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ACCOUNTING FRAUD AND THE PRICING OF CORPORATE LIABILITIES
Structural models with garbling

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Abstract. We provide a method for modelling accounting distortions and their impact on the value of corporate liabilities. Our model is able to account for both small noise (estimate errors) and large mis-representations (deliberate fraud). Such a methodology is then applied to structural pricing models, in the spirit of Merton (1974). It turns out that accounting distortions may be a relevant factor in the pricing of corporate securities: indeed, they are able to explain why credit spreads are actually larger than implied by traditional structural models, particularly on short maturities. Simulations show that such an effect is stronger for safer firms, namely those with lower leverage and asset volatility.

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1 Introduction

Very often the valuation of financial assets, as well as trading and hedging decisions, are based on signals. In a perfect information setting, signals represent a faithful portrait of the "fundamentals" driving the value of assets. Actually, information asymmetry plays a crucial role in determining the prices of corporate securities. It is then reasonable to ask whether transparency is priced in the market, that is whether the price of corporate liabilities may be affected by different degrees of noise and distortion in the signal. This question is addressed in this paper, with a particular focus on the distortion, we could say outright fraud, possibility.

The problem of transparency is paramount in the so called "structural" approach to credit risk. In these models the value of corporate securities - such as debt and equity - is recovered from a representation of the structure of the balance sheet information: for this reason, the quality of accounting information plays a key role. The seminal paper in this literature is due to Merton (1974), even though the world famous Black and Scholes (1973) was already targeted at the pricing of corporate liabilities. In structural models, corporate liabilities are evaluated by decomposing their pay-offs in linear and non-linear products, and using standard option pricing theory to price them. Equity then is a call option written on the value of assets for a strike price equal to the face value of debt: this basically follows from the limited liability feature of equity capital. Debt is subject to default risk and this is measured as a short position in a put option with the same underlying, strike price and exercise dates matching those of the call option representing equity. A low credit risk simply means that this "default put" option is far out-of-the-money and the call option representing equity is deep-in-the-money. While in the seminal Merton's paper the structure of the bond is kept very simple, assuming a zero coupon bond and the possibility of default only at maturity, successive extensions have been proposed to account for coupon bonds (Geske, 1977), covenants and seniority structures (Black and Cox, 1976), warrants and convertible debt (Bensoussan, Crouhy and Galai, 1995a-b).

While representing an elegant and informative approach to the evaluation of corporate securities, structural models do not generally provide a good fit to the real market data. Typically, reasonable values for leverage of the firm and volatility of assets produce credit spreads which are two low with respect to those observed in the market. Several answers have been proposed as possible solutions to this problem. Anderson and Sundaresan (1996) suggest that the owner of the firm may engage in a strategic rescheduling process to exploit the bankrupcy costs at the expense of bondholders. Along the same lines, Leland (1994) and Leland and Toft (1996) allow the owner of the firm to terminate the process in such a way to optimize the value of equity, again at the expense of debt.

An alternative explanation for the failure of structural models to fit the data stems from the fact that the value of the firm is not directly observed and this lack of transparency may affect the prices in the market. In this spirit Cherubini and Della Lunga (2001) propose a conservative assessment of the probability of default by using a default probability interval, in line with the MaxiMin-Expected-Utility framework in Gilboa and Schmeidler (1989). However, this approach is not able to account for another typical flaw of structural models: the strong understatement of credit spreads for short maturities. A typical credit spread term structure in the Merton model shows a hump and zero intercept. The latter feature is particularly disturbing and it is due to the main assumption on which the model was built, that is the representation of the value of the firm as an adapted diffusion process. The need to account for higher credit spreads for shorter maturities can be achieved either by allowing for a jump process in the value of the firm (Zhou, 2001), so dropping the diffusion process assumption, or by relaxing the adapted process hypothesis. The latter route was first followed by Duffie and Lando (2001), who propose a model with endogenous bankrupcy in which the market is assumed to observe a noisy signal of the value of the firm at discrete times. Other approaches based on imperfect information have been proposed: for example, Giesecke and Goldberg (2004) assume that the default threshold could not be observed.

This paper is in the branch of literature initiated by Duffie and Lando (2001), focussing on the relevance of accounting information quality for the price. This focus is also supported by the empirical evidence presented in Yu (2005). As in the Duffie and Lando paper, the price of corporate securities is determined by considering that the value of assets is observed at discrete times through an imperfect signal, representing accounting information. The novelty of our approach rests on the fact that we allow not only for noise but also for the possibility of deliberate bias in the balance sheet representation. The recent empirical evidence has documented the existence of two levels of accounting distortion: a "soft" kind of earnings management, often exploiting the room left by discretionary accruals; a "hard" one, namely a fraud leading to a large misrepresentation of the firm value (see Gao and Shrieves (2001), Johnson and Ryan (2003), Ke (2004), Cheng and Warfield (2004), Erickson *et al.* (2004), Peng and Roell (2004)).

In our setting, while credit risk is the probability that the firm may go bankrupt some time in the future, transparency risk is the probability that the firm be already in a default state, despite any good balance sheet report. The structure of the model is kept as general as possible. We only take into account an insider, called "manager", possibly including internal and external auditors linked to him by some contractual provisions, who may report the value of the firm imperfectly for different reasons. First, the information available to the manager might not be perfect, inducing him to a wrong assesment of the firm's ongoing condition. Second, the value of some balance sheet items (e.g. financial claims and intangibles) needs to be estimated, leaving some room for judgement. Third, the manager might have some private incentive to deliberately misreport his private information (and auditors may have incentive to collude with him). On one hand, a manager may be induced to hide a bad state, because: (i) he fears that such a disclosure would trigger an intervention by shareholders or creditors, interfering with his management or even leading to his dismissal; (ii) he fears that a bad realization of the firm's performance may damage his own reputation as a good manager, both within and outside the firm (career concern); (iii) his compensation is (partially) linked to the performance/market value of the firm (e.g.: stock option plans). On the other hand, a manager may decide to understate the performance of the firm, in order to: (i) reduce the tax burden; (ii) hide and divert profits from the company, to benefit himself or some related parties; (iii) engage in strategic default.

The theoretical literature on misreporting has extensively analysed the incentives faced by top managers, pointing to a basic trade-off between incentive alignment and truthful revelation: on one hand, performance-based compensation packages induce managers to maximise firm value, thus acting in the interest of shareholders; on the other hand, they introduce a clear incentive to manipulate accounting statements, in order to inflate the measure of performance used to determine their compensation (see Dye (1988), Goldman and Slezak (2003), Kadan and Yang (2004)). Other works stress the role of auditors, pointing to the need of strengthening the efficiency of the audit sector and of the supervision on it (see Kofman and Lawarrée (1993), Kaplan (2004), Baglioni and Colombo (2004)). However, this body of research has overlooked the impact of misreporting on the valuation of corporate liabilities: this is instead our focus, while we do not explicitely model the strategic interactions among agents, leading to a distortion of the information released to the market. One exception in the above literature is Fischer and Verrecchia (2000): they show that - in a model where the market is uncertain about managers' objective function - a higher reporting bias reduces the information content of reported earnings, leading to a smaller impact on share price. Our paper differs from theirs on several grounds, mainly in explicitely analysing the stochastic structure of the accounting signal and in incorporating it into structural pricing models.

Our work has been inspired by some recent cases such as Enron and Worldcom in the US, and Parmalat in Europe, in which the term "fraud" is actually more appropriate than "noise". In the aftermath of those cases the question was whether these were isolated or whether transparency risk was actually a pricing factor that ought to be taken into account in the market. Our model shows that such transparency and fraud risk may actually be a pricing factor. To give an idea as to how the market may be affected, consider a very intuitive case. Say that after Enron or Parmalat cases,[1] one accounts for a 0.5% probability that the balance sheet statement of any other firm could be distorted as well, so that the firm is actually bankrupt even though that does not show up in the report. Taking the example of a BBB firm (Parmalat was rated BBB- just before default), a reasonable estimate leads to a leverage (the ratio of debt to the value of the firm) around 40% (Standard & Poor's, 2000) and a volatility around 25%. For an investment horizon of one year, a standard structural model would assign zero default risk to the debt issued by the firm. The value of assets would be 2.5 times the value of debt (1/0.4) and the value of equity would be exactly 1.5 (0.6/0.4). But assume that in 5 cases out of 1000 one could get caught in a

[1] Or even before that: some people were actually puzzled by the balance sheet reports of Parmalat even before the fraud was discovered.

fraud such as Parmalat, so that the firm is actually in a default state. In this case the true value of equity would be zero, so accounting for that possibility makes the market value of equity drop by a percentage of 50 basis points. What about debt? In the 5 cases out of 1000 in which the firm would be actually bankrupt, the value of debt would drop to the recovery rate: assuming a figure of 40% for the recovery rate, that would imply a 30 basis points decrease in the market value of debt (0.6 times 0.005), or equivalently a 30 basis points increase in the one year credit spread. We may also compute the overall effect on the value of the firm as 50*0.6+30*0.4 = 42 basis points.

The plan of the paper is as follows. In section 2 we introduce the model in a very simple binomial setting. The case of multinomial signal is recovered as an extension of this model in section 3. In section 4 both the value of the firm and the signal are assumed to be continuous variables. In section 5 we present a new structure of signal: it is a two-tier signal in which we distinguish between distortion across different sets (such as saying that a firm is solvent, while it is bankrupt) and distortion within each set (a reporting of the value of the firm of 100 rather 99). This two-tier signal structure is applied in section 6 to represent accounting information in the standard Merton model, also allowing for the presence of covenants. Section 7 explores the quantitative effects of accounting information quality on credit spreads, by presenting some preliminary simulations of our model. Finally, Section 8 summarizes our main findings.

