# Asset Pricing in a Monetary Economy with Heterogeneous Beliefs[*]

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## Abstract

In this paper, we shed new light on the role of monetary policy in asset pricing by focusing on the case where investors have heterogeneous expectations about future monetary policy. This case is realistic, because central banks are typically less than perfectly open on their intentions. Accordingly, surveys of economists in the press reveal that they frequently disagree in their expectations. Under heterogeneity in beliefs, investors place bets against each other on the evolution of the money supply, and as a result, the sharing of wealth in the economy evolves stochastically over time, making money non-neutral. Employing a continuous-time, general equilibrium model, we establish these fluctuations to be rich in implications, in that they majorly affect the equilibrium prices of all assets, as well as inflation. In some specific cases, we are able to derive explicit formulas for important economic quantities. In particular, we find that the stock market volatility may be significantly increased by the heterogeneity in beliefs. In addition to generating interesting behavior on the part of asset prices, our model provides a natural framework in which to assess the impact of the transparency of monetary policy, a topical and controversial issue. Our model is particularly appropriate for the study of the effects of transparency, because it is intuitive that one of the key effects of increased transparency should be a drop in the amount of heterogeneity in beliefs.

# 1. Introduction

The impact of monetary policy on financial asset prices can hardly be overstated. Its impact on the value of money itself (in other words, inflation) is obvious, but it is also possibly the single most important factor affecting stock market returns. While the academic literature provides many studies documenting this (see, e.g., Thorbecke (1997)), it is not necessary to go that far: by simply browsing through financial pages in the press, it is clear that participants in financial markets attach tremendous importance to the actions of central banks. Any upcoming decision by the Federal Reserve is awaited with great anticipation and, as a news article on the BBC website[1] put it, "A mere word from Mr. Greenspan can cause the stock market and the dollar to rise and fall." The mere anticipation may suffice to cause stock market moves: for example, CNN stated on September 8, 2004:[2] "cautiousness ahead of a key address by Fed Chairman Alan Greenspan before a Congressional panel could take stocks lower Wednesday."

One key feature of central banks' policies is that they are typically hard to anticipate. As the same BBC article puts it, Alan Greenspan "is also famous for his ability to keep the markets and the politicians guessing." Part of the monetary economics literature suggests that transparency in monetary policy may compromise its effectiveness, and this school of thought seems to still exert some influence, even though there has been a trend toward more transparency. (A survey by Carpenter (2004) points that "there is a lack of consensus on whether central bank transparency is beneficial." It is interesting to note that the author of the survey is an economist at the Fed.) While many central banks have switched to following more formal rules in the recent past, by far the most important central bank in the world, the U.S. Fed has not adopted such rules and remains difficult to read. As economist Robert Barro put it,[3] "Beginning with New Zealand in 1989, a number of central banks successfully designed and followed formal rules for inflation targeting. Examples include Canada, Australia, Britain and Sweden. (...) Despite increases in transparency, the U.S. system remains opaque and relies more on the judgment and credibility of its chair."

The fact that there is imperfect information on monetary policy makes it likely that market participants hold heterogeneous beliefs on its future evolution. The presence of disagreement among traders, and its importance for understanding prices, is well established in other areas of finance. For example, a recent study by Anderson, Ghysels and Juergens (2004) shows that there exists significant disagreement among stock analysts about expected earnings, and that this heterogeneity in beliefs matters for asset pricing. It is not necessary to invoke the academic literature to verify that such heterogeneous beliefs are also present regarding monetary policy. Examples of disagreement abound in the popular press. For example, on September 24, 2004, following an interest rate hike by the Fed, CNBC surveyed several economists on their take on the

[1] http://news.bbc.co.uk/1/hi/business/business_basics/178569.stm
[2] http://money.cnn.com/2004/09/08/markets/stockswatch/
[3] Business Week, November 7, 2005

near future of US monetary policy:[4] "While most analysts still expect a rate hike November 10, some disagree as economists debate just when and whether the Fed will pause in its tightening cycle. Wachovia economist Mark Vitner said he expects the Fed to keep rates unchanged in November." We are not aware of any academic study directly documenting heterogeneous beliefs on monetary policy, but Mankiw, Reis and Wolfers (2003) establish substantial disagreement (both among professional economists and consumers) in expectations on inflation, a quantity closely related to monetary policy.

Given the impact of monetary policy on asset prices, it is surprising how little academic work has been done to incorporate money into an asset pricing framework. In almost all models of financial asset pricing, the value of financial securities and their payoffs are denominated in units of consumption goods, and money is not present. There exists a small intersection between monetary economics and asset pricing (see Bakshi and Chen (1996) and the references therein), but it does not incorporate heterogeneity in beliefs on monetary policy.

In this paper, we attempt to better understand asset prices and the effects of monetary policy by building an equilibrium model that incorporates investors' disagreement on monetary policy. To our knowledge, this analysis has not been performed in a framework that allows for a realistic modeling of asset prices (even though Basak (2003) suggests that such an analysis could be performed, and sketches a model to do so). We show that this heterogeneity in beliefs has important implications on equilibrium economic quantities (inflation, interest rates, stock returns, sharing of wealth across investors), and explore the direction and the magnitude of these effects.

We employ a general equilibrium model in continuous time. We chose to employ a continuous-time model for tractability (it makes it possible to obtain explicit formulas in many cases) and also because of the availability of well-understood benchmark models, with which we will be able to compare our results. Our model builds on the work of Bakshi and Chen (1996), who start with a discrete time framework and then take the continuous-time limit. This model is extended by Basak and Gallmeyer (1999) to an international context with two currencies, and by Lioui and Poncet (2004) and Buraschi and Jiltsov (2005) to a production economy. (None of these studies include heterogeneity in beliefs, and so there is no redundancy with this paper).

The main features of our model are as follows. Money plays a role because investors' utilities are assumed to be a function of money holdings in addition to consumption. "Money-in-the-utility-function" (MIUF) is one of the modeling strategies that are most popular with monetary economists. It can be interpreted as capturing in reduced form the fact that money renders transaction services, and that it is necessary for investors to hold cash balances (even though they forfeit interest income by doing so) in order to be able to consume. Feenstra (1986) has shown that MIUF is approximately equivalent to assuming a "cash-in-advance" or "Clower" constraint (the other modeling strategy commonly employed to generate an economic role for money, where

[4] http://www.msnbc.msn.com/id/6064674/

investors are constrained to hold cash balances proportional to the level of their consumption; see, e.g., Blanchard and Fischer (1989)). The main advantage of the MIUF formulation is that it is quite tractable and often allows for explicit computations. Money enters the economy by being endowed to the investors by the central bank. The money supply is assumed to summarize the central bank's monetary policy, a standard simplification in monetary economics. The money supply is assumed to follow an Itô process, whose drift is chosen by the central bank. The presence of a random component in the money supply is quite realistic, as it is clear that central banks only imperfectly control the money supply (due in particular to their imperfect control of the decisions of the banking sector). There are two investors in our model (both with MIUF), who observe the changes in the money supply, but have incomplete information on its dynamics. We assume that the two investors have heterogeneous beliefs so that, even though they have symmetric information, they disagree on the estimated expected money supply growth. There are two sources of uncertainty, one associated with money supply, and the other associated with aggregate consumption, which are allowed to be correlated.

Three securities are available for trading: a risky stock (equity) representing a claim on aggregate consumption, a real riskless bond (whose payments are indexed on inflation) and a nominal riskless bond (whose payments are denominated in nominal terms and not protected against inflation; thus, the nominal bond is risky in real terms). Since there are three securities for two sources of uncertainty, markets are complete. We use martingale representation technology (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) to solve the investors' optimization, and construct a representative agent with stochastic weights (Cuoco and He (1994), Basak and Cuoco (1998)) to solve for the equilibrium.

In a general framework, without making particular assumptions on the dynamics of aggregate consumption and money supply and investors' preferences, we find that the equilibrium economic quantities are all affected by the presence of heterogeneous beliefs. The most important feature of the model is that, under heterogeneous beliefs, agents place bets against each other on the money supply and so shocks in the money supply affect the distribution of wealth in the economy. Thus, money supply shocks affect consumptions: heterogeneity in beliefs makes money non-neutral. The extra risk for the investors (a type of risk often referred to as "trading risk" in finance) leads to an extra factor in the pricing of assets, that subsists even when preferences are separable. One interesting result is that, when the investors' utilities are additively separable, financial assets' risk premia are only affected by their exposure to monetary risk in the presence of heterogeneous beliefs. This shows that the heterogeneity in beliefs has a profound effect on the equilibrium, because the structure of the expressions (not just the numerical values) is affected. This result is independent of the specific shape of the investors' utilities and implies that it is possible to obtain novel, interesting implications in a tractable setup, with separable preferences (models with non-separable preferences are typically very intractable). To derive sharper implications, we move to such a setup, and analyze two examples in depth, where simplifying assumptions are made on the dynamic behavior of aggregate consumption and money supply, and on investor preferences.

