Optimization With Tail-Dependence and Tail Risk: A Copula Based Approach For Strategic Asset Allocation

Francesco Paolo Natale
Department of Economics and Management
Università Milano-Bicocca, Italy
Piazza dell'Ateneo Nuovo 1, 20126 Milano
Tel: 0039 – 02- 64483022
Fax: 0039 – 02 64483065
francesco.natale@unimib.it

Abstract

This paper proposes a method to overcome the classical drawbacks of the Monte Carlo methods for the asset allocation, namely *resampling*, deeply dependent upon the multinormal assumption. Differently from the *resampling,* the proposed approach allows to set a derivative-free barrier against joint extreme negative returns (tail-dependence) and extreme (negative) returns (univariate tail risk) not included in the multivariate normal distribution. The extremely dangerous tail-dependence between asset returns is considered by using a copula based approach instead of the multivariate normal Monte Carlo simulation. Then the proposed model has been applied on a sample of eleven euro-denominated asset classes with historical input and the consequent asset weights have been tested on multivariate Student's *t* returns and on a set of out-of-the sample real returns. A comparison has been also performed with both parametric and non-parametric simulation methods, namely *resampling* and bootstrapping. The results of this model provide evidence of a barrier against extreme negative returns occurring simultaneously. Furthermore the proposed model is totally distribution-free and therefore it does not involve any *a priori* decision on the marginal distributions for asset returns. The cost of this approach in terms of loss of Sharpe ratio, in our example, is negligible.

Keywords: *asset allocation, copula, tail index, Monte Carlo methods*

EFM Classification Code 370

Introduction

The financial planning process is characterized by three main steps:

1. identification of the strategic asset allocation, i.e. the set of efficient portfolios constituted by broad asset classes, summarized by benchmark indices;
2. identification of the best portfolio with the aim to achieve the goal of the investor in terms of wealth. This specific portfolio should be consistent with investor's preferences;
3. selection of specific assets available on the marketplace that match the strategic asset allocation or, alternatively, the definition of the asset allocation with a explicit trade-off between active risk and active return.

Each step has been carefully examined separately in the last decades.

The first step deals with both parameter uncertainty and optimization process. Various techniques have been proposed to solve the problems related to the choice of the inputs (see Jobson and Korkie (1981), Black and Littermann (1992)) and to reduce the impact of the input instability through the *resampling* (see Jorion (1992) and Michaud (1998)). The second step is indebted to the contribution of Leibowitz *et al.* (1991a), (1991b) which introduces a shortfall constraint and to the contributions of Rockafellar and Uryasev (1999) and Sentana (2001) which introduce parametric-VaR constraint and Iso-VaR constraint. It is worth emphasizing that both the shortfall constraint and the two different VaR constraints rely on the normality assumption of asset returns. Following the growing empirical evidence on the violation of the normality assumption Williams (1997) and Campbell, Huisman and Koedijk (2001) elaborated other constraints based on lognormal assumption and non-parametric assumption. The third step has grown on the wave of multimanager portfolio construction, inspired by the *Return-Based Style Analysis* of Sharpe (1992) extensively applied by Lucas (1999), Baierl and Chen (2000), Waring *et al.* (2000).

All the previous works assume that asset returns can be modeled using a parametric distribution (normal, lognormal or Student's *t*) and correlation coefficients are key inputs when setting dependences between asset classes. Unfortunately this is not the reality: extreme events can affect the results in the short and in the long term; marginal distributions can significantly differ from normal distribution (see Longin (1996)) and crashes occur more often than booms (see Peiro (1999)). Moreover constraints based on parametric assumptions are likely to fail.

An alternative approach, widely diffused in the literature[1] but less among practitioners, involves the maximization of the expected utility under the assumption of a given utility function.

In the more comprehensive case of elliptical distribution for asset returns the mean-variance approximation results is still viable for all utility functions (see Chamberlain (1983)). In contrast, not homogeneous and severely asymmetric distributions show that the mean-variance criterion does not correctly approximate the expected utility. In this case an higher moment optimization better approximates the expected utility (see Athayde and Flôres (2004)). However analytical closed solutions are available only if marginal distributions have defined functions such as the multivariate skewed Student's *t* (see Jondeau and Rockinger (2005)). Different marginal distributions do not have closed formulas to be applied yet. Recently, elegant non-parametric solutions to the optimization problem with co-skewness and co-kurtosis matrix have been proposed by Jondeau and Rockinger (2006). However they propose an approximation of the utility function given by Taylor expansion up to order four and thus they rely on a defined utility function (CARA) for the investor.

We propose an heuristic model to overcome some of the key drawbacks analyzed in the previous literature. First, our model allows to deal with very flexible and different return distributions (for each standardized innovation process) for each asset involved. For the sake of simplicity the example is restricted to a set of Student's $t(\nu)$ distributions with different degrees of freedom (ν). According to the empirical evidence daily return distributions usually show severe tails and skewness whereas monthly returns are usually fat-tailed but with negligible skewness. The former case is typical in the field of market risk management, while the latter is appropriate for

[1] A wide reference on these studies can be found in Jondeau and Rockinger (2006).

asset allocation purposes. Second, the parameters of each marginal distribution are fitted in a non-parametric way using some theoretical findings of the *Extreme Value Theory*. It is worth highlighting that a parametric estimation implies the choice of the marginal distribution by asset managers and therefore the entire process may be affected by these *a priori* decisions. We propose a method to obtain reliable estimates based on a set of Student's *t* distributions whose key parameters (a vector of different ν) are estimated with a fully non parametric method. The use of EVT also ensures a proper inclusion of the tail risk for each margin. Third, the known limits of the Pearson correlation coefficient (see Embrechts *et al.* (2002)) are overcome with the insertion of a set of Archimedean copulas whose parameters are again fitted with non-parametric relationships. The use of a rank correlation like Kendall's tau has two advantages: it ensures an higher stability of the inputs over the time and integrates the tail-dependence among asset classes.

The proposed model incorporates both the tail risk for each return distribution and the dependence among negative extremes highlighted by Longin and Solnik (2001). Tail risk is incorporated analyzing the tail index, a specific index for each asset return distribution. The theory behind tail index and its estimation (see Embrechts *et al.* (1997)) is applicable to events with no serial correlations. This feature requires the fit of a filtering process with standardized innovations given by a specific distribution. At this step we test the family of Student's *t* distribution whose vector of ν parameters is estimated with non-parametric methods. The choice of Student's *t* innovation process is the only *a priori* choice we made. However this family is very flexible and it can properly deal with fat tails. Tail dependence is an important issue that deals with multivariate distributions with joint extreme events. It is an important issue to include since it is well known that correlations raise during high-volatility market and therefore extreme negative events are likely to significantly reduce the performance of the portfolio. An explicit way to deal with tail-dependence is the extensive use of Archimedean copulas which allow to consider upper and lower tail-dependence. These topics experienced an increasing development and success in the modern finance related to the risk measurement issues (among the latest contributions see Longin (2005) and Yamai and Yoshiba (2005)), deeply concentrated on the left tail of the distributions.

Finally an example is provided using multivariate Monte Carlo simulation with different techniques and with relationships expressed by copulas. In order to provide a significant comparison the *resampling* method has also been applied and the two methods have been tested on a set of simulated asset returns. We also make an out-of-the sample test with real data in order to prove the potential of the proposed method.

The main findings are quite interesting. The financial planning theorem of the normal-lognormal distribution of asset returns should be reviewed by taking into account more realistic assumptions (related to the specific marginal distributions) and the chance to experiment extreme events. The events to be considered are those (the negative extremes) which effect the final wealth in a saving instrument like pension plan or life policy (pure endowment).

The paper proceeds as follows. Section 2 shortly discusses the concept of tail-dependence, Archimedean copula and EVT. Section 3 introduces the heuristic model that deal with tail risk and tail-dependence. Section 4 deals with the parameter estimates required by the heuristic model. Section 5 discusses the results and highlights the main difference with a naïve approach as the classical Quasi-Random Monte Carlo Simulation Asset Allocation (QRMCSAA) introduced by Michaud (1998) and greatly dependent upon the normality assumption and upon the linear correlation matrix. Section 6 concludes.

Archimedean copula, tail-dependence and EVT

Under the assumption of multivariate normality the dependence structure of the multivariate distribution is fully explained by the matrix of covariances or, alternatively, by the matrix of linear correlation coefficients. If distributions of returns significantly differ from multivariate normal then Pearson's coefficient is no longer suitable and can largely induce to misleading solutions. For distributions different from the normal the dependence structure is fully explained by a copula function. A wide reference to copula can be found in Joe (1997) and Nelsen (1999). More recently an excellent introduction to copulas and several financial applications can be found in McNeil *et al.* (2005), Rob van den Goorbergh (2004) and Embrechts *et al.* (2002).

Copulas rely on the fundamental Sklar's theorem (see Sklar (1996) and Nelsen (1999)).

Sklar's theorem states that any multivariate distribution can be factored into the marginal cumulative distributions and a copula function describing the dependence between the components. In other terms the copula is a multivariate distribution with Uniform (0,1) margins. In this way the problem of the dependence structure of the joint distribution can be completely separated from the problem of modeling marginal distributions (see Genest and Rivest (1993)).

In formula let F be an N-dimensional distribution function with marginals $X_j \sim F_j$ $(j=1, \dots . N)$. Then exists an N-copula $C:(0,1)^N$ such that for every $\boldsymbol{x} = (x_1, \dots\dots, x_N)$

$$F(x_1,\dots,x_N) = C(F_1(x_1),\dots,F_N(x_N)) \qquad (1)$$

The theorem also states that if F_j are continuous then the copula C is unique.
Since $F_j(x_j)$: (0,1), we can easily apply the inversion method to the marginal distributions $F_j(x_j)$ in order to obtain dependent non-normal distributions with different margins. In formula

$$C(u_1,\dots,u_N) = F\left(F_1^{-1}(u_1),\dots,F_N^{-1}(u_N)\right) \qquad (2)$$

Strictly related to the general concept of copula it is very useful to distinguish between:

a) implicit copulas (namely Gaussian copula $C_\rho^{Gaussian}(u_1,\dots,u_N)$ and t copula $C_\rho^{t_v}(u_1,\dots,u_N)$), both dependent upon the linear correlation matrix while the t copula also upon the degree of freedom (v);

b) Archimedean copulas, strictly related to the generating function $\varphi(.)$ defined on (0,1) → (0,∞). The function φ must be convex and decreasing. Thus $\varphi(0) = \infty$, $\varphi(1) = 0$, $\varphi'(.) < 0$ and $\varphi'(.) > 0$.

To reduce the notation we briefly discuss the bivariate case. It is worth noting that the extension from the bivariate to the multivariate case often become intractable[2]. Even though our approach involves a multivariate framework, this study, as well as Ward and Lee (2002) and Rosenberg and Schuermann (2006), approaches the problem of risk aggregation by considering risks pair-wise.

In the bivariate case the Archimedean copula has the following form:

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)) \qquad (3)$$

with $u, v \in (0,1)$. The extension to the multivariate case is straightforward. A list of known $\varphi(.)$ function is available in Frees and Valdez (1998) and Nelsen (1999). Archimedean copulas are widely used in finance, insurance and reinsurance applications since they stress the tail dependence

[2] As noted by Bouyè et al. (2001) Archimedean copula significantly reduces the number of the parameters to be estimated. Implicit (Gaussian and *t*) copula provides information about the dependence between each pair of random variables therefore, for N variables, N(N-1)/2 parameters have to be estimated. For Archimedean copula the dependence is characterized only by N-1 parameters.

between different sources of risk. We report the most famous Clayton ($C_\alpha^{Clayton}$) and Gumbel (C_α^{Gumbel}) copulas. Risk managers apply these functions to improve the upper or the lower tail dependence (see Vaz de Melo Mendes and Martins de Souza, (2004)). The generator function $\varphi(.)$ for Clayton copula is $\varphi_t^{Clayton} = \frac{1}{\alpha}\left(t^{-\alpha} - 1\right)$, thus the bivariate case can be expressed as follows:

$$C_\alpha^{Clayton}(u,v) = \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-\frac{1}{\alpha}} \qquad (\alpha>0) \qquad (4)$$

The generator function $\varphi(.)$ for Gumbel copula is $\varphi_t^{Gumbel} = \left(-\ln(t)\right)^\alpha$, thus the bivariate Gumbel copula is:

$$C_\alpha^{Gumbel}(u,v) = \exp\left\{-\left[\left(-\ln(u)\right)^\alpha + \left(-\ln(v)\right)^\alpha\right]^{\frac{1}{\alpha}}\right\} \qquad (\alpha\geq 1) \qquad (5)$$

To emphasize the importance of the joint distribution expressed by the copula function, Exhibit 1 shows 10,000 standard normal bi-variates with roughly the same (Pearson) correlation coefficient ρ=0.7. These random variables have the same marginal distributions, the same correlation matrix but different structure of dependence. We represent two implicit copulas (Gaussian and *t*(2)) and Gumbel and Clayton copulas.

(Exhibit 1)

It is well evident that Gaussian copula does not show any tail dependence, while the copula of the bivariate *t* distribution is asymptotically dependent in both upper and lower tails. Exhibit 1 shows symmetric and asymmetric tail dependence, therefore Gumbel and Clayton copulas are also known as asymmetric copulas. Clayton copula emphasizes lower tail dependence whereas Gumbel copula highlights joint positive dependence.

It is also worth noting that while Gaussian copula depends only on the linear correlation coefficient (ρ), *t*-copula depends on the linear correlation coefficient (ρ) and on the number of degrees of freedom (ν). Clayton and Gumbel copulas depend on a parameter (α) that has direct relationships with *Kendall's tau* (ρ_τ), one of the most diffused measure of dependence.

In the copula related theory, *Kendall's tau* ρ_τ is the most popular coefficient of rank correlation to be estimated instead of the classical Pearson's rho ρ. *Kendall's tau* measures the concordance between two random variable X_1 and X_2 or, in other words, it reflects the degree of some monotonic dependence. Therefore ρ_τ is a *rank* correlation. In terms of copula ρ_τ can be represented as follows:

$$\rho_\tau = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 \qquad (6)$$

where the integral above is the expected value of the random variable *C(u,v)* where *u, v ~Uni(0,1)* with joint distribution function *C*.