2 Accounting information and firm market value: a binomial example

In this section, we present a simple introductory model to help the reader to grasp the basic intuition of our work; such a model will be generalized and made more realistic in the next sections.

Let us consider a firm and focus on the information available to its "manager" and to the "market": the latter stands for any stakeholders (e.g. minority shareholders and debt-holders) not having access to the private information retained by those encharged of running the firm. We assume that insider dealing regulation prevents the manager from trading in securities issued by his own company.

We concentrate on three dates: initial (t_0), interim (t_1) and final (T).

2.1 Common knowledge information

Suppose that the value of the firm's assets (V) follows a binomial process, as described in Figure 1. The structure of such stochastic process is known by both the manager and the market.

In particular, at the initial date the following information is common knowledge:

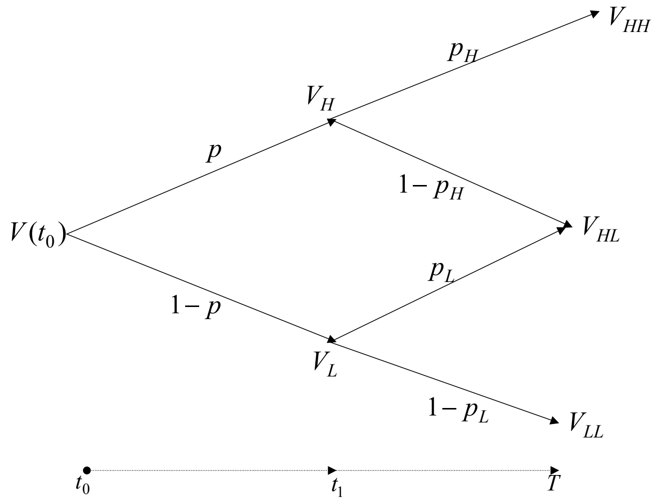

Figure 1: Common knowledge information

- the possible values of the firm at T - denoted by the vector $\mathbf{V}(T) \equiv [V_{HH} \;\; V_{HL} \;\; V_{LL}]'$;

- a probability measure satisfying the martingale property. This condition ensures existence of a set of prices ruling out arbitrage opportunities (Harrison - Kreps , 1979). In particular, we denote the transition probabilities from t_0 to t_1 by the vector $\mathbf{p} \equiv [p \;\; (1-p)]'$ and those from t_1 to T by the matrix:[2]

$$\mathbf{P} \equiv \begin{bmatrix} p_H & 1-p_H & 0 \\ 0 & p_L & 1-p_L \end{bmatrix}$$

Given such information, the "fundamental" values of the firm may be easily computed as follows:[3]

$$\mathbf{V}(t_1) \equiv \begin{bmatrix} V_H \\ V_L \end{bmatrix} = \mathbf{PV}(T) \tag{1}$$

and, exploiting again the martingale property:

$$V(t_0) = \mathbf{p}'\mathbf{V}(t_1) = \mathbf{p}'\mathbf{PV}(T) \tag{2}$$

[2] Matrix notation might seem redundant here; we use it because it makes it easier to extend the present model to the multinomial case (see the next section).

[3] For simplicity, we momentarily abstract from discounting (equivalently we set the risk-free interest rate to zero). Of course, this implies no loss of generality, as the value of assets may be assumed to be expressed in terms of a money market fund numeraire.

Finally, we assume that at the end of the project (T) everybody observes the value taken by firm's assets. Therefore, the final market value of the firm will be one of the elements of $\mathbf{V}(T)$, based upon the observation of which of the three states of the world realizes.

2.2 Accounting information as a noisy signal

Now we focus on the information available at t_1. At that time, the manager has accumulated information, relative to the ongoing performance of the firm; formally, he is supposed to observe whether the true state occurring is V_H or V_L. At this date, he is required by the regulation to disclose his private information through a balance sheet statement: this is modelled as a noisy signal $s \in [h, l]$. The basic idea is that, through the accounting information, the manager may convey to the market a more or less favorable picture of the firm economic/financial condition; then $s = h$ ($s = l$) stands for a positive (negative) scenario, as described by the balance sheet statement. More precisely, $s = h$ signals that the high state of nature (V_H) has occured at t_1, while $s = l$ signals the opposite (V_L).

However, the signal may be distorted and the market itself might not be confident in the accounting information released by the firm. The noise possibly present in the accounting information is modelled in a general way, by assuming that it may be distorted in either one of the two following ways, or both. (I) "Up-distortion": a good signal ($s = h$) is observed in the low state of the world. (II) "Down-distortion": a bad signal ($s = l$) is observed in the high state. A picture of the information structure, available to the market, is provided in Figure 2, where π_d and π_u are the conditional probabilities of a down-distortion / up-distortion respectively: $\pi_d = \Pr(s = l \,|V = V_H)$ and $\pi_u = \Pr(s = h \,|V = V_L)$. We assume that $\pi_d + \pi_u \leq 1$; as we are going to see shortly, if such inequality is strict the accounting information is valuable: in particular, the observation $s = h$ ($s = l$) leads to an upward (downward) revision of the probability of being in the high state of the world.

Possible accounting distortions are then summarized by the "garbling matrix":

$$\mathbf{\Pi} \equiv \begin{bmatrix} 1 - \pi_d & \pi_d \\ \pi_u & 1 - \pi_u \end{bmatrix}$$

Despite its simplicity, this formulation is highly flexible, as it is able to account for those cases where the accounting information is "symmetrically" garbled ($\pi_d = \pi_u$), say because of observational mistakes, as well as for those quite different situations where a deliberate manipulation of accounting data leads to an asymmetric noise (for example: $\pi_d = 0$ and $\pi_u > 0$).

Finally, the probabilities of observing each value of the signal are easily calculated as follows:

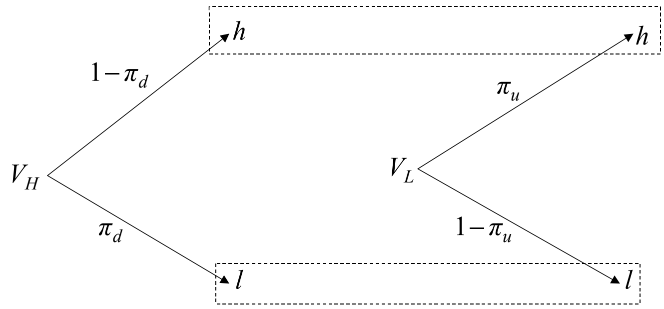

Figure 2: Accounting information as a noisy signal

$$\Pr(s = h) = p(1 - \pi_d) + (1 - p)\pi_u \tag{3}$$

and

$$\Pr(s = l) = p\pi_d + (1 - p)(1 - \pi_u) \tag{4}$$

or in matrix form:

$$\mathbf{p}'_{\mathbf{s}} \equiv \begin{bmatrix} \Pr(s = h) & \Pr(s = l) \end{bmatrix} = \mathbf{p}'\mathbf{\Pi} \tag{5}$$

2.3 Updating probabilities

At t_1, the balance sheet statement provides some information to the outsiders, relative to the ongoing performance of the firm. Suppose that a good accounting signal is released. Publicly available information does not - in general - allow to say for sure where we are at t_1: more technically, the information set includes both the nodes V_H and V_L in Figure 1. Such information may be conveniently described by the compound lottery shown in Figure 3. The market updates the probabilities of the high (low) states of the world, according to the signal received. By Bayes' rule, the posterior probability of the high state, given that the good signal ($s = h$) has been observed, is:

$$\Pr(H\,|h) \equiv \Pr(V = V_H\,|s = h) = \frac{p(1 - \pi_d)}{\Pr(s = h)} \tag{6}$$

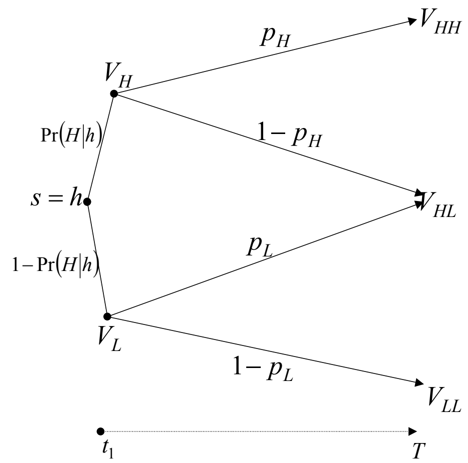

Figure 3: Market information upon observing a good signal

Of course, the same reasoning applies in the alternative case, where $s = l$ is observed. The posterior probability of the high state, given that the bad signal has been observed, is:

$$\Pr(H\,|l) \equiv \Pr(V = V_H\,|s = l) = \frac{p\pi_d}{\Pr(s = l)} \tag{7}$$

Let us define the matrix of posterior probabilities as follows:

$$\mathbf{A} \equiv \begin{bmatrix} \Pr(H\,|h) & \Pr(L\,|h) \\ \Pr(H\,|l) & \Pr(L\,|l) \end{bmatrix}$$

where obviously $\Pr(L\,|h) = 1 - \Pr(H\,|h)$ and similarly for $s = l$.

