}(t)\,dt + \sigma_3^{(3)} M_3^{(3)}(t)\,dW_3^{(3)}
\end{aligned}$$

For each of these state variables, we need to estimate the variance and covariances with the other variables. In the case of 3 modules, we would have 6 variances and 15 covariances, which can have different values. For illustration purposes we assume the following simplifications:

- The variance of the modules within each configuration is the same, i.e. $\sigma_i^{(m)} = \sigma_j^{(m)}$, and is a multiple ($\beta$) of the variance of the system: $\sigma_i^{(m)} = \beta\sigma_1^{(1)}$;
- The correlations between the module values, within each configuration, are constant (ρ);
- The correlations between the module values of different configurations are constant (ρ_m);
- The correlations between the module values and system value are constant (ρ_s);

These assumptions reduce the correlation matrix to:

$$\begin{array}{ccccccc}
 & M_1^{(1)} & M_1^{(2)} & M_2^{(2)} & M_1^{(3)} & M_2^{(3)} & M_3^{(3)} \\
M_1^{(1)} & 1 & \rho_s & \rho_s & \rho_s & \rho_s & \rho_s \\
M_1^{(2)} & \rho_s & 1 & \rho & \rho_m & \rho_m & \rho_m \\
M_2^{(2)} & \rho_s & \rho & 1 & \rho_m & \rho_m & \rho_m \\
M_1^{(3)} & \rho_s & \rho_m & \rho_m & 1 & \rho & \rho \\
M_2^{(3)} & \rho_s & \rho_m & \rho_m & \rho & 1 & \rho \\
M_3^{(3)} & \rho_s & \rho_m & \rho_m & \rho & \rho & 1
\end{array}$$

The valuation of a modular project with several state variables and a portfolio of American or Bermudan options is impracticable with lattices or finite differences schemes, due to the curse of dimensionality of such methods. Monte Carlo simulation becomes the only feasible alternative. The valuation of the portfolio of Bermudan or American options, embedded in a modular project, can be easily handled with the LSM method of Longstaff and Schwartz (2001) and extended by (Gamba, 2003).

To make the model tractable for simulation, the maximum number of modules (M) must be defined *a priori*[3].

Option to implement

Starting backwards from the maturity of the project (T), we have, for each configuration (number of modules chosen in previous steps), $m \leqslant M$ independent options to implement, at the cost of $K_i^{(m)}$. Each individual option has the following payoff[4]:

$$\Pi_{I_i^{(m)}}\left(t, M_i^{(m)}(t)\right) = \max\left[M_i^{(m)}(t) - K_i^{(m)}; 0\right] \qquad i = 1, ..., m; \quad m = 1, ..., M \tag{5}$$

and the following value:

$$F_{I_i^{(m)}}\left(t, M_i^{(m)}(t)\right) \qquad i = 1, ..., m; \quad m = 1, ..., M \tag{6}$$

Each individual option value and optimal stopping time is computed independently, using the LSM method. Since the module is required to be developed in the previous stage, which has a duration of t_d, and after T, the opportunity to implement is no longer available, the option has no value $\left(\Pi_{I_i^{(m)}}\left(t, M_i^{(m)}(t)\right) = 0\right)$ for $t > T$ and $t < t_d$.

Option to develop

The option to develop (start R&D stage) is a compound option since it gives the right to exercise, in a later stage, the option to invest. Again, we have for each configuration, $m \leqslant M$ independent options to develop, paying the development cost, $D_i^{(m)}$. Each individual option has the following payoff:

$$\Pi_{D_i^{(m)}}\left(t, M_i^{(m)}(t)\right) = \max\left[F_{I_i^{(m)}}\left(t, M_i^{(m)}(t)\right) - D_i^{(m)}; 0\right] \qquad i = 1, ..., m; \quad m = 1, ..., M \tag{7}$$

[3]The number of assets (n) depends on the maximum number of modules (M) into which the system can be split: $n = \frac{M(M-1)}{2}$.

[4]See Table 1 for a description of the variables.

and Bellman equation:

$$\begin{aligned} F_{D_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)\right) = \max\Big\{ & \Pi_{D_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)\right), \\ & e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[F_{D_i^{(m)}}\left(t_{n+1}, M_i^{(m)}(t_{n+1})\right)\right]\Big\} \\ & i = 1, ..., m; \quad m = 1, ..., M \end{aligned} \tag{8}$$

The following decision rule is used to find the optimal stopping time $(\tau_{D_i^{(m)}})$ at t_n, for the ω-th path:

$$\begin{aligned} \text{if:} \quad & \Phi_{D_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right) \leq \Pi_{D_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right) \\ & \text{then:} \quad \tau_{D_i^{(m)}}(\omega) = t_n \\ & i = 1, ..., m; \quad m = 1, ..., M \end{aligned} \tag{9}$$

The continuation value, $\Phi_{D_i^{(m)}}$ is obtained using the LSM approach:

$$\begin{aligned} \Phi_{D_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)}\Pi_{D_i^{(m)}}(t_n, t_i, \tau, \cdot)\right] \\ i = 1, ..., m; \quad m = 1, ..., M \end{aligned} \tag{10}$$

$\Phi_{D_i^{(m)}}$ is approximated by a finite number of basis functions, $\Phi^J_{D_i^{(m)}}$, which is estimated by a least squares regression. The value of the option to invest has been already computed at t_n as:

$$\begin{aligned} F_{I_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)\right) = \max\Big[& \Pi_{I_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)\right), \\ & \Phi_{I_i^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right)\Big] \end{aligned} \tag{11}$$

However, this is only valid when the R&D stage is instantaneous. If the R&D stage has a duration of t_d, the value of the option to invest must be estimated by least squares regression, using the present value of the option to invest as the dependent variable $\left(e^{-rt_d}F_{I_i^{(m)}}\left(t_n + t_d, M_i^{(m)}(t_n + t_d)\right)\right)$, and basis functions of the state variables at t_n as independent variables.

Option to split

The option to split is a mutually exclusive option: once exercised the option to split into m modules the other splitting options are killed. The control is a couple variable (τ, ζ), where τ is a stopping time in $\mathcal{T}(t, T - t_d)$ and $\zeta \in \{1, 2, ..., M\}$ is the optimal number of modules.

As Rodrigues and Armada (2005) have suggested, the algorithm proposed by Gamba (2003), to value this kind of options, is improved, skiping the calculation of the optimal stopping time of the development option, if we were valuing the best of single options. This improvement allows a faster convergence of the option value and a correct choice of the best alternative. The algorithm proposed by Gamba (2003) can produce an incorrect choice of the best alternative in the out-the-money region, where the estimated continuation value should be zero, but the least squares estimate can be slightly positive. This means that the algorithm assumes the choice of an option, when the correct choice would be another option or none of the options.

In the present case, the best of the modular configurations corresponds to the choice between a single option to develop the system and a portfolio of independent options to develop modules within each modular configuration. After modularisation, one does not have to start simultaneously the development of all the modules (in which case the improvement could be used) but, at least, one of the modules will have its development started. Otherwise it would be optimal to delay splitting.

