

# CALLABLE RISKY PERPETUAL DEBT: OPTIONS, PRICING AND BANKRUPTCY IMPLICATIONS

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ABSTRACT. Issuances of *perpetual* risky debt are often motivated by capital requirements for financial institutions. However, observed market practice indicates that actual maturity equals first possible call date. We analyze callable risky perpetual debt including an initial protection period before the debt may be called. To this end we develop European barrier option pricing formulas in a Black and Cox (1976) environment.

The total market value of debt including the call option is expressed as a portfolio of barrier options and perpetual debt with a time dependent barrier. We analyze how the issuer's optimal bankruptcy decision is affected by the existence of the call option using closed-form approximations. In accordance with intuition, our model quantifies the increased coupon and the decreased bankruptcy level caused by the embedded option. We show that the option will be exercised even at fairly low asset levels at the time of expiry.

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*Date:* This version: January 5, 2006.

*Key words and phrases.* JEL classifications G13, G32, G33.

We want to thank Petter Bjerksund, B.Espen Eckbo, Trond Døskeland, Hans K. Hvide, Thore Johnsen, Elyès Jouini, Kristian R. Miltersen, Tommy Stamland and Gunnar Stensland. Earlier versions of this paper has been presented at FIBE 2005, NHH Skinance 2005 (Hemsedal) and at internal seminars at the Norwegian School of Economics and Business Administration.

## 1. INTRODUCTION

Perpetual debt securities seldom turn out to be particularly long-lived - in spite of their *ex ante* infinite horizon. The contractual horizon gives the securities a, using regulatory language, *permanence*, which is crucial when banks and other financial institutions are allowed to include them as regulatory required risk capital. However, the contracting parties, the issuing institution and the investors in the securities, typically value financing flexibility and may thus prefer a more tractable finite horizon. In the capital markets these apparently conflicting objectives are resolved by embedding such perpetual securities, almost without exceptions, with an issuer's call-option, facilitating a finite realized horizon.

Our overall objective is to value perpetual debt securities including this option and analyze its impact on optimal terms and conditions between debt- and equityholders.

We follow the approach by Black and Cox (1976) and Leland (1994), including symmetric information, efficient market assumptions, and that the market value of the issuing company's assets follows a geometric Brownian motion. In this setup no cash is paid out from the company and all debt coupons are paid directly by the equityholders. For a given capital structure and an infinite horizon debt contract, there exists a constant optimal market value of assets where it is optimal for the equityholders to stop paying coupons and let the company go bankrupt. After introducing a finitely lived option on the debt, this bankruptcy level is no longer independent of time to expiration of the option. The bankruptcy level after expiration of the option equals the constant Black and Cox (1976)-level.

One could consider the situation where third parties trade options on the issuing company's debt. Naturally, the existence of such options would neither influence the pricing of the debt at issue or in the marketplace nor the issuing company's optimal choice of bankruptcy level. However, we consider the situation where the issuer's call option is an integrated part of the debt contract. That is, the option is written by the debtholders in favor of the equityholders. We refer to such a call option as an *embedded option*. The existence of the option will influence both the issue-at-par coupon on the debt and the issuer's bankruptcy considerations before the option's expiration date. Intuition suggests that the coupon is increased to compensate for the embedded option, whereas the bankruptcy level is decreased due to the option value - both compared to the case without an option.

We show in section 3 that the market value of infinite horizon debt is not lognormally distributed and this fact represents a challenge for the valuation of options on such instruments. The standard Black and

Scholes (1973) and Merton (1973) option pricing formulas are thus not directly applicable.

The time 0 market value of perpetual debt according to Black and Cox (1976) can be interpreted as a risk-free value of an infinite stream of coupons, from which the market value of the debtholder's net loss in case of bankruptcy is subtracted. The market value of the debtholder's net loss is equivalent to the market value of the equityholders' default-option in a limited liability company. The market value of this net loss has the required lognormal properties and can be used as a modified underlying asset replacing the market value of infinite horizon debt itself. By this reformulation the standard Black-Scholes-Merton formulas can be applied using a time 0 market value of the modified underlying asset, and a modified exercise price and volatility.

We develop pricing formulas for both plain vanilla European options and down-and-out barrier European options on infinite horizon continuous coupon paying debt. Down-and-out barrier options are relevant since the debt options may only be exercised at the future time  $T$  if the issuing company has not gone bankrupt. The asset-level which defines optimal bankruptcy is thus the barrier used in the barrier option formulas.

For analytical tractability we assume that the time dependent barrier is an exponential function. This is a straightforward way to model a time dependent barrier and a natural first attempt, but still an arbitrary choice. Our numerical examples show that our model approach yields fairly good results and confirms the use of the analytical approximation.

### 1.1. Economic interpretation and insights from our analysis.

Our valuation formulas are fairly technical but contain important economic insights. In our application of the barrier option formulas on debt with embedded options, we want in particular to discuss the impact on debt payoff and optimal bankruptcy decisions.

#### 1.1.1. *The payoff to debtholders at expiration of the embedded option.*

The payoff to debtholders when the option expires is illustrated in figure 1<sup>1</sup>. The payoff to debtholders is shown as a function of asset value,  $A_T$ , for the two alternative debt structures assuming that the absolute priority rule is followed. The leftmost part shows that in bankruptcy, debtholders receive all assets as payoff, indicated by the 45-degree line. Beyond the bankruptcy asset level, the thicker line indicates the payoff to debt with embedded option whilst the lower line represents payoff to

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<sup>1</sup>This and the next graphical presentations use the same base case parameters in Table 1 in section 5 of the paper: Time 0 asset level  $A_0 = 100$ , par value of debt  $D_0 = 70$ , expiration date of option  $T = 10$  years, volatility of assets  $\sigma = 0.20$  and riskfree interest rate  $r = 5\%$ . This implies coupon rates of 5.312 % for perpetual debt without option and 5.526 % for the equivalent with embedded option

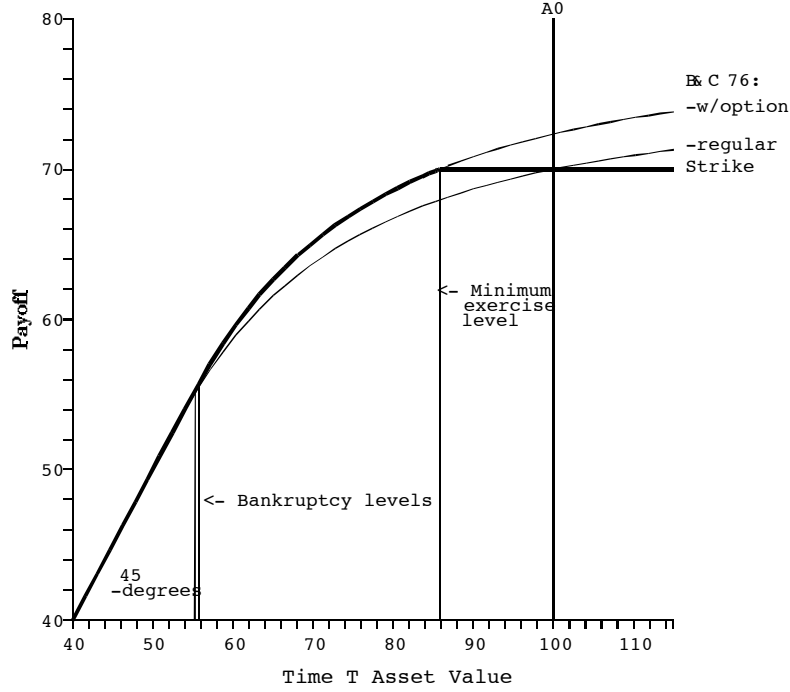


FIGURE 1. Payoff from perpetual debt with and without embedded option at time  $T$  as a function of asset level  $A_T$ . See Table 1 for parameter values.

regular perpetual debt. The bankruptcy levels for these structures are different due to the difference in coupon-levels. At time  $T$  the option does not impact optimal bankruptcy level anymore and it is only the higher coupon that causes a higher optimal long-term bankruptcy asset level.