Our assumption, that the sum of the two accounting distortions $(\pi_d + \pi_u)$ is not larger than one, is necessary and sufficient for a good balance sheet statement being interpreted as a positive signal, and viceversa for a bad one. Indeed, it is easy to verify that:

$$\Pr(H\,|l) \leq p \leq \Pr(H\,|h) \text{ iff } \pi_d + \pi_u \leq 1 \tag{8}$$

2.4 The market value of the firm

The market value of the firm at t_1 depends on the accounting information released at that time. Denoting by $\widehat{V}(h)$ the value of the firm if the good signal is observed and by $\widehat{V}(l)$ its value in the opposite case, we have:

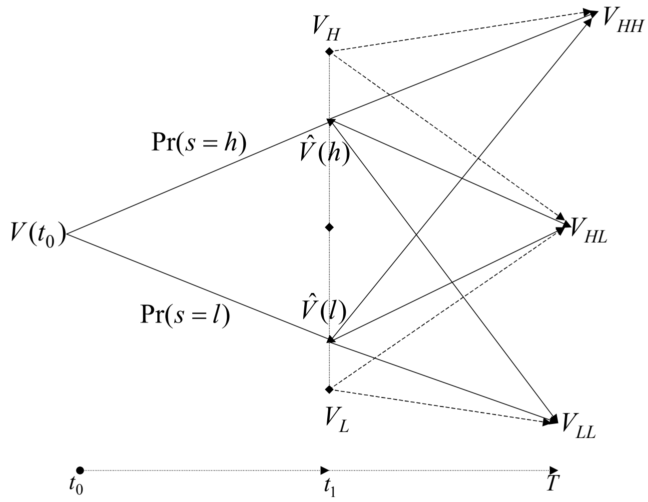

Figure 4: Possible paths of firm market value

$$\widehat{\mathbf{V}}(t_1) \equiv \begin{bmatrix} \widehat{V}(h) \\ \widehat{V}(l) \end{bmatrix} = \mathbf{A}\mathbf{V}(t_1) = \mathbf{A}\mathbf{P}\mathbf{V}(T) \tag{9}$$

The evolution over time of the firm market value, together with the range of values possibly taken at t_1, are shown in Figure 4. The exact positions of $\widehat{V}(h)$ and $\widehat{V}(l)$ crucially depend on the reliability of the accounting signal. As the accounting information becomes more accurate, the firm market value moves towards one of the extremes: V_H or V_L. In particular, $\widehat{V}(h)$ is decreasing in π_u: the upward revision - due to a good balance sheet statement - is larger the lower is the probability of an up-distortion in the accounting signal. A similar reasoning holds for $\widehat{V}(l)$.

The extreme case of perfect information may be easily formalized by setting $\mathbf{\Pi} = \mathbf{I}$ (where $\mathbf{I}$ is a two-dimension identity matrix) - or equivalently $\pi_u = \pi_d = 0$ - implying $\mathbf{A} = \mathbf{I}$ and $\widehat{\mathbf{V}}(t_1) = \mathbf{V}(t_1)$. If no distortion is present, the market fully trusts the message in the accounting statement and revises the value of the firm accordingly.

At the opposite extreme, an uninformative signal leaves the firm's market value unchanged (i.e. the same as with prior information). Indeed, if the two rows of matrix $\mathbf{\Pi}$ are identical - equivalently $\pi_u + \pi_d = 1$ - then:

$$\mathbf{A} = \begin{bmatrix} \mathbf{p}' \\ \mathbf{p}' \end{bmatrix} \text{ and } \widehat{\mathbf{V}}(t_1) = \begin{bmatrix} V(t_0) \\ V(t_0) \end{bmatrix}$$

Finally, by the law of iterated expectations you may easily check that information distortion does not affect the martingale property of the probability measure, and so fulfills the no-arbitrage pricing requirement. Notice in fact that matrix $\mathbf{A}$ has the property:

$$\mathbf{p_s'A} = \mathbf{p}' \tag{10}$$

from which we compute:

$$\mathbf{p_s'}\widehat{\mathbf{V}}(t_1) = \mathbf{p_s'AV}(t_1) = \mathbf{p'V}(t_1) = V(t_0) \tag{11}$$

which is the same "fundamental" value we found in equation 2, based upon the prior information. This result makes sure that the initial market value of the firm only depends on the information available at t_0.

3 A multinomial model

Having shown the basic intuition of the model in a simple binomial example, we provide here an extension to the case in which the firm's assets and the accounting signal may take many values. Obviously this is crucial to increase the realism of the model. All the other assumptions of the previous section remain unchanged (we will drop some of them in the next sections, in order to get further realism). We focus on the formal framework here, referring the reader to the previous section for comments and intuition.

3.1 Common knowledge information

Figure 5 shows the structure of the stochastic process followed by the firm's asset value, which is common knowledge.

The vector of possible firm's asset values at t_1 is $\mathbf{V}(t_1) \equiv [V_1(t_1)\ \ V_2(t_1)\ldots V_N(t_1)]'$ and the vector of possible values at T is $\mathbf{V}(T) \equiv [V_1(T)\ V_2(T)\ldots V_{N+1}(T)]'$; we assume - without loss of generality - that $V_i > V_{i+1}$.

The vector of transition probabilities from t_0 to t_1 is $\mathbf{p} \equiv [p_1\ p_2\ldots p_N]'$; in the structure portayed in the figure the $(NxN+1)$ matrix of transition probabilities from t_1 to T is:

$$\mathbf{P} \equiv \begin{bmatrix} p_{11} & 1-p_{11} & 0 & . & . & 0 \\ 0 & p_{12} & 1-p_{12} & 0 & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & p_{1N-1} & 1-p_{1N-1} & 0 \\ 0 & 0 & . & . & p_{1N} & 1-p_{1N} \end{bmatrix}$$

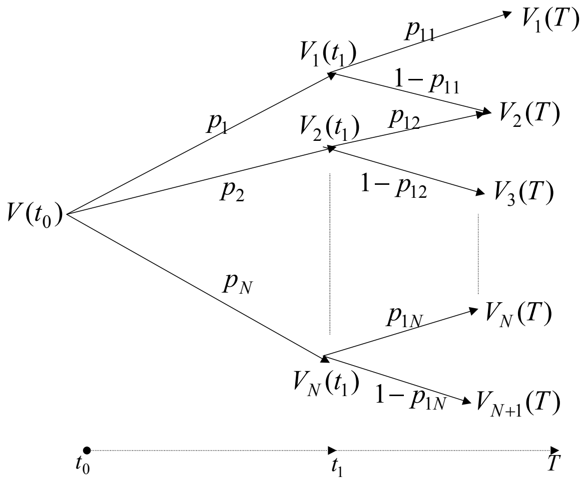

Figure 5: A multinomial model

The martingale property still holds:

$$\mathbf{V}(t_1) = \mathbf{P}\mathbf{V}(T) \tag{12}$$

and

$$V(t_0) = \mathbf{p}'\mathbf{V}(t_1) = \mathbf{p}'\mathbf{P}\mathbf{V}(T) \tag{13}$$

3.2 The accounting signal

The accounting information released at t_1 is described by a signal s; the vector $\mathbf{s} \equiv [s_1 \ s_2...s_N]'$ denotes the possible values of s, corresponding to the states $V_1(t_1), V_2(t_1), ..., V_N(t_1)$. The signal, however, may not carry perfect information on the state which is actually obtaining. Denoting by $\pi_{ij} = \Pr(s = s_j \mid V(t_1) = V_i(t_1))$ the conditional probability of the signal, we define the "garbling matrix" as:

$$\mathbf{\Pi} \equiv \begin{bmatrix} \pi_{11} & \pi_{12} & . & . & \pi_{1N} \\ \pi_{21} & \pi_{22} & . & . & \pi_{2N} \\ . & . & . & . & . \\ . & . & \pi_{ij} & . & . \\ . & . & . & . & . \\ \pi_{N1} & \pi_{N2} & . & . & \pi_{NN} \end{bmatrix}$$

where, for example, in the first row one reads the probability distribution of s, conditional on the state $V_1(t_1)$.

The unconditional probabililty of observing a value of the signal is given by:

$$\Pr(s = s_j) = \sum_{i=1}^{N} p_i \pi_{ij} \tag{14}$$

or in matrix form

$$\mathbf{p}'_s \equiv [\Pr(s = s_1) \ \ \Pr(s = s_2) ... \Pr(s = s_N)] = \mathbf{p}'\mathbf{\Pi} \tag{15}$$

Of course $\mathbf{\Pi} = \mathbf{I}$ (the N-dimension identity matrix) identifies the case of perfect information - implying $\mathbf{p}'_s = \mathbf{p}'$. At the opposite extreme, if $\pi_{ij} = \pi_j$ for all i (the elements of column j are all identical), the signal $s = s_j$ carries no information.

3.3 Updating probabilities

Once the signal is released the information is updated. The probability of state i, conditional upon observing s_j, is given by Bayes' rule:

$$a_{ji} \equiv \Pr(V(t_1) = V(t_1)_i \mid s = s_j) = \frac{p_i \pi_{ij}}{\Pr(s = s_j)} \tag{16}$$

These posterior probabilities can be reported in the "updating matrix" $\mathbf{A}$:

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & a_{2N} \\ . & . & . & . & . \\ . & . & a_{ji} & . & . \\ . & . & . & . & . \\ a_{N1} & a_{N2} & . & . & a_{NN} \end{bmatrix}$$

where, for example, in the first row one reads the probability distribution of $V(t_1)$, conditional upon the observation of s_1.

In the perfect information case we recover $\mathbf{A} = \mathbf{I}$. On the other side, if some signal value $s = s_j$ carries no infomation, we have $a_{ji} = p_i$ for all i (the elements of row j coincide with the priors).