Splitting requires the definition of the design rules, which enables the partitioning of the task structure of the system and the definition of the interfaces between the modules. It implies, therefore, that the value of each modular configuration - the sum of the m independent compound options - must be balanced with the costs needed to define the design rules. The payoff of a configuration with m modules is given by:

$$\Pi_{C^{(m)}}\left(t, M_i^{(m)}(t)\right) = \max\left[\sum_{i=1}^{m} F_{D_i^{(m)}}\left(t, M_i^{(m)}(t)\right) - R^{(m)}, 0\right] \qquad i = 1, ..., m; \quad m = 1, ..., M \tag{12}$$

The cost of the design rules is zero when the system is chosen ($R_1 = 0$). In the case of the option to develop the system, we can skip the calculation of its optimal stopping time, replacing $F_{D_1^{(1)}}$ by the payoff of the option to develop, since, if we chose the system configuration, we will start developing it immediately[5]:

$$\Pi_{C^{(1)}}\left(t, M_1^{(1)}(t)\right) = F_{D_1^{(1)}} = \Pi_{D_1^{(1)}}\left(t, M_1^{(1)}(t)\right) \tag{13}$$

The value of option to choose the best modular configuration is given by:

$$F_S\left(t, M_i^{(m)}(t)\right) = \max_{(\tau,\zeta)}\left\{e^{-r(\tau-t)}\mathbb{E}_t^*\left[F_{C^{(\zeta)}}\left(\tau, M_i^{(\zeta)}(\tau)\right)\right]\right\} \qquad i = 1, ..., m; \quad m = 1, ..., M \tag{14}$$

[5]Thus, using the improvement suggested by Rodrigues and Armada (2005).

The Bellman equation for this option is given by:

$$
\begin{aligned}
F_S\left(t_n, M_i^{(m)}(t_n)\right) = \max\Big\{ & F_{C^{(1)}}\left(t_n, M_1^{(1)}(t_n)\right), \cdots, F_{C^{(M)}}\left(t_n, M_i^{(M)}(t_n)\right), \\
& e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[F_S\left(t_{n+1}, M_i^{(m)}(t_{n+1})\right)\right]\Big\} \\
& i = 1, ..., m; \quad m = 1, ..., M
\end{aligned}
\tag{15}
$$

The following decision rule is used to find the optimal control (τ, ζ) at t_n for the ω-th path:

$$
\begin{aligned}
\text{if:} \quad & \Phi_S\left(t_n, M_i^{(m)}(t_n)(\omega)\right) \leq \max_m\left\{F_{C^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right)\right\} \\
& \text{then:} \quad (\tau, \zeta)(\omega) = (t_n, \overline{m})
\end{aligned}
\tag{16}
$$

where:

$$
\overline{m} = \arg\max_m\left\{\Pi_{C^{(m)}}\left(t_n, M_i^{(m)}(t_n)(\omega)\right)\right\}
$$

To avoid the incorrect choice of the best modular configuration in the out-of-money regions, the optimal configuration $(\overline{m})$ must only be updated if, at least, one of the options to develop (of the m-th portfolio) is exercised, i.e. the exercise value given by equation 7 is greater than the continuation value given by equation 10.

The continuation value is obtained using the LSM method:

$$
\Phi_S\left(t_n, M_i^{(m)}(t_n)\right) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)}\Pi_S(t_n, t_i, \tau, \zeta, \cdot)\right]
$$

with:

$$
\Pi_S(t_n, s, \tau, \zeta, \cdot) = \begin{cases} F_{C_m}\left(t, M_i^{(m)}(s)(\omega)\right) & \text{if } s = \tau(\omega) \text{ and } m = \zeta(\omega) \\ 0 & \text{otherwise} \end{cases}
$$

The continuation value (Φ_S) is computed by a least squares regression. It is contingent on all the assets (system and modules of each configuration) which must be used as basis functions for the least squares regression.

4 Numerical results

The implementation of the model, using simulation, requires some choices about the simulation method. Low-discrepancy sequences (quasi-Monte Carlo methods - QMC) tend to produce more accurate results with fewer paths as was shown by Rodrigues and Armada (2005). Among the alternative low-discrepancy sequences generators we have used the improvement of

Sobol sequences proposed by Silva and Barbe (2003), which is more efficient for high-dimensional problems. Brownian bridges are used to reduce the problems associated with the poorer performance of the QMC methods in high dimensional problems.

The value obtained by the LSM method converges to the true value of an American option increasing the number of basis functions, the number of paths and the number of exercise dates. Since the computational effort increases significantly with any of these parameters, we have chosen to value Bermudan options, which can be exercised 10 times per year. As basis functions we use the weighted Laguerre polynomials suggested by Longstaff and Schwartz (2001), which outperform other polynomial families (Rodrigues and Armada, 2005). For the options to implement and develop, which are contingent on a single state variable (the underlying system/module value), we use a constant and polynomials up to the 5th degree. For the option to split, which is contingent on all of the assets (six in the base case) we use a constant, weighted Laguerre polynomials up to the third degree for each of the underlying assets and pairs of cross-products of the assets (this means 40 basis in the base case).

The base case parameters are presented in Table 1.

Table 1: Base case parameters

Symbol	**Value**	**Description**
M	3	Maximum number of modules
$M_1^{(1)}$	100	System present value
$M_i^{(m)}$	$\frac{M_1^{(1)}}{m}$	Present value of module i when the system is splitted into m modules $i = 1, ..., m; \quad m = 2, ..., M$
$K_1^{(1)}$	75	System investment cost
$D_1^{(1)}$	25	System development cost
$D_i^{(m)}$	$\gamma \frac{D_1^{(1)}}{m}$	Development cost of the module i when the system is splitted into m modules $i = 1, ..., m; \quad m = 2, ..., M$
$K_i^{(m)}$	$\gamma \frac{K_1^{(1)}}{m}$	Investment cost of the module i when the system is splitted into m modules $i = 1, ..., m; \quad m = 2, ..., M$
γ	1.2	Cost multiplier
R	5	Design formulation cost per module
R_m	$\gamma \times R \times m$	Design formulation cost when the system is splitted into m modules $m = 2, ..., M$
$\sigma_1^{(1)}$	0.3	Standard deviation (risk) of the system returns
$\sigma_i^{(m)}$	$\beta\sigma_1^{(1)}$	Standard deviation (risk) of the i-th module returns $i = 1, ..., m; \quad m = 2, ..., M$

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Symbol	Value	Description
β	1.0	Risk multiplier
ρ	0.5	Correlation between module returns within each modular configuration
ρ_m	0.5	Correlation between module returns of different modular configurations
ρ_s	0.5	Correlation between module returns and system returns
T	5	Time horizon (years) of the project
t_d	1	Duration (years) of the R&D stage
r	0.05	Risk-free interest rate
$\delta_i^{(m)}$	0.03	"Dividend yield" (lost cash-flows rate) $\quad i = 1, ..., m; \quad m = 1, ..., M$

Table 2 presents some numerical results for the value of the modular project. The first thing to note is that the early exercise premium is not very significant, as expected, when the project is "out-the-money" (low module and system value) and when the cash flows lost by delaying the option are low. All the options described before are call options, which do not have an early exercise premium when the dividend yield is null. When the firm does not face any competition (which erodes the business value), or does not lose any cash flows, it is always optimal to delay the investment until the last available moment.