The more interesting issue is for which levels of  $A_T$  the option is rationally exercised. Perpetual debt with a higher coupon will always be more valuable than debt with a lower coupon. In our model, uncertainty is only included in the asset process  $A_t$ . By not exercising the option, the issuer is left with regular perpetual debt with a higher coupon than identical debt issued at time  $T$ . The explanation is that the coupons are fixed and that an element of the historical coupon is a compensation for the embedded, but expired at time  $T$ , option. The issuer is therefore willing to exercise the option at lower levels of  $A_T$  relative to the time 0 value of  $A_0$  to avoid the relatively high coupon. In figure 1, where the exercise level is par (70), the indifference asset level is appr. 86, compared to the time 0 asset level of 100.<sup>2</sup> At this

<sup>2</sup>Our analysis provide the calibrated coupon level  $c^*$  to ensure issue-at-par. The indifference level of  $A_T$  is found by using equation (3) setting  $D(A_T)$  equal the exercise level (par) and solve for  $A_T$ .

indifference level, the coupon for newly issued debt will exactly equal the original coupon for debt with option. This is valid irrespective of potential refinancing considerations which are in any case beyond our model.<sup>3</sup>

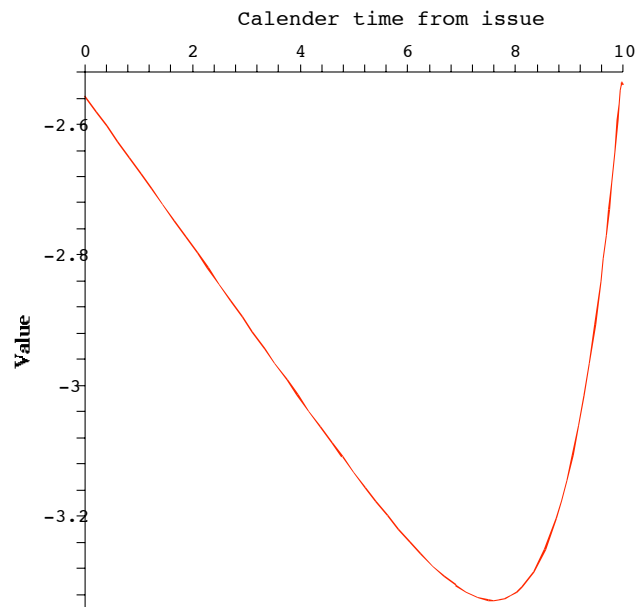


FIGURE 2. Value of the debtholders' short position barrier call option as a function of calendar time when call option expires at time  $T = 10$ . See Table 1 for parameter values.

1.1.2. *The 'smiling' bankruptcy-level.* Our analysis combines the infinite debt contract with an embedded finite option. The classical infinite setting from Black and Cox (1976) leads to a constant bankruptcy level. The market value of a finite option depends on time to expiration. After introducing a finite embedded option, the optimal bankruptcy level therefore becomes time-dependent. Basic intuition tells that the existence of an option with positive value will lower the optimal bankruptcy asset level. The value of such options is also in itself dependent on the bankruptcy risk of the issuer. To model options with inherent bankruptcy risk, it is natural to use barrier options. We have illustrated this in figure 2, again using the same parameters as above. This figure

<sup>3</sup>Mauer (1993) also claims that the value of a call-option is the value of the opportunity to repurchase a non-callable bond with the same coupon and principal. This approach is intuitive at the time when the option expires, but in a case without any exercise premium on the option, such a comparison is impossible at time of issuance simply because the coupon will incorporate the option-premium.

shows how the combined market value of all option-elements taken from expression (24) in section 4 varies over time. The graph is shown from the debtholders' side and the total market value is therefore negative. The explanation of the 'smile'-shape is that the market value of barrier options do not vary monotonically with time like regular options due to the inherent bankruptcy risk.

**1.2. Literature overview.** The related literature may broadly be separated into research on debt-based derivatives on one hand and on perpetual debt on the other hand. Kish and Livingston (1992) test for determinants of calls included in corporate bond contracts. Their findings are that the interest rate level, agency costs and bond maturity significantly affect whether a bond comes with an embedded call option. Sarkar (2001) is the closest precedent to our paper in his focus on callable perpetual bonds modelled in the tradition of Leland (1994). The main difference is that the calls are assumed to be *American* and immediately exercisable, i.e., without a protection period, and a main part of the paper thus deals with the decision when to exercise the call. The paper does neither include analytical valuation of the options nor optimal coupon- or bankruptcy levels.

Jarrow and Turnbull (1995) model various derivatives on fixed maturity debt securities, but do not include any analysis of the impact on endogenous bankruptcy decisions. Acharya and Carpenter (2002) develop valuation formulas for callable defaultable bonds with stochastic interest rate and asset value. Through decomposing the bonds into a riskfree bond less two options, they explore how the call option impacts optimal default in line with our results. They analyze fixed maturity bonds and the hedging aspects of callable bonds through the options' impact on bond duration, but without developing exact valuation formulas for the specific bonds. Toft and Prucyk (1997) develop modified equity option expressions based on Leland (1994) for leveraged equity and various capital structure and bankruptcy assumptions. The infinite horizon property of equity makes it comparable to our work although the specific issues related to embedded options on debt are not handled directly. Rubinstein (1983) is related to our approach with the use of a modified asset process, in his term, a "displaced diffusion process", to modify the standard Black-Scholes approach.

In the perpetual debt pricing tradition, starting with Black and Cox (1976), our research is related to the paper by Emanuel (1983) which develops a valuation of perpetual preferred stock, based on the option-methodology of Black-Scholes. Preferred stock can be viewed as perpetual debt for analytical purposes. Emanuel's analysis allows unpaid dividends to accumulate as arrearage due to the junior position of the claim, which is relevant for financial institutions, but beyond the scope of the current paper. He does not cover options on preferred stock as

such. Sarkar and Hong (2004) extends Sarkar (2001) and analyze the impact from callability on the duration of perpetual bonds and find that a call reduces the optimal bankruptcy level and thus extends the duration of a bond, similarly to our intuition and results.

**1.3. Outline of the paper.** Our main contributions are to develop useful options and barrier option formulas for perpetual debt contracts. In doing this, we handle both the lack of lognormal distribution of market values of debt and the finite option expiration included in an infinite security. We thus expand the results of Black and Cox (1976) to integrate an issuer's call option into the pricing, setting of coupons and defining the optimal bankruptcy level for a given capital structure. Our final contribution is a complete valuation expression for callable perpetual continuous coupon paying debt which then is exemplified through numerical examples. The set of formulas form a basis for improved understanding of the pricing of such securities and their impact on the optimal bankruptcy level of the issuing company.

The structure of the paper is as follows: In section 2 we present the model and the basic results. In section 3 the option formulas are developed, section 4 contains the complete expressions for perpetual debt with embedded options. Section 5 covers the numerical examples and section 6 concludes the paper. Supporting technical derivations and results are enclosed in three appendices.

## 2. THE MODEL AND BASIC RESULTS

We consider the standard Black-Scholes-Merton economy and impose the usual perfect market assumptions:

- All assets are infinitely separable and continuously tradeable.
- No taxes, transaction cost, bankruptcy costs, agency costs or short-sale restrictions.
- There exists a known constant riskless rate of return  $r$ .

We study a limited liability company with financial assets and a capital structure consisting of two claims, infinite horizon debt and common equity. We assume that the market value of the portfolio of assets of the firm at time  $t$ , denoted by  $\hat{A}_t$ , is given by the stochastic differential equation

$$(1) \quad d\hat{A}_t = r\hat{A}_t dt + \sigma\hat{A}_t dW_t,$$

under the equivalent martingale measure, where  $W_t$  is a standard Brownian motion and the time 0 market value  $\hat{A}_0 = A$ , a given constant. The constant parameter  $\sigma$  is interpretable as the volatility of the portfolio of assets. We assume no payouts to any claimholder, and thus that the coupons on debt are paid directly by the equityholders and not from the company's assets.