In a first example, we assume that investors exhibit separable, logarithmic expected utility preferences, which considerably simplifies the computations. The main advantage of this case is that an explicit computation of the price of money (expressed in units of the consumption good), expected inflation and nominal interest rates is possible. (In general, these quantities are characterized by a backward stochastic differential equation whose solution requires complex numerical techniques, such as those employed by Basak and Gallmeyer (1999)). The term structure of nominal interest rates can also be computed explicitly. In spite of the relative simplicity of this case, the implications are rich. Nominal interest rates are driven by investors' expectations of future monetary policy – including the short rate, which is surprising under logarithmic utility (which implies myopic behavior). The most robust implication is that the heterogeneity in beliefs increases the volatility of inflation: when a positive shock to the money supply occurs, not only does the extra amount of money cause inflation, but those investors who expect higher money supply growth and higher future inflation "win their bet" and so their weight in the economy increases, which generates extra inflation. Real asset prices, on the other hand, are unaffected in the logarithmic case. This provides the main motivation for our second example, in which heterogeneity in beliefs affects real asset prices.

In our second main example, we examine the case where investors have separable, constant relative risk aversion (CRRA) utility functions. In this case, it is not possible to solve explicitly for the price of money and the nominal interest rate, but it is possible to provide an explicit formula for the stock price and its volatility. It is rare to be able to explicitly compute a stock price that is not "trivial" (for example, equal to the amount of aggregate dividends) and exhibits interesting properties, as is the case here. In particular, heterogeneity in beliefs on monetary policy generates a much higher volatility for the stock. Real interest rates are also increased. The implications on the equity premium are ambiguous; nonetheless, for some plausible parameter values, the equity premium is much increased over a standard model.

In addition to furthering our understanding of asset prices, our model provides a natural framework in which to assess the impact of the transparency of monetary policy, and how much transparency and openness on the part of central banks are optimal – a topical and controversial issue. Our model is particularly appropriate for the study of the effects of transparency, because it is intuitive that one of the key effects of increased transparency should be a drop in the amount of heterogeneity in beliefs: the more transparent monetary policy is, the less subjective prior beliefs enter forecasts, and the more similar individual forecasts should be. Hence, our model suggests that increased transparency could potentially, among other implications, reduce the volatility of inflation and stock prices, as well as the level of real interest rates.

The rest of the paper is organized as follows. Section 2 describes our model, and Section 3 characterizes the equilibrium in a general setup. Section 4 and 5 are devoted to two special cases, where investors have, respectively, separable logarithmic and separable CRRA preferences. Section 6 concludes, and the Appendix provides all proofs.

# 2. The Economic Setup

We consider a continuous-time, pure exchange, finite horizon ($[0, T]$), economy populated with two investors $i = 1, 2$, possibly heterogeneous in their beliefs, preferences and endowments. The uncertainty is generated by a two-dimensional Brownian motion, $w = (w_\varepsilon, w_M)^\top$. There is a single consumption good that serves as the numeraire. In addition to their consumption, agents derive utility from holding money balances, which captures in reduced form the transaction services rendered by money. We assume sufficient regularity for all stochastic differential equations and investors' optimization problems to have a solution.

## 2.1. The Aggregate Consumption, Money Supply and Information Structure

The aggregate endowment of consumption $\varepsilon$ and the money supply $M$ are assumed to be positive and to follow Itô processes:

$$\begin{aligned} d\varepsilon(t) &= \varepsilon(t)[\mu_\varepsilon(t)dt + \sigma_\varepsilon(t)dw_\varepsilon(t)], \\ dM(t) &= M(t)[\mu_M(t)dt + \sigma_{M\varepsilon}(t)dw_\varepsilon(t) + \sigma_{MM}(t)dw_M(t)], \end{aligned}$$

where $w = (w_\varepsilon, w_M)^\top$ is a standard 2-dimensional Brownian motion. While $w_\varepsilon$ (the consumption risk) and $w_M$ (the "pure" monetary risk) are independent, this formulation allows for correlation between aggregate consumption and money supply. The (instantaneous) correlation is given by $\rho(t) = \sigma_{M\varepsilon}(t)/(\sigma_{M\varepsilon}(t)^2 + \sigma_{MM}(t)^2)^{1/2}$. For simplicity, we sometimes denote the total volatility of the money supply by $\sigma_M(t) = (\sigma_{M\varepsilon}(t)^2 + \sigma_{MM}(t)^2)^{1/2}$.

The investors commonly observe the processes $\varepsilon$ and $M$, but have incomplete information and heterogeneous beliefs on the dynamics of the money supply.[5] They observe the volatility coefficients $\sigma_{M\varepsilon}$, $\sigma_{MM}$ (from the quadratic variation and covariation with $\varepsilon$), but must estimate $\mu_M$. We denote agent $i$'s estimate of $\mu_M$ by $\mu_M^i$. Even though investors have symmetric information on $\mu_M$, due to their heterogeneous prior beliefs they may disagree in their estimates of $\mu_M$. We denote the (normalized) difference in their estimates by $\bar{\mu}_M(t) = (\mu_M^1(t) - \mu_M^2(t))/\sigma_{MM}(t)$. Agent $i$'s innovation process (or estimate for the monetary risk factor $w_M$) is given by:

$$w_M^i(t) = \int_0^t \frac{1}{\sigma_{MM}(s)} \left( \frac{dM(s)}{M(s)} - \mu_M^i(s)ds - \sigma_{M\varepsilon}(s)dw_\varepsilon(s) \right),$$

the estimate that reconciles the investor's estimate of $\mu_M$ with his observation of the money supply. The investors' innovation processes are related by

$$dw_M^1(t) = dw_M^2(t) + \bar{\mu}_M(t)dt. \tag{2.1}$$

[5] It would be easy to additionally incorporate heterogeneity in beliefs on the aggregate endowment growth. Here, we choose to focus on heterogeneity in beliefs on the money supply growth, a novelty of this work, and so, in order to not obscure its implications, we assume homogeneous beliefs on consumption growth. The effects of heterogeneous beliefs on aggregate consumption growth are well-understood (e.g., Basak (2003)).

By Girsanov's theorem, $w_M^i$ is a Brownian motion under agent $i$'s subjective beliefs, and his perceived dynamics for the money supply are as follows:

$$dM(t) = M(t)[\mu_M^i(t)dt + \sigma_{M\varepsilon}(t)dw_\varepsilon(t) + \sigma_{MM}(t)dw_M^i(t)].$$

It will be verified that, in equilibrium, the price of money $p$, expressed in units of the consumption good (the numeraire), follows an Itô process with dynamics:

$$\begin{aligned} dp(t) &= p(t)[\mu_p(t)dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M(t)] \\ &= p(t)[\mu_p^i(t)dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M^i(t)], \qquad i = 1, 2, \end{aligned}$$

under the "objective" probability and as perceived by the investors, respectively. The expected inflation, as perceived by investor $i$, is denoted by $\pi^i(t) = -\mu_P^i(t)$, consistent with the past literature.

## 2.2. Investment Opportunities

There are three securities available for continuous trading. The first is a zero-net supply, riskless (in real terms) bond paying-off the real interest rate $r$. Its price follows

$$dB(t) = B(t)r(t)dt.$$

There is also a zero-net supply, nominally riskless bond paying-off the nominal interest rate $R$. An application of Itô's lemma shows that the real price of the nominal bond, $B_m$, has dynamics:

$$\begin{aligned} dB_m(t) &= B_m(t)[(\mu_p(t) + R(t))dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M(t)] \\ &= B_m(t)[(\mu_p^i(t) + R(t))dt + \sigma_{p\varepsilon}(t)dw_\varepsilon(t) + \sigma_{pM}(t)dw_M^i(t)], \qquad i = 1, 2. \end{aligned}$$

Thus, the nominal bond is risky in real terms. Finally, there is a risky stock representing a claim on the aggregate consumption $\varepsilon$, with a total supply of one share, whose price $S$ has dynamics:

$$\begin{aligned} dS(t) + \varepsilon(t)dt &= S(t)[\mu_S(t)dt + \sigma_{S\varepsilon(t)}dw_\varepsilon(t) + \sigma_{SM}(t)dw_M(t)] \\ &= S(t)[\mu_S^i(t)dt + \sigma_{S\varepsilon(t)}dw_\varepsilon(t) + \sigma_{SM}(t)dw_M^i(t)], \qquad i = 1, 2. \end{aligned}$$

Agents facing the same price processes $S$, $B_m$ and equation (2.1) imply that the perceived expected returns are related by

$$\mu_p^1(t) - \mu_p^2(t) = \sigma_{pM}(t)\bar{\mu}_M(t), \qquad \mu_S^1(t) - \mu_S^2(t) = \sigma_{SM}(t)\bar{\mu}_M(t). \tag{2.2}$$

Assuming that the (endogenous) volatility coefficients $\sigma_{p\varepsilon}$, $\sigma_{pM}$, $\sigma_{S\varepsilon}$, $\sigma_{SM}$ are nonzero, the market is complete and investor $i$ faces unique market prices of risk, $\theta_\varepsilon^i$ and $\theta_M^i$, associated with aggregate consumption and pure monetary uncertainty, respectively. $\theta_\varepsilon^i$ and $\theta_M^i$ solve:

$$\begin{pmatrix} \sigma_{p\varepsilon}(t) & \sigma_{pM}(t) \\ \sigma_{S\varepsilon}(t) & \sigma_{SM}(t) \end{pmatrix} \begin{pmatrix} \theta_\varepsilon^i(t) \\ \theta_M^i(t) \end{pmatrix} = \begin{pmatrix} \mu_p^i(t) + R(t) - r(t) \\ \mu_S^i(t) - r(t) \end{pmatrix}.$$

(2.2) implies that the investors only disagree on the market price of pure monetary risk, and face the same market price of consumption risk, $\theta_\varepsilon \equiv \theta_\varepsilon^1 = \theta_\varepsilon^2$. Agent $i$'s perceived state-price density is given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \theta_\varepsilon(t)dw_\varepsilon(t) + \theta_M^i(t)dw_M^i(t)]. \tag{2.3}$$

No-arbitrage implies that, under standard regularity conditions, the price of any asset is given by a present value formula. In particular, the stock and money prices are given by

$$\begin{aligned} S(t) &= \frac{1}{\xi^i(t)} E_t^i \left[ \int_t^T \xi^i(s)\varepsilon(s)ds \right], \\ p(t) &= \frac{1}{\xi^i(t)} E_t^i \left[ \int_t^T \xi^i(s)R(s)p(s)ds \right], \qquad i = 1, 2, \end{aligned} \tag{2.4}$$

where $E_t^i$ denotes the time-$t$ conditional expectation under agent $i$'s beliefs. The first equation is standard. The second is intuitive, if one thinks of one unit of money as an asset worth $p$, and paying-off a continuous dividend equal to the nominal interest rate, $R$ (worth $Rp$ in real terms).