Table 2: Value of a modular project

S	δ	T	σ	ρ	**European**	**American**	**Prob. not invest**	**Prob. system**	**Prob. Modularisation**
90	0.05	5.0	0.20	0.25	6.024	6.377	0.69	0.15	0.16
90	0.05	5.0	0.20	0.75	6.981	7.436	0.69	0.15	0.16
90	0.05	5.0	0.30	0.25	15.375	16.734	0.55	0.18	0.27
90	0.05	5.0	0.30	0.75	17.427	19.087	0.57	0.18	0.24
90	0.10	5.0	0.20	0.25	1.554	2.082	0.88	0.07	0.05
90	0.10	5.0	0.20	0.75	1.808	2.481	0.88	0.07	0.05
90	0.10	5.0	0.30	0.25	6.716	8.711	0.72	0.12	0.15
90	0.10	5.0	0.30	0.75	7.611	9.970	0.74	0.13	0.14
100	0.05	5.0	0.20	0.25	9.771	10.266	0.56	0.21	0.23
100	0.05	5.0	0.20	0.75	11.296	11.922	0.57	0.21	0.22
100	0.05	5.0	0.30	0.25	20.872	22.517	0.47	0.22	0.31
100	0.05	5.0	0.30	0.75	23.662	25.609	0.49	0.22	0.29
100	0.10	5.0	0.20	0.25	2.859	3.844	0.80	0.11	0.09
100	0.10	5.0	0.20	0.75	3.325	4.520	0.80	0.11	0.09
100	0.10	5.0	0.30	0.25	9.701	12.360	0.64	0.16	0.19
100	0.10	5.0	0.30	0.75	10.993	14.127	0.66	0.16	0.18
110	0.05	5.0	0.20	0.25	14.506	15.123	0.44	0.27	0.29
110	0.05	5.0	0.20	0.75	16.716	17.522	0.46	0.26	0.28
110	0.05	5.0	0.30	0.25	26.997	28.924	0.39	0.26	0.35
110	0.05	5.0	0.30	0.75	30.596	32.949	0.42	0.26	0.33
110	0.10	5.0	0.20	0.25	4.722	6.318	0.70	0.17	0.13
110	0.10	5.0	0.20	0.75	5.479	7.401	0.70	0.17	0.13

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S	δ	T	σ	ρ	**European**	**American**	**Prob. not invest**	**Prob. system**	**Prob. Modu-larisa-tion**
110	0.10	5.0	0.30	0.25	13.214	16.723	0.57	0.21	0.23
110	0.10	5.0	0.30	0.75	14.975	19.087	0.58	0.21	0.21

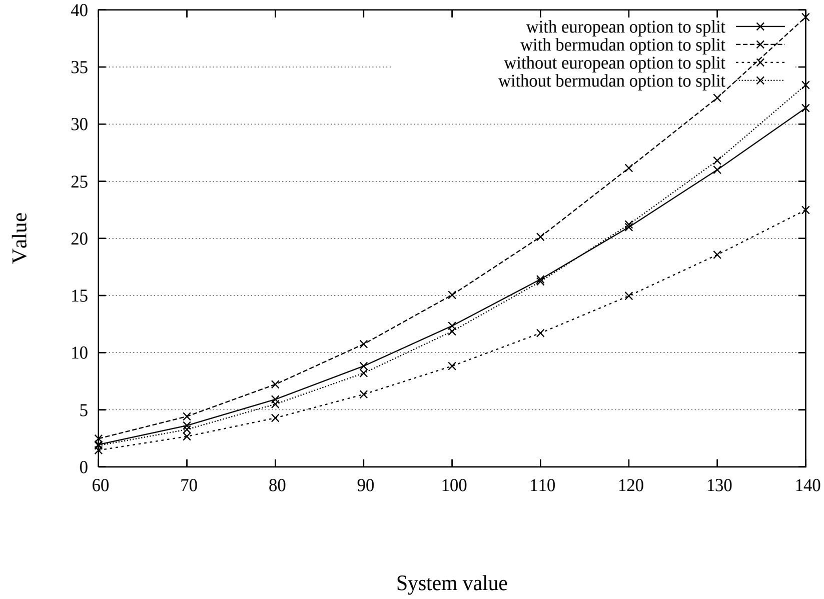
The riskless interest rate is 0.05, the system investment cost is 75, the system development cost is 25 and the design rules costs per module is 5. The multipliers of the volatility and costs are 1 and 1.2 respectively. The underlying system value s, the volatility σ, the correlation coefficient between the modules returns ρ, the shortfall rate of return δ, and the time to maturity of the option T are as indicated in the table. The simulation was done with 50000 paths. The random number generator MRG31k3p routine of (L'Ecuyer and Touzin, 2000) was re-initialised for every option with the seed 12345 and Moro normal variates were used. The regression was performed using Numerical Recipes SVDFIT (Press, Teukolsky, Vetterling and Flanney, 1992), with weigthed Laguerre polynomials.

The higher the value of the system (and of the modules, since we assume a linear relation), the higher the value of the modular project and the higher the early exercise premium (Figure 1(a)). When the project is deep in-the-money, i.e. the business value exceeds significantly the investment costs, the probability[6] of investment is higher, both in the interconnected system and modular configurations: for some of the cases where it would be optimal no to invest when the option to split is not available, we should invest in a modular configuration or in an interconnected system (Figure 1(b)). Since we are assuming that the costs of the modular configurations are higher than those of the system (cost multiplier of 1.2), the investment in the system increases more than the investment in the modular configuartions.