3.4 Firm market value

We are now in a position to evaluate the impact of the signal on the firm's market value. Let us define $\widehat{\mathbf{V}}(t_1) \equiv \left[\widehat{V}_1(t_1)\ \widehat{V}_2(t_1)...\widehat{V}_N(t_1)\right]'$, where $\widehat{V}_j(t_1)$ is the updated value upon observing s_j. So at time t_1 we have $\widehat{\mathbf{V}}(t_1) = \mathbf{AV}(t_1) = \mathbf{APV}(T)$ (as in equation 9 above).

In the particular case of perfect information, we have $\widehat{\mathbf{V}}(t_1) = \mathbf{V}(t_1)$. At the opposite extreme we find the case of no information, where all the rows of matrix $\mathbf{\Pi}$ are identical, implying statistical independence between the signal and the state of nature: then all the rows of matrix $\mathbf{A}$ turn out to be equal to $\mathbf{p}'$ and $\widehat{\mathbf{V}}(t_1) = \mathbf{V}(t_0)\mathbf{e}$,where $\mathbf{e}$ is the N-dimension unit vector.

Finally, notice that matrix $\mathbf{A}$ retains the property $\mathbf{p}'_s\mathbf{A} = \mathbf{p}'$, implying that $\mathbf{p}'_\mathbf{s}\widehat{\mathbf{V}}(t_1) = V(t_0)$ (in other words, equations 10-11 still hold).

4 Continuous variables

We now extend the analysis to the case in which both the underlying asset, i.e. the value of the firm, and the signal are continuous random variables. For the rest, the setting of the model is the same as before. The valuation date is t_0 and on the two futures dates $\{t_1, T\}$ a signal is released and the final value of the firm is revealed. In what follows we assume a probability space $\{\Omega, \Im_t, P\}$, representing the overall information produced in the model at times $\{t_{0,}t_1, T\}$. The idea is that the σ-algebra $\Im_t$ is generated by the value of the firm and by a signal. The information set available to the general public in the market is instead $\Im^S_{t_1} \subseteq \Im_{t_1}$, where $\Im^S_{t_1}$ denotes the σ-algebra generated by the signal alone. Put in other terms, only the signal rather than the variable is observed by agents in the market.

4.1 Common knowledge information

We begin with a description of the information which is assumed to be common knowledge. Complete information is attained at time T, so that the value of the firm $V(T)$ is a random variable taking values in the Borel-set and measurable with respect to $\{\Omega, \Im_T\}$. Before that, people know that information about the value of the firm $V(t_1)$ - i.e. a signal $s(t_1)$ - is released at time t_1. As for the probability measure, people are endowed with a probability prior $p(y)$ representing the probability density function of $V(t_1) = y$ and a conditional density $p_{1T}(x \mid y)$ describing the probability of $V(T) = x$ given that $V(t_1) = y$. The probability measure is endowed with the martingale property as required in the standard arbitrage-free pricing setting. This means that in the case of full information, in which $\Im^S_{t_1} = \Im_{t_1}$ and $V(t_1)$ is directly observed by the general public, we must have

$$V(t_1) = \int_0^\infty x p_{1T}(x \mid y)\, dx \tag{17}$$

and at time t_0 accordingly it must be

$$V(t_0) = \int_0^\infty y p(y)\, dy \tag{18}$$

4.2 The signal

We assume that the signal observed at time t_1 is noisy, so that $\Im^S_{t_1} \subseteq \Im_{t_1}$. The signal is described by a conditional density $\pi(s \mid y)$ representing the probability of observing signal $s(t_1) = s$ conditional on $V(t_1) = y$.

The unconditional probability density of observing signal s is then given by

$$g(s) = \int_0^\infty \pi(s \mid y)\, p(y)\, dy \tag{19}$$

In one extreme case the signal carries no information content, namely if $\pi(s \mid y) = g(s)$: statistical independence between signal and true state. At the other extreme, full information content is represented by $\pi(s \mid y) = \delta(y - s)$, where $\delta(.)$ is the *Dirac delta* function (see Laffont, 1989): by definition, $\delta(y - s) = 0$ for $y \neq s$, and $\int_0^\infty \delta(y - s)\, dy = 1$; moreover - by the sifting property of the delta functional - it is $g(s) = p(s)$ and the probability density of observing the signal is the same as the prior density of the firm value.

4.3 Updating probabilities

Once a signal $s(t_1) = s$ is observed, the probability that the state $V(t_1) = y$ obtains is updated according to Bayes rule

$$f(y \mid s) = \frac{p(y)\, \pi(s \mid y)}{g(s)} \tag{20}$$

Notice that if the signal carries no information content we have $f(y \mid s) = p(y)$.

In the opposite case, if the signal carries full information, we have

$$f(y \mid s) = \frac{p(y)\, \delta(y - s)}{p(s)} \tag{21}$$

which vanishes for $y \neq s$ and it boils down to the delta function for $y = s$. Then we may write $f(y \mid s) = \delta(y - s)$.

4.4 The market value of the firm

By exploiting the martingale property, the value of the firm conditional upon observing the signal $s(t_1) = s$ may be computed as

$$\widehat{V}(t_1) = \int_0^\infty y f(y \mid s)\, dy \tag{22}$$

It is easy to check that, if the signal carries no information, the above value becomes $\widehat{V}(t_1) = V(t_0)$: the market price does not move between t_0 and t_1.

To the contrary, the case of perfect information yields $\widehat{V}(t_1) = \int_0^\infty y\delta(y - s)\, dy = s$: the message $s(t_1) = s$ is fully believed, and the firm market value is revised accordingly.

An example of continuous signal is in Duffie and Lando (2001). In their model the observed signal is $s = V(t_1)\, U$. The variable U is log-normally distributed with mean equal to 1, and it is uncorrelated with the true value $V(t_1)$. Under this construction, the signal is then meant to introduce unbiased noise in the observation of the process. Possible extensions would be to include a bias or assume some correlation with the unobserved value $V(t_1)$. These distorsions of the nature of the signal would be unavoidable in cases in which the signal is the outcome of strategic behavior of some "insider" in the game.

5 Two-tier signal: measurement error and fraud

The information released through the accounting statements might be distorted for two different reasons: measurement errors and deliberate fraud. While the former lead to a small - possibly unbiased - noise in the accounting signal, the latter typically produces a large misrepresentation of the firm performance. Distinguishing between these two sources of distortions is important, since their consequences on the valuation of the firm are presumably quite different. The novelty of our approach relies in combining the technical tools, introduced so far, to design a two-tier signal able to account for both the two levels of misreporting. In particular, the model with continuous variables is modified by introducing a signal with a mixed structure, having both a continuous and a discrete component: the former accounts for measurement errors, while the latter accounts for fraud. The common knowledge information is the same as in the previous section, so we directly focus on the design of the accounting signal.

5.1 The signal

Assume the following signal structure. Partition the support of $V(t_1)$ in a set of intervals $\{A_1, A_2, ..., A_N\}$, such that $\bigcup_{i=1}^{N} A_i = \Re_+$. Select $\boldsymbol{\pi}_i = [\pi_{i1}\ \pi_{i2}...\pi_{ij}...\pi_{iN}] \in [0,1]^N$ and such that $\boldsymbol{\pi}_i \mathbf{e} = 1$ for every i. The probability of observing a signal $s(t_1) = s$ in a set A_j, given that the value $V(t_1) = y$ is in set A_i, is defined as

$$\pi_{ij} \equiv \pi\left(s \in A_j \mid y \in A_i\right) \tag{23}$$

By analogy to the discrete model, we may refer to $\boldsymbol{\pi}_i$ as a row in the "garbling matrix".

Defining the prior probabilities as $\mathbf{p} \equiv [p(A_1)\ \ p(A_2)...p(A_i)...p(A_N)]'$ with $p(A_i) = \int_{A_i} p(y)\, dy$ we may carry out a model similar to that with the multinomial signal. The probability of observing $s \in A_j$ and $y \in A_i$ is then given by

$$\Pr\left(s \in A_j \cap y \in A_i\right) = p(A_i)\, \pi_{ij} \tag{24}$$

and the marginal probability is

$$\Pr\left(s \in A_j\right) = \sum_{i=1}^{N} p(A_i)\, \pi_{ij} \tag{25}$$

Notice that in this way we have specified the structure of the signal by exclusively focussing on its membership with respect to the sets A_j. Formally, "fraud" is defined as $s \in A_j$ and $y \in A_i$ with $i \neq j$. In other terms, "fraud risk" is the distortion possibly present in this membership structure. In the limit case, this distortion might be such that all the rows of the garbling matrix are identical ($\pi_{ij} = \pi_j$ for all i), leading to $\Pr(s \in A_j) = \pi_j$: we label this case as "no info across sets". The opposite case of "perfect info across sets" is obtained when the garbling matrix is an identity matrix, leading to $\Pr(s \in A_j) = p(A_j)$.