## 2.3. Investors' Optimization

Investor $i$ is endowed with $a^i > 0$ share of the stock, and $b^i > 0$ share of the money supply (with $a^1 + a^2 = b^1 + b^2 = 1$), so that his initial wealth is given by $W^i(0) = a^i S(0) + b^i p(0)M(0)$. He then chooses his consumption rate $c^i \geq 0$, money balance $m^i \geq 0$ and portfolio policy $\pi^i = (\pi_0^i, \pi_B^i, \pi_S^i)^\top$, where $\pi_0^i$, $\pi_B^i$ and $\pi_S^i$ denote the amounts of the numeraire invested by agent $i$ in the real riskless bond, the nominal bond and the stock, respectively, so as to maximize his cumulated lifetime expected utility $E^i\left[\int_0^T u^i(c^i(t), p(t)m^i(t))dt\right]$. Agent $i$'s utility $u^i$ of consumption and real money holding is assumed to be strictly increasing, strictly concave and three times continuously differentiable in both of its arguments, and to satisfy standard Inada conditions.[6] We define the gradient by $Du^i(c^i, pm^i) \equiv \left(\frac{\partial u^i}{\partial c^i}, \frac{\partial u^i}{\partial pm^i}\right)$; under our (standard) assumptions on $u^i$, it has an inverse, denoted by $J^i(\cdot,\cdot)$. An admissible policy $(c^i, m^i, \pi^i)$ is one such that the resulting wealth process $W^i(t) = p(t)m^i(t) + \pi_0^i(t) + \pi_B^i(t) + \pi_S^i(t)$ is bounded from below, satisfies $W^i(T) \geq 0$ and obeys the dynamic budget constraint:

$$\begin{aligned} &dW^i(t) = \left[W^i(t)\, r(t) - c^i(t)\right] dt \\ &+\pi_B^i(t)\left[\left(\mu_p^i(t) + R(t) - r(t)\right) dt + \sigma_{p\varepsilon}(t)\, dw_\varepsilon(t) + \sigma_{pM}(t)\, dw_M^i(t)\right] \\ &+\pi_S^i(t)\left[\left(\mu_S^i(t) - r(t)\right) dt + \sigma_{S\varepsilon}(t)\, dw_\varepsilon(t) + \sigma_{SM}(t)\, dw_M^i(t)\right] \\ &+p(t)\, m^i(t)\left[\left(\mu_p^i(t) - r(t)\right) dt + \sigma_{p\varepsilon}(t)\, dw_\varepsilon(t) + \sigma_{pM}(t)\, dw_M^i(t)\right] \\ &+b^i p(t)\, M(t)\left[\left(\mu_M^i(t) + \sigma_{pM}(t)\,\sigma_{MM}(t) + \sigma_{P\varepsilon}(t)\sigma_{M\varepsilon}(t)\right) dt + \sigma_{M\varepsilon}(t)dw_\varepsilon(t) + \sigma_{MM}(t)\, dw_M^i(t)\right] \end{aligned}$$

[6] That is, denoting the first derivatives of $u^i$ by $u_c^i \equiv \frac{\partial u^i}{\partial c}$, $u_m^i \equiv \frac{\partial u^i}{\partial m}$, $u_c^i(0) = u_m^i(0) = \infty$ and $u_c^i(\infty) = u_m^i(\infty) = 0$.

The first three lines of the dynamic budget constraint are standard, capturing the impact of consumption and gains from trading in securities. The fourth line is equal to $m^i(t)(dp(t) - p(t)r(t)dt)$ and accounts for the changes in the value of investor $i$'s money holding due to inflation. The last line (equal to $b^i(p(t)+dp(t))dM(t)$) captures investor $i$'s endowment of the money supply.

Standard martingale techniques (Cox and Huang (1989), Karatzas et al. (1987)) imply that investor $i$'s problem is equivalent to the following static problem:

$$\max_{c^i, m^i} E^i \left[ \int_0^T u^i \left( c^i(t), p(t) m^i(t) \right) dt \right]$$

$$\text{s.t.} \quad E^i \left[ \int_0^T \xi^i(t) \left( c^i(t) + R(t) p(t) m^i(t) \right) dt \right] \leq \xi^i(0) a^i S(0) + b^i E^i \left[ \int_0^T \xi^i(t) R(t) p(t) M(t) dt \right]. \tag{2.5}$$

The static budget constraint (2.5) takes into account, in the left hand side, the present value of the investor's lifetime consumption, as well as the cost of holding a money balance in the amount given by $m^i$; this cost is given by the interest that would be paid to borrow that amount (or, equivalently, the interest that is lost by holding a money balance instead of investing), i.e., $Rm^i$ in units of money, or $Rpm^i$ in units of consumption good. Similarly, receiving the endowment of money $b^iM$ is equivalent to receiving a stochastic endowment at a rate of $b^iRpM$, the income that would be obtained by investing the endowment, and this stochastic endowment appears in the right hand side, together with the value of the stock that agent $i$ is endowed with.

Proposition 2.1 characterizes investor $i$'s optimal policy.

**Proposition 2.1.** *Investor $i$'s optimal consumption $\hat{c}^i$ and money holding $\hat{m}^i$ solve*

$$u_c^i(\hat{c}^i(t), p(t)\hat{m}^i(t)) = y^i \xi^i(t), \tag{2.6}$$

$$u_m^i(\hat{c}^i(t), p(t)\hat{m}^i(t)) = y^i \xi^i(t) R(t), \tag{2.7}$$

*where $y^i$ is such that the investor's static budget constraint holds with equality, i.e.,*

$$E^i \left[ \int_0^T \xi^i(t) \left( \hat{c}^i(t) + R(t) p(t) \hat{m}^i(t) \right) dt \right] = \xi^i(0) a^i S(0) + b^i E^i \left[ \int_0^T \xi^i(t) R(t) p(t) M(t) dt \right].$$

In addition to characterizing agent $i$'s optimal consumption, equations (2.6)-(2.7) reveal that the nominal short rate is given by the ratio of the marginal utility of holding money to the marginal utility of consumption. This is intuitive: by holding the money balance $m^i$, agent $i$ gives up investing this amount of money and receiving interest on it, at a rate of $R$, which is thus the implicit cost of holding money. At the optimum, this cost should be proportional to the marginal benefit of holding money, or the value of the services rendered by holding money.

*The case of separable preferences.* For tractability and clarity of our intuitions, we often make the assumption that both investors' preferences are separable with respect to consumption and money holdings, i.e.,

$$u^i(c, pm) = v^{ic}(c) + v^{im}(pm), \qquad i = 1, 2,$$

and refer to this case as the "separable case."

## 3. Equilibrium

**Definition 3.1 (Competitive Equilibrium).** *An* equilibrium *is a price system $(r, R, S, p)$ and admissible policies $(c^i, m^i, \theta^i)$, $i = 1, 2$ such that: (i) agents choose their optimal policies given their beliefs; and (ii) markets for consumption, money and securities clear, i.e.,*

$$\begin{aligned} c^1(t) + c^2(t) &= \varepsilon(t), & (3.1)\\ m^1(t) + m^2(t) &= M(t), & (3.2)\\ \pi_B^1(t) + \pi_B^2(t) &= 0, & (3.3)\\ \pi_S^1(t) + \pi_S^2(t) &= S(t), & (3.4)\\ W^1(t) + W^2(t) &= S(t) + p(t)M(t). & (3.5)\end{aligned}$$

For convenience in the exposition of our results, we introduce a representative investor with utility defined by

$$U(c, pm; \lambda) \equiv \max_{c^1+c^2=c,\, m^1+m^2=m} u^1(c^1, pm^1) + \lambda u^2(c^2, pm^2), \tag{3.6}$$

where the relative weight $\lambda > 0$ is allowed to be stochastic. We denote the partial derivatives of $U$ by $U_c \equiv \frac{\partial U}{\partial c}$, $U_{cc} \equiv \frac{\partial^2 U}{\partial c^2}$, etc..

Proposition 3.1 provides a characterization of equilibrium asset prices in our economy.