Its success can be mainly attributed to two reasons. First, known relationships between the unknown parameter of several Archimedean copulas and the *Kendall's tau* are available. In the case of elliptical distributions ρ_τ has important direct relationship (7) with the linear correlation (ρ) (see Embrechts *et al.* (2002)). For this reason ρ remains the most important parameter in the Gaussian copula and it is as important as ν in the case of *t*-copula.

$$\rho_\tau = \frac{2}{\pi} ArcSin(\rho) \quad (7)$$

Exhibit 3 highlights the relationships between the ρ_τ and the parameter α of the Archimedean copulas.

The second reason entails the drawbacks of the linear (Pearson) correlation coefficient ρ. We summarize those in the list below[3]:

1. ρ between two random variables is strictly invariant under increasing linear transformations but it is non invariant under nonlinear strictly increasing transformations[4]. In other words ρ depends on the copula and on the marginals;
2. it is possible to calculate ρ if and only if the variance of the desired distribution is known. However several common marginal distributions in insurance and finance are infinite-variance and thus the concept of linear correlation is not applicable. For example the Cauchy distribution, diffused in modeling asset returns, has infinite variance and therefore ρ can not be calculated.
3. ρ is more unstable than the correspondent ρ_τ. Its sensibility to outliers can be again shown through an example. We draw a sample of 100 bivariate Student's *t* variates with 3 degree of freedom (heavy tailed distribution) with a linear correlation of +0.7. Inverting the relationship (7) it is straightforward to verify an implicit *Kendall's tau* of +0.4936. This simulation is performed 5000 times and both correlations (ρ and ρ_τ) are calculated[5]. Exhibit 2 highlights the different level of stability for the two different estimators.

(Exhibit 2)

To summarize ρ addresses linear dependences for the family of the elliptical distributions which includes a wide range of potential return distributions. Other common random variables can incidentally have the same Pearson's rho coefficients but their dependence structures could not be fully explained by ρ.

Kendall's tau overcomes the fallacies of Pearson's rho described above. Exhibit 2 shows the higher stability of *Kendall's tau* in the case of *t(3)* random variables (heavily tailed). Furthermore it is easy to demonstrate that *Kendall's tau* is stable under strictly increasing nonlinear transformations[6]. It is also possible to link infinite-variance marginal distributions through direct relationships (see Exhibit 3) between α parameter of Archimedean copula and ρ_τ.

The coefficient of upper (λ_U) and lower tail dependence (λ_L) is a measure of pairwise dependence in the tails or in the case of extremal events. Some authors describe this coefficient in terms of limiting conditional probabilities of *quantile exceedances* (McNeil *et al.* (2005), pg. 208). If there exists a positive association between extreme events of X_1 and X_2, then the conditional probability $\Pr\{X_1 > F_1^{-1}(1-k) | X_2 > F_2^{-1}(1-k)\}$ is greater than zero and decreases as $k\downarrow 0$ in the upper case. The coefficients of upper λ_U and lower λ_L tail dependence are respectively defined as:

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[3] For an extensive discussion about these fallacies see Embrechts, McNeil and Straumann (2002).

[4] It is really easy to verify this fallacy by drawing two correlated (ρ) vectors or normally distributed random variables (X_1 and X_2) and then taking Exp(X_1) and Exp(X_1). This transformation is nonlinear strictly increasing. ρ of the two transformed variables is not the same whereas ρ_τ remains unchanged.

[5] A sample of bivariate Student's *t*(ν) is simulated in two steps (see algorithm 3.10 in McNeil *et al.* (2005)): 1) simulation of multivariate normal variates using Cholesky decomposition; 2) introduction of a mixing variable W drawn from an Inverse Gaussian - IG - distribution (W~IG(1/2 ν, 1/2 ν)).

[6] Again, simulate two random vectors of correlated normally distributed random variables (X_1 and X_2), take the Exp(X_1) and Exp(X_2) and then calculate the *Pearson's rho* and *Kendall's tau* in both cases (for an analytical proof see Proposition 5.29 of McNeil *et al.* (2005)).

$$\lambda_U(k) = \lim_{k\to 0} \Pr\{X_1 > F_1^{-1}(1-k) | X_2 > F_2^{-1}(1-k)\} \quad (8)$$

$$\lambda_L(k) = \lim_{k\to 1} \Pr\{X_1 \le F_1^{-1}(1-k) | X_2 \le F_2^{-1}(1-k)\} \quad (9)$$

If λ_U and $\lambda_L = 0$ random variables are asymptotically independent, otherwise a form of upper or lower dependence exists. Fortunately for symmetric copula (eg. Gaussian and *t*-copula) $\lambda_U = \lambda_L$. For Archimedean copulas with closed form these coefficients have straightforward solutions dependent on the copula α-coefficients. For this reason the parameter α is also known as degree of dependency. Exhibit 3 shows λ_U and λ_L directly related to the *Kendall's tau* through the parameter α for different copulas.

(Exhibit 3)

In the last column of Exhibit 3 λ is a function of ρ_τ[7]. Clayton copula turns out to have only lower tail dependence whereas Gumbel copula emphasizes the upper tail dependence without any weight on the lower tail dependence. Exhibit 4 shows different coefficients λ of tail dependence. Notice that Exhibit 4 does not distinguish between upper and lower dependence. We plot for positive ρ_τ since Clayton copula exists only for positive α, strictly related to positive ρ_τ.

In the *t*-copula the higher the number of degree of freedom the lower the dependence (*t*-copula tends to be nearer to the Gaussian copula). However even for zero correlation and high ν, a positive λ can be found. Clayton copula has the heaviest (lower) tail dependence whereas the heaviest (upper) tail dependence is a feature of Gumbel copula.

(Exhibit 4)

Spearman's rho is another well known measure of rank correlation. It is less used than *Kendall's tau* since direct relationships with copulas (direct and Archimedean) are not always available[8]. For those reasons we concentrate our model on *Kendall's tau* as measure of dependence.

The last argument to be introduced is the well-known theory of extremes or *Extreme Value Theory* (EVT). Numerous theoretical studies and empirical financial/insurance applications are available about this important topic. This argument is gaining an increasing attention in any field of the financial theory since the distributions of asset returns tend to be different from the Gaussian and events on the (left) tail are a matter of concern for both practitioners and supervisors. A solid background on EVT is in Embrechts *et al.* (1997) and in McNeil *et al.* (2005). Extremes can be modeled in two different ways: modeling the maximum or minimum of a portfolio of random variables, and modeling the largest values over some high threshold. For our purposes the second approach is more suitable than the first since the first requires a large amount of data, typically daily returns divided in blocks (e.g. months). The second approach is called *Peaks Over Threshold* (POT) method and attempts to focus only on events greater than some large preset threshold. Negative (positive) exceedances are defined as the observations of the returns lower (greater) than a given threshold. Longin and Solnik (2001) show that the cumulative distribution of the exceedances $\left(F_R^\theta\right)$

[7] Notice that a closed formula is available for Pearson's rho (ρ). However, for the sake of comparison, we derive direct relationships with *Kendall's tau* (ρ_τ). The relationship between ρ_S and ρ for multivariate *t* is not available. This is another reason to consider only *Kendall's tau* (ρ_τ).

[8] For example the relationship between *t*-copula and *Spearman's rho* is still unknown (see Hult and Lindskog, (2002)). Moreover we do not have a direct formula to link the parameter α of the Gumbel copula to the *Spearman's rho* (see Frees and Valdez (1998)).

over a threshold θ is exactly known if the parent distribution of returns (F_R) is also known. However in many financial applications the distributions of returns are not known and therefore an asymptotic distribution is required.

Balkema, De Haan (1974) and Pickands (1975) demonstrate that the only non-degenerate distribution able to approximate the distribution of return exceedances $\left(F_R^\theta\right)$ is the Generalized Pareto Distribution (GPD). This limit distribution is given by:

$$G_R^\theta(x)=\begin{cases}1-\left(1+\tau\left(\frac{x-\theta}{\sigma}\right)\right)^{-\frac{1}{\tau}} & \tau\neq 0\\ 1-Exp\left(\frac{-(x-\theta)}{\sigma}\right) & \tau=0\end{cases} \tag{10}$$

where θ is the threshold (location parameter), σ is the dispersion (scale) parameter and τ is the tail index (or shape parameter), intrinsic in the original distribution of the returns (F_R). The tail index τ is the most important parameter because it describes the behavior of the extreme returns. The function (10) is called 'Generalized' since it can assume different shapes according to the tail index τ. If the distribution has a power-declining tail, it is an ordinary Pareto distribution and it has $\tau > 0$ (fat-tailed). Otherwise if the distribution has an exponentially-declining tail or it is thin-tailed ($\tau = 0$) it is exponential. If the return distribution has negative τ (no tail) therefore it is a Pareto type II distribution. Thin-tailed distributions include normal, exponential, gamma, lognormal. Short-tailed distributions include beta and uniform. Fat-tailed distributions are widely studied in financial applications. Precisely a distribution with τ greater than 0.5 is consistent with the Stable Paretian. $\tau = 1$ implies a Cauchy distribution and $0 < \tau < 0.5$ is typical for the Student's *t* distribution. It also worth noting that there is a direct relationship between the tail index τ and the highest existing moment k ($k=1/\tau$). Thus, the normal distribution has infinite moments (1/0) whereas Cauchy distribution has only the first moment (1/1) but is infinite-variance. For the Student's *t* k is greater than 2 and equal to the number of degree of freedom ν. It also means that there is an inverse and simple relationship between the dof (ν) of the Student's *t* and its tail index ($\tau = 1/\nu$) and also that the Student's $t(\nu)$ has at least ν moments.

Parameter estimation is made with two approaches. The first is parametric while the second is based on non-parametric estimators. The former relies on the assumption that extreme returns are exactly drawn from the extreme value distribution and it typically requires a large amount of data (daily returns). The latter has no distributional assumption and it is based, like Schmid-Trede test, on the order statistics of the parent variable R (see Embrechts *et al.* (1999) and Resnick and Starica, (1997)) and particularly on the time series of returns ranked in increasing order (R'). The most used estimators are Pickands's estimator, given by

$$\tau_{Pickands}=-\frac{1}{\ln(2)}\ln\frac{R'_{N-q+1}-R'_{N-2q+1}}{R'_{N-2q+1}-R'_{N-4q+1}} \tag{11}$$

and Hill's estimator whose formula is given by

$$\tau_{Hill}=\frac{1}{q-1}\sum_{i=1}^{q-1}\ln R'_{N-i}-\ln R'_{N-q} \tag{12}$$

It is widely accepted that Hill's estimator is consistent and the most efficient estimator (see Longin, (2005) and Embrechts *et al.* (1999)). However it is applicable only with positive τ while Pickands's estimator is suitable for any τ. Normalized Pickands's statistics is asymptotically normally distributed with mean τ and variance $\tau^2\left(2^{-2\tau+1}+1\right)/\left[2\left(2^{-\tau}-1\right)Log(2)\right]^2$ while Normalized Hill's estimator is asymptotically normally distributed with mean τ and variance $\tilde{\tau}^2$ Hill's estimator is significantly more efficient than Pickands's[9] in the classical interval of τ for monthly asset returns. For this reason our model greatly relies on Hill's estimator. The choice of the appropriate θ and therefore the number of exceedances q is an important issue well investigated in the extreme value theory. An high q leads to a small number of exceedances with inefficient parameter estimates whereas a low q leads to many exceedances giving biased parameter estimates. This can be viewed as the classical problem of optimizing the trade-off between bias and inefficiency. Jansen and De Vries (1991) and Danielsson *et al.* (2000) propose a Monte Carlo simulation in order to minimize the mean square error (MSE) of the simulated tail index τ[10]. Their method finds the minimum of the U-shaped relationship between MSE and q (number of exceedances) of the simulated data (see Longin (2005) for a detailed description).

The heuristic model

The classical asset allocation models rely on the normal assumption for each asset return. Under this assumption elegant solutions for the optimization problem are available with and without the inclusion of the investor preferences, i.e. utility function. However asset returns are rarely normally distributed and therefore the classical optimization methods are rejected by the empirical evidence. Interesting alternative approaches have been proposed by Harvey and Siddique (2000) and Jondeau and Rockinger (2005), (2006). They suggest the inclusion of higher than second moments in order to significantly improve the asset allocation and also calculate how costly can be the departure from the classical normality assumption. Unfortunately analytical forms for the optimal portfolios are available only in few cases. The authors use a multivariate skewed Student's *t* framework with a time-varying correlation matrix (Dynamic Conditional Correlation – DCC – multivariate GARCH introduced by Engle in 2002) in order to find closed formulas for the optimal portfolio. Bradley and Taqqu (2001) report another significant example of explicit solutions, assuming Stable Paretian distribution $S(\alpha, \beta, \sigma, \mu)$ for all asset classes[11].

However those authors do not considered three important features:

1. the presence of autocorrelation and heteroskedasticity that can significantly underestimate the standard deviation. Notice that this problem should be solved with the DCC-MVGARCH model. However Rob van den Goorbergh (2004) provides examples of alternative approaches able to overperform the DCC-MVGARCH model;
2. marginal distributions can be different from the skewed-*t* and Stable Paretian (a significant example of testing univariate marginal distributions with different time horizons is given

[9] It can be verified that for Pickands's estimator the minimum of its variance can be obtained for τ=0.599847 and the correspondent variance is 3.02674. Hill's variance is strictly increasing for $\tau > 0$ (no local minimum) while its variance on the minimum of the Pickands's estimator is 0.359816 (more than 8 times lower than Pickands's variance). The estimators are equivalently consistent (they have the same variance) if τ=2.06691.

[10] A detailed description of this Monte Carlo simulation is reported in the Appendix of Longin (2005).

[11] Notice that he Gaussian distribution can be viewed as a special case of the Stable Paretian. In particular Gaussian distribution corresponds to the $S(2, 0, \sigma, \mu)$ distribution. α is the index of stability or tail exponent and controls the decay in the tails of the distribution. The other parameters (σ, β, μ) control respectively scale, skewness and location. Notice also that the Stable Paretian has known variance if and only if $\alpha = 2$ (Gaussian case). With $0 < \alpha < 2$ (the classical case in portfolio theory reveals $1 < \alpha < 2$) and both skewness and location parameter β equal to zero, closed formulas for optimal portfolios are available minimizing the scale parameter σ equivalent to the classical risk parameter (standard deviation) in the Normal case.

in Levy and Duchin (2004)[12]) therefore analytical solutions are not available. Moreover the same distributional assumption for each margin seems to be too limiting;

3. correlation matrix might not capture the entire dependence among asset classes. Particularly tail dependence may be greatly underestimated in the (Pearson) correlation coefficient framework.