We may now specify the model further to describe the information content of the signal within the same set. To this purpose, define a set of kernel functions $\varphi_{ij}(y,s) : A_i \times A_j \to \Re_+$ such that

$$\int_{A_j} \varphi_{ij}(y,s)\, ds = 1 \tag{26}$$

Then define the conditional density $\pi(s \mid y)$ as follows:

$$\pi(s \mid y) \equiv \pi_{ij}\varphi_{ij}(y,s) \ \text{ for } y \in A_i \text{ and } s \in A_j \tag{27}$$

This choice ensures that:

$$\int_{A_j} \pi(s \mid y)\, ds = \pi_{ij} \tag{28}$$

$$\int_0^{\infty} \pi(s \mid y)\, ds = \sum_{j=1}^{N} \pi_{ij} \int_{A_j} \varphi_{ij}(y,s)\, ds = \sum_{j=1}^{N} \pi_{ij} = 1 \tag{29}$$

We label as "measurement error" the noise possibly present in the information provided by the signal within the same set. In the limit case the signal conditional density - defined on a pair A_i, A_j - does not depend on the specific value of the firm within the set A_i: $\varphi_{ij}(y,s) = \varphi_{ij}(s)$ for $y \in A_i$ and $s \in A_j$; we refer to this case as "no info within a set". To the opposite, we have "perfect info within a set" if - for some j - $\varphi_{jj}(y,s) = \delta(y-s)$ for $y \in A_j$ and $s \in A_j$, where $\delta(.)$ is the *Dirac delta* function.[4]

The marginal probability density of the signal is then given by:

$$g(s) = \int_0^{\infty} \pi(s \mid y)\, p(y) dy = \sum_{i=1}^{N} \pi_{ij} \int_{A_i} \varphi_{ij}(y,s)\, p(y)\, dy \ \text{for } s \in A_j \tag{30}$$

5.2 Updating probabilities

Suppose you observe a specific value of the signal $s(t_1) = s$. The probability updating function is computed as:

$$f(y \mid s) = \frac{p(y)\, \pi_{ij}\varphi_{ij}(y,s)}{\sum_{i=1}^{N} \pi_{ij} \int_{A_i} \varphi_{ij}(y,s)\, p(y)\, dy} \ \text{for } y \in A_i \text{ and } s \in A_j \tag{31}$$

and it is easy to verify that

[4] Note that the case of perfect info within a set is meaningful only when y and s lie in the same set.

$$\int_0^\infty f(y \mid s)\,dy = \sum_{i=1}^N \int_{A_i} f(y \mid s)\,dy = \sum_{i=1}^N \frac{\pi_{ij}\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy}{\sum_{i=1}^N \pi_{ij}\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy} = 1 \tag{32}$$

5.3 The market value of the firm

By using equations (22) and (31), we compute the value of the firm conditional upon observing the signal $s(t_1) = s \in A_j$:

$$\begin{aligned} \widehat{V}(t_1) &= \int_0^\infty y f(y \mid s)\,dy \qquad (33) \\ &= \sum_{i=1}^N \frac{\pi_{ij}\int_{A_i} y\varphi_{ij}(y,s)\,p(y)\,dy}{\sum_{i=1}^N \pi_{ij}\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy} \end{aligned}$$

and the price is a weighted integral average computed using updated probabilities across the sets A_i.

To ease notation, it is natural to define

$$\overline{V}(A_i,t_1) \equiv \frac{\int_{A_i} y\varphi_{ij}(y,s)\,p(y)\,dy}{\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy} \tag{34}$$

the integral mean of $V(t_1)$ in each set A_i (conditional upon observing $s(t_1) = s \in A_j$).

Define also

$$f(A_i \mid s) \equiv \int_{A_i} f(y \mid s)\,dy = \frac{\pi_{ij}\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy}{\sum_{i=1}^N \pi_{ij}\int_{A_i}\varphi_{ij}(y,s)\,p(y)\,dy} \tag{35}$$

the posterior probability that $y \in A_i$, given $s(t_1) = s \in A_j$.

We may finally rewrite

$$\widehat{V}(t_1) = \sum_{i=1}^N f(A_i \mid s)\,\overline{V}(A_i,t_1) \tag{36}$$

This notation highlights the two-tier structure of the accounting signal. In principle we may conceive models with no fraud (in which case $f(A_j \mid s \in A_j) = 1$), allowing for the possibility of measurement errors; or models without measurement errors (in which case $\overline{V}(A_j,t_1) = s \in A_j$), allowing for a positive chance of fraud.[5]

[5] Again, note that the absence of measurement errors is relevant only when y and s lie in the same set (absence of fraud). This does not rule out the chance that they might lie in different sets (presence of fraud).

Example 1 *Let us make an example here, in order to get a better understanding of the model. Consider the particular case where: (i) absent fraud, the accounting signal is immune from measurement errors (perfect info within a set); (ii) in presence of fraud, the signal is completely unrelated to true value of the firm (no info within a set). Formally, for $y \in A_i$ and $s \in A_j$:*

$$\varphi_{jj}(y,s) = \delta(y-s) \text{ if } i=j \tag{37}$$
$$\varphi_{ij}(y,s) = \varphi_{ij}(s) \text{ if } i \neq j \tag{38}$$

This example refers to the case where we are mainly concerned about the risk of accounting fraud, leading to a large misreporting. Then, in absence of fraud we are willing to consider any measurement error as negligible; to the contrary, in presence of fraud we consider the information content of the accounting signal (if any) as completely useless.

The posterior density function of the firm value, conditional upon observing the signal $s(t_1) = s \in A_j$, turns out to be:

$$f(y \mid s) = \frac{\pi_{jj}\delta(y-s)\,p(y)}{g(s)} \text{ for } y \in A_j \tag{39}$$
$$f(y \mid s) = \frac{\pi_{ij}\varphi_{ij}(s)\,p(y)}{g(s)} \text{ for } y \in A_i,\ i \neq j \tag{40}$$

where the marginal density of the signal is:

$$g(s) = \pi_{jj}p(s) + \sum_{i \neq j} \pi_{ij}\varphi_{ij}(s)p(A_i) \tag{41}$$

Using the above definitions we obtain:

$$\overline{V}(A_j, t_1) = s \tag{42}$$

$$\overline{V}(A_i, t_1) = \frac{\int_{A_i} yp(y)\,dy}{p(A_i)} \text{ for all } i \neq j \tag{43}$$

Note that, in the set where fraud is absent, the expectation of the firm value is identical to the released accounting signal, as any measurement error has been assumed away. To the contrary, if fraud is there the expected value of the firm - whithin each set - is based only on prior information, confirming that the accounting information disclosed at t_1 has no value.

We also compute

$$f(A_j \mid s \in A_j) = \frac{\pi_{jj}p(s)}{\pi_{jj}p(s) + \sum_{i \neq j} \pi_{ij}\varphi_{ij}(s)p(A_i)} \tag{44}$$
$$f(A_{i \neq j} \mid s \in A_j) = \frac{\pi_{ij}\varphi_{ij}(s)p(A_i)}{\pi_{jj}p(s) + \sum_{i \neq j} \pi_{ij}\varphi_{ij}(s)p(A_i)} \tag{45}$$

Finally the value of the firm, given that a signal $s \in A_j$ is observed, may be computed by using equation (36), which yields:

$$\widehat{V}(t_1) = \frac{1}{\pi_{jj} p(s) + \sum_{i \neq j} \pi_{ij} \varphi_{ij}(s) p(A_i)} \left[\pi_{jj} p(s) s + \sum_{i \neq j} \pi_{ij} \varphi_{ij}(s) \int_{A_i} y p(y)\, dy \right] \tag{46}$$

6 Structural models of debt and equity with garbling

In this section we apply the theory described above to extend the main structural models proposed in the finance literature to the case in which the information concerning the value of the firm is distorted because of a garbling effect. While up to now the analysis was carried out with respect to the market value of the firm - focussing on the technicalities of the construction of the signal - here we explicitly include leverage and investigate how the value of the firm is split into equity and debt. We also account for a sequence of signals. Our task is to keep the specification of garbling as general and simple as possible, in order to apply it to some structural credit risk models that are extensively used in applied work. Most importantly, we would like to make the model suited to recover, in future research, the amount and direction of these distorsions from market data.

Let us describe the basic structure of the model, at least in its simplest form. At time t_0 a firm is issuing debt to finance a project that will be completed at time T, when it will be worth $V(T)$. The structure of the funding is a zero coupon bond maturing at the same date, for a nominal amount equal to D. In order to issue debt, the firm has to share some information with the market. Information concerning the shape of the distribution function of the project is assumed to be common knowledge. In our setting we assume then that the market agrees that the value of assets follows a geometric brownian motion

$$dV(t) = \mu V(t)\, dt + \sigma V(t)\, dz \tag{47}$$

with μ and σ constant drift and diffusion parameters and dz a Wiener process. This means that $V(T)$ is log-normally distributed. We also rule out estimation risk and model risk, assuming that the parameters of the process are common knowledge, and that the no-arbitrage condition holds. Then, both the volatility parameter σ and the market price of risk λ are assumed to be common knowledge, so that the drift of the assets is recovered from the usual no-arbitrage restriction $\mu = r + \lambda\sigma$ where r is the instantaneous risk-free rate, assumed to be constant.

This is the basic setting of a classical structural debt pricing model. Our model departs from that literature in two ways. First, information on the value of assets is assumed to arrive at discrete times $\{t_0, t_1, ..., t_N\}$, that is when the balance sheet reports are issued. Second, the value of assets is not directly observed by the market, but must be inferred from a garbled signal $s(t_k)$, $k = 0, 1, 2, ..., N$. Only at the final date T the value of the firm will be observed. Before that, the signal may be simply "noisy" due to imperfect observation, or it may even be distorted because of fraudolent behavior by firm's managers. In particular we focus on the latter possibility, referring the reader to the paper by Duffie and Lando (2001) for a model in which the signal observed at discrete times is noisy, but unbiased.