In this setting there is a level of  $A_t$  where it is optimal for the equityholders to stop paying debt coupons and declare bankruptcy. In the classic case this level is independent of time. Our initial exercise is to price a finitely lived option on infinite horizon debt. Due to the finite horizon of the embedded option the optimal bankruptcy level depends on remaining time to expiration of the option. In order to capture this aspect we introduce an increasing bankruptcy asset level  $B_t$ , modelled by  $B_t = Be^{\gamma t}$ , for a given time 0 level  $B$  and a constant  $\gamma$ . The time of bankruptcy is given by the stopping time  $\tau$  defined as

$$\tau = \inf\{t \geq 0, \hat{A}_t = B_t\}$$

where  $\hat{A}_t$  is given in expression (1).

By modifying the asset process this stopping time can equivalently be expressed as

$$\tau = \inf\{t \geq 0, A_t = B\},$$

where  $A_t$  is

$$(2) \quad dA_t = (r - \gamma)A_t dt + \sigma A_t dW_t,$$

This is the process in equation (1) with a negative drift adjustment of  $\gamma$ . Although  $\gamma$  determines the curvature on the bankruptcy level, it can formally be interpreted as a constant dividend yield on  $A_t$ . Again formally, this transformation allows us to work in the simpler setting of a constant bankruptcy level  $B$ , although no economic fundamentals have been changed. In section 5, we numerically compare our analytical approximation with the actual optimal barrier.

It is shown by Black and Cox (1976) and Leland (1994) that the time 0 market value of infinite horizon debt with continuous constant coupon payment is

$$(3) \quad D(A) = \frac{cD}{r} - \left(\frac{cD}{r} - B\right)\left(\frac{A}{B}\right)^{-\beta},$$

where  $c$  is the constant coupon rate,  $D$  is the par value of the debt-claim and  $cD$  is the continuous coupon payment rate. The ratio  $(\frac{A}{B})^{-\beta}$  can be interpreted as the *current market value* of one monetary unit paid upon bankruptcy, i.e., when the process  $A_t$  hits the bankruptcy level  $B$ .

Here  $\beta$  is the positive<sup>4</sup> solution to the quadratic equation

$$(4) \quad -\frac{1}{2}\sigma^2\beta(\beta + 1) + \beta(r - \gamma) + r = 0$$

given by

$$(5) \quad \beta = \frac{r - \gamma - \frac{1}{2}\sigma^2 + \sqrt{(r - \gamma - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$

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<sup>4</sup>Mjøs and Persson (2005) motivate the choice of the positive solution for  $\beta$ .



In the special case where  $\gamma = 0$  and thus the bankruptcy level  $B$  is constant, we denote  $\beta$  by  $\alpha$ , and calculate

$$\alpha = \frac{2r}{\sigma^2}.$$

The expression for the market value of debt carries a nice intuition. Observe that  $\frac{cD}{r}$  is the current market value of infinite horizon default-free debt. Upon bankruptcy the debt investor loses infinite coupon payments which at the time of bankruptcy has market value  $\frac{cD}{r}$ . On the other hand the debt investor receives the remaining assets which equals  $B$ . We can therefore interpret  $(\frac{cD}{r} - B)$  as the debt issuer's *net loss* upon bankruptcy. The time 0 market value of the net loss  $(\frac{cD}{r} - B)(\frac{A}{B})^{-\beta}$  therefore represents the reduction of the time 0 total market value of debt due to default risk.

Regular perpetual risky debt can be characterized by the fundamental specification of the net loss and the seniority of the claim upon bankruptcy.

The value of equity as the residual claim on the assets is in this setting determined by

$$(6) \quad E(A) = A - D(A) = A - \frac{cD}{r} + (\frac{cD}{r} - B)(\frac{A}{B})^{-\beta}$$

It is shown by Black and Cox (1976) in their original case without embedded options that the optimal bankruptcy level for a given capital structure  $(E, D)$  from the perspective of the equityholders (found by differentiating expression (6) with respect to  $B$ ) is  $\frac{\beta}{\beta+1} \frac{cD}{r}$ . For future use we denote this level by  $\bar{A}$  for the special case where  $\gamma = 0$ , so

$$(7) \quad \bar{A} = \frac{\alpha}{\alpha + 1} \frac{cD}{r}.$$

### 3. OPTION FORMULAS FOR FINITE OPTIONS ON INFINITE DEBT CLAIMS

We develop formulas for options and barrier-options by the standard approach for barrier options from financial economics. As shown below, the market value of the underlying asset, an infinite horizon debt contract, is not lognormally distributed. We solve this problem by reinterpreting the underlying asset.

Our formulas are developed for any general, but not necessarily optimal, bankruptcy barrier, which make them in our setting applicable both to option contracts between third parties as well as to options embedded in debt contracts.

We denote by  $T$  the exercise-date for these *European*-type options.

**3.1. The debt dynamics.** The underlying asset of all option contracts is the infinite horizon debt contract of Black and Cox (1976).

However, we reformulate (compared to expression (3)) the market value at time  $t$  of this contract as follows:

$$(8) \quad D_t = \frac{cD}{r} - JF_t,$$

where

$$F_t = \left(\frac{A_t}{\bar{A}}\right)^{-\alpha}.$$

Here  $J$  equals the net loss as discussed above. Our option pricing formulas may readily be used for other debt contracts with different net loss (e.g. as a result of different seniority) just by alternative specifications of  $J$ . Observe that in expression (8) we use the parameters  $\alpha$  and  $\bar{A}$  in place of  $\beta$  and  $B$  because no options are present in the underlying asset after time  $T$ .

An application of Itô's lemma on  $F$  using expression (2) yields

$$(9) \quad dF_t = (r + \alpha\gamma)F_t dt - \alpha\sigma F_t dW_t,$$

which we recognize as a geometric Brownian motion. It has drift parameter  $r + \alpha\gamma$  and volatility parameter  $-\alpha\sigma$ . Furthermore,  $F_t$  is a function of  $A_t$ , and can therefore also be interpreted as a tradable asset in the time-period  $[0, T]$ .

Another application of Itô's lemma on expression (8) shows that

$$(10) \quad \frac{dD_t}{D_t} = \left(r + \alpha\gamma - \frac{cD}{D_t} \left(1 + \frac{2\gamma}{\sigma^2}\right)\right) dt + \frac{2}{\sigma} \left(\frac{cD}{D_t} - r\right) dW_t$$

which is not a geometric Brownian motion (the right-hand side depends on  $D_t$ ), and is thus not lognormally distributed. Options on  $D_t$  can therefore not be valued using standard option pricing formulas.

**3.2. Plain Vanilla call and put options.** Compared to the payoff from regular options, the payoff at maturity  $T$  from options and barrier-options on perpetual debt are non-linear, not piecewise linear, functions of  $A_T$ . The payoff at maturity  $T$  is illustrated in figure 3

Denote the time  $T$  cash flow of a European *call* option on  $D_T$  with exercise price  $K$  and expiration at time  $T$  by  $C_T^D(A_T, K)$ . Therefore from expression (8)

$$(11) \quad \begin{aligned} C_T^D(A_T, K) &= (D_T - K)^+ = \left(\frac{cD}{r} - JF_T - K\right)^+ \\ &= J(X - F_T)^+ = JP_T^F(F_T, X), \end{aligned}$$

where the *modified exercise price* is

$$X = \frac{\frac{cD}{r} - K}{J}.$$

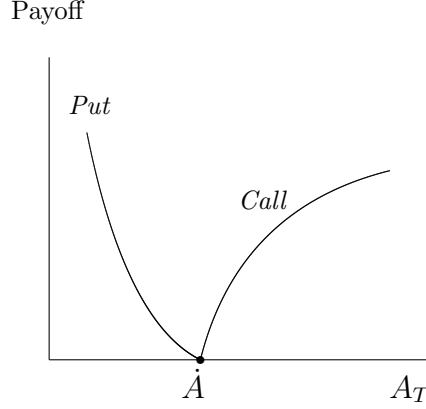


FIGURE 3. The payoff at maturity for plain vanilla put and call options on infinite coupon paying debt as functions of the time  $T$  market value of the firm  $A_T$ .