We try to remove these limitations using simulation techniques. In our model we can deal with different marginal distributions, even though belonging to the same elliptical family, and for a structure of dependences given by a set of bivariate Clayton copulas. Clayton copulas have been externally imposed in order to correctly consider the lower tail-dependence, if present, or otherwise overweight the probability of joint extreme negative returns. Following the example of McNeil and Frey (2000) we propose a technique consisting in first filtering the data, second applying extreme value techniques to the tails of the standardized innovations and third relating these innovation processes with bivariate Clayton copulas. We then extend the methodology of Filtered Historical Simulation (FHS) proposed by Barone-Adesi and Giannopoulos (2001) and recently tested by Giannopoulos and Tunaru (2005) by rescaling our standardized innovations matrix to obtain pathways of multivariate returns.

To prove the potential distortions of the elegant methods with closed formulas we propose an example of asset allocation based on a set of eleven euro-denominated indexes. Furthermore we provide a comparison of our method with the widely diffused *resampling* method proposed by Michaud (1998). Michaud's method involves a multivariate normal Monte Carlo simulation for asset returns whose parameters are calibrated on the historical vectors of average returns, average standard deviations and the correlation coefficient matrix. Then the standard constrained optimization is performed and the resulting weights are stored in a matrix for each simulation. Key variables in the *resampling* method are the length T of the vector of each simulated asset return and the number of simulations M. These parameters are typically left to the asset managers. However the former should be related to the time period of the investor while the latter is set in relationship with the precision of the asset weights to be achieved. It is worth noting that Michaud's approach does not consider tail dependences and extreme (negative) returns (tail risk), not assumed in the classical multinormality assumption.

The proposed heuristic method can be analytically divided in eight steps:

1. Estimation of the filtering model in order to eliminate autocorrelation and heteroskedasticity in the marginal distribution of the n asset returns involved in the asset allocation. This is done through an ARMA/GARCH model with the innovations given by a Student's t with dof (ν) estimated using log-likelihood. Notice that this step is required in order to remove relevant autocorrelation and heteroskedasticity of the time series for the application of EVT. We reached the following model specification:

$$\mu_{i,t} = \phi\mu_{i,t-1} + \varphi\varepsilon_{i,t-1} + C_i + \varepsilon_{i,t} \tag{13a}$$

$$\sigma_{i,t}^2 = \kappa_i + \alpha_i\sigma_{i,t-1}^2 + \beta_i\varepsilon_{i,t-1}^2 \tag{13b}$$

$$\varepsilon_{i,t} = \sigma_{i,t}x_{i,t} \quad x_t \sim t(\nu). \tag{13c}$$

   Estimated models for each margin will be used in the step 5 to simulate paths of standard deviations separately $\overline{\sigma}^{(i)} = \left[\sigma_1^{(i)},...,\sigma_T^{(i)}\right]$ for (i=1….n).

2. Fit the shape parameter τ_i (tail index) and the number of appropriate exceedances q_i^{opt} of the Generalized Pareto Distribution on the left tail of the standardized residuals distribution

[12] For a large sample of equities, indexes and bonds authors find the supremacy of logistic, gamma, lognormal and extreme value.

(once historical returns have been filtered) with a non-parametric approach based on the Hill's estimator. This step implies the identification of the minimum of the U-shaped relationship between the MSE and the number of exceedances q. With this minimum we can easily identify the optimal number of exceedances q_i^{opt} and the correspondent tail index (τ_i). For this estimation we follow the non-parametric method suggested by Jansen and De Vries (1991) and Danielsson *et al.* (2001). Once the minimum MSE tail index and the correspondent number of exceedances have been found, we can infer the appropriate parent distribution of the standardized innovations (see the next step)[13].

3. Simulation of a vector of T standardized innovations, drawn independently, for each asset class using a Student's t distribution whose dof (ν_i) has been derived by the tail index τ_i estimated in 2 with the simple relationship $\nu_i = 1/\tau_i$. Notice that these distributions are different for each margins and therefore they incorporate specific characteristics for each innovation process. With this approach we obtain a multivariate distribution with n marginals $X^{(i)} = \left[x_1^{(i)},...,x_T^{(i)}\right]$ (for i=1…n) of T standardized innovations parameterized on the specific tail index (τ_i).
4. Simulation of the dependencies through the Clayton copula. This is done using the inversion method in (2). As noted above this copula significantly emphasizes the lower tail dependence while maintaining the same (Pearson) correlation coefficient ρ. The tail-dependence coefficient (λ) is estimated on *Kendall's tau (ρ_τ)* matrix[14]. With this step we obtain a multivariate distribution with n marginals ${}^{cop}X^{(i)} = \left[{}^{cop}x_1^{(i)},...,{}^{cop}x_T^{(i)}\right]$ (for i=1…n) of T standardized innovations, parameterized on the specific tail index (τ_i) for each asset class. Each margin is related according to the specific coefficient of lower tail-dependence (λ_L). At this stage we use the *pair-wise* method introduced by Ward and Lee (2002) and extensively applied by Rosenberg and Schuermann (2006).
5. Simulation of n vectors of conditional standard deviations $\overline{\sigma}^{(i)} = \left[\sigma_1^{(i)},...,\sigma_T^{(i)}\right]$ and reintroduction of the copula-related standardized innovations ${}^{cop}X^{(i)}$ in the ARMA/GARCH process estimated in (13a) and (13b) to obtain a sample of T simulated returns with margins given by distributions related to the tail index τ_i and with the dependence structure given by the α-parameterized Clayton copula. The reintroduction is done in two steps. First, rescaling the n vectors of T simulated conditional standard deviations $\overline{\sigma}^{(i)}$ given by the conditional volatility model in (13b). In this way we obtain n vectors of simulated innovations ${}^{cop}\overline{\varepsilon}^{(i)} = \left[\left({}^{cop}\varepsilon_1^{(i)} = {}^{cop}x_1^{(i)} \cdot \sigma_1^{(i)}\right),\left({}^{cop}\varepsilon_2^{(i)} = {}^{cop}x_2^{(i)} \cdot \sigma_2^{(i)}\right),...,\left({}^{cop}\varepsilon_T^{(i)} = {}^{cop}x_T^{(i)} \cdot \sigma_T^{(i)}\right)\right]$ (for i=1….n). Second, the n vectors ${}^{cop}\overline{\varepsilon}^{(i)}$ have been reintroduced in the ARMA process in (13a) in order to obtain ${}^{cop}\overline{\mu}^{(i)} = \left[{}^{cop}\mu_1^{(i)},...,{}^{cop}\mu_T^{(i)}\right]$ for each i asset.
6. Optimization *à la Markowitz* using quadratic programming on the matrix ${}^{cop}\overline{\mu} = \left[{}^{cop}\overline{\mu}^{(1)},...,{}^{cop}\overline{\mu}^{(n)}\right]$ where ${}^{cop}\overline{\mu}^{(i)} = \left[{}^{cop}\mu_1^{(i)},...,{}^{cop}\mu_T^{(i)}\right]$ and on the covariance matrix recalculated in each simulation. We optimize in the presence of positivity constraints and then store the optimal weight for each asset class ($W^{(i)}$) related to the asset allocation along the efficient frontier. The number of portfolios along the efficient frontier has to be chosen.
7. Repetition of steps 3, 4, 5 and 6 M times in order to obtain M matrix of weights $W^{(i)}$.

[13] Notice that the scale parameter σ in (28) is not required since in this conditional variance approach we do not directly simulate the returns but we simulate specific standardized innovations, given the tail index. In a unconditional approach we could need random variates drawn from a GPD, therefore the scale parameter σ should be also estimated.

[14] Note that this approach of estimation is known as '*Method of Moments*' (see McNeil et al. (2005), section 5.5.1) and it is appropriate in the case of few observations. Otherwise the classical MLE estimators for the parameter α can be applied but it requires a huge amount of data, rarely available with a monthly framework.

8. Calculation of the main statistics of these weights along the efficient frontier in order to have estimates of the uncertainty of the allocation.

Our model is empirical, distribution-free[15], only related to the copula-parameters and to the tail indexes, both estimated with non-parametric techniques. Such a solution should emphasize the lower tail-dependence between return distributions and therefore implicitly overestimates the joint tail risk by underweighting those asset classes with high lower tail-dependence, all other things (e.g. covariance matrix) being equal. Notice also that the classical quadratic programming for the calculation of the efficient frontier is suitable for many marginal distributions belonging to the elliptical family since this family has very important properties listed below. Among the others it is important to highlight that:

a) any linear combination of elliptically distributed variates is elliptical itself with the same characteristic generator (see Valdez and Chernih (2003));
b) if all univariate marginals are elliptical then the minimization of variance for a given level of expected return leads to the same results given by the minimization of other more sophisticated risk measures (see Embrechts *et al.* (2002))

Next session reports the detailed description of the input estimation methods for the eleven asset classes involved in the example. The *resampling* method has also been performed on the same asset classes in order to evaluate the cost of assuming multinormality for asset returns and the correlation matrix as the only input required in the algorithm.

Finally, we perform two different tests in order to have empirical evidence about the potential of our method. First, we simulate a new sample of 20.000 Student's $t(\nu_{\text{sim}})$ (with ν_{sim}=3,….,20) multivariate returns with an implicit correlation matrix given by historical *Kendall's tau* (ρ_τ). Then we use each simulated sample to obtain several statistics on the returns of some identified portfolios along the efficient frontier whose weights come from our model and from the *resampling* method. Second, we perform an out-of-the sample test using the real historical returns of the eleven asset classes from a subsequent period and then we analyze the results.

We expect a smaller distance between the upper bound (e.g. the 1% percentiles of the portfolio returns) and the lower bound (e.g. the 99% percentiles of the portfolio returns) with our model. We also expect more thin-tailed return distributions with our model than in the *resampling* model for each portfolio.

Parameter estimates

Exhibit 5 shows the descriptive statistics of the monthly returns from January 1991 to December 2004 for a total of 168 observations, including market crashes and bubbles. Parameters estimates reported in Exhibit 5 indicate positive average monthly returns for ten markets and negative average monthly returns for the Japanese market. Notice also the minimum statistics ranging from – 12.5% for UK market to -31.3% for Hong Kong market.

(Exhibit 5)

Return distributions are slightly asymmetric and leptokurtic (excluding S&P 500, CAC40 and Nikkei). However it is well known that the fourth moment is a weak measure of leptokurtosis. Its weaknesses can be listed as follows: a) it does not exist for every distribution; b) it accounts jointly for peakedness and fat tails; c) it is extremely sensible with respect to outliers.

Since the fourth moment is a poor measure of leptokurtosis we perform the new distribution-free Schmid-Trede test (2004) in order to provide consistent estimates for peakedness and fat tails

[15] We let data decide what is the right $\nu=1/\tau$ parameter for each standardized innovation process. Therefore we do not externally impose any distribution with the consequent parameters.

separately. This test relies on the Hogg selector statistics based on quantiles (see Hogg and Lenth (1984)) and allows to unambiguously test the fat-tail behavior (T-test), the peakedness (P-test) of every distribution and the leptokurtosis of the sample data (L-test), regardless to the scale and location. Nevertheless, as pointed out by the authors, critical values are available only for large sample size[16]. For finite sample (less than 2000 observations) the critical value has to be calculated with a Monte Carlo approach. Therefore we perform a Monte Carlo simulation, following the method described by the authors, in order to obtain critical values in our case of 168 observations. The simulation is repeated 100,000 times. The critical values $\hat{P}, \hat{T}, \hat{L}$ for 168 observations at 95% are 1.5238 (P-test), 1.15104 (T-test), 2.4805 (L-test), while at 99% the critical values are 1.4621 (P-test), 1.4461 (T-test), 2.3378 (L-test). In our sample the null hypothesis H_0[17] is always rejected (excluding the P-test for Nikkei) revealing interesting features about indexes.

Returns show significant peakedness (P-test) and fat taildness (T-test) even though with different intensities. For example the S&P shows a low level of sample kurtosis explained by an high and significant peakedness and an higher than the correspondent normal level of fat taildness. It is worth mentioning that the taildness of S&P is remarkable but not as tremendous as the Hang Seng while S&P has an higher peakedness than the Hang Seng. Nikkei index is less peaked than the normal (the normality assumption is accepted at 95% and rejected at 99%), but its tails are fatter than the correspondent normal tails. This decomposition allows to concentrate on the tails of the Nikkei distribution even though its sample kurtosis is lower than the normal. This specific effect explains the higher and significant level of leptokurtosis badly captured by the sample kurtosis in Exhibit 5. FTSE index has also a sample kurtosis lower than the normal while the more precise decompositions reject the hypothesis of peakedness and taildness equal to the normal-ones. The most leptokurtic index is the Singapore Straits and it is explained with its fattest tails. The least leptokurtic index is the Nikkei and it is mainly due to the absence of peakedness.

We also test the hypothesis of univariate normality for each return distribution with Jarque-Bera tests. These tests accept the normality assumption for five indices (at 95%) whereas the more appropriate investigation of P, T and L tests allow to reject the assumption of normality for all indices.

A test for the multinormality is also conducted. The best known is Mardia's test of multivariate skewness and kurtosis. Specifically Mardia's test is based on the Mahalanobis distance of data vector from its sample mean and it allows to reject the hypothesis of the normality if the sample has no significant skew and the measure of kurtosis deviates from expectancy only randomly. Both multivariate tests strongly reject the hypothesis of multinormality[18].

An investigation for autocorrelation in the vectors of returns is also performed. The Ljung-Box-Pierce Q-test for a departure from randomness up to 20 lags for the vectors of returns and for the its squared is conducted. We did not find significant departures from randomness for the returns and for volatilities. We found only some significant persistence in volatilities. This is also consistent with the empirical evidence about asset returns with monthly observations. Differently from more frequent data, monthly returns are seldom serially correlated.