A comment is in order concerning the default rules assumed in this model. Here we do not explicitely analyse endogenous default strategies. The firm can go bankrupt only at the end of the business process or, in a simple extension of the model, at discrete times in which the value of the firm, notified through a signal, falls below a given level. Moreover, our model does not incorporate bankruptcy costs, which are an essential ingredient in the theory of strategic default (see Anderson and Sundaresan, 1996).

To give a more clear description of the idea and the main task of our approach, as well as its novelty, consider the possibility that a signal $s(t_k) > D$ be issued, even though the true state is $V(t_k) < D$, to hide a possible state of financial distress at the advantage of management careers, or even the entrepeneur private wealth (as in the Parmalat case). It may also happen that signal $s(t_k) < D$ be issued, while in the true state it is $V(t_k) > D$, for example to solicitate debt rescheduling. Our task is to evaluate the effect of these distortions on the value of corporate securities, that is equity and debt. The idea is that if this possibility of mis-representation is taken into account by the market, it should be priced in corporate securities. The natural effect would be that garbled information would cause securities reactions to news to be more sluggish than they would be under the assumption of perfect information. Furthermore, as in most cases the value of these securities is observed in liquid markets, it should be indeed possibile to gauge the degree of distortion perceived by the market, by simply comparing the observed changes in prices with those predicted under the full information model.

6.1 Default at maturity: Merton model with garbling

In the seminal paper by Merton (1974) default is assumed to be possible only at the end of the contract, when the value of assets $V(T)$ is observed. Defaultability of debt is then represented by the non-linear pay-off at maturity $D(T) = \min(D, V(T))$, meaning that if the value of the firm is not sufficient to cover repayment of the debt, the creditors will be allowed, only then and not before, to take over the firm at no cost. The value of corporate debt can be decomposed as $D(T) - \max(D - V(T), 0)$, that is a default-free bond and a short position in a put option, written on the asset for a strike equal to the nominal value of debt: it is this short position, also called default put option, that measures the default risk in the price. The call option with same underlying and strike $E(T) = \max(V(T) - D, 0)$ represents the value of equity. It may be verified that Modigliani-Miller theorem holds in this setting.

A clarifying note is in order concerning the notation. Merton rescaled all of the results in the model by the value of assets, leading to the definition of a key variable, called the quasi-debt-to-firm value ratio (or quasi-leverage) defined as

$$\delta(t) = \frac{\exp(-r(T-t)D)}{V(t)} \tag{48}$$

In our model we find more natural to rescale everything by the discounted value of debt (the quasi-value of debt), so that as the underlying asset (the value of the firm) we prefer to use $v(t) \equiv 1/\delta(t)$: all of the prices expressed in lower case characthers will be assumed to be rescaled in the same way.

In the Merton model information about the value of assets is assumed to be observed in continuous time, and it is used by the traders to set up hedging or trading strategies in continuous time, exploiting the buit-in option feature of equity and defaultable bonds. As described above, our assumption is that information is represented by a garbled signal that arrives at discrete times. Before describing the garbling structure of the model, let us notice that if this signal, say $s(t_k)$ were perfect, so that $s(t_k) = v(t_k)$ we could compute the value of equity and debt using the standard Black and Scholes formula

$$\begin{aligned} e(v(t_k), t_k) &\equiv \frac{E(v(t_k), t_k)}{\exp(-r(T-t_k))D} = v(t_k)N(d_1) - N(d_2) && (49) \\ d(v(t_k), t_k) &\equiv \frac{D(v(t_k), t_k)}{\exp(-r(T-t_k))D} = 1 + v(t_k)N(-d_1) - N(-d_2) && (50) \\ d_1 &= \frac{\ln(v(t_k)) + \sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Notice that the value of debt can also be represented as

$$d(v(t_k), t_k) = 1 - [-v(t_k)N(-d_1) + N(-d_2)] \tag{51}$$

emphasising the nature of credit risk as a short position in a put option. It may be also worth noting that this option can be rewritten as

$$d(v(t_k), t_k) = 1 - N(-d_2)\left[1 - v(t_k)\frac{N(-d_1)}{N(-d_2)}\right] \tag{52}$$

in which credit risk is decomposed into *default probability* ($DP = N(-d_2)$) and *loss given default* ($LGD = 1 - v(t_k)N(-d_1)/N(-d_2)$), the current credit risk jargon (Cossin and Pirrotte, 2000).

Let us now specify the garbling structure of the signal observed at time t_k. The structure is kept as simple as possible, so we only assume a bivariate outcome. We partition the positive real line into two sets $A_1 = (1, \infty)$ and $A_2 = [0, 1]$ and define two possible signals $s(t_k) \in A_1$ and $s(t_k) \in A_2$. Of course, only one of the two signals is observed, hopefully in state A_1, meaning that the current value of assets is sufficient to cover the discounted value of debt. As in the bivariate signal case, we assume to have the garbling matrix

$$\mathbf{\Pi} = \begin{bmatrix} 1-\pi_d & \pi_d \\ \pi_u & 1-\pi_u \end{bmatrix}$$

and we define the prior probability of observing $v(t_k) \in A_i$, $i = 1, 2$, as $p_k(A_i)$.

As for the structure of the signal inside the sets $A_1$and A_2, we follow the same logic as in Example 1 above. In particular, in absence of fraud we assume perfect information within the set where the signal is; to the contrary, no information within the set is assumed in case of fraud. The idea is that any possible mis-representation of the values that the signal may possibly take on in the sets A_1 or A_2 is of minor relevance to our pricing problem. So, if for example a signal $s \in A_1$ is released, truthfully revealing that $v(t_k) > 1$, we do not care if the signal is distorted towards other values in the same set. However, one may have very strong incentives to mis-represent the value of the firm if he observes $v(t_k) \leq 1$, meaning that the value of assets may not be sufficient to repay the debt, and that is very relevant to us.

Suppose a signal in set A_1 is observed: $s(t_k) = h \in A_1$. Absent fraud, we have $\varphi_{11}(v(t_k), s) = \delta(h-s)$, so: $\pi(s \mid v(t_k)) = (1-\pi_d)\delta(h-s)$, for $v(t_k) = h$ and $s(t_k) \in A_1$. In presence of fraud, we have $\varphi_{21}(v(t_k), s) = \varphi(s)$, so: $\pi(s \mid v(t_k)) = \pi(s) = \pi_u\varphi(s)$, for $v(t_k) \in A_2$ and $s(t_k) \in A_1$.

Following the procedure illustrated in the previous section we compute

$$\overline{e}(A_1, t_k) = e(h, t_k) \text{ and } \overline{e}(A_2, t_k) = \frac{\int_{A_2} e(y, t_k)\, p_k(y)\, dy}{p_k(A_2)} \tag{53}$$

$$\overline{d}(A_1, t_k) = d(h, t_k) \text{ and } \overline{d}(A_2, t_k) = \frac{\int_{A_2} d(y, t_k)\, p_k(y)\, dy}{p_k(A_2)} \tag{54}$$

Also, we have that

$$f(A_1 \mid h) = \frac{p_k(h)(1-\pi_d)}{p_k(h)(1-\pi_d) + p_k(A_2)\pi_u\varphi(s)} \tag{55}$$

We may finally write

$$\widehat{e}(v(t_k), t_k \mid h) = f(A_1 \mid h)\, e(h, t_k) + (1 - f(A_1 \mid h))\, \overline{e}(A_2, t_k) \tag{56}$$

The same holds for the value of debt, which gives

$$\widehat{d}(v(t_k), t_k \mid h) = f(A_1 \mid h)\, d(h, t_k) + (1 - f(A_1 \mid h))\, \overline{d}(A_2, t_k) \tag{57}$$

and finally for the value of the firm. Remember in fact that by Modigliani-Miller theorem we have $e(h, t_k) + d(h, t_k) = h$ and $\overline{e}(A_2, t_k) + \overline{d}(A_2, t_k) = \overline{v}(A_2, t_k)$ so that we may compute

$$\widehat{v}\left(v\left(t_k\right), t_k \mid h\right) = f\left(A_1 \mid h\right) h + \left(1 - f\left(A_1 \mid h\right)\right) \overline{v}\left(A_2, t_k\right) \tag{58}$$

Notice that, as both equity and debt are worth less in the bad state than in the good one, the effect of garbling is to prevent equity and debt from reacting completely to the announcement of a value $v\left(t_1\right) = h$. There is always a small probability that the good signal be deceptive, so that the worse scenario does actually take place. This possibility endowes the model with the usual peculiarity of raising the credit spread curve particularly in the short end, as in the Duffie and Lando approach.

The same analysis can be carried out assuming that a signal in the set A_2 is observed: $s\left(t_k\right) = l \in A_2$. In order to evaluate debt and equity, it is relevant to account for the possibility that one could have incentive to cheat and signal a value l even in the case $v\left(t_k\right) > 1$, if for example he wants to engage in a strategic debt service game in order to reschedule his obligations. The overall effect on equity and debt would then lead to

$$\widehat{v}\left(v\left(t_k\right), t_k \mid l\right) = f\left(A_2 \mid l\right) l + \left(1 - f\left(A_2 \mid l\right)\right) \overline{v}\left(A_1, t_k\right) \tag{59}$$

An important practical consequence of this model is that by looking at the response of equity and debt we may recover not only the probability update measuring the degree of confidence of the market in the signal released, but also the implied "average" value of the opposite signal that was not observed. So, if signal h is released, by observing the reaction of equity and debt we may recover both $f\left(A_1 \mid h\right)$ and the average value $\overline{v}\left(A_2, t_k\right)$ in such a way as to match the prices observed. If signal l is released instead, market prices enable to recover $f\left(A_2 \mid l\right)$ and the value $\overline{v}\left(A_1, t_k\right)$.