Similarly, the time  $T$  cash flow of a European *put* option on  $D_T$  with exercise price  $K$  and expiration at time  $T$  is

$$(12) \quad \begin{aligned} P_T^D(A_T, K) &= (K - D_T)^+ = \left(K - \frac{cD}{r} + JF_T\right)^+ \\ &= J(F_T - X)^+ = JC_T^F(F_T, X). \end{aligned}$$

We have shown <sup>5</sup> that one call option on debt with exercise price  $K$  is equivalent to  $J$  put options on  $F_T$  with a modified exercise price  $X$ . Similarly, one put option on debt with exercise price  $K$  is equivalent to  $J$  call options on  $F_T$  with a modified exercise price  $X$ .

These relationships are fundamental to the development of all option and barrier option formulas in this paper. These relationships allow us to use standard option approach on the debt-options and barrier options we consider and as such they are fundamental to the results.

In general our formulas will depend on the 10 parameters

$$\sigma, r, K, T, B, \gamma, \bar{A}, A, c, D.$$

For notational simplicity we write the expressions as functions of  $A$  and  $K$  only. Option pricing formulas follow in the propositions below.

**Proposition 1.** *The time zero market prices of European put and call options on infinite horizon continuous coupon paying debt claims as described above are*

$$(13) \quad \begin{aligned} P_0^D(A, K) &= JC_0^F(F_0, X) \\ &= J\left(\frac{A}{\bar{A}}\right)^{-\alpha} e^{\alpha\gamma T} N(d_1) - JXe^{-rT} N(d_2) \end{aligned}$$

<sup>5</sup>A similar call/put-relationship was also pointed out by Sarkar (2001)(page 510).

and

$$(14) \quad \begin{aligned} C_0^D(A, K) &= JP_0^F(F_0, X) \\ &= \left(\frac{cD}{r} - K\right)e^{-rT}N(-d_2) - JF_0N(-d_1), \end{aligned}$$

where

$$d_1 = \frac{\ln(\frac{\bar{A}}{A}) - \frac{1}{\alpha}(\ln X) + (r + \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2 = d_1 - \frac{2r}{\sigma}\sqrt{T}.$$

*Proof.* We have shown how one call [put] option on  $D_T$  equivalently can be seen as  $J$  put [call] options on  $F_T$  with a modified exercise price. Under no-arbitrage assumptions these options must have (pairwise) the same market value at any point in time before expiration. Options on  $F_T$  can immediately be calculated by the Black-Scholes-Merton formulas including constant dividend yield, by using  $F_0 = (\frac{A}{\bar{A}})^{-\alpha}e^{\alpha\gamma T}$  as the time 0 market value of the underlying asset,  $|\alpha\sigma| = \alpha\sigma$  as the volatility parameter<sup>6</sup> and  $X$  as the exercise price.  $\square$

**3.3. Down-and-out put and call options.** The time  $T$  cashflows of down-and-out put and call options on infinite debt-claims with barrier  $B$  for the asset-process  $A_t$  and exercise price  $K$  are

$$(15) \quad P_T = \left(K - \frac{cD}{r} + J\left(\frac{A}{\bar{A}}\right)^{-\alpha}\right)1\{m_T^A > B\},$$

and

$$(16) \quad C_T = \left(\frac{cD}{r} - J\left(\frac{A}{\bar{A}}\right)^{-\alpha}\right)1\{m_T^A > B\},$$

where  $1\{\cdot\}$  represents the usual indicator function and  $m_T^A = \min\{A_t; 0 \leq t \leq T\}$ .

From the expression for the payoffs of the plain vanilla call and put, (12) and (11), we see that the following value of  $A$ :

$$(17) \quad \dot{A} = \left(\frac{J}{\frac{cD}{r} - K}\right)^{\frac{1}{\alpha}} \bar{A} = \left(\frac{1}{X}\right)^{\frac{1}{\alpha}} \bar{A}$$

produces payoffs of zero both for the plain vanilla put and call options.

The payoff at maturity from barrier options are not only dependent on the asset level  $A_T$ , but also the (bankruptcy) barrier  $B$ , because all payoff is lost for asset levels below  $B$ . We define barriers  $B < \dot{A}$  as 'low' barriers and barriers  $B > \dot{A}$  as 'high' barriers and analyze the high- and low-cases separately.

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<sup>6</sup>Option prices on assets with negative volatility, as  $F_t$ , are, in this setting, calculated by inserting the absolute value of the volatility parameter into the option pricing formula, see e.g., the recent article by Aase (2004).

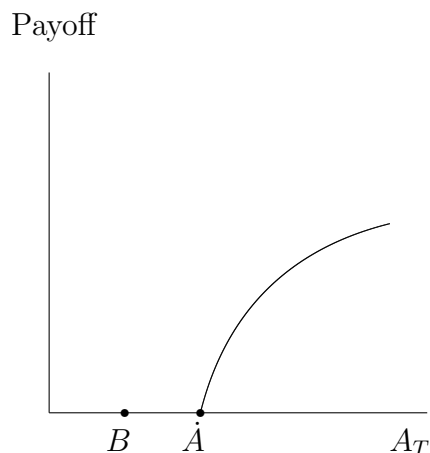


FIGURE 4. The payoff at maturity for a barrier *call* option with 'low barrier' on infinite coupon paying debt as function value of the firm  $A_T$  and the bankruptcy asset level,  $B$ .

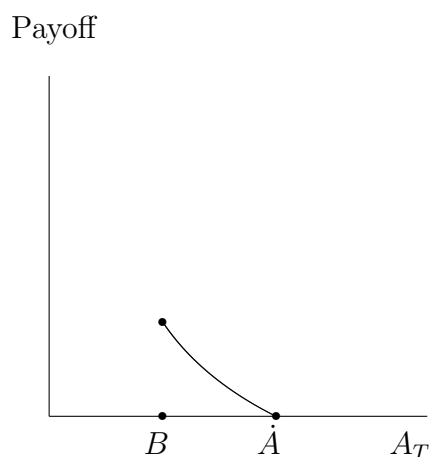


FIGURE 5. The payoff at maturity for a barrier *put* option with 'low barrier' on infinite coupon paying debt as function value of the firm  $A_T$  and the bankruptcy asset level,  $B$ .

**3.4. Case 1: Down-and-out options with 'low' barriers.** We consider initially the case where  $B < A$  and the payoffs from the options are as shown in figure 4 and 5.

**Proposition 2.** *The time zero market values of the described barrier put and call options with barrier  $B$  and exercise price  $K$  are, respectively*

$$(18) \quad P_L^{do}(A, K) = M_0(A, K) - \left(\frac{B}{A}\right)^{\left(\frac{2(r-\gamma)}{\sigma^2}-1\right)} M_0\left(\frac{B^2}{A}, K\right),$$

and

$$(19) \quad C_L^{do}(A, K) = C_0^D(A, K) - \left(\frac{B}{A}\right)^{\left(\frac{2(r-\gamma)}{\sigma^2}-1\right)} C_0^D\left(\frac{B^2}{A}, K\right),$$

where

$$M_0(S, K) = P_0^D(S, K) - \left(J\left(\frac{S}{A}\right)\right)^{-\alpha} e^{\alpha\gamma T} N(f_1) - \left(\frac{cD}{r} - K\right) e^{-rT} N(f_2),$$

$$f_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{B}{A}\right) + \left(r + \gamma + \frac{1}{2}\sigma^2\right)T\right),$$

and

$$f_2 = f_1 - \frac{2r}{\sigma}\sqrt{T}.$$

*Proof.* We follow the general approach by Björk (2004) for pricing down-and-out contracts in this setting. The first exercise is to calculate the time 0 market value of the payoffs 'chopped-off' at the lower barrier. In the case of the call option the time  $T$  payoff of the 'chopped-off' claim is

$$(D(A_T) - K)^+ 1\{A_T > B\}.$$

In the case where  $B < \dot{A}$ ,

$$(D(A_T) - K)^+ 1\{A_T > B\} = (D(A_T) - K)^+,$$

i.e., the payoff from the 'chopped-off' claim equals the payoff from the plain vanilla call since the barrier is in a region of  $A_t$  where this option does not have any payoff anyway. The market value of this is given in expression (14).