**Proposition 3.1.** *Assume that an equilibrium exists. Then, the investors' state-price densities are given by*

$$\xi^1(t) = U_c(\varepsilon(t), p(t)M(t); \lambda(t)), \qquad \xi^2(t) = \frac{\lambda(0)U_c(\varepsilon(t), p(t)M(t); \lambda(t))}{\lambda(t)}, \tag{3.7}$$

*and their consumptions and real money holdings by*

$$\begin{aligned} (c^1(t), p(t)m^1(t)) &= J^1(U_c(\varepsilon(t), p(t)M(t); \lambda(t)), U_m(\varepsilon(t), p(t)M(t); \lambda(t))),\\ (c^2(t), p(t)m^2(t)) &= J^2\left(\frac{U_c(\varepsilon(t), p(t)M(t); \lambda(t))}{\lambda(t)}, \frac{U_m(\varepsilon(t), p(t)M(t); \lambda(t))}{\lambda(t)}\right),\end{aligned}$$

*where $\lambda$ has dynamics*

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M(t) dw_M^1(t) = -\bar{\mu}_M(t)\left(\frac{\mu_M(t) - \mu_M^1(t)}{\sigma_M(t)} dt + dw_M(t)\right) \tag{3.8}$$

*and $\lambda(0) = 1/y^2$, where $y^2$ satisfies investor 2's static budget constraint with equality (equation (2.5)).*[7]

[7] The investors' budget constraints are equivalent, and only determine the ratio $y^1/y^2$. Without loss of generality, we set $y^1 = 1$.

*The nominal interest rate and money price are given by*

$$R(t) = \frac{U_m(\varepsilon(t), p(t)M(t); \lambda(t))}{U_c(\varepsilon(t), p(t)M(t); \lambda(t))} \tag{3.9}$$

$$p(t) = \frac{1}{U_c(\varepsilon(t), p(t)M(t); \lambda(t))} E_t^1 \left[ \int_t^T U_m(\varepsilon(s), p(s)M(s); \lambda(s)) p(s) ds \right] \tag{3.10}$$

The main difference with a standard model with homogeneous beliefs is that the two investors effectively face different state-prices densities; this is not surprising if one recalls the interpretation of a state-price density as a state-price per unit of probability, and that under heterogeneous beliefs the investors assign different probabilities to the possible states. Thus, the equilibrium allocation is not Pareto efficient, and only solves the representative agent's problem if the agents' relative weight $\lambda$ is allowed to be stochastic (making the representative agent's preferences state-dependent) (Cuoco and He (1994)).

The relative weight of the two agents in the economy is driven by the heterogeneity in beliefs $\bar{\mu}_M$. The intuition for this is simple: the more optimistic investor (with the higher estimate for $\mu_M$) invests in a portfolio that is more positively correlated with the money supply than the more pessimistic investor. Effectively, he is betting (against the pessimistic investor) that the money supply will grow by a lot. If the realization of the Brownian motion representing pure monetary risk is high, he wins his bet, and his weight in the economy increases, at the expense of the other investor. In this case, not only does he hold more money, he also consumes more, even if preferences are separable. This is in contrast with a model without heterogeneity in beliefs, where pure monetary risk would affect consumptions only indirectly, via the effect of money holdings on marginal utility of consumption (an effect that disappears in the separable case).

The nominal interest rate provides a measure of the value of the services rendered by holding money (relative to the marginal utility of consuming), and is driven primarily by the quantity of money in the economy, relative to aggregate consumption. While a similar interpretation holds for the price of money itself, its value takes into account the future "dividends" from holding money (i.e., the future transaction services rendered by money), and as a result its price follows a backward stochastic differential equation ((3.10)) that is difficult to analyze.

Proposition 3.2 provides more explicit characterization of the real asset prices.

**Proposition 3.2.** *In equilibrium, the market prices of aggregate consumption risk and pure monetary risk are given by*

$$\theta_\varepsilon(t) = A_c(t)\varepsilon(t)\sigma_\varepsilon(t) + A_m(t)p(t)M(t)\left(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)\right),$$

$$\theta_M^1(t) = A_m(t)p(t)M(t)\left(\sigma_{MM}(t) + \sigma_{pM}(t)\right) - A_\lambda(t)\lambda(t)\bar{\mu}_M(t) \qquad \theta_M^2(t) = \theta_M^1(t) - \bar{\mu}_M(t),$$

*where* $A_j(t) \equiv -U_{cj}(\varepsilon(t), p(t)M(t); \lambda(t))/U_c(\varepsilon(t), p(t)M(t); \lambda(t)),\ j \in \{c, m, \lambda\}$.

*The real interest rate is given by*

$$\begin{aligned} r(t) &= A_c(t)\varepsilon(t)\mu_\varepsilon(t) + A_m(t)p(t)M(t)\left(\mu_M^1(t) + \mu_p^1(t) + \sigma_{MM}(t)\sigma_{pM}(t) + \sigma_{p\varepsilon}(t)\sigma_{M\varepsilon}(t)\right) \\ &\quad + \frac{1}{2}B_{cc}(t)\varepsilon(t)^2\sigma_\varepsilon(t)^2 + \frac{1}{2}B_{mm}(p(t)M(t))^2\left[(\sigma_{MM}(t)+\sigma_{pM}(t))^2 + (\sigma_{p\varepsilon}(t)+\sigma_{M\varepsilon}(t))^2\right] \\ &\quad + B_{cm}(t)\varepsilon(t)p(t)M(t)\sigma_\varepsilon(t)\left(\sigma_{p\varepsilon}(t)+\sigma_{M\varepsilon}(t)\right) \\ &\quad + \frac{1}{2}B_{\lambda\lambda}(t)\lambda(t)^2\left(\bar{\mu}_M(t)\right)^2 - B_{m\lambda}\lambda(t)p(t)M(t)\left(\sigma_{MM}(t)+\sigma_{pM}(t)\right)\bar{\mu}_M(t), \end{aligned} \tag{3.11}$$

*where* $B_jk(t) \equiv -U_{cjk}(\varepsilon(t), p(t)M(t); \lambda(t))/U_c(\varepsilon(t), p(t)M(t); \lambda(t))$, $j, k \in \{c, m, \lambda\}$.

While the market price of aggregate consumption risk is only indirectly affected by the disagreement (via its effect on $\lambda$), the market prices of pure monetary risk are directly affected.

If for ease of exposition we assume that both investors' preferences are separable, implying that all the cross-partial derivatives $U_{cm}$, $U_{cmm}$, $U_{ccm}$, $U_{cm\lambda}$ are zero (as well as $A_m$), we note that the market price of pure monetary risk is only non-zero in the presence of heterogeneity in beliefs. This is intuitive: under homogeneous beliefs, a shock affecting only the money supply only has an impact on the investors' money balances, not on their consumptions, and so is not correlated with the marginal utility of consumption. Thus, exposure to such risks is not rewarded by a higher expected return. In the presence of heterogeneous beliefs, however, investors effectively place opposite bets on the evolution of the money supply. The investor who wins the bet receives wealth from the other investor, and his share of consumption increases. Thus, in this case pure monetary shocks impact agents' consumptions and so exposure to this type of risk is priced. The more optimistic agent facing a higher market price of monetary risk entices him to invest in a portfolio that is more positively correlated with monetary uncertainty, which is the mechanism he uses to "bet" that the money supply will grow a lot. Under separable preferences, algebraic manipulation shows that $A_\lambda(t) = -A^1(t)/(\lambda(t)(A^1(t) + A^2(t)))$, where $A^i = -u^i_{cc}/u^i_c$ is agent $i$'s absolute risk aversion, and so we have: $\theta^1_M(t) = (A^1(t)/(A^1(t) + A^2(t)))\bar{\mu}_M(t)$, $\theta^2_M(t) = -(A^2(t)/(A^1(t) + A^2(t)))\bar{\mu}_M(t)$. Under heterogeneity in beliefs, investors must take the opposite sides of the same bet, and so it is necessary to reward monetary risk-taking in proportion with their risk-aversions (to take a bet of the same size, the more risk-averse agent needs to be rewarded more.)

When preferences are not separable, this interpretation remains essentially true, but there is an additional effect: independently of heterogeneity in beliefs, investors' marginal utilities of consumptions are affected by monetary shocks. This indirect effect of monetary shocks is captured by the terms in $A_m = -U_{cm}/U_c$: investors receive extra return to make up for their utility of consumption being affected by monetary shocks.

The real interest rate is also affected by the disagreement about expected monetary policy, which holds whether or not preferences are separable. If one recalls the familiar interpretation of the real interest rate as minus the instantaneous expected rate of change of marginal utility, this is not surprising. The heterogeneity in beliefs, via its impact on $\lambda$, increases the variability of consumption, because on top of the fundamental uncertainty, agents' consumptions are affected by their betting against each other. In the expression for the interest rate (3.11), the first three lines are almost as in a standard model (risk aversion times consumption growth plus one half times consumption variance times prudence), but made more complicated by the inclusion of monetary risk in addition to consumption risk. The terms in the last line, however, arise only under heterogeneity in beliefs, reflecting the impact of disagreement risk on the investors' precautionary savings motive: in general, the extra risk under heterogeneity in beliefs entices investors to save more, and the interest rate must be lowered to counteract this tendency (so that markets clear).