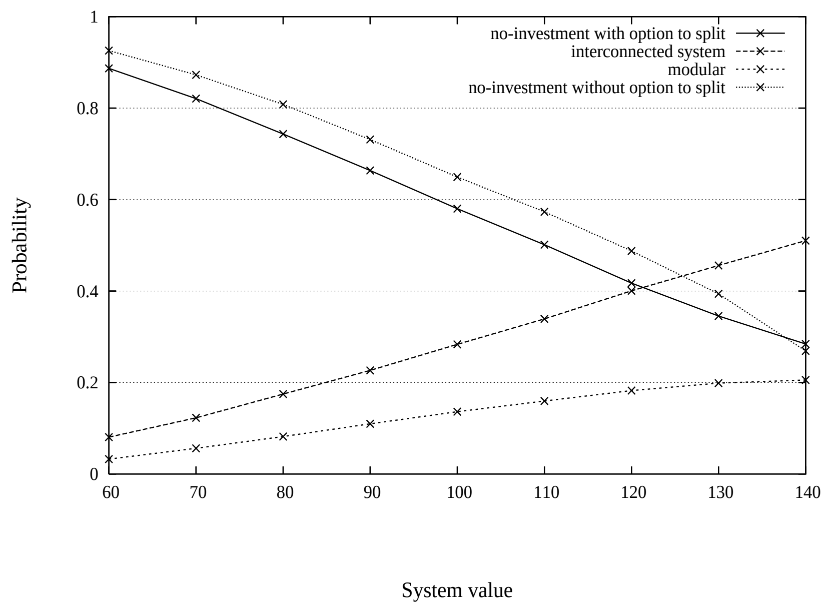
Figure 2 shows the impact of the cost multiplier. As the costs of investment, devolpement and design formulation increase, the value of option to delay investment in a modular project converges to the value of the option to delay investment in an interconnected system (Figure 2(a)). The investment in a modular project can double the investment in an interconnected system without eroding totally the value of the modular project. Other variables can affect this "allowance" to increase costs with modularisation, namely the risk of the modules compared to the risk of the system (Figure 4). A small reduction of the costs implied by modularisation (which is an unlikely situation) increases significantly the probability of choosing a modular project and, in contrast, the costs of modularisation must increase significantly to eliminate any probability of choosing a modular project (Figure 2(b)).

The value of the option to delay becomes worthless (as any option) if uncertainty disappears (Figure 3(a)). As uncertainty increases, the probability of choosing a modular project also increases (Figure 3(b)). Although the portfolio of modules has, for the parameters chosen, a lower volatility than the system, for correlations lower than one, a higher risk leads to a higher investment in a modular configuration, not only due to a higher probability of investment but also because it can be worthwile to invest in a modular configuration rather than in an interconnected system. Even in the case

[6] All the probabilities presented are risk neutral probabilities.

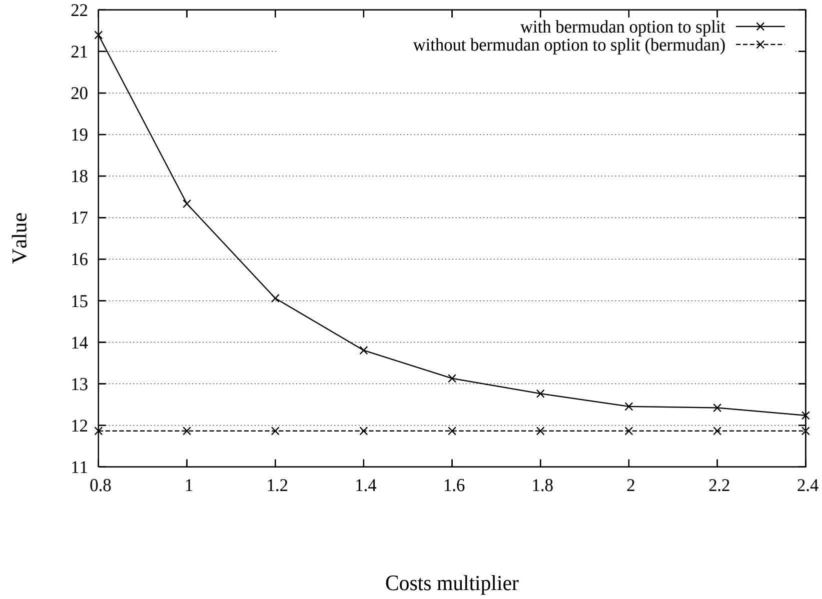


(a) Value of the modular project

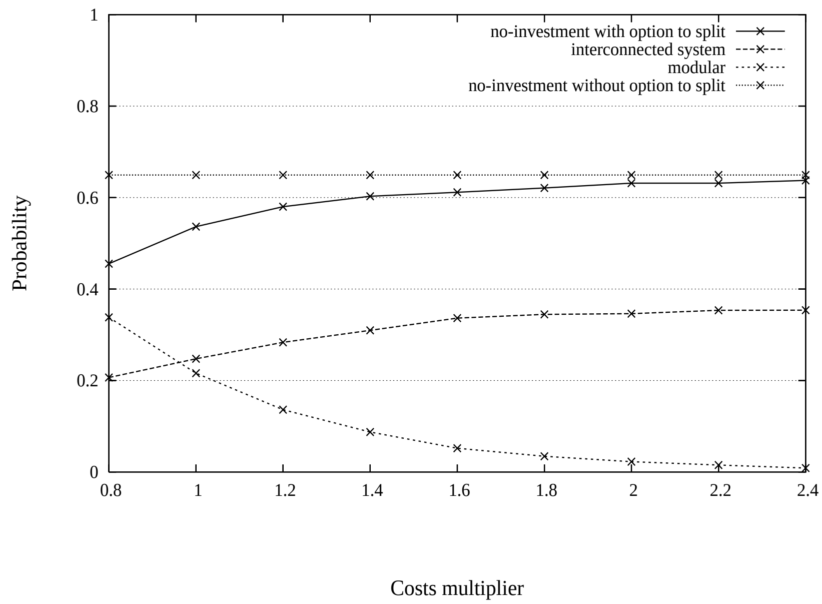


(b) Probabilities associated with each strategy

Figure 1: The valuation of the modular project - the effect of the system/module values