In the case of the put option we must calculate the time 0 market value of the 'chopped-off' claim with the pay-off

$$(K - D(A_T))^+ 1\{A_T > B\}.$$

First observe that

$$(K - D(A_T))^+ 1\{A_T > B\} = (K - D(A_T))^+ - (K_1 - D(A_T))^+ - K_2 1\{A_T \leq B\},$$

i.e., as a difference between two put options from which a constant  $K_2$  is subtracted for values of  $A_T$  less than  $B$ . Here  $K_1$  is a modified exercise price calculated as follows: We need the second put option to have zero payoff for values of  $A_T > B$ , and we therefore choose the exercise price, denoted by  $K_1$ , so that  $\dot{A} = B$ . From expression (17) this is

$$K_1 = \frac{cD}{r} - J\left(\frac{B}{A}\right)^{-\alpha}.$$

i.e. exactly when the process  $A_t$  hits the barrier. The constant  $K_2$  represents the net difference in the payoff of a long position in the first

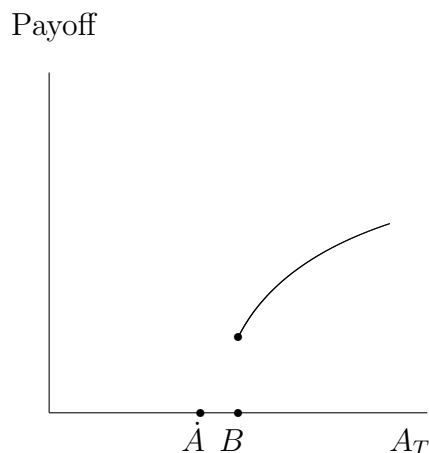


FIGURE 6. The payoff at maturity for barrier call option with 'high barrier' on infinite coupon paying debt as function value of the firm  $A_T$  and the bankruptcy asset level,  $B$ .

and a short position in the second option for values of  $A_T$  less than  $B$ , i.e.,  $(K - D(A_T))^+ - (K_1 - D(A_T))^+$  for  $A_T \leq B$ . A simple calculation yields

$$K_2 = K - \frac{cD}{r} + J\left(\frac{B}{A}\right)^{-\alpha}.$$

The above identity is then verified.

The market value of the above claim is easily calculated and the result given by the formula  $M_0(A, K)$  above.

The result now follows immediately from Theorem 18.8 in Björk (2004).  $\square$

### 3.5. Case 2: Down-and-out call option with 'high' barrier.

Next we consider the case where

$$B > A.$$

and the payoffs from the options are as shown in figure 6.

**Proposition 3.** *The time zero market values of the described barrier call option with barrier  $B$  and exercise price  $K$  is*

$$(20) \quad C_H^{do}(A, K) = G_0(A, K) - \left(\frac{B}{A}\right)^{\left(\frac{2(r-\gamma)}{\sigma^2}-1\right)} G_0\left(\frac{B^2}{A}, K\right),$$

where

$$G_0(S, K) = \left(\frac{cD}{r} - K\right)e^{-rT} N(-f_2) - J\left(\frac{S}{A}\right)^{-\alpha} e^{\alpha\gamma T} N(-f_1),$$

*Proof.* The time  $T$  payoff of the chopped claim is

$$(D(A_T) - K)^+ 1\{A_T > B\}.$$

This can be written as

$$(D(A_T) - K_1)^+ + K_3 1\{A_T > B\},$$

where  $K_3 = -K_2$  and  $K_1$  and  $K_2$  are given in the proof of Proposition 2.

The market value of the above claim is easily calculated and is given by the formula  $G_0(A, K)$  above.

The final formula follows immediately from Theorem 18.8 in Björk (2004).  $\square$

The down-and-out put option with high barrier has market value identical to zero because any payoff would be in the region  $A_T < \bar{A} < B$ , and this option does not give any payoff when  $A_T$  is below the barrier.

#### 4. ISSUE-AT-PAR COUPON INCLUDING EMBEDDED OPTION

The option formulas in the previous section are applicable in the situation where third parties trade options on any corporate perpetual debt. In such situations the existence of an option in the marketplace will neither influence the pricing of the debt nor the issuing company's own optimal choice of bankruptcy level. In particular, all the option pricing formulas above can be applied for this purpose by using  $B = \bar{A}$  and  $\gamma = 0$ . Recall that  $\bar{A}$  represents the constant optimal bankruptcy level in the case of infinite horizon debt claims with no embedded call option.

In this section we analyze the case with an issuer's European call option which is an integrated part of the debt contract, i.e., the option is written by the debtholders in favor of the equityholders. Thus, the existence of the option will influence both the issue-at-par coupon and the bankruptcy level before the option's expiration date. Intuition suggests that the coupon is increased to compensate for the added option, whereas the optimal bankruptcy level is decreased due to the value of the option - both compared to the case without an option.

We analyze a company with a simple capital structure and define the net loss,  $J = \frac{cD}{r} - \bar{A}$ . Let  $D_T^c$  denote the time  $T$  value of perpetual debt including an embedded option to repay debt at par value  $D$  at the time of expiration  $T$  of the option, given that the company has not gone bankrupt before. Therefore,

$$D_T^c = \begin{cases} \min(D_T, D) & \text{for } \tau > T \\ 0 & \text{otherwise,} \end{cases}$$



where  $D_T$  is given by expression (8) and  $\tau$  is the time of bankruptcy as defined previously. This expression can be rewritten as

$$(21) \quad D_T^c = \begin{cases} D_T - \max(D_T - D, 0) & \text{for } \tau > T \\ 0 & \text{otherwise.} \end{cases}$$

This shows how  $D_T^c$  equals  $D_T$  minus the pay-off from a *call-option* on the debt. Alternatively, the reformulation  $D_T^c = D - \max(D - D_T, 0)$  for  $\tau > T$  shows how the time  $T$  payoff can be divided into the par value of debt minus a *put-option* on debt with exercise price equal par.

From expression (21) and standard financial pricing theory the time 0 market value  $D_0^c$  of infinite horizon debt including an embedded option to repay debt at par value can be written as

(22)

$$D_0^c(A) = E^Q[(D_T - 0)^+ e^{-rT} 1_{\{\tau > T\}}] - E^Q[e^{-rT} (D_T - D)^+ 1_{\{\tau > T\}}] \\ + E^Q\left[\int_0^{\tau \wedge T} cD e^{-rs} ds\right] + E^Q[Be^{-r\tau} 1_{\{\tau \leq T\}}],$$

where  $E^Q[\cdot]$  denotes the expectation under the equivalent martingale measure. Observe that the two first terms of this expression represents the current market value of a long European down-and-out barrier call option with exercise price 0 and a short European down-and-out barrier call option with exercise price par, both expiring at time  $T$ . Here the third term represents the time 0 market value of coupon payments before time  $T$  and the last term the time 0 market value of the compensation in case of bankruptcy before time  $T$ .

We use the expression (34) in appendix B to reformulate the two last terms

$$(23) \quad L(A) = E^Q\left[\int_0^{\tau \wedge T} cD e^{-rs} ds\right] + E^Q[Be^{-r\tau} 1_{\{\tau \leq T\}}] \\ = \left(\frac{cD}{r} - \left(\frac{cD}{r} - B\right)\left(\frac{A}{B}\right)^{-\beta} - C_H^\beta(A, 0),\right.$$

where the barrier call option with high barrier, denoted  $C_H^\beta(A, D)$ , is given in expression (40) in appendix C. Expression (23) shows that the two last terms in expression (22) can be interpreted as the time 0 market value of regular perpetual debt minus the time 0 market value of a barrier call option with a high barrier and exercise price 0.