Furthermore we perform the ARCH test statistics to check for heteroskedasticity in the vectors of returns. We found evidence of heteroskedasticity in the vectors of returns, even though with different intensity. The introduction of EVT for modeling the tail risk requires to completely remove the persistence in variance. This is done by fitting a general ARMA/GARCH model to filter the returns in order to obtain uncorrelated standardized innovations and to remove

[16] The asymptotic distribution may be used for calculating critical values for hypothesis test if n>2000. In this case the authors provide the quantiles of the test statistics, dependent only on the level of confidence *p*.
[17] Recall that the P-test is 'the sample is peaked as a normal distribution'; the T-test is 'the sample has tails comparable with the normal distribution' and the L-test is 'the sample is leptokurtic as a normal distribution'.
[18] See Von Eye A. and Bogat A. (2004) for further details on Mardia's tests.

heteroskedasticity[19]. However, since the main aim of the paper is not to estimate the best model for all kind of financial time series but it is to provide evidence of the importance of the tail dependence and of the tail risk in the asset allocation choices with tail-dependent assets, we decide to keep the filtering model simple and to introduce a general ARMA/GARCH filtering model with a Student's $t(\nu)$ innovation process with ν estimated using log-likelihood function. Innovations given by Student's t theoretically incorporate significant extreme events and, in the form of skewed-*t* also a relevant skewness. However the skewness not significantly different from zero does not allow to test a parameterized skewed-*t*. We test several filtering models in order to obtain a good equilibrium between parsimony and the absence of autocorrelation in the standardized residuals and in the squared standardized residuals and the model described in (13a), (13b) and (13c) met all these requirements. Student's t has been used since it can incorporate either Gaussian innovations when ν tends to be high or excess kurtosis in the time series of returns. Exhibit 6 reports the estimated coefficients for each vector in our sample.

(Exhibit 6)

LBQ and ARCH tests before and after the ARMA/GARCH filtering for each index are reported in the Appendix A.

The second step requires the estimation of the appropriate tail index for the left tail and the correspondent number of exceedances. Tail index τ_i for each distribution '*i*' is estimated on the standardized residuals of the filtering model resulting from step 1. for each distribution '*i*' The method of estimation is non-parametric. The identification of the optimal value of q (number of exceedances), namely q_i^{opt}, requires a Monte Carlo simulation. We simulate 168 innovations drawn from different Student's t distributions with degrees of freedom equal to 2, 3, 4 and 5. The fatness (tail index) of these distributions corresponds to 1/2, 1/3, 1/4, 1/5. The higher the number of the degree of freedom (the lower the tail index) the lower the number of extreme innovations. Then we estimate the tail index τ_i using the Hill's estimator with different values of q ranging from 2 to 36 (roughly 20% of the observations in our sample). We repeat this simulation 10,000 times. Therefore we get 10,000 observations of the tail index estimates for each distribution i and each value of q. For each distribution i we compute the MSE of the series and then we choose the value of q (called q_i^{opt}) which minimizes the U-shaped MSE function. q_i^{opt} is the number of extremes over the threshold θ that minimizes the trade-off between bias ad inefficiency for each distribution. Having q_i^{opt} for each distribution *i,* we compute the statistics $\left[\tau_{Hill}\left(q_i^{opt}\right)-\tau_i\right]/\sigma_i$ where τ_{Hill} is the vector of the Hill's estimators calculated on the real data with different q, τ_i is the theoretical estimate of the tail index given by $1/\nu$ for a Student's $t(\nu)$ and σ_i is the standard error of this estimate. We then calculate the p-values associated to these statistics and retain the estimates of $\tau_{Hill}\left(q_i^{opt}\right)$ with the lowest p-value. The simulation and the estimation is computed only for the Hill's estimator since it is widely recognized that Hill's estimator is more efficient and more stable than the Pickands's estimator (see Embrechts *et al.* (1999)). However the correspondent Pickands's estimator is also reported in Exhibit 8. As noted above we do not need the other parameters in (10) since we simulate from the related parent (elliptical) function and not from the GPD. Exhibit 7 and Exhibit 8 summarize the sample estimates for both Hill's and Pickands's estimators and the results for the tail index estimates related to the Monte Carlo method explained above.

(Exhibit 7)

[19] Notice that a similar two-step GARCH-EVT approach has been applied in the risk management field by McNeil and Frey (2000) for a more reliable estimation of VaR and Expected Shortfall (ES). The authors first filter the returns in order to have *i.i.d.* innovations. Second they fit a GPD distribution by maximum likelihood estimators for the innovations and then estimate VaR and ES.

(Exhibit 8)

Notice the concordance of the tail index estimation with the *T*-test in Exhibit 5. Asset classes with high *T*-test tend to have higher estimated tail index on the standardized innovations. It is also worth highlighting that despite the rejection of the hypothesis of Student's *t* standardized innovations in several cases (see Exhibit 5), Hill's estimates are always positive. This can be attributable to the poor fit of the whole distribution and to a better fit of the left tail.

The simulation of each margin is straightforward once the tail indexes for the standardized innovation distributions have been properly estimated. The step from the simulation of eleven independent univariate margins to a copula dependent multivariate distribution of asset returns can be divided in two parts. The first part entails the Clayton copula α-parameter estimation. This is done studying the matrix of the Pearson's rho (ρ) and then computing the implicit *Kendall's tau* (ρ_τ). Once the *Kendall's tau* has been estimated we obtain the parameter α of the Clayton copula. Exhibit 9 reports the estimated ρ in the upper-right triangle and the implicit ρ_τ in the lower-left triangle.

(Exhibit 9)

The second part of this step can be described as follows. We simulate a matrix of *T* rows and the number of assets as columns *i*=(1,….11) in the way described above. Then we de-standardized the innovations by rescaling each simulated margin with the simulated standard deviation as described in (13c). We reinsert the de-standardized innovations in the ARMA process estimated in (13a) in order to obtain a matrix of α-parameterized copula returns with embedded tail risk. Recall that the filtering model assumes that $\varepsilon_t^{(i)} = {}^{cop}x_t^{(i)} \cdot \sigma_t^{(i)}$ where $\varepsilon_t^{(i)}$ is the vector of the simulated de-standardized innovations for each asset '*i*', ${}^{cop}x_t^{(i)}$ is the vector of the standardized innovations (simulated margins) after the Clayton copula transformation, $\sigma_t^{(i)}$ is the univariate conditional standard deviation for each asset generated from the filtering process in (13b). In this way we produce a matrix of dependent returns with a dependence structure given by the specific α-parameterized copula.

A similar approach involves the bootstrapping of the standardized innovations parameterized using the α-Clayton copula and the replacement of these innovations in the filtering process estimated in (13b). This method has been extensively applied in the field of the Filtered Historical Simulation – FHS - (see Giannopoulos and Tunaru (2005), Barone-Adesi and Giannopoulos (2001) to estimate different risk measures for a portfolio consisting of linear and non-linear securities. However this methodology requires a huge amount of historical data to incorporate all possible movements of asset returns. This in not the case when we deal with monthly observations but could be appropriate with daily returns.

An alternative approach could be to fit a multivariate GARCH model, eg. DCC –MVGARCH model proposed by Engle in 2002. The DCC model is a multivariate GARCH model with time-varying correlations. It assumes a (conditionally) joint normal distribution for the return innovations. Note that this assumption implies normal conditional margins, and a normal conditional copula, which is fully explained by the correlation coefficients. The DCC –MVGARCH model estimates simultaneously both the margins and the dynamic matrix of correlations. Van den Goorbergh (2004) demonstrated that a copula based approach systematically overperforms the easier DCC-MVGARCH approach in a risk management framework.

The simulation, analogously with the *resampling* approach proposed by Michaud, requires the setting of the length of the innovation vectors *T* (length of the multivariate returns in the *resampling*). We decide to simulate a matrix with 60 rows (5 years of monthly returns). This parameter is arbitrarily chosen but should be consistent with the horizon of the asset allocation and with the frequency of data.

For each simulation a matrix of autoregressive tail-dependent returns is obtained. We apply the classical constrained minimization *à la Markowitz* to this matrix obtaining an estimation of the efficient frontier and of the composition in terms of weights for each portfolio along the efficient frontier[20]. Notice that the final distribution of portfolio returns is a linear combination of dependent Student's t whose ν parameter is different for each margin. With these assumptions the classical minimization solution is still viable[21].

These steps obviously require the repetition of the simulation M times and the recalculation of the optimal weights each time. We store each optimal weight for each asset class for each portfolio along the efficient frontier and then we average this matrix in order to obtain a unique optimal weight for each portfolio.

The last step involves a comparison of the simulated portfolios, calculated using both our proposed approach and the classical *resampling* approach. The test is conducted using a set of simulated returns and an out-of-the sample set of real returns. We use a set of simulated returns drawn from a multivariate Student's t with ν dof (for ν=3,…30). With multivariate Student's $t(\nu)$ extreme returns have the same probability to occur because τ is the same for each margin ($\tau=1/\nu$, for ν=3,…30). Tail-dependence is variable and dependent upon the *Kendall's tau* matrix. Notice that in this Student's $t(\nu)$ simple case the tail-dependence is symmetric ($\lambda_U=\lambda_L$). Equating the relationships in Exhibit 3 on different copulas (t_ν and α-Clayton) and imposing ν=3,…30, we can solve numerically the equation $\lambda_L^{t_\nu} = \lambda_L^{Clayton}$ for ρ_τ. We can easily verify that Clayton copula has higher tail-dependence than multivariate Student's $t(\nu)$ for all ρ_τ in our sample. In other words if we consider the correlation matrix in Exhibit 9 Clayton copula has always higher λ_L than t_ν copula, assuming that the margins are the same $t(\nu)$[22].

We report some relevant statistics for the description of the simulated portfolio returns in both cases. The out-of-the sample test is concentrated on the absolute difference between standard deviations of some selected portfolios.

Results

As stated before we perform both our asset allocation method and the *resampling* method setting T=60 and M=1000 for 200 portfolios.

The results are well summarized by the area plots for the weights. Exhibit 10 highlights the results in our proposed model while Exhibit 11 shows the allocation of the wealth along the efficient frontier among the eleven asset classes, according to the *resampling* procedure. For the sake of comparison in Appendix B we report the composition of the portfolios along the efficient frontier à la *Markowitz*, even though it is well known that it is affected by numerous drawbacks and therefore not diffused among the practitioners.

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[20] The number of efficient portfolios along the efficient frontier has been set to 200.

[21] Recall that Student's $t(\nu)$ has at least ν moments, thus imposing $\nu > 2$ the variance always exists and the minimization approach *à la Markowitz* rests valid.

[22] Multivariate Student's $t(\nu)$ in d-dimensions can be decomposed in d univariate Student's $t(\nu)$ with the a t_ν copula and a given correlation matrix. Therefore the multivariate Student's $t(\nu)$ can be considered as a special case of the t_ν copula in which the marginals and the t_ν copula have the same ν. For this reason λ_U and λ_L in the multivariate Student's $t(\nu)$ case are a function of the unique ν and ρ_τ. Numerical solutions of the equation $\lambda_L^{t_\nu} = \lambda_L^{Clayton}$ in Exhibits 3 for different ν give the following relationships: for ν=3 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.1247, for ν=4 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.1092, for ν=5 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.0973,….., for ν=10 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.0634, for ν=15 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.0474, for ν=20 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.0379, for ν=25 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.0316, for ν=30 $\lambda_L^{t_\nu} < \lambda_L^{Clayton}$ if ρ_τ>0.02718. Clearly the higher ν the lower the tail-dependence of the t_ν copula, the lower also the ρ_τ in the Clayton copula to have the same lower tail-dependence.

(Exhibit 10)

(Exhibit 11)

Exhibit 10 shows a greater stability of the asset classes along the efficient frontier and portfolios more similar to the equally weighted in the middle of the efficient frontier.

To provide further investigation about the effect of the proposed method we simulate a sample of 20,000 unconditional multivariate Student's $t(\nu)$ monthly returns and then we calculate the most important quantiles for the portfolio return distribution for the 200 efficient portfolios. Exhibit 12 represents the case of multivariate $t(4)$ returns. The lower quantiles are severely affected by extreme returns in case of significant lower tail dependence. For homogeneity we also represent the upper quantiles for the $t(4)$ case.

(Exhibit 12)

It is important to highlight that potential returns can have heavier tails than Student's *t*. To complete the description we also report the main statistics for some relevant portfolios along the efficient frontier in Exhibit 13. We compute the P, T and L tests as indicated by Schmid and Trede (2003) in order to have a precise distinction between peakedness and taildness of the return distribution for each portfolio. Notice that in this case the formula for critical values does not require a simulation but is only dependent upon the sample size (20,000 observations for each portfolio) and the level of confidence (p).

(Exhibit 13)

In this simple case of returns drawn from a multivariate $t(4)$ the τ-EVT α-Clayton portfolio returns are unambiguously less heavy-tailed than in the sophisticated *resampling* method. This effect leads to a potential lowest return higher in the τ-EVT α-Clayton method than in the *resampling* method and to a wider bounds in the worst (1%, 5%, 10%) cases for *resampled* portfolios. These results are efficiently summarized by the kurtosis of each portfolio along the efficient frontier and, more precisely, by the T-tests.

For other multivariate $t(\nu)$ simulations we only indicate the main results in Exhibit 14. Particularly we concentrate on: standard deviation, sample kurtosis, T-test and Sharpe ratio of the middle portfolio (portfolio n.100). Empirical evidence does not change if we simulate potential returns from other multivariate Student's $t(\nu)$ distributions. Our method allows to strongly reduce the negative effect of both potential extreme negative returns and lower tail-dependence.

(Exhibit 14)

Finally we test our model with out-of-the sample data. We test the portfolio returns for some portfolios with real monthly returns of the eleven assets from 01/2005 to 09/2006. Our model allows to relatively reduce the standard deviations for all portfolios along the efficient frontier. The reduction can be measured as the absolute difference between the two standard deviations for several portfolios. These differences range from 0.0003 (200^{th} portfolio) to 0.0018 (20^{th} portfolio). Nevertheless this approach is not costless. The cost of adopting this method can be interpreted as the difference between the average return for each portfolio or, better, as the difference between the Sharpe ratios for each portfolio. Assuming a risk free rate of 2% per year it is trivial to note that the difference is always negative ranging from a minimum of -0.0210 to a maximum of -0.0060 for the $t(4)$ case. However our method is aimed to build an implicit tail-risk and negative tail dependence constraint and therefore this implicit cost represents the insurance premium for an efficient asset allocation aimed to reduce the negative effects of extreme negative returns and negative tail-dependence. It is worth noting that the cost tends to be negligible with real data. The out-of-the sample test reveals that the real portfolio returns are both lower and higher than the correspondent resampled portfolios. The difference is positive and decreasing for portfolios from 80th to 200th (from +0.03% to 0% for the last portfolio) but negative and decreasing for portfolios from 1st to 80th (from -0.11% to -.01% for the 80th portfolio). This effect could be attributed to FTSE and SMI. These asset classes have a significantly different role in the first part of the efficient frontier.

Conclusions

In this paper we present a Monte Carlo method for asset allocation purposes aimed to reduce the effects of negative extreme returns and lower tail dependence among asset classes.