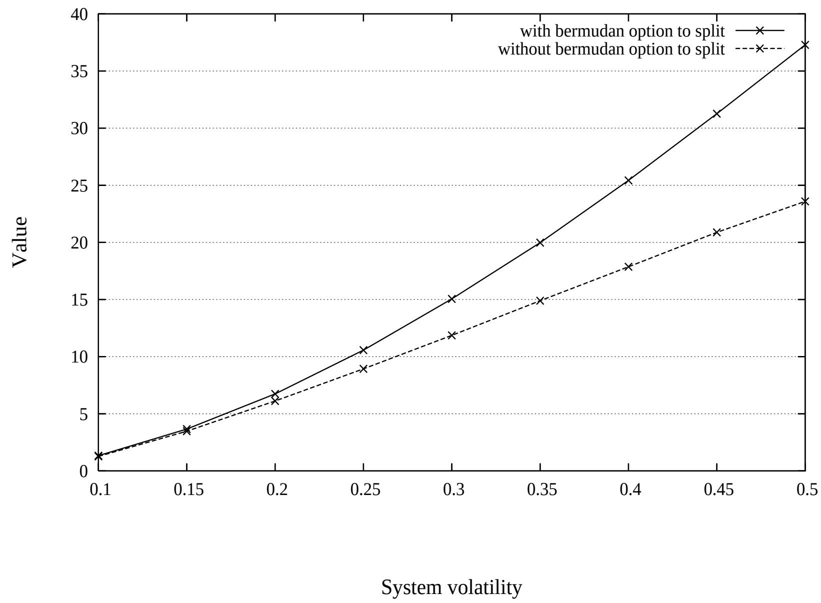


(a) Value of the modular project

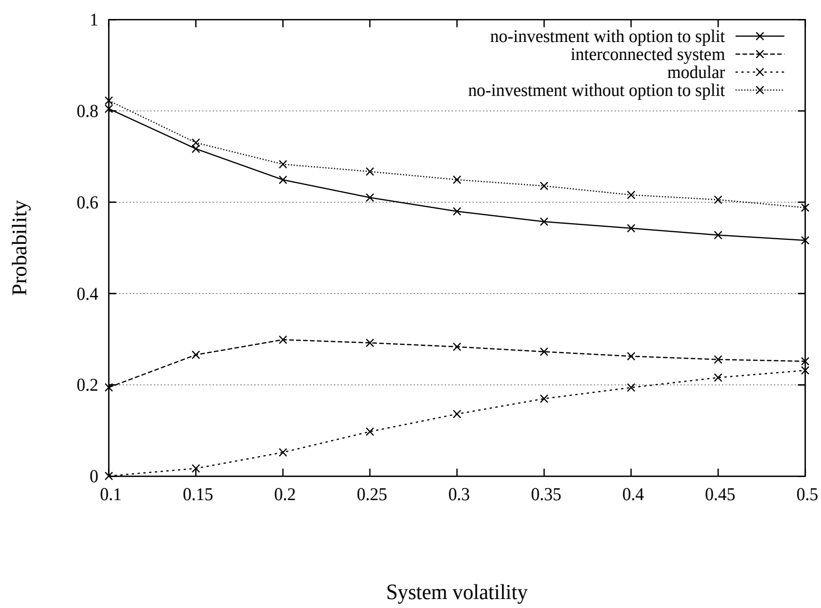


(b) Probabilities associated with each strategy

Figure 2: The valuation of the modular project - the effect of the costs multiplier



(a) Value of the modular project



(b) Probabilities associated with each strategy

Figure 3: The valuation of the modular project - the effect of the system volatility

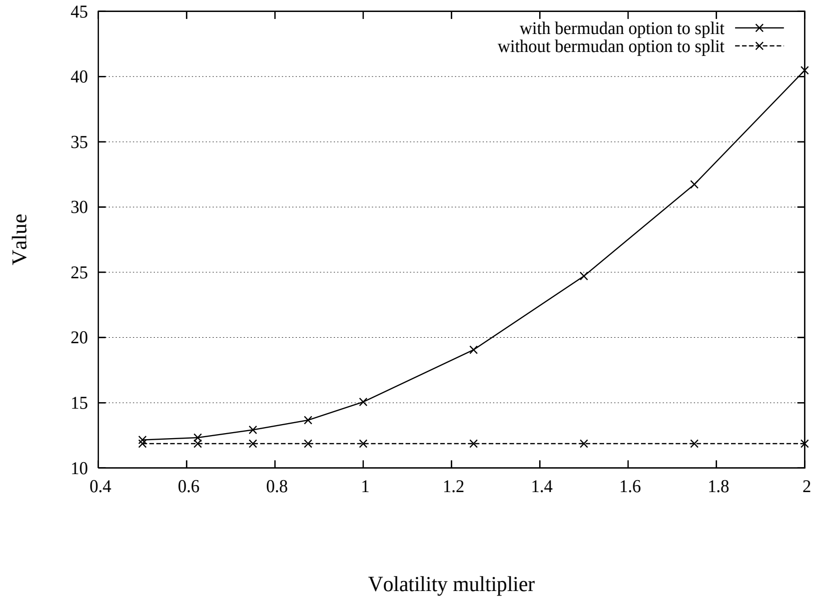
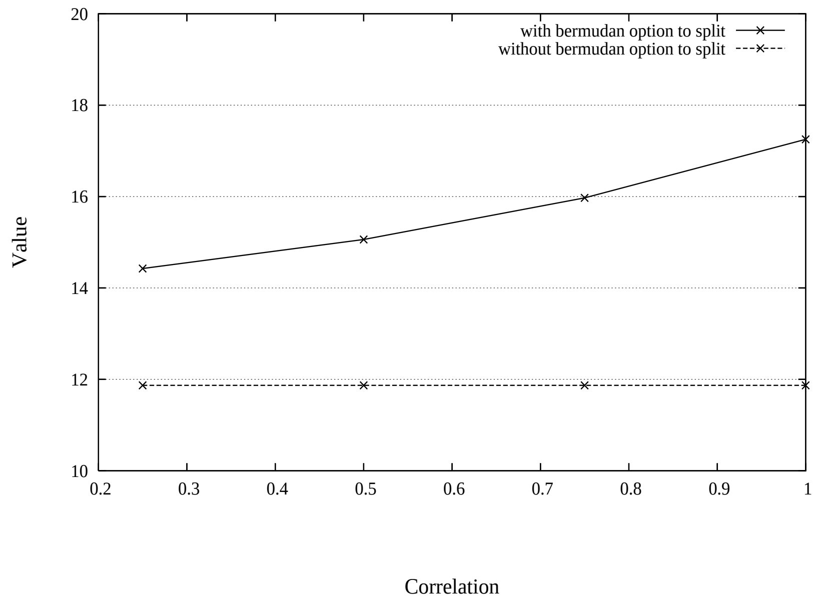


Figure 4: Value of the modular project vs Volatility multiplier

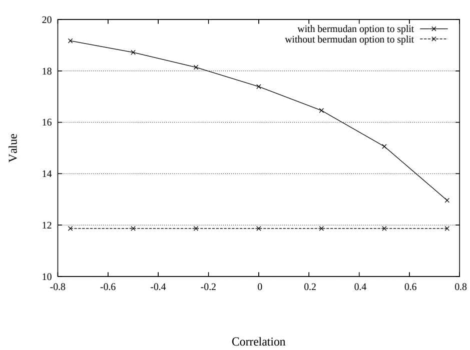
where the uncertainty of each module is lower than the system uncertainty, there is some incentive to modularise unless it is significantly lower (Figure 4).

The correlations have also some effect in the optimal modular strategy (Figure 5). The first thing to note is that the correlation between the modules within each configuration increases the value of a modular project (Figure 5(a)). When the aggregate value of the system is unchanged by modularisation, as in the model of Baldwin and Clark (2000), modularisation benefits from a lower correlation between the modules. In our model, if we assume that the volatility of the portfolios of the modules is equal to the volatility of the system, we find a similar result (Figure 6). Is this case, a lower correlation implies that, due to the diversification effect, each module has a higher volatility. As we will show later, if we consider the net value of splitting, the relation suggested by Baldwin and Clark (2000) is found in our model. Even in the case when of perfect correlation between the modules of the same configuration, the modular project still has an higher value than the non-modular project (Figure 6). Aditional assumptions are needed to make modularisation worthless.