**Proposition 4.** *The time 0 value of infinite horizon continuous coupon-paying debt claims including an embedded option to repay debt at par value  $D_0^c$  is*

$$(24) \quad D_0^c(A) =$$

$$\begin{cases} C_H^{do}(A, 0) - C_L^{do}(A, D) + L(A), B > (1 - \frac{r\bar{A}}{cD})^{\frac{1}{\alpha}} \bar{A} \\ C_L^{do}(A, 0) - C_L^{do}(A, D) + L(A), B \leq (1 - \frac{r\bar{A}}{cD})^{\frac{1}{\alpha}} \bar{A}, \end{cases}$$

where  $C_H^{do}$  and  $C_L^{do}$  are derived in Proposition 3 and 2, respectively, and  $L(A)$  is given in expression (23) above.

This proposition follows immediately from equation (22) for  $D_0^c(A)$ . We need however, for the first barrier option with exercise price equal to zero, to distinguish between the case of a high or low barrier relative to the zero payoff asset level,  $\bar{A}$ , cf. expression (17) for  $\bar{A}$  in our case for  $J = \frac{cD}{r} - \bar{A}$ .

Our expression for  $D_0^c(A)$  can be interpreted as follows: The first term represents the time 0 value of infinite horizon debt issued at time  $T$  at the then prevailing market terms. The second term represents the time 0 market value of a call option on debt at time  $T$ . This possibility to refinance at improved market terms at time  $T$  is exactly the purpose of the embedded option in the time 0 debt contract. The last term represents the time 0 market value of all cashflows before time  $T$ .

## 5. NUMERICAL ESTIMATION

In this section we want to analyze the relationship between the bankruptcy barrier parameters,  $B$  and  $\gamma$ , and the calibrated coupon  $c$  for perpetual risky debt with embedded call option as expressed in equation (24). The purpose of the calibration is to achieve  $D_0^c(A) = D$ , i.e., that the debt with embedded option can be issued at par. In order to calibrate the bankruptcy barrier parameters  $B$  and  $\gamma$ , we implement a binomial tree. This approach also provides a calibrated coupon. The values of  $B$  and  $\gamma$  are then used in the closed form solution to calculate the coupon  $c$ . The coupon from the closed form solution is then compared to the coupon from the binomial tree which serves as an overall benchmark for our approach.

In our binomial approach we apply the parameters in Table 1 and run 100,000 steps per year. The chosen level of asset volatility is taken from Leland (1994), whereas the level of riskfree interest rate is common in similar illustrations. The time to expiration of the option resembles the protection periods in most publicly listed perpetual bonds issued by financial institutions.

From the binomial tree approach we obtain both the time 0 level  $B$  and the terminal level  $B_T$  of the bankruptcy barrier. Consistent with our assumed analytical form of the bankruptcy barrier  $B_t = Be^{\gamma t}$  we calculate  $\gamma = \frac{1}{T} \ln(\frac{B_T}{B})$ . Observe that by this formulation  $\gamma$  only depends on the time 0 and time  $T$  values of the optimal barrier and not on intermediary values.

The binomial approach calculates  $B_t$  for all  $t$ . To test the assumed functional form  $B_t = Be^{\gamma t}$ , we use Ordinary Least Squares (OLS) to

$A$	100	total asset value at time 0
$D$	70	face value of debt
$T$	10	expiration date of option
$\sigma$	0.2	volatility of total assets
$r$	5 %	riskfree interest rate

TABLE 1. Base case parameters.

estimate  $\gamma$  based on the complete sequence of numerically calculated values of  $B_t$ . The regression equation is  $\ln(B_t) = \ln(B) + \gamma t$ .

Figure 7 shows the development of  $B_t$  as a function of elapsed time to expiration both from the binomial approach, analytical approximation and OLS-regression. The latter is shown by the dotted line. A visual inspection indicates that the approximated barrier is reasonably close to the numerically derived bankruptcy barrier. The OLS-regression is included to check to which extent  $\ln(B_t)$  is a linear function of time. It also serves as an alternative estimation of  $\gamma$ , also using the intermediate values of  $B_t$ . By its' inherent nature, the OLS-approach will underestimate the value of  $B$ .

<i>Alternative solutions:</i>		$B$	$\gamma$	$c$	$A$
Analytical	- regular (B&C'76)	n/a	n/a	5.312 %	53.12
	- with option	n/a	n/a	5.564 %	55.64
Binomial	- with option	53.70	0.00285	5.526 %	55.26

TABLE 2. Calibrated values of the coupon  $c$  using the base case parameters and the numerically calculated pre-expiry barrier described by  $B$  and  $\gamma$ .

The results in Table 2 supports our intuition that an embedded option increases the calibrated coupon (from 5.312% to 5.564 %). For a given coupon-level, the initial bankruptcy-level with an option is lower than the bankruptcy level without an option (53.70 vs. 55.26). As an overall assessment, we find that the analytical results are close to the results from the binomial approach. The OLS-approach yields  $\gamma = 0.00296$  and  $B = 53.33$ , and a  $R^2 = 80.1\%$ . The high value of  $R^2$  indicates that the linear approximation of  $\ln(B_t)$  is reasonable. The estimated  $\gamma$  from the two approaches are very close (The difference is in the magnitude of  $\frac{1}{10000}$ .) The estimated value of  $B$  is, as expected, somewhat below the numerically calculated level (The difference is in the magnitude of 0.38.) We conclude that these findings support the use of the closed form solutions based on an approximated barrier.

Observe in Figure 7 that both the numerically calculated and the modelled bankruptcy levels are below the constant long-run level  $\bar{A}$ .

Additional analysis shows that for large  $T$ , the effect of the option disappears and  $B$  approaches  $\bar{A}$ .

Our simplified model and set of assumptions do not encapsulate actual market terms and conditions and the absolute size of the results are thus misleading. In particular, most perpetual debt-issues come with a floating coupon linked to a market-interest rate, e.g., a fixed margin over 3-month LIBOR<sup>7</sup> in US-dollars.

The coupon-level for debt with embedded option is driven by two counterbalancing forces, compared to the base-case without options. On the one hand, the value of the option-premium increases the coupon, on the other hand, the reduced optimal bankruptcy-level before the expiration decreases the coupon. This is also shown in Figure 2.

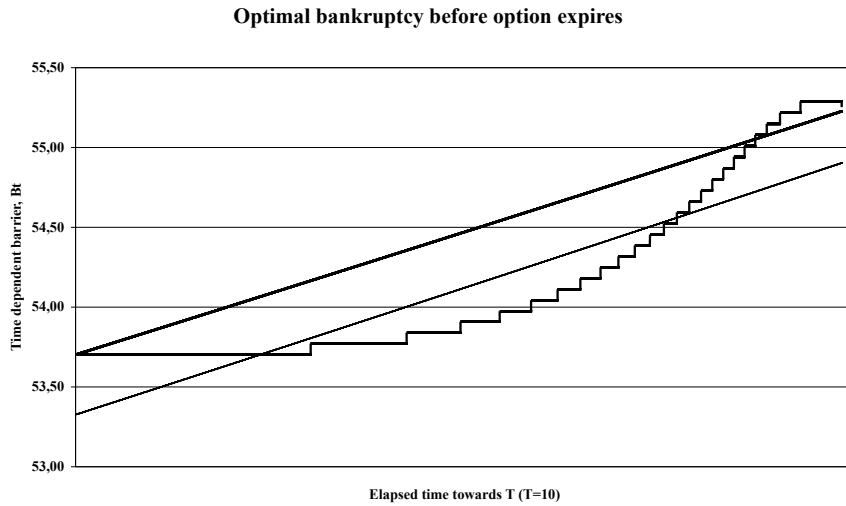


FIGURE 7. The numerically calculated optimal, and the modelled bankruptcy asset level  $B_t$  as a function of elapsed time until expiration of the call option embedded in perpetual debt. The dotted line represents an OLS-regression of the numerically calculated  $B_t$ -values. See Table 1 for parameter values.