Our method is based on non-parametric estimations and therefore it does not imply any *a priori* choice for the asset manager. This feature improves the efficiency of this method since other diffused methodologies have shown their limits on the distributional assumptions. The only distributional assumption we made is strictly related to the standardized innovations. However these are not chosen by the asset managers but are derived from the *Extremal Value Theory* properties. This theory provides simple relationships between the tail index (τ) of the Generalized Pareto Distribution for the returns in excess over a threshold θ (number of exceedances) and the parent return distribution. Furthermore we introduce a more realistic structure of dependencies whose aim is to stress the negative tail dependence among asset classes in order to build an implicit barrier against joint negative extreme returns. Again, the parameters of the multivariate copula (α-vector) are estimated using a distribution-free methodology.

We have shown that EVT and copula-related theory can help not only risk managers but also asset managers. The study of extreme quantiles can support the asset allocation decisions leading to safer portfolios. Furthermore theory on copulas can significantly improve the quantitative asset allocation considering more complex dependencies than those 'linear', described by the Pearson's rho (ρ).

We have also simulated a realistic framework with multivariate Student's *t* returns to demonstrate the efficiency of the method and to compute the cost of this implicit protection. This cost (only valued in terms of loss of average return and in terms of Sharpe ratios), in our simple simulation, seems to be negligible. However further researches are required on this important topic. For example other not strictly financial elements can justify the adoption of our proposed method, i.e. reputational costs and research costs related to the loss of current customers.

Finally we have tested our model on a set of out-of-the sample returns. The model can reduce the standard deviations of the portfolios along the efficient frontier, even though with different intensities. The decline is variable and dependent upon the absolute risk of the portfolio. The higher the absolute risk the lower the reduction of the volatility. However the volatility decrease is not coupled with a correspondent decline of the real return. The differences between portfolio returns tend to be negative for the safer portfolios and positive for the riskier part of the efficient frontier.

We leave for further research the introduction of a dynamic adjustment in the α-vector for the Clayton copula. Notice that these features require a substantial shift from the classical mean-covariance framework to a more sophisticated method. However these tools are quite common in the risk and capital management and never applied, to our knowledge, in the asset management.

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APPENDIX A

Pre-filtering Ljung-Box-Pierce Q-test and Engle's ARCH test

Ljung-Box-Pierce Q-test for vectors of returns

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ
1	2.5762	0.2964	2.4336	0.0280	0.7842	1.1388	0.1171	2.3287	0.2543	2.3978	2.0378
p-value	*0.1085*	*0.5862*	*0.1188*	*0.8672*	*0.3759*	*0.2859*	*0.7323*	*0.1270*	*0.6141*	*0.1215*	*0.1534*
2	3.8351	0.3165	3.0029	1.2053	0.8738	1.2993	5.2634	3.8304	0.2721	3.2506	2.1056
p-value	*0.1470*	*0.8537*	*0.2228*	*0.5474*	*0.6460*	*0.5222*	*0.0720*	*0.1473*	*0.8728*	*0.1969*	*0.3490*
3	3.8417	1.3143	3.9255	1.5623	3.4537	2.5751	5.6056	4.4629	0.8325	3.3632	2.4970
p-value	*0.2791*	*0.7258*	*0.2696*	*0.6680*	*0.3268*	*0.4619*	*0.1325*	*0.2156*	*0.8417*	*0.3390*	*0.4758*
4	4.7599	2.9395	4.5692	1.8164	3.5656	2.6982	6.4703	4.5586	2.1755	3.3659	3.1693
p-value	*0.3128*	*0.5680*	*0.3344*	*0.7695*	*0.4680*	*0.6095*	*0.1667*	*0.3357*	*0.7035*	*0.4986*	*0.5299*
5	5.1346	3.0132	4.9819	2.0583	4.8459	3.4982	6.4851	6.0827	3.5182	3.3670	8.4941
p-value	*0.3997*	*0.6980*	*0.4181*	*0.8410*	*0.4350*	*0.6237*	*0.2618*	*0.2983*	*0.6206*	*0.6436*	*0.1310*
10	17.4380	8.5239	8.9882	9.0326	14.8710	6.8328	17.8170	10.9320	9.8558	10.5390	11.7570
p-value	*0.0652*	*0.5778*	*0.5332*	*0.5290*	*0.1369*	*0.7411*	*0.0581*	*0.3628*	*0.4532*	*0.3946*	*0.3017*
15	22.7120	15.9970	18.3450	14.8590	17.9180	13.5750	25.3170	22.1340	11.3290	12.6210	25.7770
p-value	*0.0904*	*0.3823*	*0.2449*	*0.4616*	*0.2670*	*0.5580*	*0.0458*	*0.1043*	*0.7289*	*0.6316*	*0.0404*
20	30.4490	19.1410	22.1800	17.3170	23.0530	15.5360	31.4810	25.1770	14.5220	13.8120	28.0160
p-value	*0.0629*	*0.5127*	*0.3308*	*0.6323*	*0.2862*	*0.7450*	*0.0492*	*0.1948*	*0.8031*	*0.8399*	*0.1090*

Ljung-Box-Pierce Q-test for vectors of squared returns

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ
1	8.9037	6.9263	2.8220	0.0386	6.3849	0.4567	8.6803	0.0168	11.8150	11.9230	41.3450
p-value	*0.0028*	*0.0085*	*0.0930*	*0.8442*	*0.0115*	*0.4992*	*0.0032*	*0.8969*	*0.0006*	*0.0006*	*0.0000*
2	18.5810	17.5280	6.0508	11.9330	6.6759	0.4844	27.8490	6.3389	24.9680	20.7780	50.3010
p-value	*0.0001*	*0.0002*	*0.0485*	*0.0026*	*0.0355*	*0.7849*	*0.0000*	*0.0420*	*0.0000*	*0.0000*	*0.0000*
3	21.2850	18.9750	6.3324	12.2570	6.8343	0.4858	30.4870	6.7647	25.1190	23.2660	50.6470
p-value	*0.0001*	*0.0003*	*0.0965*	*0.0066*	*0.0774*	*0.9220*	*0.0000*	*0.0798*	*0.0000*	*0.0000*	*0.0000*
4	21.2880	18.9770	6.7397	12.7460	7.2099	1.2712	30.6750	6.8314	25.2750	23.2860	50.7330
p-value	*0.0003*	*0.0008*	*0.1503*	*0.0126*	*0.1252*	*0.8663*	*0.0000*	*0.1451*	*0.0000*	*0.0001*	*0.0000*
5	21.4300	23.6130	7.0523	13.5370	7.6141	2.2308	32.0650	15.1750	25.2750	24.9380	50.8690
p-value	*0.0007*	*0.0003*	*0.2168*	*0.0188*	*0.1788*	*0.8164*	*0.0000*	*0.0096*	*0.0001*	*0.0001*	*0.0000*
10	35.6700	30.9470	8.9014	18.6580	12.6120	5.4996	44.2400	29.6880	28.7860	29.6830	51.1540
p-value	*0.0001*	*0.0006*	*0.5415*	*0.0448*	*0.2462*	*0.8554*	*0.0000*	*0.0010*	*0.0013*	*0.0010*	*0.0000*
15	38.6780	42.8540	10.3710	27.6250	18.4260	11.1630	53.2510	41.9450	31.5470	30.1040	55.8010
p-value	*0.0007*	*0.0002*	*0.7958*	*0.0240*	*0.2410*	*0.7410*	*0.0000*	*0.0002*	*0.0074*	*0.0116*	*0.0000*
20	52.4290	47.7340	14.4900	28.8270	20.1830	14.7230	54.5120	50.9330	34.1520	32.5660	60.0690
p-value	*0.0001*	*0.0005*	*0.8048*	*0.0912*	*0.4466*	*0.7921*	*0.0000*	*0.0002*	*0.0251*	*0.0376*	*0.0000*

Engle ARCH-test for vectors of returns

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH
1	8.7366	6.7906	2.7683	0.0379	6.2612	0.4479	8.5067	0.0165	11.5800	11.7200	40.4900
p-value	*0.0031*	*0.0092*	*0.0962*	*0.8457*	*0.0123*	*0.5033*	*0.0035*	*0.8979*	*0.0007*	*0.0006*	*0.0000*
2	16.1150	14.6500	5.3435	11.5800	6.5570	0.3582	22.7540	6.2473	19.9200	17.5310	40.2650
p-value	*0.0003*	*0.0007*	*0.0691*	*0.0031*	*0.0377*	*0.8360*	*0.0000*	*0.0440*	*0.0000*	*0.0002*	*0.0000*
3	15.4040	14.7820	5.3171	12.2840	6.9605	0.3222	22.6690	6.7252	20.8260	18.0240	44.8910
p-value	*0.0015*	*0.0020*	*0.1500*	*0.0065*	*0.0732*	*0.9558*	*0.0000*	*0.0812*	*0.0001*	*0.0004*	*0.0000*
4	18.4790	15.7660	6.3708	14.8510	7.3710	1.1042	24.1250	7.5312	22.4420	19.2290	45.9690
p-value	*0.0010*	*0.0033*	*0.1731*	*0.0050*	*0.1175*	*0.8936*	*0.0001*	*0.1103*	*0.0002*	*0.0007*	*0.0000*
5	19.0220	19.7260	6.8158	15.4810	8.1067	2.1514	24.7490	14.3890	22.6680	20.3640	45.6470
p-value	*0.0019*	*0.0014*	*0.2347*	*0.0085*	*0.1505*	*0.8278*	*0.0002*	*0.0133*	*0.0004*	*0.0011*	*0.0000*
10	25.2100	28.3390	8.0673	20.9270	12.9340	5.2875	35.4060	27.9610	27.5150	23.3830	44.6440
p-value	*0.0050*	*0.0016*	*0.6223*	*0.0216*	*0.2274*	*0.8712*	*0.0001*	*0.0018*	*0.0022*	*0.0094*	*0.0000*
15	26.5050	32.2970	9.4679	26.8300	18.7130	9.9191	35.7800	32.4660	32.7900	26.5380	47.3410
p-value	*0.0330*	*0.0059*	*0.8518*	*0.0302*	*0.2270*	*0.8248*	*0.0019*	*0.0056*	*0.0050*	*0.0327*	*0.0000*
20	34.9120	38.1170	13.5510	26.2250	19.6380	11.7880	35.8430	34.8550	37.6190	31.6490	47.4910
p-value	*0.0206*	*0.0086*	*0.8525*	*0.1585*	*0.4808*	*0.9232*	*0.0160*	*0.0209*	*0.0099*	*0.0472*	*0.0005*

Post-filtering Ljung-Box-Pierce Q-test and Engle's ARCH test

Ljung-Box-Pierce Q-test for vectors of standardized innovations

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ
1	0.0110	0.5767	0.4516	0.0088	0.6886	0.0003	3.4967	0.0020	0.3687	0.5529	1.1060
p-value	*0.9167*	*0.4476*	*0.5016*	*0.9251*	*0.4066*	*0.9861*	*0.0615*	*0.9647*	*0.5437*	*0.4571*	*0.2930*
2	0.9360	0.5767	1.3477	0.8608	1.2382	0.3338	7.1771	2.3589	0.5270	0.8520	1.2091
p-value	*0.6262*	*0.7495*	*0.5097*	*0.6502*	*0.5384*	*0.8463*	*0.0276*	*0.3075*	*0.7684*	*0.6531*	*0.5463*
3	0.9549	1.3160	2.4195	0.8791	4.4804	1.6510	7.1887	2.8673	1.7977	0.8525	2.1339
p-value	*0.8122*	*0.7253*	*0.4900*	*0.8305*	*0.2141*	*0.6479*	*0.0661*	*0.4125*	*0.6154*	*0.8369*	*0.5451*
4	1.3374	2.2339	2.6623	1.3249	4.5842	1.6902	7.4544	2.8676	2.4477	1.4364	2.6395
p-value	*0.8550*	*0.6928*	*0.6158*	*0.8571*	*0.3327*	*0.7925*	*0.1137*	*0.5802*	*0.6540*	*0.8379*	*0.6199*
5	1.8028	2.2983	3.3400	1.3262	5.1349	2.6570	7.6803	3.0799	3.7018	1.7466	6.9364
p-value	*0.8757*	*0.8065*	*0.6477*	*0.9322*	*0.3996*	*0.7527*	*0.1748*	*0.6877*	*0.5931*	*0.8830*	*0.2254*
10	11.0700	7.3449	8.7719	7.3955	15.9200	6.4199	17.5960	7.3324	10.7620	9.1980	8.8103
p-value	*0.3521*	*0.6925*	*0.5539*	*0.6877*	*0.1020*	*0.7788*	*0.0622*	*0.6937*	*0.3763*	*0.5134*	*0.5502*
15	17.3980	12.0380	18.5030	11.4590	19.4530	11.5840	23.7380	13.6630	11.9180	10.9450	18.7880
p-value	*0.2956*	*0.6762*	*0.2371*	*0.7194*	*0.1939*	*0.7102*	*0.0697*	*0.5512*	*0.6852*	*0.7565*	*0.2235*
20	22.3710	14.6360	22.2560	12.4640	24.8390	13.7820	26.4310	15.8190	17.5540	12.7740	20.5930
p-value	*0.3207*	*0.7969*	*0.3268*	*0.8992*	*0.2077*	*0.8414*	*0.1520*	*0.7278*	*0.6167*	*0.8869*	*0.4214*