These results show that, under heterogeneous beliefs, there is a "spillover" effect of the monetary sphere into the real side of the economy, while models assuming homogeneous beliefs might erroneously conclude that there is no effect (with money being neutral under separable preferences) or only a limited, indirect effect. In later sections, we will be able to assess the size of this effect and its importance for asset pricing.

In general, it is not possible to solve explicitly for the price of money and the price of the stock (although this is possible in some setups that we will investigate in the next sections). It is possible, however, to provide meaningful expressions for the stock risk premium and the expected inflation. An alternative way to state the first result in Proposition 3.2 is an appropriately modified version of the familiar CCAPM:

$$\begin{aligned}
\mu_S^1(t) - r(t) &= A_c(t)\varepsilon(t)\sigma_\varepsilon(t)\sigma_{S\varepsilon}(t) \\
&\quad + A_m(t)p(t)M(t)(\sigma_{S\varepsilon}(t)(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)) + \sigma_{SM}(t)(\sigma_{MM}(t) + \sigma_{PM}(t)) \\
&\quad - A_\lambda(t)\lambda(t)\bar{\mu}_M(t)\sigma_{SM}(t).
\end{aligned}$$

The first term is standard, the second arises from correlations between the stock and the real value of money balances and disappears in the separable case. The last term reflects heterogeneity in beliefs: the more bullish agent 1 is about money supply growth relative to agent 2, the higher his perceived expected return for the stock (assuming positive correlation between stock and money supply). This is intuitive: the higher stock expected return entices him to invest more in it, making his portfolio more positively correlated with the money supply; this is how the more bullish investor "places his bet" that the money supply is going to grow a lot. This makes it clear that heterogeneity in beliefs on the money supply appears as an additional factor in the pricing of financial assets, even in the separable case. More interestingly, although it may not be obvious, a similar result applies to inflation. When its value is measured in real terms, the nominally riskless bond is a risky asset and its expected return must also be consistent with the CCAPM, which implies:

$$\begin{aligned}
\pi^1(t) &= R(t) - r(t) - A_c(t)\varepsilon(t)\sigma_\varepsilon(t)\sigma_{P\varepsilon}(t) \\
&\quad - A_M(t)p(t)M(t)(\sigma_{p\varepsilon}(t)(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)) + \sigma_{pM}(t)(\sigma_{MM}(t) + \sigma_{MM}(t))) \\
&\quad + A_\lambda(t)\lambda(t)\bar{\mu}_M(t)\sigma_{pM}(t).
\end{aligned}$$

This expression emphasizes that, due to the inflation risk that is born by nominal bondholders, the nominal interest rate differs from the real riskless rate by more than expected inflation; it also includes a number of risk premia. The intuition on the effect of heterogeneity in beliefs on the stock return still applies here. It is, however, impossible to make an unambiguous prediction on the effect of heterogeneity in beliefs on inflation, because of its ambiguous effect on the real interest rate.

The most important insight of this section is that, under heterogeneous beliefs on monetary policy, agents place bets against each other on the money supply and so shocks in the money supply affect the distribution of wealth in the economy (proxied for by $\lambda$). Relative to the homogeneous beliefs case where the distribution of wealth is fixed, this generates extra risk for the investors (a particular kind of "trading risk"), leading to an extra factor in the pricing of assets, that subsists even when preferences are separable.

## 4. The Case of Separable Logarithmic Preferences

In this section, we make a number of simplifying assumptions, Assumptions 4.1 and 4.2 below. This allows us to explicitly solve all quantities in our economy.

**Assumption 4.1.** *Both investors are have separable, logarithmic preferences, i.e.*

$$u^i(c^i, pm^i) = \phi \log(c^i) + (1-\phi)\log(pm^i), \qquad i = 1, 2,$$

*where* $\phi \in (0,1)$.

**Assumption 4.2.** *All dynamics coefficients for the aggregate consumption and the money supply, as well as the two investors' estimates for these, are constant, i.e.,* $\mu_\varepsilon$, $\sigma_\varepsilon$, $\mu_M$, $\sigma_{MM}$, $\sigma_{M\varepsilon}$, $\mu_M^1$, $\mu_M^2$ *are constant.*

Assumption 4.1 implies that the representative agent utility function can be computed explicitly. We have:

$$U(c, pm; \lambda) = \phi \left[\log\left(\frac{c}{1+\lambda}\right) + \lambda \log\left(\frac{\lambda c}{1+\lambda}\right)\right] + (1-\phi)\left[\log\left(\frac{pm}{1+\lambda}\right) + \lambda \log\left(\frac{\lambda pm}{1+\lambda}\right)\right]$$

Assumption 4.2 implies that both the aggregate consumption and the money supply follow geometric Brownian motions, both under the objective probability and under each investor's beliefs. It also implies that the disagreement process, $\bar{\mu}_M$, is a constant as well. The fact that $\mu_M^1$ and $\mu_M^2$ are constant means that the investors do not update their beliefs as more observations of the money supply become available. This is not an appealing assumption, but adds considerable tractability and allows us to derive many explicit results. In addition, in real-life it is debatable whether most investors update their beliefs at all, and how fast they do it. Models with rational Bayesian updating in continuous-time seem to overstate the speed at which investors update their beliefs. Our case could be viewed as an approximation of a case where agents update slowly (or of an overlapping generations model where new, inexperienced investors continuously enter the market, leading to constant aggregate heterogeneity in beliefs in the economy). In fact, we can solve the model with Bayesian updating in the case where agents have normally distributed prior beliefs on $\mu_M$ (with an identical variance across agents, implying that $\bar{\mu}_M$ is deterministic); equilibrium expressions are affected and made more complicated, but our qualitative insights remain unchanged.[8]

## 4.1. Characterization of Equilibrium

Proposition 4.1 reports equilibrium allocations and prices under Assumptions 4.1 and 4.2.

**Proposition 4.1.** *In equilibrium, the consumption and money holdings, state-price densities, money price and nominal interest rate are as follows:*

$$
\begin{aligned}
c^1(t) &= \frac{\varepsilon(t)}{1+\lambda(t)}, \qquad c^2(t) = \frac{\lambda(t)\varepsilon(t)}{1+\lambda(t)}, \\
m^1(t) &= \frac{M(t)}{1+\lambda(t)}, \qquad m^2(t) = \frac{\lambda(t)M(t)}{1+\lambda(t)}, \\
\xi^1(t) &= \phi\frac{1+\lambda(t)}{\varepsilon(t)}, \qquad \xi^2(t) = \phi\frac{\phi a^2+(1-\phi)b^2}{\phi a^1+(1-\phi)b^1}\frac{1+\lambda(t)}{\lambda(t)\varepsilon(t)}, \qquad (4.1) \\
p(t) &= \frac{1-\phi}{\phi}\frac{F^1(t)+\lambda(t)F^2(t)}{1+\lambda(t)}\frac{\varepsilon(t)}{M(t)}, \\
R(t) &= \frac{1+\lambda(t)}{F^1(t)+\lambda(t)F^2(t)}, \qquad (4.2)
\end{aligned}
$$

*where*

$$
F^i(t) = \frac{\exp[(\sigma_M^2-\mu_M^i)(T-t)]-1}{\sigma_M^2-\mu_M^i}, \qquad i=1,2, \qquad (4.3)
$$

*and the weighting process is given by*

$$
\lambda(0) = \frac{\phi a^2+(1-\phi)b^2}{\phi a^1+(1-\phi)b^1}, \qquad \frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M dw_M^1(t).
$$

Under logarithmic preferences, both investors' consumptions and money holdings are proportional to their shares of the wealth in the economy ($1/(1+\lambda)$ for agent 1 and $\lambda/(1+\lambda)$ for agent 2). At time 0, the agents' weights in the economy are proportional to their initial endowments (i.e., $\lambda(0) = (\phi a^2+(1-\phi)b^2)/(\phi a^1+(1-\phi)b^1)$) but, under heterogeneous beliefs, they evolve stochastically as agents place bets on the evolution of the money supply against each other, and the winning investor becomes wealthier.

One of the main difficulties in our model is that, in general, the price of money solves a backward stochastic differential equation ((3.10)) involving the future values of $p$. Such an

[8] The model remains tractable because, under deterministic dynamics for $\bar{\mu}_M$, the stochastic weighting $\lambda$ is lognormally distributed, as it is under Assumption 4.2, and prices can still be computed explicitly.

equation typically is very difficult, if not impossible, to solve explicitly. The key simplification that makes the logarithmic case tractable is that, due to the properties of the logarithmic function, the price of money "separates out" of the utility of real money holdings. The marginal utility of real money holdings is inversely proportional to the money price $p$, and so the product $Rp = (U_m/U_c)/p$ does not depend on $p$. As a result, the future values of $p$ disappear from the equation for $p$, and the computation can be performed explicitly.

Both money price and nominal interest rate are equal to weighted averages (more specifically, in the case of the nominal interest rate, a harmonic mean) of their values in an (otherwise identical) homogeneous beliefs economy populated only with agents of each type, 1 and 2; unsurprisingly, the weights used in this averaging are the agents' weakth shares $(1/(1+\lambda)$ and $\lambda/(1+\lambda))$. In the homogeneous beliefs economy, the price of money – or the present value of future services rendered by money, measured by the marginal utility of money holdings – is given by $p(t) = ((1-\phi)/\phi)\varepsilon(t)F^i(t)/M(t)$, where $F^i(t)$ is as in equation (4.3) or, equivalently, $F^i(t) = E^i[\int_t^T (1/M(s))ds]/(1/M(t))$. $F^i(t)$ can be viewed as a measure of the expected future marginal utilities from money holdings (relative to the current marginal utility). Unsurprisingly, the relative price of money is decreasing in the current quantity of money (and increasing in the aggregate endowment), but it is also decreasing in $\mu_M^i$: the more the money supply is expected to grow in the future, the less valuable money is now.