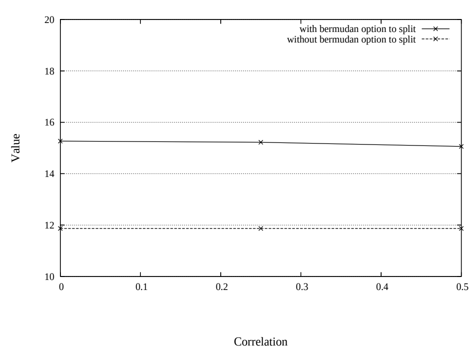
The correlations between modules of different configurations have a small impact on the project value (Figure 5(b)), whereas a higher correlation between the modules and the system reduces the value of the modular project (Figure 5(b)). The higher this correlation, more similar are the modular



(a) Correlation within each modular configuration



(b) Correlation between the system and modules



(c) Correlation between modules of different configurations

Figure 5: The valuation of the modular project - the effect of the correlations

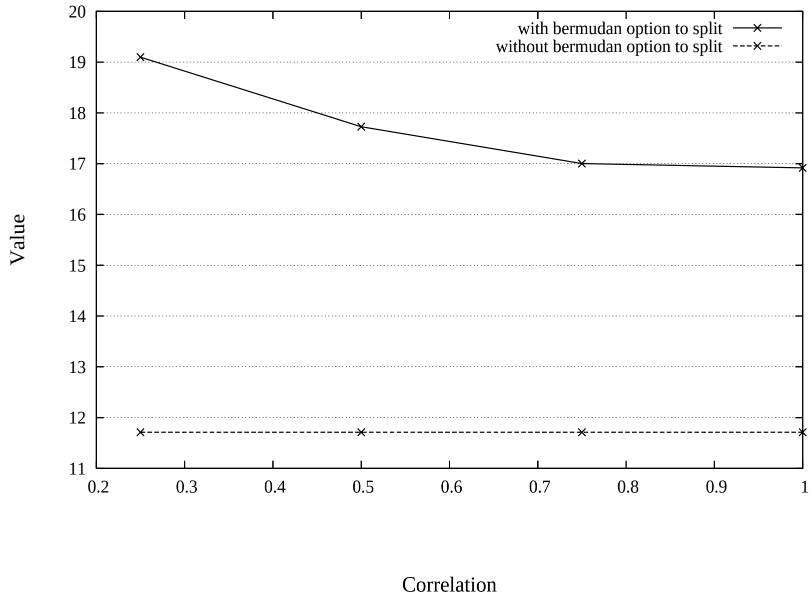


Figure 6: Value of the modular project vs Correlation within each modular configuration - Equal portfolio volatility

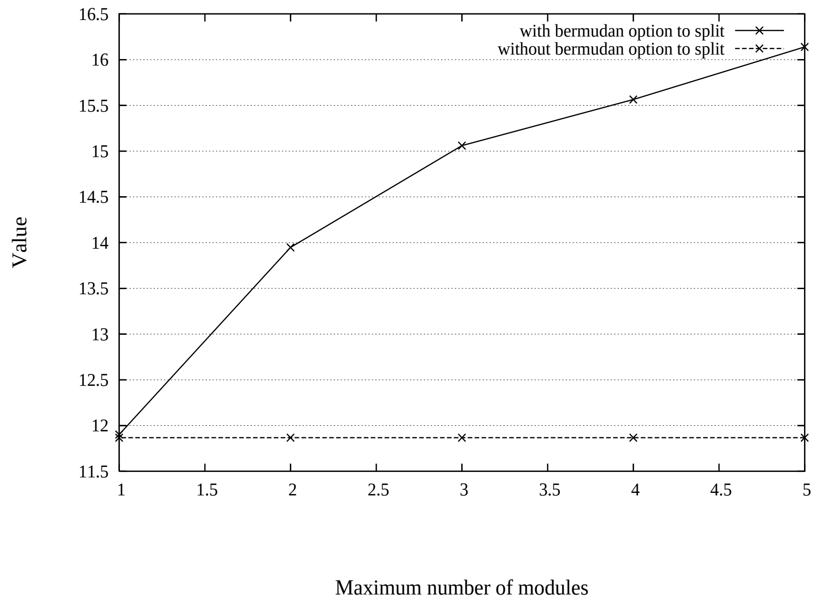
configuration and the interconnected system, thus decreasing the value of modularity.

In the extreme case where all the (system and modules) volatilities are equal and all the underlying assets (system and modules in all possible configurations) are perfectly correlated, the incentive to choose a modular project vanishes, unless there is a "size" advantage, i.e. higher relative values or lower relative costs of the modules induce some incentive to modularise (Figure 8).

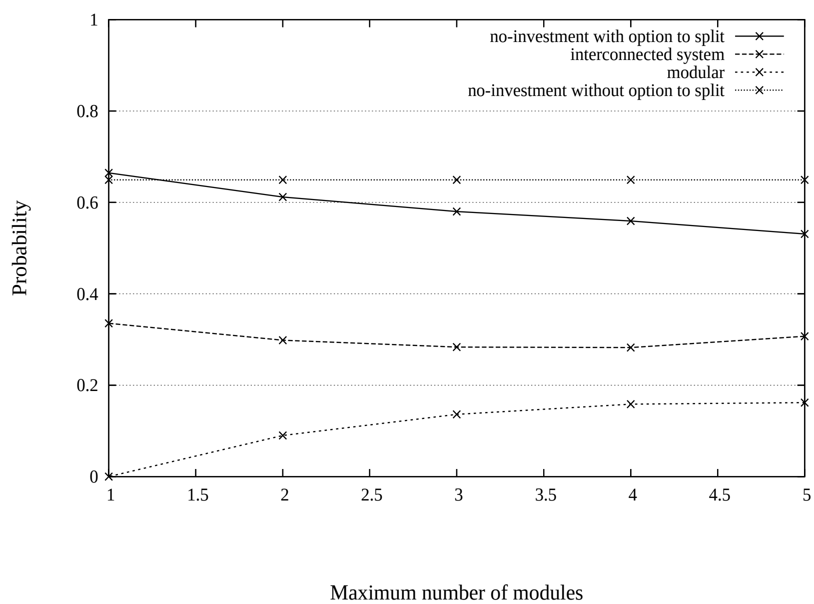
Although there are cases where modularisation does not increase value (higher relative costs, lower relative values or perfectly correlated assets, for example), usually the possibility of higher degrees of modularisation provides a higher project value (Figure 7). For the base case parameters, where it is assumed that splitting costs increase linearly with the number of modules, it happens at a decreasing rate (Figure 7(a)). After a certain point (in our case with four potential modules) the probability of investing in an interconnected system increases, without making the project worst, because there is a wider choice of modular configurations (Figure 7(b)).