<sup>7</sup>LIBOR: London Interbank Offered Rate

## 6. CONCLUDING REMARKS AND FURTHER RESEARCH

We show how a European embedded option in perpetual debt impacts both the value of debt and the issuer's rational economic behavior. Specifically, this impacts the bankruptcy decision, level of debt coupons and the optimal exercise of the option. We derive closed form solutions based on an approximation of the optimal bankruptcy level before the option expires. These expressions perform well compared to numerical solutions both for bankruptcy levels and optimal coupon-levels.

The equityholders pay for the embedded option through a higher fixed coupon on the perpetual debt, compared to regular perpetual debt. At the time of issuance both debt-alternatives are issued with coupon-rates which, for analytical purposes, secures that the market value equal par value. The equityholders determine the optimal bankruptcy-level which is different for two reasons; an increased coupon and the existence of the option. The increased coupon raises the optimal long-term bankruptcy-level. Since the value of the option varies over time, the optimal bankruptcy-level pre-expiration of the option will also be time dependent.

The market values of perpetual debt with and without option are different after expiration in the situation when the option has not been exercised. A higher coupon in the first case reflects the historical cost of the expired option and is a major motivation for the exercise of such options. This higher option causes exercises also in significantly worse states compared to the situation at time of issue. It is common in the marketplace to contractually agree that coupons are even "stepped-up" post-expiry to further incentive exercise.

Alternative structures:	$J$	$\bar{A}$
A: A single class of debt	$\frac{c_D D}{r} - \bar{A}$	$\left( \frac{(c_D D - \bar{A}) \bar{A}^\alpha}{\frac{c_D D}{r} - K} \right)^{\frac{1}{\alpha}}$
B: Two classes of debt:		
- <i>Senior debt:</i>	$\frac{c_s D_s}{r} - \min(\bar{A}, D_s)$	$\left( \frac{(c_s D_s - \min(\bar{A}, D_s)) \bar{A}^\alpha}{\frac{c_s D_s}{r} - K_s} \right)^{\frac{1}{\alpha}}$
- <i>Junior debt:</i>	$\frac{c_j D_j}{r} - (\bar{A} - D_s)^+$	$\left( \frac{(c_j D_j - (\bar{A} - D_s)^+) \bar{A}^\alpha}{\frac{c_j D_j}{r} - K_j} \right)^{\frac{1}{\alpha}}$

TABLE 3. Liquidation loss and zero payoff parameters of alternative infinite horizon debt claims.

For our analytical purpose, we have developed some European option and barrier option pricing formulas on perpetual debt. These formulas are quite general and may be used for valuing both embedded and third-party options. Furthermore, the formulas may be applied to other

classes of perpetual debt as indicated in Table 3 which shows how some stylized alternative debt structures impact the net loss expressions  $J$  and the zero payoff points for options,  $\hat{A}$ .

Our model can be extended along a number of dimensions such as introducing frictions (taxes, bankruptcy costs), different priorities (hybrid/preferred stock, see e.g., Mjøs and Persson (2005)), and American options.

APPENDIX A. PRESENT VALUE OF 1 PAYABLE AT FIRST HITTING TIME BEFORE A FINITE HORIZON.

In this appendix we collect some technical results. Consider the Itô process

$$(25) \quad X_t^z = zt + W_t,$$

where  $z$  is a constant, and the stopping time

$$\tau = \inf\{t \geq 0, X_t = b\}$$

where  $b$  is a constant. Define another constant

$$w = \sqrt{z^2 + 2r},$$

where  $r$  represent the constant riskfree interest rate. We are concerned about the present value of one currency unit payable at the first hitting time of a lower boundary if this occurs before the horizon  $T$  and define

$$V_0 = E^Q[e^{-r\tau} 1\{\tau \leq T\}].$$

where  $E^Q[\cdot]$  denotes the expectation under the equivalent martingale measure. E.g., Lando (2004) shows that

$$V_0 = e^{b(z-w)} Q^w(\tau \leq T),$$

where

$$(26) \quad Q^w(\tau \leq T) = N\left(\frac{b-wT}{\sqrt{T}}\right) + e^{2wb} N\left(\frac{b+wT}{\sqrt{T}}\right),$$

represents the cumulative probability distribution of  $\tau$  as a function of the parameter  $w$ . The above result can be rewritten as

$$(27) \quad V_0 = e^{b(z-w)} N\left(\frac{b-wT}{\sqrt{T}}\right) + e^{b(z+w)} N\left(\frac{b+wT}{\sqrt{T}}\right).$$

The constants  $z$  and  $b$  for our problem, see section 2, which may be plugged into expression (27), are:

$$(28) \quad z = \frac{1}{\sigma} \left( r - \gamma - \frac{1}{2} \sigma^2 \right)$$

and

$$(29) \quad b = \frac{1}{\sigma} \ln\left(\frac{B}{A}\right).$$

The special case of a constant lower boundary  $\gamma = 0$  leads to the simple expression

$$(30) \quad V_0 = \frac{A}{B} N\left(\frac{\ln(\frac{B}{A}) - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + \left(\frac{B}{A}\right)^\alpha N\left(\frac{\ln(\frac{B}{A}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

In the case with  $\gamma > 0$ , the revised expression becomes

$$(31) \quad V_0 = \left(\frac{A}{B}\right)^{-\beta + \frac{2}{\sigma}w} N(-n_1) - \left(\frac{A}{B}\right)^{-\beta} N(n_2),$$

where

$$n_1 = \frac{-(r - \gamma - \frac{1}{2}\sigma^2 - \sigma^2\beta)T + \ln(\frac{B}{A})}{\sigma\sqrt{T}}$$

and

$$n_2 = \frac{(r - \gamma - \frac{1}{2}\sigma^2 - \sigma^2\beta)T + \ln(\frac{B}{A})}{\sigma\sqrt{T}}$$

and  $\beta$  is given in (5).

APPENDIX B. A DECOMPOSITION OF THE MARKET VALUE OF  
INFINITE HORIZON BLACK AND COX (1976) AND  
LELAND (1994) DEBT

We now denote by  $D_s$  the time  $s$  market value of infinite horizon coupon-paying debt, thus  $D_s = D(A_s)$ , where  $D(A_s)$  is given in expression (3)<sup>8</sup>:

$$(32) \quad D_s = \frac{cD}{r} - \left(\frac{cD}{r} - B\right)G_s$$

where

$$G_s = \left(\frac{A_s}{B}\right)^{-\beta}.$$

Using Itô's lemma and equation (4), we calculate the dynamics of  $G_t$  as

$$(33) \quad dG_t = rG_t dt - \sigma\beta dW_t.$$

As an exercise, we want to calculate the time 0 market value  $D_0$  based on the market value at some fixed future time  $T$   $D_T$  where  $D_T$  is given by the expression above for  $s = T$ . We apply standard valuation-techniques from, e.g., Duffie (2001) and calculate

$$(34) \quad D_0 = E^Q[e^{-rT}D_T 1_{\{\tau > T\}}] + E^Q\left[\int_0^{\tau \wedge T} cDe^{-rs} ds\right] + E^Q[Be^{-r\tau} 1_{\{\tau \leq T\}}],$$

where  $E^Q[\cdot]$  denotes the expectation under the equivalent martingale measure. The first term represents the time 0 of time  $T$  infinite horizon debt, provided that bankruptcy has not occurred before time  $T$ . The second term represents the market value of the coupons to debtholders until whatever comes first of time  $T$  or bankruptcy. The final term represents the market value of the debtholders compensation  $B$  given that bankruptcy occurs before time  $T$ .

We will now verify that the right hand side of expression (34) equals the Black and Cox (1976) and Leland (1994) result found by using  $s = 0$  in expression (32). The right hand side of equation (34) is therefore a way to decompose the initial market value of debt for the purpose of the analysis in section 4.

<sup>8</sup>Observe that this notation is different from the notation used in expression (8).