The nominal short rate is the payoff an investor is willing to give up in order to hold a money balance, which equals, at the agent's optimum, the marginal utility of holding money relative to marginal utility of consumption. In the logarithmic case, we have $R(t) = ((1-\phi)/\phi)\varepsilon(t)/(p(t)M(t))$. Substituting $p(t)$, we see that, in the homogeneous belief economy with agents of type $i$, $R(t) = 1/F^i(t)$: the nominal interest rate is purely driven by expectations on the future growth of money, and is deterministic. While this may appear surprising, the real interest flow to a nominally riskless bond holder is given by $R(t)p(t) = ((1-\phi)/\phi)\varepsilon(t)/M(t)$; the nominal short rate makes up for fluctuations in the price of money that are due to future expectations of money growthand, as a result, the real return to nominal bondholders is independent of future expectations. In our heterogeneous beliefs economy, the nominal short rate is additionally affected by the distribution of wealth in the economy, which fluctuates in response to monetary shocks. Interestingly, despite the myopic behavior associated with logarithmic utility, future expectations of money supply growth play a key role in the pricing of monetary assets.

Real prices, however, are largely unaffected. The real short rate, stock price and stock dynamics are as in a standard economy:

$$r(t) = \mu_\varepsilon - \sigma_\varepsilon^2, \quad S(t) = (T-t)\varepsilon(t), \quad \mu_S(t) = \mu_\varepsilon, \quad \sigma_{S\varepsilon}(t) = \sigma_\varepsilon, \quad \sigma_{SM}(t) = 0.$$

The market prices of risk, in contrast, are affected by heterogeneity in beliefs, and are as provided by Proposition 4.2.

**Proposition 4.2.** *In equilibrium, the market prices of risk are as follows:*

$$\theta_\varepsilon(t) = \sigma_\varepsilon(t), \qquad \theta^1_M(t) = \frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}_M, \qquad \theta^2_M(t) = -\frac{1}{1+\lambda(t)}\bar{\mu}_M$$

While the market price of consumption risk remains as in a standard economy, due to the heterogeneity in beliefs, the market prices of monetary risk are individual-specific. The investor who expects a higher money supply growth rate faces a higher market price of monetary risk, as he expects any asset with a return positively correlated with the money supply to have a higher return than the other investor does. From the viewpoint of consumption smoothing, such an asset is not very valuable to him, because if the money supply increases by a lot, he will become wealthier and so he is not willing to pay a lot for this asset. Thus, he faces a high expected return for the asset. In the case of homogeneous beliefs, agents' shares of consumption are fixed, and pure monetary risk is not correlated with agents' consumptions. Thus, the market price of monetary risk is zero.

## 4.2. The price of money and inflation

The dynamics of the price of money are as follows. The expected inflation is:

$$\begin{aligned}\pi^1(t) \;=\;& \mu^1_M - \mu_\varepsilon - \sigma^2_M + \rho\sigma_\varepsilon\sigma_M \\ &+ \frac{1}{F^1(t)+F^2(t)\lambda(t)}\left[\frac{F^2(t)-F^1(t)}{1+\lambda(t)}\lambda(t)\bar{\mu}_M\left(\frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}_M - \sigma_M\right)\right. \\ &\left. -(\mu^1_M - \sigma^2_M)F^1(t) - (\mu^2_M(t) - \sigma^2_M)F^2(t)\lambda(t) + (1+\lambda(t))\right]\end{aligned}$$

and the diffusion coefficients are:

$$\sigma_{p\varepsilon}(t) = \sigma_\varepsilon - \sigma_{M\varepsilon}, \qquad \sigma_{pM}(t) = -\sigma_{MM} - \frac{F^2(t)-F^1(t)}{F^1(t)+F^2(t)\lambda(t)}\left(\frac{\lambda(t)}{1+\lambda(t)}\right)\bar{\mu}_M.$$

While its effect on expected inflation is ambiguous, the heterogeneity in beliefs is revealed to increase the volatility of the price of money. Assuming a positive correlation between money supply and economic activity ($\rho \geq 0$), the higher the heterogeneity in beliefs, the more the price of money drops in response to unexpected increases in the money supply (as the product $(F^2(t) - F^1(t))\bar{\mu}_M(t)$ is positive, and increases with heterogeneity in beliefs). This is intuitive: when such a shock occurs, not only does the quantity of money increases, but the investor who expects higher money growth wins his bet, and so his weight in the economy increases; since he is also the one who places a lower value on the future services rendered by money (and thus expects higher inflation), the price of money decreases further. As is the case for most quantities in this economy, the price of money is essentially a weighted average of the two investors' private valuations; when a positive shock to the money supply occurs, both agents' valuations of money drop, and the weight of the lower of the two valuations increases. These two effects reinforce each other, leading to more volatile inflation.

## 4.3. Nominal interest rates

The dynamic behavior of the nominal short rate is dramatically affected by heterogeneity in beliefs as, in this setup, it would be deterministic under homogeneous beliefs. Its dynamics follow the Itô process: $dR(t) = \mu_R(t)dt + \sigma_R(t)dw_M^1(t)$, where

$$\begin{aligned}
\mu_R(t) &= R(t)^2 \frac{F^2(t)(F^2(t) - F^1(t))}{(1 + \lambda(t))^3} \lambda(t)^2 \bar{\mu}_M^2 \\
&+ \frac{R(t)^2}{1 + \lambda(t)} \left\{ \exp[(\sigma_M^2 - \mu_M^1)(T - t)] + \lambda(t) \exp[(\sigma_M^2 - \mu_M^1)(T - t)] \right\}, \\
\sigma_R(t) &= \frac{F^2(t) - F^1(t)}{(F^1(t) + F^2(t)\lambda(t))^2} \lambda(t)\bar{\mu}_M.
\end{aligned}$$

The drift appears to be ambiguously related to heterogeneity in beliefs (and other variables), but could potentially generate interesting behavior on the part of the nominal interest rate. By substituting (4.2), we can express $\mu_R$ as a function of $R$ itself; for some parameter values, this function is decreasing, meaning that our model can generate mean-reversion of the nominal interest rate. The diffusion coefficient reveals that the nominal short rate is positively correlated with pure monetary uncertainty, and its volatility is made stochastic and increased by the heterogeneity in beliefs. When a positive shock occurs to the money supply, the price of money drops as the investor expecting higher money growth becomes more wealthy, and the nominal interest rate increases to make up for this, and ensure that the real interest received from nominal bonds ($Rp$) remains commensurate with the quantity of money in the economy. In our model, the behavior of the nominal interest rate may appear puzzling, but the real return from nominally riskless bonds is negatively correlated with money supply and inflation, as is consistent with both intuition and empirical studies (see, e.g., Bakshi and Chen (1996) and the references therein).

In this separable logarithmic case, nominal zero-coupon yields for all future maturities can be computed explicitly. Proposition 4.3 characterizes the term structure in our economy.

**Proposition 4.3.** *The nominal time $t$-price of a nominal discount bond with maturity $\tau$, paying one unit of money at time $t + \tau$, is given by:*

$$B_m(t, \tau) = \frac{1}{F^1(t) + \lambda(t)F^2(t)} \left\{ F^1(t + \tau) \exp[(\sigma_M^2 - \mu_M^1)\tau] + \lambda(t)F^2(t + \tau) \exp[(\sigma_M^2 - \mu_M^2)\tau] \right\}.$$

*The zero-coupon yield with maturity $\tau$ is given by: $R(t, \tau) = -\frac{1}{\tau} \ln B_m(t, \tau)$.*

It is difficult to provide general results on the term structure in this model, and how it is affected by heterogeneity in beliefs, in particular because when heterogeneity in beliefs increases, it could be either that the more optimistic agent becomes more optimistic, or the less optimistic agent becomes even less optimistic, or both, and the effect on the term structure is different in each case. Accordingly, most comparative statics are ambiguous. Nonetheless, numerical analyses suggest that, when the more optimistic agent is more wealthy, nominal zero-coupon yields increase with heterogeneity in beliefs, and when he is less wealthy, yields decrease with heterogeneity in beliefs. This can be explained intuitively. We can rewrite the price of a discount bond as follows:

$$B_m(t,\tau) = \frac{M(t)}{F^1(t) + \lambda(t)F^2(t)} \left\{ F^1(t+\tau)E_t^1\left[\frac{1}{M(t+\tau)}\right] + \lambda(t)F^2(t+\tau)E_t^2\left[\frac{1}{M(t+\tau)}\right] \right\}.$$

In this setup, real interest rates are constant, and so the only type of risk affecting these nominal bonds is that related to money supply growth. Assume that agent 1 is more optimistic. If he is more wealthy (low $\lambda$), the quantity that plays the most important role in pricing the bond is $E_t^1[1/M(t+\tau)]$, driven by his expectation of money supply growth $\mu_M^1$. When heterogeneity in beliefs increases, $\mu_M^1$ generally increases, $E_t^1[1/M(t+\tau)]$ decreases, and agent 1's valuation of the bond's payoff (one unit of money at time $t+\tau$) decreases: since there is expected to be more money available in the future, this payoff is less valuable, and the bond's price drops, leading to an increase in its yield. A symmetric argument applies when the more pessimistic agent is more wealthy, and thus plays a dominant role in setting bond prices. When heterogeneity in beliefs increases and his expectation of money supply drops, his valuation of future nominal payoffs increases and yields decrease. As is the case for other quantities in this economy, the price of a discount bond is essentially a weighted average of the two agents' valuations of the corresponding payoffs. Other observations that can be made based on numerical simulations are that the yield curve is generally increasing, and that the higher the heterogeneity in beliefs, the less steep its slope is (to the point that, in some cases, the yield curve can be decreasing for high heterogeneity in beliefs).