6.2 Covenants: Black and Cox model with garbling

An important extension of the Merton model, particularly consistent with the assumption that some imperfect signals of the value of the firm can be observed before the maturity of debt, is the possibility that default could occur before that date. Black and Cox (1976) were the first to amend the model in this direction. The idea is that default may occur before maturity if some covenant written on debt is triggered. The covenant is typically referred to the relative size of the value of the firm with respect to the amount of debt. Following the Black and Cox approach, the covenant in its simplest form is represented as the inequality

$$\widehat{v}\left(t_k\right) \leq \kappa \leq 1 \tag{60}$$

So, when the value of the firm is signalled to be too low with respect to the discounted value of debt, default is triggered. The presence of covenants of course reduces the default risk component of debt and the credit spreads. The reduction of risk depends on the value of parameter κ; the model can be proved to converge to the standard Merton model as κ gets close to zero. If the value of the firm were perfectly observed in continuous time, as assumed in the original model, the covenant would tendentially eliminate the risk of default as κ gets close to 1. Of course, given the parameter, the effect of default risk reduction also depends on the monitoring frequency and the information content of the signal. It is the latter factor that we are going to explore with our model.

A comment is in order concerning the differences with respect to the Merton model and the impact of our assumption of observing the signal at discrete times. If the covenant could be monitored in continuous time, the value of equity would be a call barrier option of the down-and-out type with zero rebate: that is, the option granted by the equity would cease to exist as soon as the default barrier were activated. In the case of continuous monitoring of the covenant the pricing formula for equity is readily available in the standard option pricing literature

$$\begin{aligned} e\left(v\left(t_k\right), t_k; \kappa\right) &= e\left(v\left(t_k\right), t_k\right) \\ &\quad - \left[v\left(t_k\right)\left(\frac{\kappa}{v\left(t_k\right)}\right)^{2\alpha} N\left(\xi\right) - \left(\frac{\kappa}{v\left(t_k\right)}\right)^{2\alpha-2} N\left(\xi - \sigma\sqrt{T-t_k}\right)\right] \\ \xi &= \frac{\ln\left(\kappa^2/v\left(t_k\right)\right) + \sigma^2\left(T-t\right)}{\sigma\sqrt{T-t}} \\ \alpha &= \frac{1}{2} + \frac{r}{\sigma^2} \end{aligned} \tag{61}$$

Our assumption is instead that the barrier could be monitored, through the signal, only at discrete dates $\{t_0, t_1, ..., t_N\}$. This makes equity a discrete (or partial) barrier option in which typically the barrier is observed at fixed intervals of time, say every quarter or every semester. A closed form solution to this pricing problem was found by Heynen and Kat (1996). However, evaluation involves the computation of joint normal distributions in dimension $N+1$ which is not available in closed form. For this reason, it may be useful to resort to approximations suggested in the literature. Broadie, Glasserman and Kou (1997) propose a strategy based on the displacement of the barrier in the formula above: so, denoting τ the time interval between monitoring dates, they suggest

$$\begin{aligned} e\left(v\left(t_k\right), t_k; \kappa, \tau\right) &\simeq e\left(v\left(t_k\right), t_k; \widetilde{\kappa}\right) \\ \widetilde{\kappa} &\equiv \kappa \exp\left(-0.5826\right)\sigma\sqrt{\tau} \end{aligned} \tag{62}$$

We now proceed to describe the signal structure. The model itself suggests a natural way to partition the support of the unobserved variable into two sets $A_1 = (\kappa, \infty)$ and $A_2 = [0, \kappa]$. The structure of the problem also suggests a particularly simple structure of the garbling matrix

$$\mathbf{\Pi} = \begin{bmatrix} 1 & 0 \\ \pi_u & 1 - \pi_u \end{bmatrix}$$

The first row of the matrix tells us that no signal will be released in A_2 if the true state is in A_1, because in that case default would be immediately triggered and the value of equity would tumble to zero[6].

Apart from the garbling matrix, the structure of the signal is the same as in the Merton model above. If a signal h in set A_1 is observed, we have $\pi(s \mid v(t_k)) = \delta(h-s)$, for $v(t_k) = h$ and $s(t_k) \in A_1$ (no fraud); we also have $\pi(s \mid v(t_k)) = \pi(s) = \pi_u \varphi(s)$, for $v(t_k) \in A_2$ and $s(t_k) \in A_1$ (fraud). We then compute:

$$\overline{e}(A_1, t_k) = e(h, t_k; \kappa, \tau) \text{ and } \overline{e}(A_2, t_k) = 0 \tag{63}$$

and

$$f(A_1 \mid h) = \frac{p_k(h)}{p_k(h) + p_k(A_2)\pi_u \varphi(s)} \tag{64}$$

The value of equity is then immediately recovered as

$$\widehat{e}((v(t_k), t_k; \kappa, \tau) \mid h) = f(A_1 \mid h)\, e(h, t_k; \kappa, \tau) \tag{65}$$

Let us now come to debt. First of all, notice that if the true state were in the default region A_2 the debt would be worth

$$d(v(t_k), t_k \mid v(t_k) \in A_2) = v(t_k) \tag{66}$$

As for the good states in set A_1, remember that by Modigliani Miller theorem we have

$$d(v(t_k), t_k \mid v(t_k) \in A_1) = v(t_k) - e(v(t_k), t_k; \kappa, \tau) \tag{67}$$

We may now compute

$$\begin{aligned}
\overline{d}(A_1, t_k) &= h - e(h, t_k; \kappa, \tau) \\
\overline{d}(A_2, t_k) &= \overline{v}(A_2, t_k) = \int_0^{\kappa} \frac{p_k(y)}{p_k(A_2)} y\, dy
\end{aligned}$$

and finally

$$\widehat{d}(v(t_k), t_k \mid h) = f(A_1 \mid h)[h - e(h, t_k; \kappa, \tau)] + (1 - f(A_1 \mid h))\, \overline{v}(A_2, t_k) \tag{68}$$

Notice that the posterior distribution can be extracted directly by observing the value of equity and comparing it with the value that would obtain if the information were true. The posterior can then be plugged into the debt valuation formula to recover the implied value $\overline{v}(A_2, t_k)$. If κ is very close to 1, this implied value has a straightforward interpretation, as it represents the average recovery

[6] This model implicitely assumes that covenants cannot be renegotiated. This rules out a downward bias in reporting in order to trigger a renegotiation of the debt contract.

rate that would be collected if the signal were deceiving and the true state of the firm were in the bad scenario represented by set A_2.

In this model the features that we have discovered in the standard Merton approach are highlighted. Again, equity is assumed to be sluggish to respond to news concerning the value of the firm. The credit spreads are higher than in the standard Black and Cox model even if the signal points to a value of the firm well above the default barrier: there is always a possibility that that signal may be deceiving and the true state of the firm is of financial distress.

Example 2 *An example of application can point out the meaning of the model and highlight the differences with respect to similar approaches that insist, like ours, on the unobservability of the value of the firm. Consider a signal which is observed in continuous time. We set*

$$\begin{aligned} ds(t) &= \mu_s s(t)\,dt + \sigma_1 s(t)\,dz_1 && (69) \\ dV(t) &= \mu(t)V(t)\,dt + \rho\sigma_2 V(t)\,dz_1 + \sigma_2\sqrt{1-\rho^2}V(t)\,dz_2 && (70) \end{aligned}$$

with ρ a parameter describing the instantaneous correlation between the true process and the signal. Assume that the information available is limited to the σ-algebra generated by the signal. Only the signal is observed in continuous time. This would lead directly to a continuous time model which would be the exact "garbled" version of the Black and Cox one. To gauge the simplicity of the model, assume further that the covenant structure of debt is such that $\kappa = 1$, that is default is triggered whenever the signal points to a value of the firm lower than the discounted value of debt ($v(t) \leq 1$). As we said before, this model would yield, under perfect information, $s(t) - e(s(t), t; 1) = 1$: corporate debt would be risk-free, because it would be repaid as soon as the value of the firm would reach the default barrier represented by its discounted value. In this model, though, the credit spread would be zero. In the model with garbling we would have instead

$$\begin{aligned} \widehat{e}(v(t), t; 1 \mid s(t) \in A_1) &= f(A_1 \mid s)\, e(s(t), t) && (71) \\ \widehat{d}(v(t), t \mid s(t) \in A_1) &= 1 - (1 - f(A_1 \mid s))\,[1 - \overline{v}(A_2, t)] && (72) \end{aligned}$$

and the credit spread would be

$$spread = -\ln\left[1 - (1 - f(A_1 \mid s(t)))\,[1 - \overline{v}(A_2, t)]\right] / (T - t) \qquad (73)$$

7 Some illustrative simulations

In order to highlight the impact of the "garbling effect" on the credit spreads, we provide here some simulations of the model introduced in Section 6.1, comparing our results with those obtained under the original Merton (1974) model. Actually, instead of carrying out a true simulation of the model, which would require assuming a specific value for all the parameters involved (garbling matrix included), we prefer to stick to a simple illustrative example, where only a few assumptions have to be made.