By inserting the expression for the market value of  $D_T$  from (32), and evaluating the integral, we get that

$$D_0 = e^{-rT} \frac{cD}{r} Q(\tau > T) - \left( \frac{cD}{r} - B \right) E^Q [e^{-rT} G_T 1_{\{\tau > T\}}] \\ + \frac{cD}{r} [1 - e^{-rT} Q(\tau > T) - E^Q [e^{-rT} 1_{\{\tau \leq T\}}]] + B E^Q [e^{-rT} 1_{\{\tau \leq T\}}],$$

or

$$(35) \quad D_0 = \frac{cD}{r} - \left( \frac{cD}{r} - B \right) [E^Q [e^{-rT} G_T 1_{\{\tau > T\}}] + E^Q [e^{-rT} 1_{\{\tau \leq T\}}]],$$

This expression can be simplified by the use of the following lemma.

**Lemma 1.**  $E^Q [e^{-rT} G_T 1_{\{\tau > T\}}] + E^Q [e^{-rT} 1_{\{\tau > T\}}] = \left( \frac{A}{B} \right)^{-\beta}$ .

From this lemma it is clear that

$$D_0 = \frac{cD}{r} - \left( \frac{cD}{r} - B \right) \left( \frac{A}{B} \right)^{-\beta},$$

i.e., the traditional Black and Cox (1976) result one immediately gets by using  $s = 0$  in (32).

*Proof of Lemma 1.* First observe that from expression (26) follows that

$$(36) \quad Q^z(\tau > T) = 1 - Q^z(\tau \leq T) = N\left(\frac{zT - b}{\sqrt{T}}\right) - e^{2zb} N\left(\frac{b + zT}{\sqrt{T}}\right).$$

The first term of the equality in the lemma  $E^Q [e^{-rT} G_T 1_{\{\tau > T\}}]$  can be calculated by using the standard change of measure technique. Observe that

$$E^Q [e^{-rT} G_T] = \left( \frac{A}{B} \right)^{-\beta} = G_0,$$

due to the martingale properties of  $G_t e^{-rt}$ . Define another equivalent probability measure  $\hat{Q}$  by

$$\frac{d\hat{Q}}{dQ} = \frac{G_T}{E^Q [e^{-rT} G_T]} = e^{(r - \frac{1}{2}(\beta\sigma)^2)T - \beta\sigma W_T}.$$

From Girsanov's theorem,  $d\hat{W}_t = dW_t + \sigma\beta dt$  under  $\hat{Q}$ . Under this measure, the dynamics of  $A_t$  is  $dA_t = (r - \gamma - \beta\sigma^2)A_t dt + \sigma A_t d\hat{W}_t$  and the drift process of the corresponding  $X_t$  process is

$$\hat{z} = \frac{r - \gamma - \frac{1}{2}\sigma^2 - \sigma^2\beta}{\sigma}.$$

The first term of Lemma 1 can now be expressed as

$$E^Q [e^{-rT} G_T 1_{\{\tau > T\}}] = G_0 E^Q \left[ \frac{d\hat{Q}}{dQ} 1_{\{\tau > T\}} \right] = G_0 \hat{Q}(\tau > T).$$

Inserting  $\hat{z}$  into equation (36) we get

$$\hat{Q}^{\hat{z}}(\tau > T) = N(n_1) - \left( \frac{B}{A} \right)^{2\left(\frac{r - \gamma - \frac{1}{2}\sigma^2}{\sigma^2} - \beta\right)} N(n_2).$$

Thus, we get

$$(37) \quad E^Q[e^{-rT}G_T 1_{\{\tau > T\}}] = \left(\frac{A}{B}\right)^{-\beta} N(n_1) - \left(\frac{A}{B}\right)^{\beta-2\left(\frac{r+\gamma-\frac{1}{2}\sigma^2}{\sigma^2}\right)} N(n_2).$$

In appendix A above we calculate  $V_0 = E^Q[e^{-rT} 1_{\{\tau > T\}}]$  for the case of  $\gamma > 0$  in equation (31).

By adding expression (37) and expression (31) Lemma 1 is proved.  $\square$

### APPENDIX C. DERIVATION OF BARRIER CALL ON INFINITE DEBT WITH ADJUSTED DRIFT.

Our starting point is the processes  $D_t$  and  $G_t$  (with dynamics given in expression (33)) from expression (32) in Appendix B.

Mimicking the arguments in section 3 we arrive at the call option pricing formula

$$(38) \quad C_0^\beta(A, K) = \left(\frac{cD}{r} - K\right)e^{-rT} N(-\hat{d}_2) - J\left(\frac{A}{B}\right)^{-\beta} N(-\hat{d}_1),$$

where

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{B}{A}\right) - \frac{1}{\beta}\left(\ln\left(\frac{cD}{r} - K\right) - \ln(J)\right) - \left(r - \gamma - \sigma^2\left(\beta + \frac{1}{2}\right)\right)T\right)$$

and  $\hat{d}_2 = \hat{d}_1 - \beta\sigma\sqrt{T}$  or

$$\hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{B}{A}\right) - \frac{1}{\beta}\left(\ln\left(\frac{cD}{r} - K\right) - \ln(J)\right) - \left(r - \gamma - \frac{1}{2}\sigma^2\right)T\right).$$

Our next step is to derive a  $\hat{G}_0(A, K)$  function similar to what we did in section 3.5.

The major steps are to identify  $K_1 = B$  and  $K_3 = B - K$  as used in the proof of proposition 3. Then we calculate

$$(39) \quad \hat{G}_0(A, K) = \left(\frac{cD}{r} - K\right)e^{-rT} N(-f_2) - J\left(\frac{A}{B}\right)^{-\beta} N(-f_3),$$

where

$$f_3 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{B}{A}\right) - \left(r - \gamma - \sigma^2\left(\beta + \frac{1}{2}\right)\right)T\right).$$

and  $f_2$  is given below expression (19). The final step is to use Björk's formula 18.8 to arrive at the down and out barrier call option formula

$$(40) \quad C_H^\beta(A, K) = \hat{G}_0(A, K) - \left(\frac{B}{A}\right)^{\left(\frac{2(r-\gamma)}{\sigma^2}-1\right)} \hat{G}_0\left(\frac{B^2}{A}, K\right).$$

## REFERENCES

- K. K. Aase. Negative volatility and the survival of the western financial markets. *WILMOTT magazine*, 2004.
- V. A. Acharya and J. N. Carpenter. Corporate bond valuation and hedging with stochastic interest rates and endogeneous bankruptcy. *Review of Financial Studies*, 15(5):1355–1383, Winter 2002.
- T. Björk. *Arbitrage Theory in Continuous Time. Second edition*. Oxford University Press, London, 2004.
- F. Black and J. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31:351–367, 1976.
- F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, May-June 1973.
- D. Duffie. *Dynamic Asset Pricing Theory. Third Edition*. Princeton University Press, Princeton, NJ, USA, 2001.
- D. Emanuel. A theoretical model for valuing preferred stock. *Journal of Finance*, 38:1133–1155, 1983.
- R. A. Jarrow and S. M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50(1):53–85, 1995.
- R. J. Kish and M. Livingston. Determinants of the call option on corporate bonds. *Journal of Banking and Finance*, 16:687–703, 1992.
- D. Lando. *Credit Risk Modeling: Theory and Applications*. Princeton University Press, Princeton, NJ, USA, 2004.
- H. E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49:1213–1252, 1994.
- D. C. Mauer. Optimal bond call policies under transaction costs. *Journal of Financial Research*, XVI(1):23–37, 1993.
- R. C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, Spring 1973.
- A. Mjøs and S.-A. Persson. Bundled financial claims: A model of hybrid capital. Technical report, Norwegian School of Economics, 2005. Unpublished.
- M. Rubinstein. Displaced diffusion option pricing. *Journal of Finance*, 38:213–217, 1983.
- S. Sarkar. Probability of call and likelihood of the call feature in a corporate bond. *Journal of Banking and Finance*, 25:505–533, 2001.
- S. Sarkar and G. Hong. Effective duration of callable corporate bonds: Theory and evidence. *Journal of Banking and Finance*, 28:499–521, 2004.
- K. B. Toft and B. Prucyk. Options on leveraged equity: Theory and empirical tests. *Journal of Finance*, 52:1151–1180, 1997.  
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