## 5. The Case of Separable Power Preferences

In this Section, we maintain Assumption 4.2, but replace Assumption 4.1 with Assumption 5.1. While an explicit computation of all monetary quantities is not possible any more, interesting implications for real asset prices can be derived.

**Assumption 5.1.** *Both investors have separable, constant relative risk aversion preferences:*

$$u\left(c^i, pm^i\right) = \phi\frac{\left(c^i\right)^{1-\alpha}}{1-\alpha} + (1-\phi)\frac{\left(pm^i\right)^{1-\beta}}{1-\beta}, \qquad i = 1, 2,$$

*where $\alpha, \beta > 1$ and $\phi \in (0,1)$.*

Assumption 5.1 implies that the representative agent utility function is given by

$$U(c, pm; \lambda) = \phi\left(1+\lambda^{\frac{1}{\alpha}}\right)^{\alpha}\frac{(c)^{1-\alpha}}{1-\alpha} + (1-\phi)\left(1+\lambda^{\frac{1}{\beta}}\right)^{\beta}\frac{(pm)^{1-\beta}}{1-\beta}.$$

## 5.1. Characterization of equilibrium

Proposition 5.1 characterizes the equilibrium.

**Proposition 5.1.** *In equilibrium, the consumption and money holdings, state-price densities, money price and nominal interest rate are as follows:*

$$c^1(t) = \frac{\varepsilon(t)}{1+\lambda(t)^{\frac{1}{\alpha}}}, \qquad c^2(t) = \frac{\lambda(t)^{\frac{1}{\alpha}}\varepsilon(t)}{1+\lambda(t)^{\frac{1}{\alpha}}}, \tag{5.1}$$

$$m^1(t) = \frac{M(t)}{1+\lambda(t)^{\frac{1}{\beta}}}, \qquad m^2(t) = \frac{\lambda(t)^{\frac{1}{\beta}}M(t)}{1+\lambda(t)^{\frac{1}{\beta}}},$$

$$\xi^1(t) = \phi\frac{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^\alpha}{\varepsilon(t)^\alpha}, \qquad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\xi^1(t), \tag{5.2}$$

$$p(t) = \frac{\phi}{1-\phi}\frac{\varepsilon(t)^\alpha}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^\alpha}E_t^1\left[\int_t^T\frac{\left(1+\lambda(s)^{\frac{1}{\beta}}\right)^\beta}{M(s)^\beta}p(s)^{1-\beta}ds\right], \tag{5.3}$$

$$R(t) = \frac{1-\phi}{\phi}\frac{\left(1+\lambda(t)^{\frac{1}{\beta}}\right)^\beta}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^\alpha}\frac{\varepsilon(t)^\alpha}{p(t)^\beta M(t)^\beta},$$

*where the weighting process is given by*

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M dw_M^1(t)$$

*and* $\lambda(0)$ *solves agent 2's static budget constraint:*

$$E^2\left[\int_0^T \xi^2(t)\left(c^2(t)+R(t)p(t)m^2(t)\right)dt\right] = \xi^2(0)a^iS(0)+b^iE^2\left[\int_0^T\xi^2(t)R(t)p(t)M(t)dt\right].$$

Consumption and money balances are somewhat similar to the logarithmic case: agents share aggregate consumption and money supply according to their weights in the economy ($1/(1+\lambda)$ and $\lambda/(1+\lambda)$), that evolve stochastically, depending on the behavior of the money supply, and whose beliefs prove to be the most accurate. The sharing is not directly proportional to wealth, however, but depends on risk aversions. Unlike in the logarithmic case, the price of money does not "separate out" of the utility; the marginal utility of money holdings depends on the price of money, so does the real value of the interest received from nominal short term bonds, and future values of $p$ appear in the equation for the price of money ((5.3)), which cannot be solved explicitly. Similarly, the price of money affects the nominal interest rate and as a result, none of the monetary asset prices can be solved in closed-form. Real asset prices, however, can be solved explicitly. Proposition 5.2 provides the market prices of risk and the real short rate.

**Proposition 5.2.** *In equilibrium, the market prices of risk are as follows*

$$\theta_\varepsilon(t) = \alpha\sigma_\varepsilon, \qquad \theta_M^1(t) = \frac{\lambda(t)^{\frac{1}{\alpha}}}{1+\lambda(t)^{\frac{1}{\alpha}}}\bar{\mu}_M, \qquad \theta_M^2(t) = -\frac{1}{1+\lambda(t)^{\frac{1}{\alpha}}}\bar{\mu}_M$$

*and the real interest rate is given by*

$$r(t) = \alpha\mu_\varepsilon - \frac{1}{2}\alpha(\alpha+1)\sigma_\varepsilon^2 + \frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)\frac{\lambda(t)^{\frac{1}{\alpha}}}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^2}\bar{\mu}_M^2.$$

As in the logarithmic preferences case, the market price of consumption risk is as in a standard economy, but the market prices of monetary risk are individual-specific; the investor who expects higher money supply growth also faces a higher market price of monetary risk. Unlike in the logarithmic case, however, the interest rate is also affected, and is increased by the heterogeneity in beliefs. This is due to the investors' subjective assessment of their consumption growth being increased by the possibility to "bet" against one another: the higher the heterogeneity in beliefs, the more each agent believes the other is "wrong", and the more he expects to win from his bet, raising his expected consumption growth and, correspondingly, the real interest rate. Mathematically, one can see from equation (5.1) that expected consumption increases with heterogeneity in beliefs: for example, $c^1$ is convex in $\lambda$, and so expected consumption increases under higher volatility for the stochastic weighting.

## 5.2. The stock price and volatility and the equity premium

For the remainder of this section, we assume, with little loss of generality, that the investors' relative risk aversion, $\alpha$, is an integer.[9] This makes an explicit computation of the stock price possible, as reported in Proposition 5.3.

**Proposition 5.3.** *If $\alpha$ is an integer, the stock price is given by*

$$S(t) = \frac{\varepsilon(t)}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^\alpha}\sum_{i=0}^{\alpha}\binom{\alpha}{i}\lambda(t)^{\frac{i}{\alpha}}\frac{\exp[f(i)(T-t)]-1}{f(i)}, \tag{5.4}$$

*where $\binom{\alpha}{i}$ denotes the binomial coefficient $\alpha!/((\alpha-i)!i!)$ and $f(i) = (1-\alpha)\left(\mu_\varepsilon - \frac{1}{2}\alpha\sigma_\varepsilon^2\right) - \frac{\alpha i - i^2}{2\alpha^2}\bar{\mu}_m^2$. The stock volatility coefficients are given by*

$$\sigma_{S\varepsilon}(t) = \sigma_\varepsilon, \qquad \sigma_{SM}(t) = \left(\frac{\lambda(t)^{\frac{1}{\alpha}}}{1+\lambda(t)^{\frac{1}{\alpha}}} - \frac{\sum_{i=0}^{\alpha}\binom{\alpha}{i}\lambda(t)^{\frac{i}{\alpha}}\frac{i}{\alpha}\frac{\exp[f(i)(T-t)]-1}{f(i)}}{\sum_{i=0}^{\alpha}\binom{\alpha}{i}\lambda(t)^{\frac{i}{\alpha}}\frac{\exp[f(i)(T-t)]-1}{f(i)}}\right)\bar{\mu}_M.$$

The stock price is reduced by the presence of heterogeneity in beliefs. The intuition is similar to that for the increased real interest rate: both agents expect to profit from the heterogeneity in beliefs and "win their bets", leading to higher expected consumption growth, and a lower

[9] In the case where $\alpha$ is not an integer, one could use a Taylor series approximation around the results presented here.

value placed on future dividends; hence a decrease in the stock price. The total stock volatility ($(\sigma_\varepsilon^2 + \sigma_{SM}(t)^2)^{1/2}$) is unambiguously raised by the heterogeneity in beliefs. Under homogeneous beliefs, the stock volatility would equal that of aggregate consumption, but the heterogeneity in beliefs generates stochastic volatility, as the stock price becomes affected by pure monetary risk. Even though this risk is uncorrelated with aggregate consumption and dividends, as agents bet against each other, "trading risk" affects consumptions, stochastic discount factors and eventually the stock price. The sign of $\sigma_{SM}$ depends on which investor, the more optimistic or the more pessimistic, is the more wealthy. If the more optimistic investor is more wealthy, $\sigma_{SM} > 0$, and conversely when the more pessimistic agent is more wealthy. The less wealthy investor, due to his lower consumption and much steeper marginal utility is the one driving the stock price volatility, despite the fact that his weight in the economy is lower (for relative risk aversion $\alpha$ more than one). When he is "right" about the money supply (e.g., he is relatively optimistic and there is a positive shock in the money supply), his expected future consumptions increase, leading to a decreased valuation for future dividends and a lower stock price. The stock return in our model could be either positively or negatively correlated with the money supply, but is more likely to be positively correlated with it (which is consistent with empirical evidence), as $\mathrm{cov}_t(dS/S, dM/M) = \sigma_M(\rho\sigma_\varepsilon + (1-\rho^2)^{1/2}\sigma_{SM}(t))dt$.