Ljung-Box-Pierce Q-test for vectors of squared standardized innovations

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ	Stat-LBQ
1	0.7379	0.0239	0.8338	0.5945	0.1448	0.8072	0.0846	1.8600	0.1644	0.5042	0.2426
p-value	*0.3903*	*0.8771*	*0.3612*	*0.4407*	*0.7036*	*0.3690*	*0.7712*	*0.1726*	*0.6851*	*0.4777*	*0.6223*
2	4.0063	0.6191	0.9348	6.8240	0.4438	0.8277	0.1466	3.3185	0.2756	0.9247	0.6661
p-value	*0.1349*	*0.7338*	*0.6266*	*0.0330*	*0.8010*	*0.6611*	*0.9293*	*0.1903*	*0.8713*	*0.6298*	*0.7167*
3	4.0291	0.8838	1.2450	6.9390	0.4450	0.8892	1.7395	3.4229	0.2822	0.9256	0.8453
p-value	*0.2583*	*0.8293*	*0.7422*	*0.0739*	*0.9308*	*0.8280*	*0.6282*	*0.3309*	*0.9633*	*0.8193*	*0.8386*
4	4.4503	3.4930	1.9024	7.5451	2.3049	1.6210	1.8771	3.6276	0.4649	1.1961	0.8518
p-value	*0.3485*	*0.4790*	*0.7537*	*0.1097*	*0.6799*	*0.8050*	*0.7584*	*0.4588*	*0.9768*	*0.8787*	*0.9314*
5	6.3006	5.2230	1.9821	7.6113	3.1298	2.8077	2.0237	5.0198	0.5070	1.2547	1.6164
p-value	*0.2781*	*0.3893*	*0.8516*	*0.1790*	*0.6800*	*0.7296*	*0.8459*	*0.4135*	*0.9919*	*0.9395*	*0.8993*
10	12.5110	10.7130	4.1698	9.1154	5.2478	6.1428	8.4880	7.5880	4.1045	4.3176	2.2559
p-value	*0.2523*	*0.3803*	*0.9394*	*0.5212*	*0.8740*	*0.8031*	*0.5813*	*0.6690*	*0.9425*	*0.9319*	*0.9940*
15	13.9510	18.3540	6.8927	15.1820	12.8650	12.0030	13.5000	16.6510	10.5780	6.7901	6.3352
p-value	*0.5293*	*0.2445*	*0.9606*	*0.4384*	*0.6127*	*0.6788*	*0.5637*	*0.3402*	*0.7819*	*0.9632*	*0.9736*
20	21.6880	22.7730	13.8880	17.4790	15.7480	16.0560	15.3180	20.2420	14.2120	9.6200	14.1130
p-value	*0.3577*	*0.3001*	*0.8362*	*0.6217*	*0.7322*	*0.7131*	*0.7579*	*0.4429*	*0.8196*	*0.9746*	*0.8247*

Engle ARCH-test for vectors of standardized innovations

	S&P 500	CAC 40	TSX	Hang Seng	MIB30	Nikkei	AEX	Straits	Madrid SE	FTSE 100	SMI
Lag	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH	Stat-ARCH
1	0.7239	0.0235	0.8190	0.5827	0.1418	0.7917	0.0830	1.8238	0.1611	0.4963	0.2376
p-value	*0.3949*	*0.8783*	*0.3655*	*0.4453*	*0.7065*	*0.3736*	*0.7733*	*0.1769*	*0.6882*	*0.4811*	*0.6260*
2	3.7709	0.5967	1.0862	6.4158	0.4708	0.6601	0.2012	2.8613	0.2478	1.4396	0.6232
p-value	*0.1518*	*0.7420*	*0.5809*	*0.0404*	*0.7903*	*0.7189*	*0.9043*	*0.2392*	*0.8835*	*0.4869*	*0.7323*
3	4.2356	0.8081	1.2312	6.6446	0.5063	0.6518	1.7793	2.8447	0.2686	2.0337	0.8362
p-value	*0.2371*	*0.8475*	*0.7455*	*0.0841*	*0.9175*	*0.8845*	*0.6195*	*0.4162*	*0.9658*	*0.5654*	*0.8408*
4	6.0001	3.6911	1.8208	8.2931	2.4794	1.4362	2.0343	3.4565	0.4729	1.9587	0.8471
p-value	*0.1991*	*0.4494*	*0.7687*	*0.0814*	*0.6483*	*0.8379*	*0.7295*	*0.4845*	*0.9761*	*0.7434*	*0.9320*
5	8.4610	5.4073	1.8042	8.5734	3.1196	2.8415	1.9179	4.7197	0.5465	2.1471	1.4851
p-value	*0.1326*	*0.3682*	*0.8755*	*0.1273*	*0.6816*	*0.7244*	*0.8604*	*0.4510*	*0.9903*	*0.8284*	*0.9148*
10	13.3880	10.8070	4.5640	9.3823	5.4673	6.0061	8.2398	7.8902	3.7484	5.7450	2.1991
p-value	*0.2028*	*0.3727*	*0.9183*	*0.4963*	*0.8579*	*0.8148*	*0.6054*	*0.6396*	*0.9580*	*0.8362*	*0.9946*
15	14.2410	16.8680	7.2544	14.7690	12.7910	11.0770	12.6560	17.0350	12.4040	9.2504	6.0028
p-value	*0.5074*	*0.3268*	*0.9502*	*0.4682*	*0.6185*	*0.7471*	*0.6289*	*0.3168*	*0.6482*	*0.8641*	*0.9797*
20	21.3910	20.9180	15.5230	16.7160	15.3650	13.1390	14.9420	20.3960	15.9500	10.5130	12.2020
p-value	*0.3744*	*0.4020*	*0.7458*	*0.6713*	*0.7552*	*0.8713*	*0.7797*	*0.4334*	*0.7197*	*0.9579*	*0.9089*

APPENDIX B: Optimal weights in the classical optimization with positivity constraints (see 1)

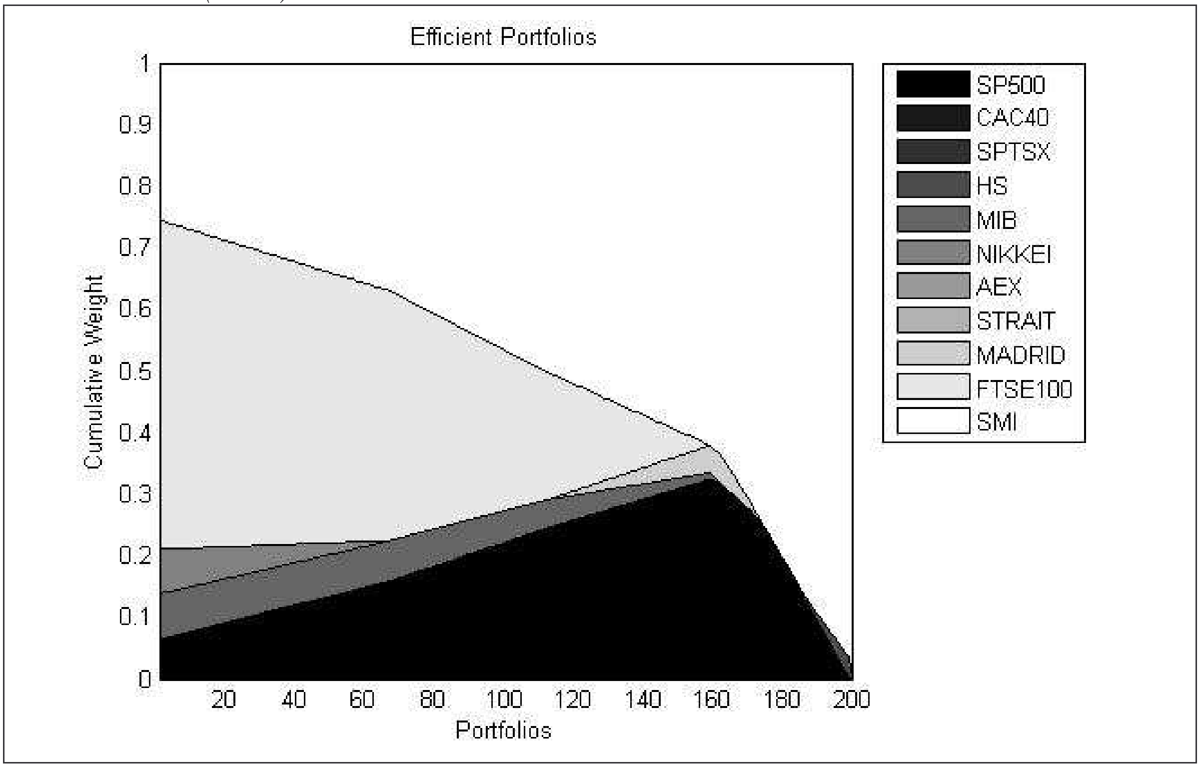


Exhibit 1: 10.000 standard normal random variables with ρ=0.7 and different structure of dependence. Parameters of copulas (α) are calculated using direct relationships between ρ, Kendall's tau (ρ_τ) and the parameter of the specific copula.

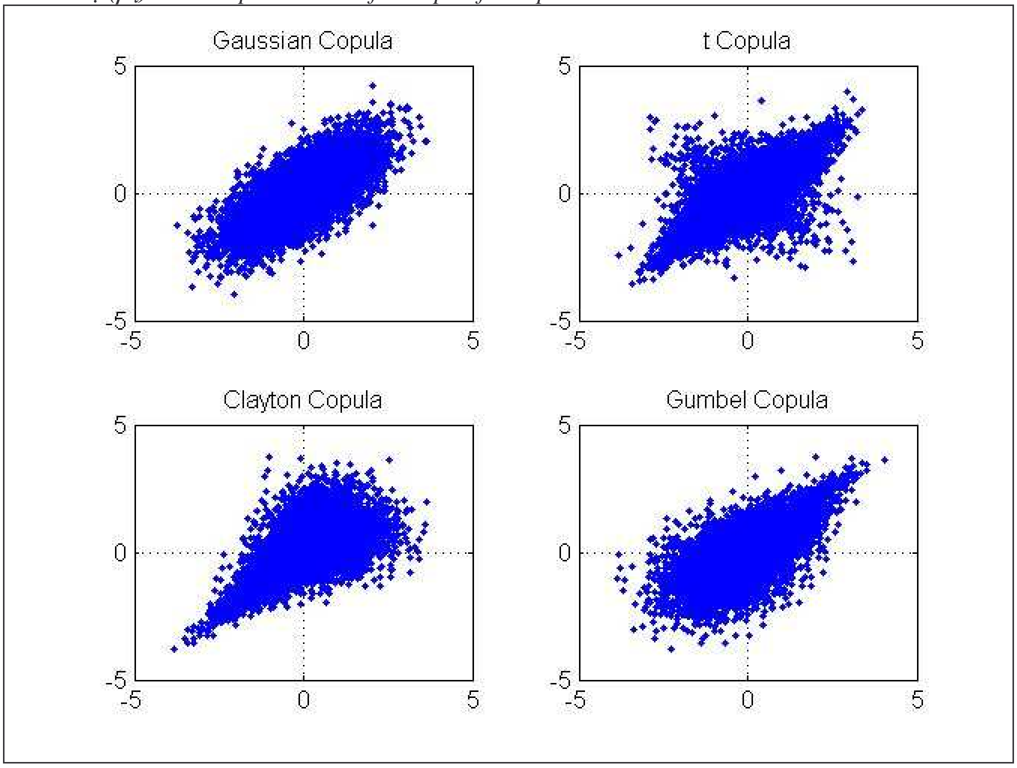


Exhibit 2: Student's t bivariate sample with 3 dof and linear correlation +0.7. Estimates of ρ and ρ_τ

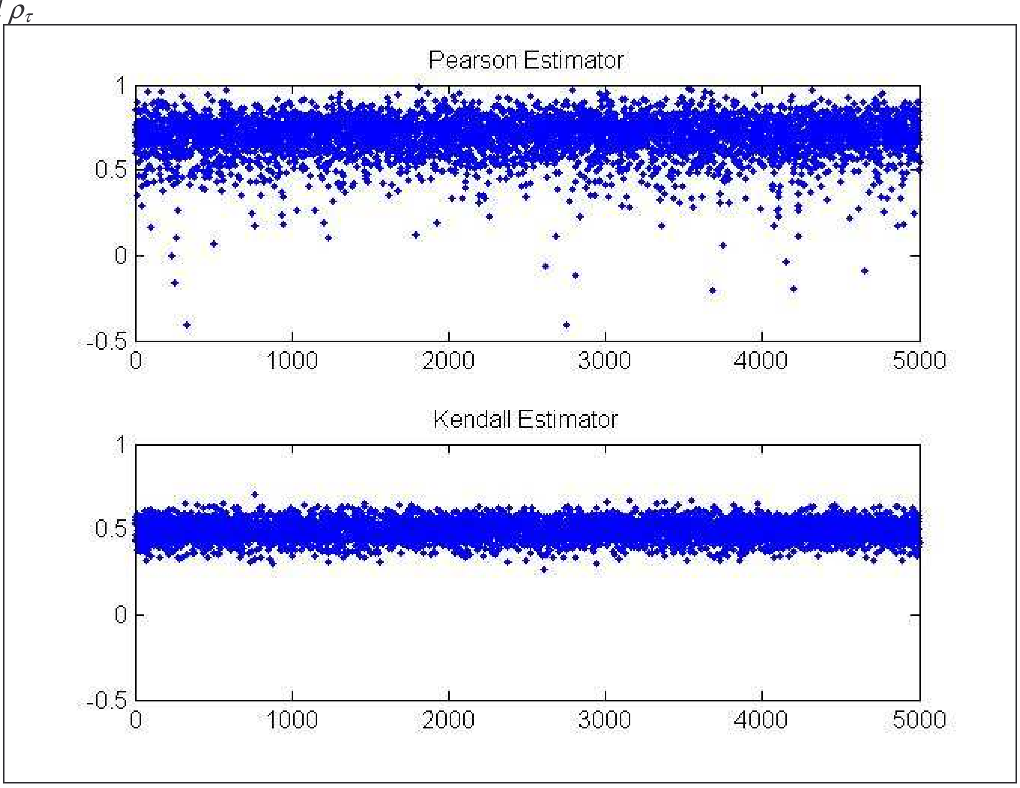


Exhibit 3: Relevant coefficients for implicit and Archimedean copulas

Copula type	ρ_τ	$\lambda_L ; \lambda_U$
$C_\rho^{Gaussian}$	$\rho_\tau = \frac{2}{\pi} ArcSin(\rho)$	$\lambda_L = \lambda_U = 0$
$C_\rho^{t_\nu}$	$\rho_\tau = \frac{2}{\pi} ArcSin(\rho)$	$\lambda_L = \lambda_U = t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)\left(1-Sin\left(\frac{\pi\rho_\tau}{2}\right)\right)}{1+Sin\left(\frac{\pi\rho_\tau}{2}\right)}}\right)$
$C_\alpha^{Clayton}$	$\rho_\tau = \frac{\alpha}{\alpha+2}$	$\lambda_L = 2^{\left(\frac{\rho_\tau - 1}{2\rho_\tau}\right)}$ $(\rho_\tau > 0)$
C_α^{Gumbel}	$\rho_\tau = 1 - \frac{1}{\alpha}$	$\lambda_U = 2 - 2^{(1-\rho_\tau)}$