5 The net value of splitting

When we analyse carefully the sources of the value of the modular project, we come to the conclusion that one of the main sources is not related to the

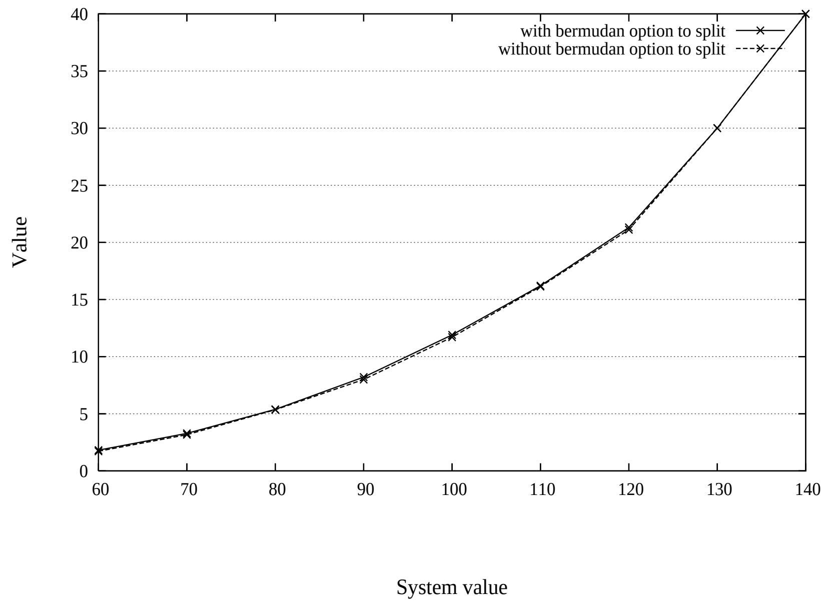


(a) Value of the modular project

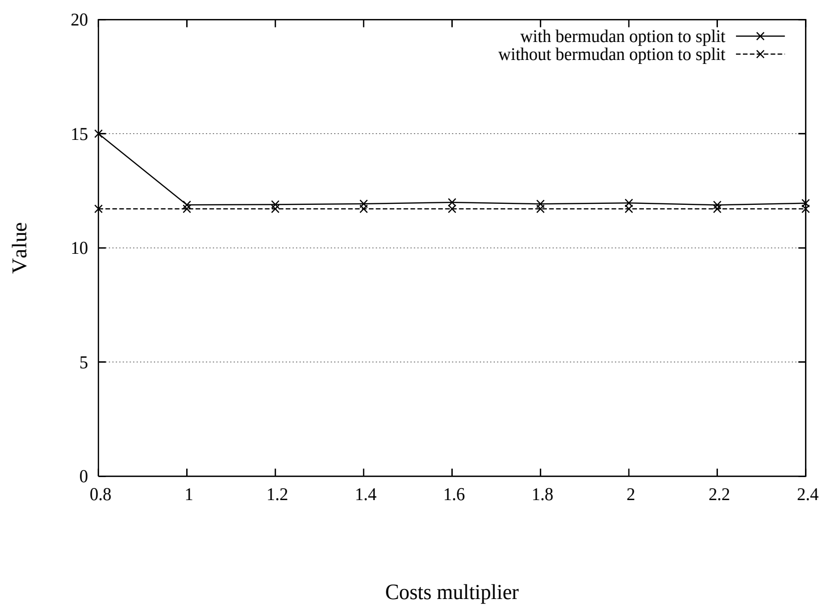


(b) Probabilities associated with each strategy

Figure 7: The valuation of the modular project - the effect of the increasing the maximum number of modules



(a) System/modules values



(b) Costs multiplier

Figure 8: The valuation of the modular project - perfectly correlated assets

modular nature of the project. The option to abandon those modules which reveal to have a lower value than the investment cost, by not exercising the option to implement or develop it, has been identified as the main source of modular value. In the case of an interconnected system, there is a single option available: to develop the system or abandon it. The real value source is, then, the difference between these two: the difference between the value of a portfolio of options and the value of a single option.

In the proposed model, and in the results presented until now, the value of the modular project has an additional component. In a modular configuration, the development and implementation of all modules, may be required to make the system work and have value, i.e. , a single module would only have value if all the others of the configuration are implemented. If the aggregate value of modules is different from the interconnected system value or, although being equal, are not perfectly correlated, there is another value source related to the option to choose the best of two products: a non-modular product and a product, although modular, only has value as a hole. This is similar to the option to chose the best of two alternatives for the same non-modular product[7]. In this context, this type of project can be named as "quasi-modular". This source of value does not relate to the modular nature of the project, and thus can not be attributed to modularity.

The "net value of splitting" can, then, be computed as the difference between the modular project value and the value of the same project when the implementation of all the modules is required.

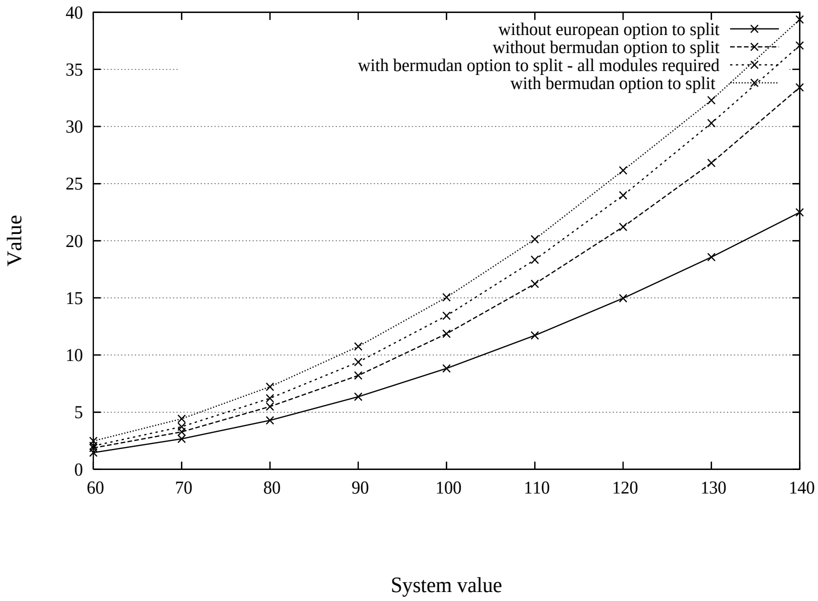
As we mentioned above, the main source of the value of a modular project is, sometimes, the option to choose between a modular project and a "quasi-modular" project (Figures 9 and 10). As expected, the value of modularity increases with the moneyness of the project (Figure 9). It is worthwhile to note that, even for the assumptions made about the relationship between the risk (variance) of the system and the risk of the portfolio of modules, we can observe the same effect as the one suggested by Baldwin and Clark (2000): a higher correlation of the modules within each configuration leads to a lower modular value, which is null for perfectly correlated modules (Figure 10).