The magnitude of this effect is significant (although, most likely, not sufficient to explain the stock volatility levels observed in real-life markets). For example, assuming the following parameter values: $\sigma_M$ =3.5355%, $\mu_\varepsilon$ =2.06%, $\sigma_\varepsilon$ =1.49% (all as in Basak and Gallmeyer (1999)), $\rho = 0.4$, $T = 50$ years, $\alpha = 2$, $\lambda = 1/3$ (i.e., the more optimistic investor is more wealthy, given that $\bar{\mu}_M > 0$), for $\bar{\mu}_M = 1$, the total stock volatility is 7.76% (vs. 1.49% in the absence of heterogeneity in beliefs). This amount of heterogeneity in beliefs seems plausible and not an extreme case: $\bar{\mu}_M = 1$ means that $\mu_M^1 - \mu_M^2 = 3.24\%$, a value that should be compared to a historical average of $\mu_M$ =4.89% (as reported in Basak and Gallmeyer (1999)), and that is sufficient to cause a five-fold increase in stock volatility.

To study the stock expected return, it is more intuitive to focus on the equity premium, which follows readily from our earlier results. Adopting the "full information" perspective of an observer who would know the true expected money supply growth,[10] the equity premium equals

$$\mu_S(t) - r(t) = \alpha\sigma_\varepsilon^2 + \frac{\sigma_{SM}(t)}{\sigma_{MM}}\left\{\mu_M - \left[\frac{1}{1+\lambda(t)^{1/\alpha}}\mu_M^1 + \frac{\lambda(t)^{1/\alpha}}{1+\lambda(t)^{1/\alpha}}\mu_M^2\right]\right\}. \tag{5.5}$$

The first term provides compensation for consumption risk, and equals the value of the equity premium in a standard economy without heterogeneity in beliefs. The term in the square bracket is essentially a wealth-weighted average of the agents' beliefs on money supply growth. Thus, how the equity premium in our model departs from this standard value depends on two things, the sign of $\sigma_{SM}(t)$ (studied above) on the one hand, and the difference between the average beliefs of

[10] In other words, we compute the stock expected return under the objective, "historical" probability measure that generates actual data, as opposed to the agents' subjective beliefs. This is the value of the equity premium that should be taken to the data to test our model.

the investors and the true value of money supply growth on the other hand. The equity premium is higher than in a standard economy if either the investors who are optimistic on money supply growth are more wealthy (so that $\sigma_{SM} > 0$) and the wealth-weighted average beliefs on $\mu_M$ are lower than the true value, or if the pessimistic investors are more wealthy and the average beliefs are higher. This is intuitive. Take the case where $\sigma_{SM} > 0$. If the agents' average beliefs are lower than the true value of $\mu_M$, it is more likely that agents will see (given their overly pessimistic beliefs) positive shocks to the money supply, leading to increases in the stock price. And conversely when $\sigma_{SM} < 0$. Thus, our model could potentially play a role in explaining the equity premium puzzle, as the second term in the right hand-side of (5.5) is typically of a much higher magnitude than the first, standard term. For example, for the above parameters, assuming that the more optimistic agents have correct beliefs (i.e., $\mu_M^1 = \mu_M$ =4.89%), the heterogeneity in beliefs makes the equity premium more than 70 times higher than in a standard model (2.83% vs. 0.04%).

## 6. Conclusion

This paper investigates the effects of heterogeneity in beliefs on future monetary policy on the pricing of money and financial assets. Under heterogeneous beliefs, investors use security markets to place bets on the future money supply against each other; thus, the weights of the two agents in the economy evolve stochastically, depending on which agent proves the better forecaster. This has profound implications on asset pricing, because it makes agents' consumptions and stochastic discount factors more volatile and correlated with the money supply (and in some cases, can break the neutrality of money). In particular, inflation is made more volatile: when a positive shock affects the money supply, not only does the increase generate inflation, but it also gives more weight to those agents who had higher expectations for money supply growth (as they win their bet) and so expected higher inflation in the first place; this generates yet extra inflation. Stock prices are also affected. In particular, the stock volatility is significantly increased by the heterogeneity in beliefs, as it creates a much greater dependence between monetary policy and the stock market. On top of its implications for asset pricing, our model sheds some new light on the issue of how much transparency is optimal on the part of central banks, because it is intuitive that greater transparency should decrease the amount of heterogeneity in beliefs on monetary policy, which, according to our model, would reduce stock market and inflation volatility (as well as decrease real interest rates). Natural extensions of our model would include assuming more sophisticated and realistic dynamics for the aggregate consumption and money supply, taking into account agents updating their beliefs over time, and a more thorough investigation of the implications for the price of money (that obeys a backward stochastic differential equation that is, in general, quite intractable) and interest rates. With more realistic, less tractable assumptions, our model could complement the growing, recent literature on the interrelation between monetary policy and bond yields (e.g., Piazzesi (2005)). All of these extensions would likely require the use of sophisticated numerical techniques.

## Appendix: Proofs

**Proof of Proposition 2.1:** The proof is an adaptation of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) to the MIUF case, and is similar to Basak and Gallmeyer (1999). *Q.E.D.*

**Proof of Proposition 3.1:** It can be easily checked that, for $\lambda = u_c^1/u_c^2$, the solution of the representative agent's problem coincides with the equilibrium allocation (as agents 1 and 2's optimality conditions and clearing in the good market hold). The stochastic weighting dynamics (3.8) then follow from applying Itô's lemma to the state-price density dynamics (2.3). The envelope theorem (applied to the optimization problem in the definition of the representative agent (3.6)) shows that $U_c = u_c^1 = y^1\xi^1$, and the expressions for the state-price densities follow. The investors' consumptions and money holdings obtain by applying Proposition 2.1. (3.9) and (3.10) follow, respectively, from agents' first-order conditions and from (2.4). *Q.E.D.*

**Proof of Proposition 3.2:** The expressions follow from applying Itô's lemma to the state-price density in (3.7). *Q.E.D.*

**Proof of Proposition 4.1:** The expressions obtain readily by substituting logarithmic utility into the expressions in Proposition 3.1. *Q.E.D.*

**Proof of Proposition 4.2:** The expressions follow from applying Itô's lemma to the state-price density in (4.1). *Q.E.D.*

**Proof of Proposition 4.3:** The bond pays one unit of money, worth $p(t+\tau)$ in real terms, at time $t+\tau$. Applying the standard present value formula, its nominal price is given by:

$$\begin{aligned} B_m(t,\tau) &= \frac{1}{p(t)} E_t^1\left[\frac{\xi^1(t+\tau)}{\xi^1(t)} p(t+\tau)\right] \\ &= \frac{M(t)}{F^1(t)+\lambda(t)F^2(t)}\left\{F^1(t+\tau)E_t^1\left[\frac{1}{M(t+\tau)}\right] + F^2(t+\tau)E_t^1\left[\frac{\lambda(t+\tau)}{M(t+\tau)}\right]\right\}, \end{aligned}$$

where the second equality follows by substituting the results in Proposition 4.1. The expectations can be computed explicitly due to both $1/M$ and $\lambda/M$ being lognormally distributed. *Q.E.D.*

**Proof of Proposition 5.1:** The expressions obtain readily by substituting power utility into the expressions in Proposition 3.1. *Q.E.D.*

**Proof of Proposition 5.2:** The expressions follow from applying Itô's lemma to the state-price density in (5.2). *Q.E.D.*

**Proof of Proposition 5.3:** The stock price obeys a standard present value formula:

$$\begin{aligned} S(t) &= \frac{1}{\xi^1(t)} E_t^1 \left[ \int_t^T \xi^1(s)\varepsilon(s) ds \right] \\ &= \frac{\varepsilon(t)^\alpha}{(1+\lambda(t)^{\frac{1}{\alpha}})^\alpha} E_t^1 \left[ \int_t^T (1+\lambda(s)^{\frac{1}{\alpha}})^\alpha \varepsilon(s)^{1-\alpha} ds \right] \\ &= \frac{\varepsilon(t)^\alpha}{(1+\lambda(t)^{\frac{1}{\alpha}})^\alpha} \int_t^T E_t^1 \left[ (1+\lambda(s)^{\frac{1}{\alpha}})^\alpha \right] E_t^1 \left[ \varepsilon(s)^{1-\alpha} \right] ds, \end{aligned}$$

where the second equality follows from substituting the results in Proposition 5.1, and the third equality from Fubini's theorem, and the fact that $\lambda$ and $\varepsilon$ are independent. Using the fact that, when $\alpha$ is an integer,

$$(1+\lambda(s)^{\frac{1}{\alpha}})^\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \lambda(s)^{\frac{i}{\alpha}},$$

algebraic manipulation yields (5.4). Applying Itô's lemma then yields the volatility coefficients. *Q.E.D.*

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