Exhibit 4: Tail dependence coefficients (ρ_τ) for several copulas

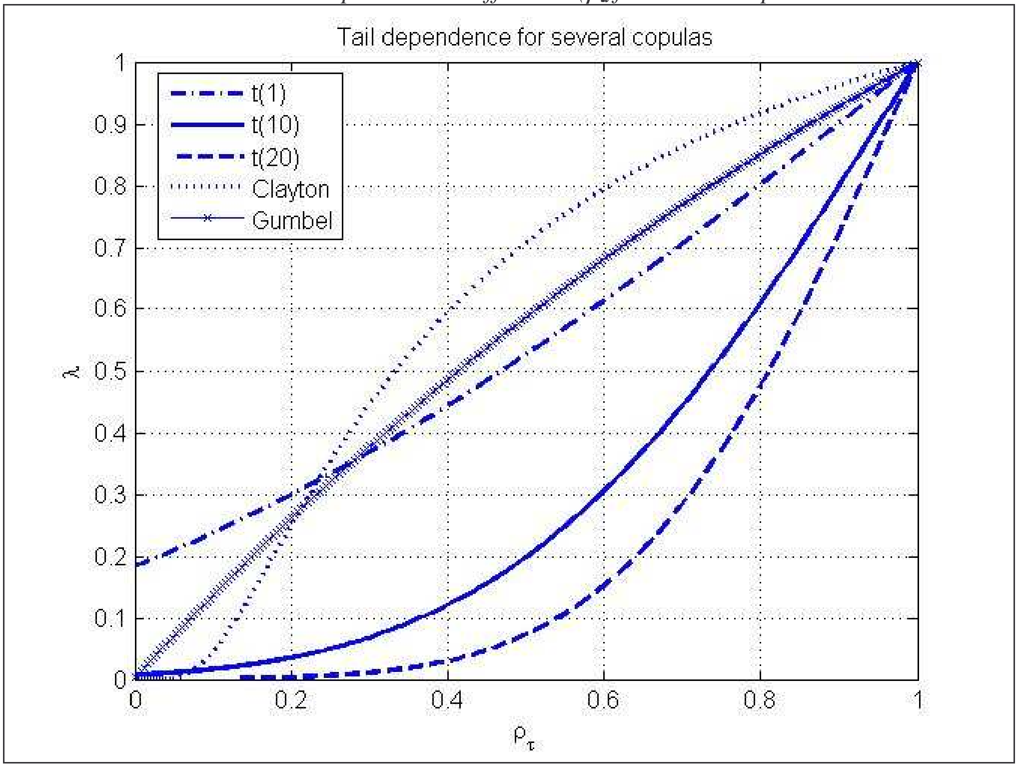


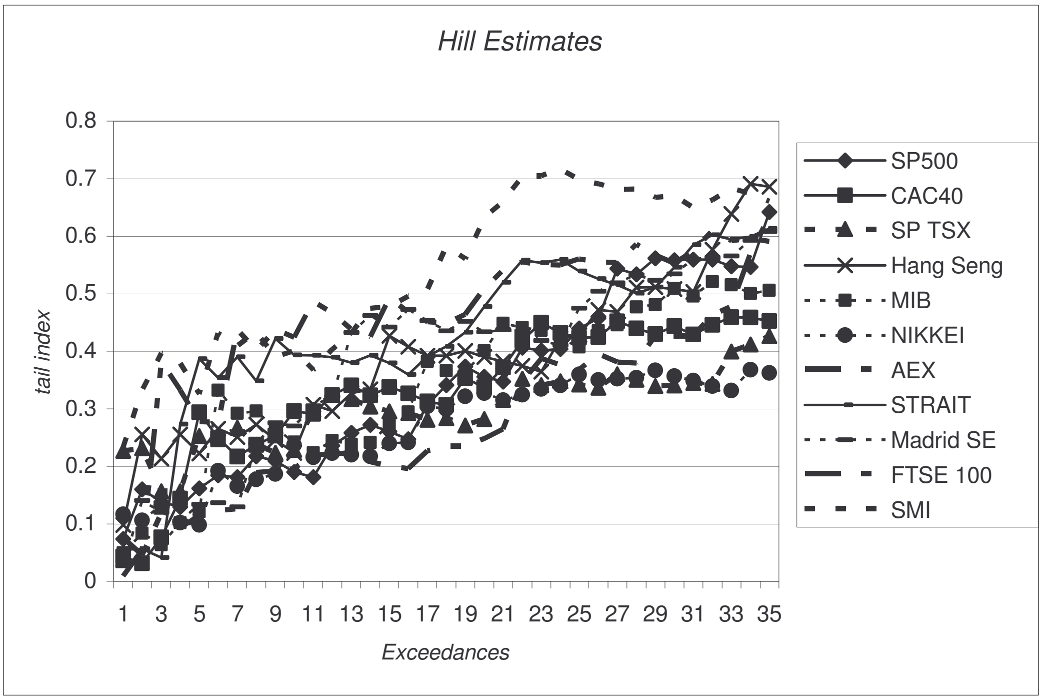
Exhibit 5: Descriptive Statistics

Descriptive Statistics - Monthly asset returns											
Univariate Statistics											
	S_P500	*SMI*	*CAC40*	*TSX*	*Hang_Seng*	*Mib30*	*Nikkei*	*AEX*	*Straits*	*Madrid*	*FTSE*
Minimum	-0.1318	-0.1899	-0.1688	-0.2653	-0.3136	-0.1786	-0.1730	-0.2494	-0.2376	-0.2154	-0.1251
Date	Aug-98	Aug-98	Sep-02	Aug-98	Oct-97	Sep-01	Dec-00	Jul-02	May-98	Aug-98	Jul-02
Maximum	0.1263	0.2024	0.1412	0.1458	0.2707	0.2315	0.2220	0.1406	0.3607	0.2020	0.1235
Mean	0.0085	0.0104	0.0070	0.0080	0.0100	0.0057	-0.0002	0.0086	0.0064	0.0081	0.0059
Std. Deviation	0.0507	0.0507	0.0575	0.0607	0.0854	0.0705	0.0725	0.0586	0.0790	0.0608	0.0453
Skewness	-0.2401	-0.6037	-0.3021	-0.5358	-0.0027	0.2105	0.1138	-0.8933	0.1651	-0.3120	-0.4191
Kurtosis	3.0529	5.9853	3.0754	4.5375	4.5910	3.5641	2.9532	5.3063	5.8149	3.9492	2.8643
Normality tests											
Jarque-Bera test	1.6736	69.5744	2.6483	23.2949	16.0753	2.9881	0.4784	57.8772	53.1028	8.2025	5.3052
p-value	0.4331	0.0000	0.2662	0.0000	0.0003	0.2245	0.7873	0.0000	0.0000	0.0166	0.0705
Schmid and Trede test											
P-test	2.0969	1.6457	1.6367	1.6598	1.9883	1.7257	1.5171	1.6986	1.787	1.9188	1.9185
T-test	1.6636	1.9887	1.8997	1.7103	2.0545	2.0638	1.7456	1.9614	2.4306	1.8678	1.6057
L-test	3.4883	3.2729	3.1092	2.8388	4.0848	3.5616	2.6482	3.3315	4.3434	3.584	3.0804
Multivariate Statistics											
Mardia's Test	Coefficient	*p-value*									
Multivariate skewness	18.9718	0.0000									
Multivariate kurtosis	181.0544	0.0000									

Exhibit 6: Parameter estimates for the ARMA/GARCH models

Parameter	*S_P500*	*SMI*	*CAC40*	*TSX*	*Hang_Seng*	*Mib30*	*Nikkei*	*AEX*	*Straits*	*Madrid*	*FTSE*
C	0.0032	0.0024	0.0176	0.0036	0.00103	0.0077	(0.0002)	0.0243	0.0055	0.0102	0.0015
(t-ratio)	0.6867	0.9125	0.7681	0.7681	0.2278	0.7199	(0.2145)	3.3586	0.8375	0.2056	0.3538
ϕ	0.6146	0.8213	(0.9421)	0.6728	0.8533	(0.0928)	(0.7609)	(0.9408)	0.4198	(0.2152)	0.8239
(t-ratio)	1.0293	4.1197	(5.1466)	1.6453	1.3415	(0.0755)	(4.5922)	(14.8029)	0.6259	(0.0366)	1.6185
φ	(0.5253)	(0.74311)	0.9232	(0.5872)	(0.8356)	0.02130	0.8709	0.8920	(0.3329)	0.2038	(0.7938)
(t-ratio)	(1.0036)	(3.1075)	4.3783	(1.2847)	(1.2394)	0.0172	7.1540	9.6063	(0.4741)	0.0346	(1.4505)
κ	0.0001	0.0008	0.0002	0.0007	0.0002	0.0001	0.0036	0.0006	0.0001	0.0015	0.0001
(t-ratio)	1.1519	1.2857	0.8669	1.1304	0.6385	1.5363	0.0011	2.0478	0.7428	1.8606	0.7878
α	0.8037	0.4858	0.7358	0.6165	0.8631	0.5665	0.1801	0.3110	0.8776	0.2930	0.6462
(t-ratio)	7.7271	1.6469	4.7310	2.7037	9.8423	3.7821	19.5750	1.7946	12.5507	0.9777	4.8661
β	0.1407	0.1990	0.1886	0.1895	0.1008	0.2996	0	0.5311	0.1003	0.2988	0.3128
(t-ratio)	1.8685	1.7564	1.7913	1.7133	1.9053	2.3549	0	3.5244	1.7655	1.4080	2.4472
DoF	200	6.3871	200	25.921	6.3793	12.028	71.737	38.887	6.131	8.6457	200
(t-ratio)	NA	2.3940	NA	0.5435	1.7228	0.8811	0.1647	0.383	1.7406	1.905	NA

Exhibit 7: Non parametric estimators for τ



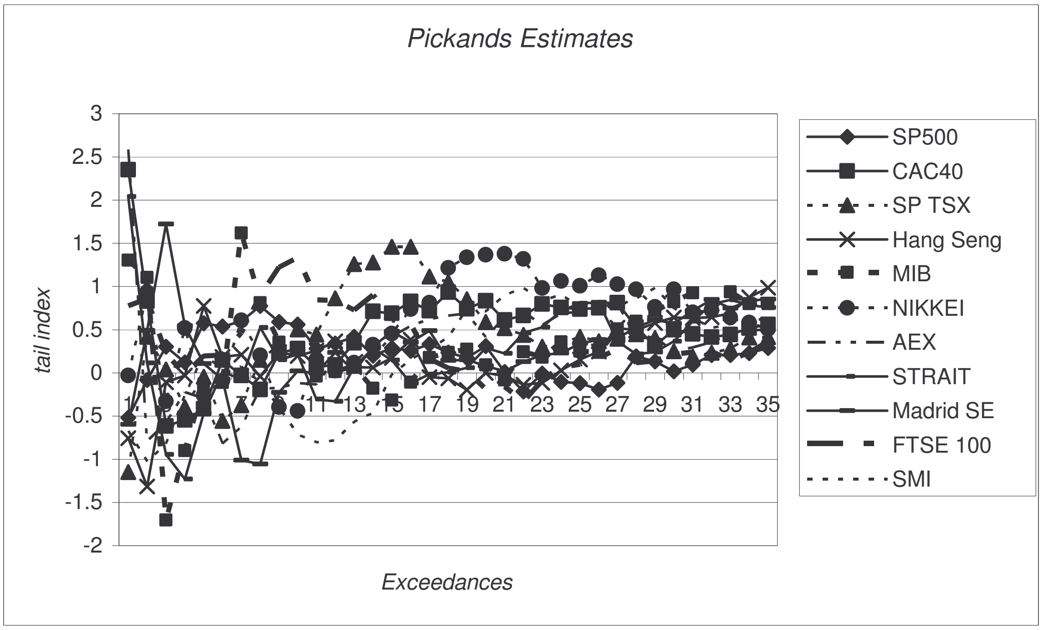


Exhibit 8: Optimal value for the tail index (τ) through the Hill's and Pickands's estimator

	t(2)	t(3)	t(4)	t(5)	
MSE	6.8741	3.0734	1.7313	1.1652	
q^{opt}	16	9	6	5	
Hill's estimator for tail index (τ_i) - *p-values* are in parenthesis					Pickands's estimator
S_P500	0.2601				0.2887
p-value	*(0.2607)*				
CAC40	0.3375				0.6898
p-value	*(0.3553)*				
TSX	0.2952				0.4591
p-value	*(0.3136)*				
HS		0.2724			0.0404
p-value		*(0.3891)*			
MIB			0.1280		0.1435
p-value			*(0.22466)*		
NIKKEI			0.0982		-0.1275
p-value			*(0.12077)*		
AEX		0.4009			-0.1802
p-value		*(0.39332)*			
STRAITS			0.3867		-0.2674
p-value			*(0.37477)*		
MADRID SE				0.1046	0.4443
p-value				*(0.26341)*	
FTSE100	0.2009				0.6133
p-value	*(0.1317)*				
SMI				0.3790	-0.263
p-value				*(0.35684)*	

Exhibit 9: Kendall's tau and Pearson's rho

The upper-right triangle shows the Pearson's rho. The lower-left triangle shows the implicit Kendall's tau

	S_P500	CAC40	TSX	Hang_Seng	Mib	Nikkei	AEX	Straits	Madrid	FTSE	SMI
S_P500	**1.000**	0.72	0.80	0.66	0.53	0.51	0.72	0.65	0.65	0.80	0.69
CAC40	0.508	**1.000**	0.66	0.55	0.61	0.43	0.83	0.52	0.74	0.78	0.68
TSX	0.590	0.445	**1.000**	0.69	0.58	0.49	0.65	0.65	0.65	0.68	0.57
Hang_Seng	0.494	0.406	0.527	**1.000**	0.42	0.40	0.57	0.76	0.55	0.63	0.52
Mib	0.369	0.473	0.398	0.316	**1.000**	0.33	0.57	0.41	0.70	0.52	0.40
Nikkei	0.353	0.291	0.342	0.265	0.249	**1.000**	0.44	0.44	0.46	0.47	0.44
AEX	0.536	0.629	0.458	0.452	0.435	0.295	**1.000**	0.56	0.73	0.80	0.76
Straits	0.469	0.364	0.464	0.551	0.313	0.271	0.412	**1.000**	0.51	0.61	0.47
Madrid	0.474	0.534	0.427	0.400	0.513	0.295	0.549	0.330	**1.000**	0.70	0.60
FTSE	0.600	0.567	0.479	0.468	0.384	0.300	0.600	0.415	0.491	**1.000**	0.71
SMI	0.470	0.468	0.348	0.377	0.292	0.258	0.513	0.291	0.391	0.488	**1.000**

Exhibit 10: Optimal portfolio weights with the tail dependence (τ) and with α-parameterized Clayton copula