To make the example realistic, we use a typical set of data inspired to a firm endowed with a BBB rating, that is at the lower end of the "investment grade" scale. Following Moody's data, we assume a tipical leverage of 40%, so that we set $v(t) = 2.5$: the value of assets is two times and a half as high as nominal debt (good state: A_1). As for the volatility figure, some calibration exercise (Huang and Huang, 2003, Cherubini and Della Lunga, 2001) suggests a range between 20% and 25%. We then set in the base model a volatility of assets equal to 25%. As for the bad state (A_2), we choose a value of assets $v(t) = 0.4$, which may correspond to the market assessment of the average recovery rate. Throughout the exercise we set $f(A_2 \mid s \in A_1) = 0.005$. This means that market assumes that in 5 cases out of 1000 there may be mis-reporting hiding a state of financial crisis of the firm.

First of all, in the base scenario described above we compute the credit spread curve induced by garbling. The result is reported in Figure 6. The results suggest two comments. The first is that the effect of garbling is very relevant for short maturities, that is where structural models traditionally fail. Second, garbling induces a specific shape of the credit spread term structure: a positive spread for short maturities is followed by a decrease down to a minimum, after which it increases again getting closer and closer to the standard model (curiously enough, this shape is typically found in the money market segment). This specific shape may be considered as a possible indicator of a garbling effect. Indeed, the reason for this shape has to do with the fact that accounting for mis-reporting causes the price of debt to decrease by a fixed percentage amount (0,3% in our example) across all of the maturities: computing the corresponding interest rates generates this negative-hump shape; moreover, the difference between the two curves - model with and model without garbling - shows a monotone decreasing pattern as maturity increases.

We then report the relative impact of garbling for different levels of leverage. The results are reported in Figure 7. The figure shows that the effect of garbling on the credit spread is less relevant for highly leveraged firms. In cases in which the value of assets is low, the credit spread already takes into account the possibility of default. It is in cases in which the value of assets is very high with respect to debt that the possibility of mis-reporting may make the difference. As the value of assets grows higher and higher the credit spread tends to an asymptotic value, which in our case is around 35 basis points.

About the same asymptotic behavior emerges if we allow asset volatility to vary. In this case, the effect of garbling is less relevant for high volatility projects, for which the credit spreads in the Merton model allow for a relevant default probability. However, it is well known that such high volatility levels are typically not consistent with historical default rates. As we said before, calibrations in the literature suggest a level hardly above 20% as a reasonable estimate. It is exactly in this volatility range that the credit spread in the model with garbling reaches a flat level, which in our case is around 30 basis points for the one year maturity. The results are reported in the Figure 8.

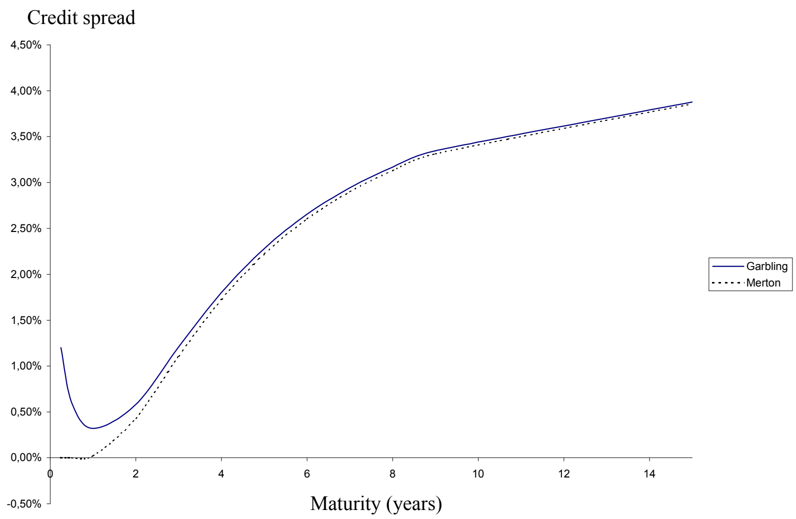

Figure 6: The term structure of credit spreads

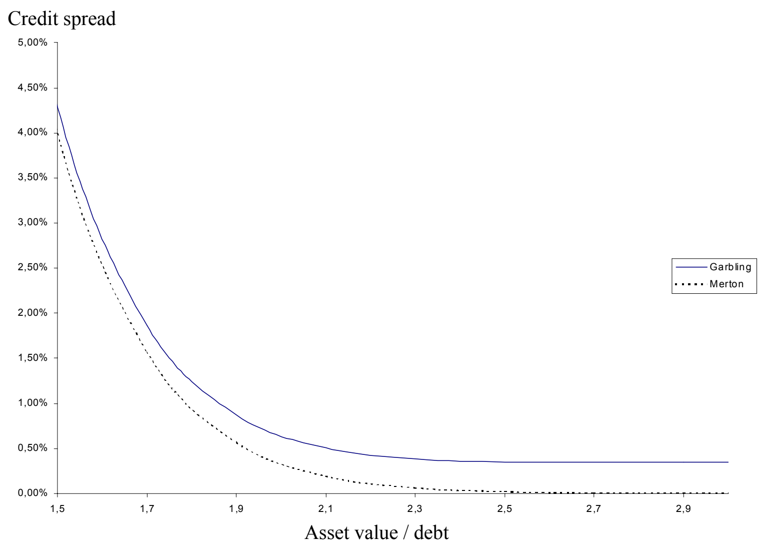

Figure 7: Credit spreads and leverage

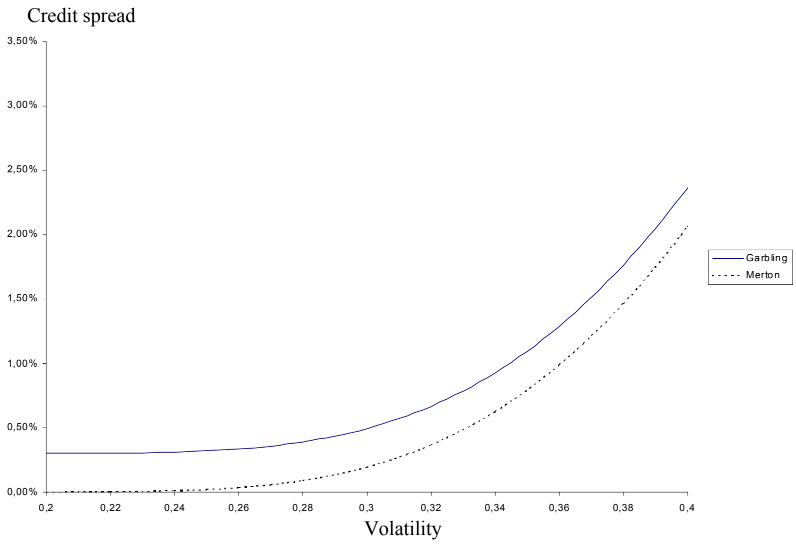

Figure 8: Credit spreads and asset volatility

To conclude, Figure 9 shows the impact of the garbling intensity on the 1 year credit spread. The posterior probability of being in the default state, in spite of favorable reporting, is allowed to range from 1 to 10 out of 1000. The credit spread increases linearly from about 10 to more than 60 basis points, in front of a credit spread of 2 basis points in the standard Merton model.

8 Concluding remarks

Let us summarize here our main contributions.

First, we provide a new method for modelling accounting distortions and their impact on the value of corporate securities. In particular, our two-tier signal is a very flexible device, able to account for both small noise (say estimate errors) and large mis-representations (say deliberate fraud).

Such a methodology is then applied to standard structural pricing models, following Merton's approach. It turns out that accounting distortions are a relevant factor in pricing corporate securities: indeed, they are able to explain why credit spreads are actually larger than implied by traditional structural models, particularly on short maturities. Some (preliminary) simulations also show that such a "garbling" effect is stronger for "safer" firms, namely those with lower leverage and asset volatility: in these cases, allowing for the possibility of accounting fraud makes the difference with more traditional credit risk models.

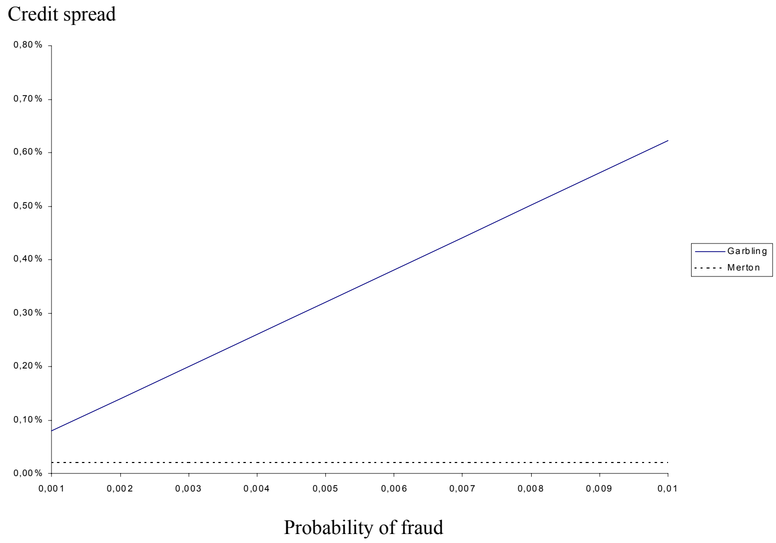

Figure 9: Credit spreads and information distortion

The methodology presented here may also be applied to evaluate the quantitative impact of any factor affecting the reliability of accounting information, like corporate governance and regulation: such goal would require a fully developed set of simulations (which were omitted here for space considerations). Last but not least, the model may be used to infer from market data the confidence of market participants in the information disclosed by firms and by security analists. We leave such goals for future research.

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