6 Concluding remarks

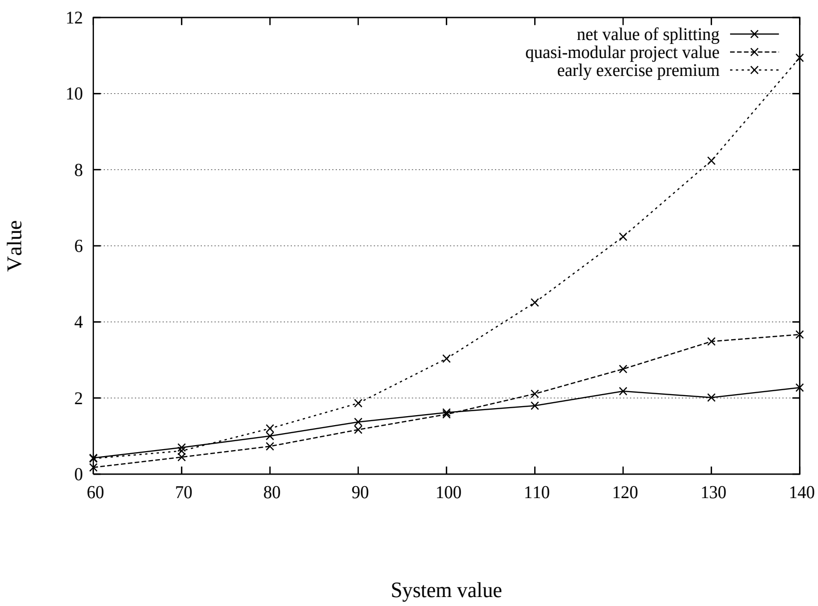
In this paper, we have studied the benefits of splitting a project into modules from a real options perspective. Building upon the Baldwin and Clark (2000) approach to modularity, we proposed a more general model which includes the possibility of delaying the option to split, incorporated variable correlations between the system and the modules, and assumed a multi-staged product development. This is a more realistic model, which can only

[7]The model of Childs, Ott and Triantis (1998) values this type of projects, comparing parallel and sequential development.

Figure 9: Net value of splitting - the effect of the modules/system value

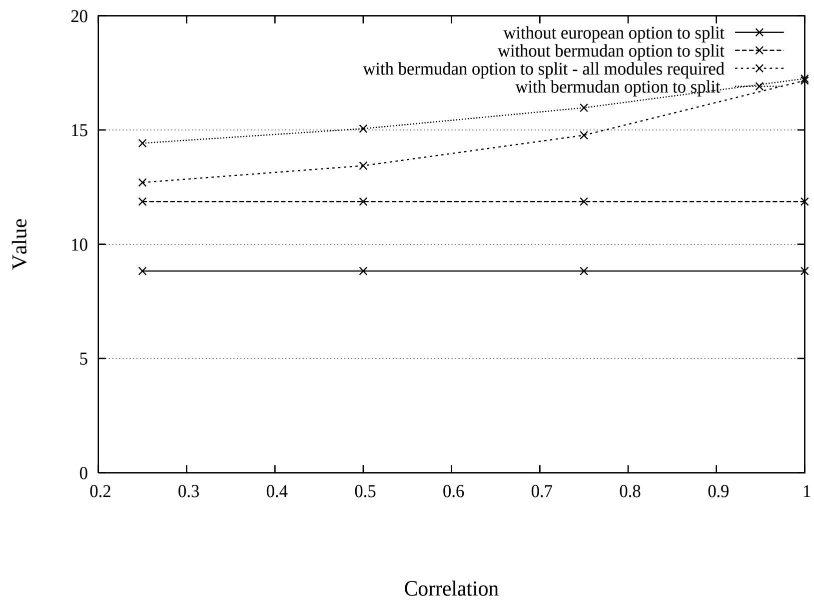


(a) Value of the modular project

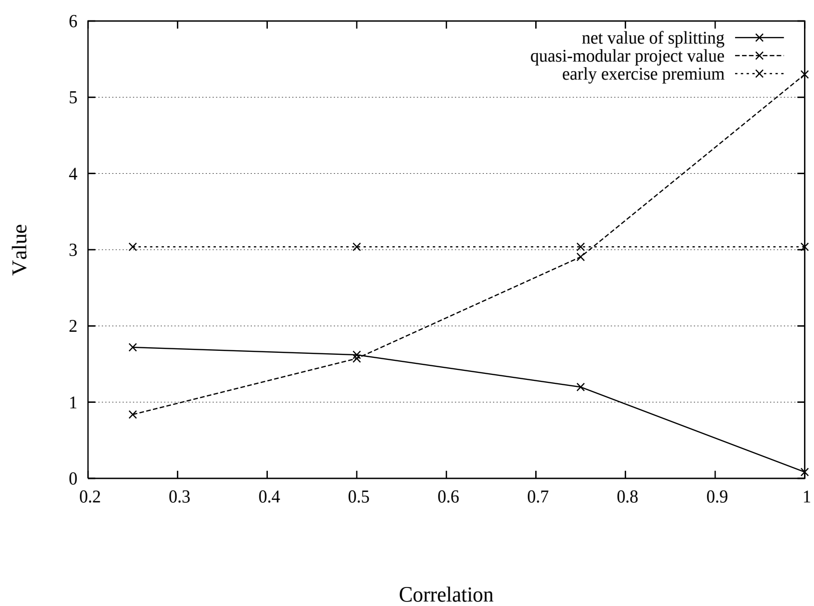


(b) Value sources

Figure 10: Net value of splitting - the effect of the correlation within each modular configuration



(a) Value of the modular project



(b) Value sources

be valued by simulation, due the curse of dimensionality of other alternative methods. Using the LSM method proposed by Longstaff and Schwartz (2001) and extended by Gamba (2003) to value portfolio of real options, we have studied the impact of changing some of the variables which influence the optimal modular strategy.

The probability of choosing a modular project rather than an interconnected system depends on the relative values, costs and risk of each modular configuration. Any of these variables can, for extreme values, eliminate any advantage of modularisation. Even if it is plausible that modularisation implies higher costs of investment, they can increase significantly until modularisation becomes worthless. When the modules are perfectly correlated with each other and with the system and have an identical risk, there is no incentive to modularise, unless there is a "size" advantage, related to lower costs or higher values of the modules. The effects of the correlations on the value of modularity, depends on the relative risks of the system and the portfolio of modules. Higher degrees of modularisation increases the value of the project but at a decreasing rate.

A significant part of the value added by modularity is related to the option to choose the best of an interconnected system and a portfolio of modules: when they are all required to be implemented, this is similar to the option to choose the best of two alternative systems. In our view, this cannot be attributed to modularity and, thus, we analysed the "net value" of splitting, which is the difference between the value of a modular project and the value of the project with the option mentioned above.

Our approach can be extended in many ways, without significant difficulties. This is, indeed, one of the major advantages of a simulation approach. The effect of other modular operators, namely the possibility of running several experiments for each module (the substitution operator), the effect of competition and the analysis of some particular cases, like the asymmetric modules or the case when some of the modules are "required" to make the system work, are some the extensions that can be added.

Finally, we believe that this paper also shows the power and flexibility of the LSM method in the valuation of portfolio of real options contingent on multiple underlying assets.

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