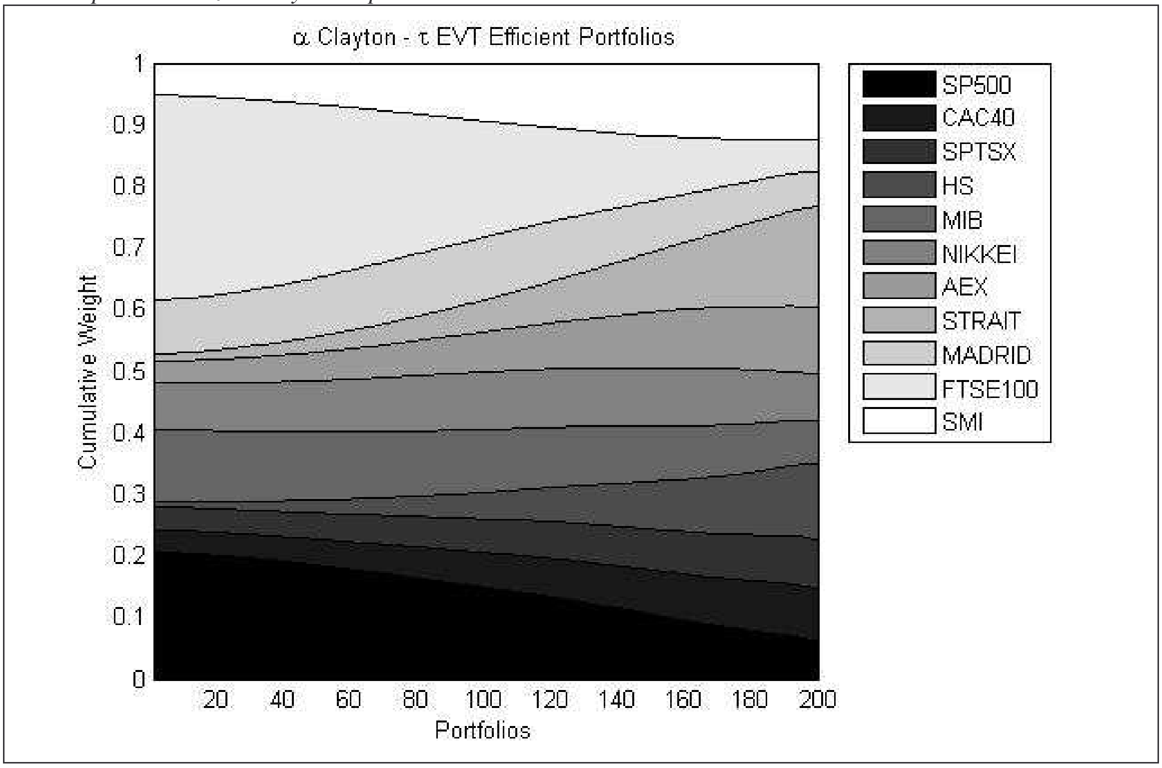


Exhibit 11: Optimal portfolio weights with the resampling (QRMCSAA) method

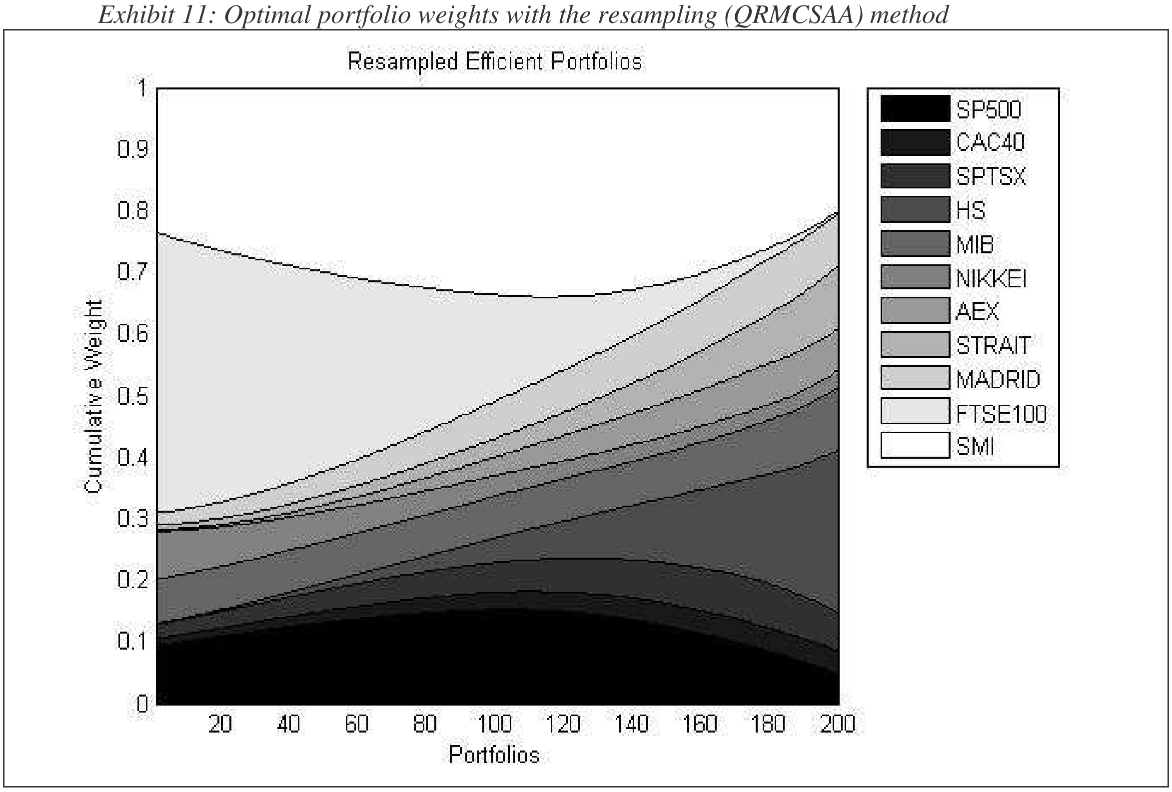


Exhibit 12: Upper and lower bounds for both resampling and τ-EVT α-Clayton methods

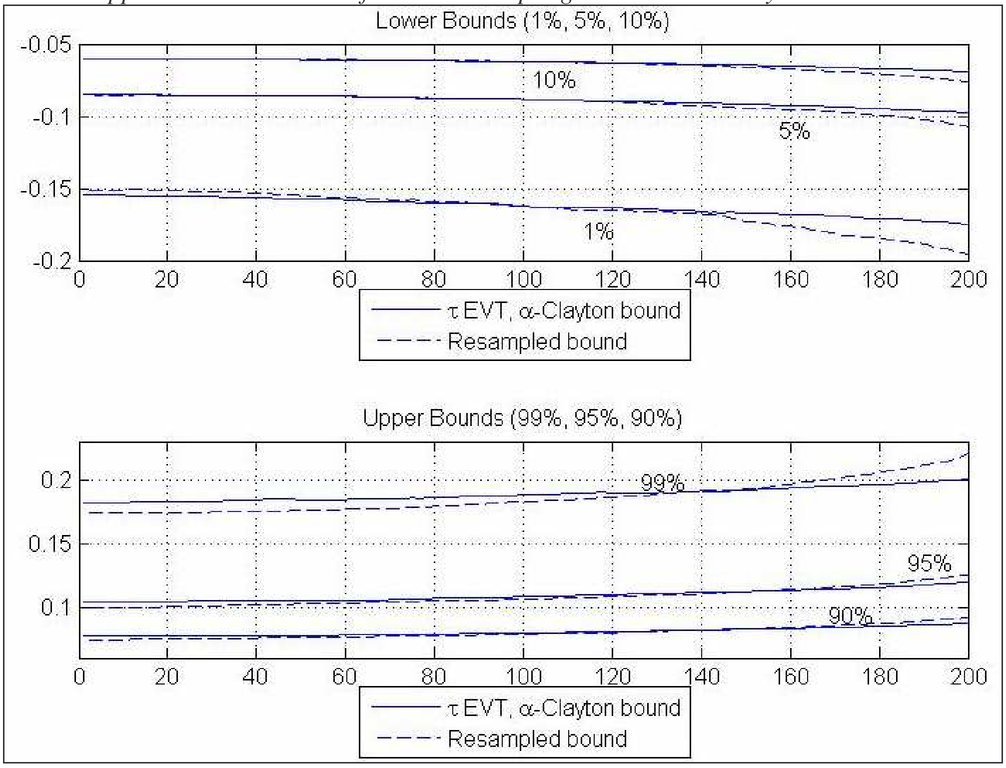


Exhibit 13: Main statistics for several portfolios with τ-EVT α-Clayton simulation and with resampling

τ-EVT α-Clayton efficient portfolios (Main Statistics)

Portfolios (#)	*1*	*20*	*40*	*60*	*80*	*100*	*120*	*140*	*160*	*180*	*200*
Mean	0.0065	0.0065	0.0066	0.0067	0.0067	0.0068	0.0069	0.0070	0.0071	0.0072	0.0073
Median	0.0060	0.0061	0.0062	0.0063	0.0064	0.0065	0.0066	0.0068	0.0069	0.0070	0.0071
Trimmed mean(1%)	0.0064	0.0064	0.0065	0.0066	0.0066	0.0067	0.0068	0.0069	0.0070	0.0071	0.0072
Min	(0.5948)	(0.5967)	(0.5996)	(0.6026)	(0.6179)	(0.6469)	(0.6744)	(0.6832)	(0.7224)	(0.7599)	(0.7838)
Max	0.6310	0.6335	0.6381	0.6434	0.6499	0.6578	0.6681	0.6812	0.6960	0.7131	0.7364
Stand. Dev.	0.0608	0.0610	0.0614	0.0619	0.0625	0.0632	0.0640	0.0651	0.0662	0.0677	0.0695
Skewness	0.0418	0.0390	0.0347	0.0292	0.0220	0.0142	0.0061	(0.0018)	(0.0090)	(0.0163)	(0.0225)
Kurtosis	9.0640	9.0779	9.1016	9.1309	9.1724	9.2314	9.2051	9.2004	9.2026	9.2066	9.2116
P test	1.8056	1.8018	1.8050	1.8083	1.8057	1.8070	1.8028	1.8068	1.8063	1.8038	1.7967
T test	2.0513	2.0416	2.0551	2.0534	2.0697	2.0653	2.0610	2.0623	2.0645	2.0607	2.0745
L test	3.7512	3.7553	3.7573	3.7701	3.7408	3.7417	3.7245	3.7448	3.7339	3.7135	3.7036
Sharpe ratio	0.0793	0.0792	0.0796	0.0812	0.0813	0.0812	0.0813	0.0817	0.0820	0.0813	0.0814

Resampled efficient portfolios (Main Statistics)

Portfolios (#)	*1*	*20*	*40*	*60*	*80*	*100*	*120*	*140*	*160*	*180*	*200*
Mean	0.0070	0.0073	0.0075	0.0078	0.0080	0.0082	0.0084	0.0085	0.0085	0.0085	0.0084
Median	0.0066	0.0069	0.0072	0.0076	0.0079	0.0080	0.0081	0.0082	0.0083	0.0086	0.0084
Trimmed mean(1%)	0.0069	0.0072	0.0075	0.0077	0.0080	0.0082	0.0083	0.0084	0.0085	0.0084	0.0084
Min	(0.6256)	(0.6313)	(0.6376)	(0.6448)	(0.6535)	(0.6645)	(0.6786)	(0.6968)	(0.7233)	(0.7716)	(0.8469)
Max	0.6444	0.6405	0.6391	0.6406	0.6456	0.6551	0.6708	0.6959	0.7321	0.7832	0.8685
Stand. Dev.	0.0615	0.0617	0.0621	0.0627	0.0634	0.0644	0.0656	0.0671	0.0692	0.0720	0.0768
Skewness	0.0736	0.0656	0.0563	0.0461	0.0355	0.0252	0.0158	0.0081	0.0023	(0.0029)	(0.0062)
Kurtosis	9.4794	9.4155	9.3712	9.3458	9.3372	9.3422	9.3628	9.3990	9.4561	9.5467	9.7469
P test	1.8196	1.8199	1.8178	1.8197	1.8249	1.8241	1.8221	1.8177	1.8086	1.8020	1.7852
T test	2.0776	2.0842	2.0817	2.0849	2.0716	2.0706	2.0659	2.0726	2.0748	2.0722	2.0903
L test	3.7325	3.7155	3.7358	3.7365	3.7771	3.7672	3.7554	3.7485	3.7476	3.7380	3.7556
Sharpe ratio	0.0863	0.0908	0.0945	0.0976	0.1000	0.1016	0.1023	0.1014	0.0988	0.0945	0.0874

Critical Values with 20,000 observations are: for P-test 1.7330 (99%) and 1.7249 (95%); for T test 1.7325 (99%) and 1.7241 (95%); for L test 2.9691 (99%) and 2.9505 (95%)

Exhibit 14: Main statistics for 100th portfolio with τ-EVT α-Clayton simulation and with resampling

τ-EVT α-Clayton efficient portfolios (Main Statistics)

Multivariate	t(3)	t(4)	t(5)	t(6)	t(7)	t(8)	t(9)	t(10)	t(15)	t(20)	t(25)	t(30)
Std.Dev.	0.0779	0.0643	0.0603	0.0576	0.0542	0.0524	0.0516	0.0509	0.0505	0.0485	0.0477	0.0472
Kurtosis	17.98	9.2314	6.3349	5.3188	4.878	4.4641	4.0714	3.8816	3.816	3.3224	3.2013	3.2156
T-test	2.2212	2.0641	1.9452	1.9013	1.8931	1.8544	1.8327	1.8252	1.7927	1.7566	1.7316	1.7328
Sharpe ratio	0.0654	0.0812	0.0947	0.0951	0.0952	0.0963	0.0972	0.0978	0.0987	0.0991	0.0993	0.1054

Resampled efficient portfolios (Main Statistics)

Multivariate	t(3)	t(4)	t(5)	t(6)	t(7)	t(8)	t(9)	t(10)	t(15)	t(20)	t(25)	t(30)
Std.Dev.	0.0783	0.0644	0.0612	0.0594	0.0559	0.0543	0.0532	0.0531	0.0522	0.0501	0.0494	0.0487
Kurtosis	17.97	9.3422	7.4322	5.4771	4.9712	4.5255	4.1364	3.9956	3.8361	3.3096	3.2116	3.2344
T-test	2.2298	2.0706	1.9561	1.9122	1.8801	1.8573	1.8445	1.8419	1.7948	1.7718	1.7477	1.7605
Sharpe ratio	0.0824	0.1016	0.1184	0.1219	0.1261	0.1195	0.1246	0.1275	0.1288	0.1328	0.1279	0.148