A General Characterization of the Early Exercise Premium*

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Abstract

Under the (weak) assumption of a Markovian underlying price process, and based on its first passage time density to the exercise boundary, an alternative characterization of the early exercise premium is proposed. This new representation is automatically consistent with the *value-matching condition*, possesses appropriate asymptotic properties, and is valid for any parameterization of the exercise boundary. A non-linear but one-dimensional integral equation is obtained for the optimal stopping time density, which is shown to be easily recovered from the underlying asset price transition density function. Several exercise boundary specifications are tested under the standard geometric Brownian motion assumption and for the CEV model.

I. Introduction

The absence of a closed-form pricing solution for the American put (or call, but on a dividend-paying asset) stems from the fact that the option price and the early exercise boundary must be determined simultaneously as the solution of the same free boundary problem set up by McKean (1965). Consequently, the vast literature on this subject, which is reviewed for instance in Barone-Adesi (2005), has proposed only numerical solution methods and analytical approximations.

The numerical methods include the finite difference schemes introduced by Brennan and Schwartz (1977), and the binomial model of Cox, Ross, and Rubinstein (1979). These methods are both simple and convergent, in the sense that accuracy can be improved by incrementing the number of time or state space steps. However, they are also too time-consuming and do not provide the comparative statics attached to an analytical solution.

One of the first quasi-analytical approximations is due to Barone-Adesi and Whaley (1987), who use the quadratic method of MacMillan (1986). Despite its high efficiency and the accuracy improvements brought by subsequent extensions (see for example, Ju and Zhong (1999)), this method is not convergent. Another non-convergent approach is proposed by Johnson (1983) and Broadie and Detemple (1996). These papers provide lower and upper bounds for American options, which are based on regression coefficients that are estimated through a time-demanding calibration to a large set of option contracts. Moreover, as argued by (Ju (1998), p. 642), this econometric approach can generate less accurate hedging ratios, because the regression coefficients are optimized only for pricing purposes. More recently, Sullivan (2000) approximates the option value function through Chebyshev polynomials and employs a Gaussian quadrature integration scheme at each discrete exercise date. Although the speed and accuracy of the proposed numerical approximation can be enhanced via Richardson extrapolation, its convergence properties are still unknown.

Concerning convergent pricing methodologies, Geske and Johnson (1984) approximate the American option price through an infinite series of multivariate normal distribution functions. Although the pricing accuracy can be increased as more terms are added, only the first few terms are considered and a Richardson extrapolation scheme is employed in order to reduce the computational burden.[1] Another convergent method, which is also fast and accurate, is the randomization approach of Carr (1998), which also uses Richardson extrapolation. It must be noted, however, that one of the main disadvantages of extrapolation schemes is the indetermination of the sign for the approximation error.

Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), and Jamshidian (1992) proposed the so-called "integral representation method", which provides a quasi-analytical characterization of the early exercise premium through an integral equation. The numerical efficiency of this approach depends on the specification adopted for the early exercise boundary. For instance, Ju (1998) derives fast and accurate approximate solutions based on a multipiece exponential representation of the early exercise boundary. Adopting simpler parameterizations of the exercise boundary (which is assumed to be constant or of exponential type), Ingersoll (1998) and Sbuelz (2004) are able to decompose the American put price into a down-and-out European put and a non-deferrable rebate. Hence, they provide closed-form approximations that are fast to implement but not very accurate.

As argued by (Carr (1998), p. 616) and as shown by the numerical experiments run by Broadie and Detemple (1996) and Ju (1998), the most efficient and accurate analytical pricing methods correspond to the econometric approach of Broadie and Detemple (1996); the randomization method of Carr (1998); and the multipiece exponential boundary approximation of Ju (1998). But, given the lower accuracy of the Broadie and Detemple (1996) method with respect to the computation of hedging ratios, the last two approaches seem to be the more promising ones to date. Notice, however, that all the studies already mentioned are based on the Black and Scholes (1973) geometric Brownian motion assumption, and most of them differ only in the specification adopted for the exercise boundary. Kim and Yu (1996) and Detemple and Tian (2002) constitute two notable exceptions: they extend the "integral representation method" to alternative diffusion processes. However, and in opposition to the standard geometric Brownian motion case, such an extension does not offer a closed-

[1] Chung and Shackleton (2007) generalize the Geske-Johnson method through a two-point scheme based not only on the inter-exercise time dimension but also on the time to maturity of the option contract.

form solution for the integral equation characterizing the early exercise premium (even for the simplest early exercise boundary specifications), which undermines the computational efficiency of this approach.

Based on the optimal stopping approach initiated by Bensoussan (1984) and Karatzas (1988), the main purpose of this paper is to derive an alternative characterization of the American option price that is valid for any continuous representation of the exercise boundary and for any Markovian diffusion process describing the dynamics of the underlying asset price. The proposed characterization possesses at least three advantages over the extended integral representation of (Kim and Yu (1996), equations 10 or 13): 1) it is automatically consistent with the *value-matching condition*; 2) it converges to the perpetual American option price as the option maturity tends to infinity; and 3) it is more efficient for the same level of accuracy. Although knowledge of the first passage time density of the underlying price process to the exercise boundary is required by the proposed pricing solution, it is shown that such optimal stopping time density can be recovered easily from the transition density function. Hence, the proposed characterization of the American option price requires only an efficient valuation formula for its European counterpart, as well as knowledge of the underlying asset price transition density function.

To exemplify the proposed pricing methodology, several parameterizations of the early exercise boundary are tested under the usual geometric Brownian motion assumption and the Constant Elasticity of Variance (CEV) model. Special attention will be devoted to this last framework since it is consistent with two well-known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized stock volatility (*leverage effect*), as documented for instance in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*), which is observed, for example, by Dennis and Mayhew (2002).

The following sections of this paper are organized as follows. Based on the optimal stopping approach, section II separates the American option into a non-deferrable rebate and a European down-and-out option. In section III, such a barrier option approach is shown to be equivalent to the usual decomposition between a European option and an early exercise premium. Moreover, an alternative, quasi-analytical, and more general characterization is offered for the early exercise premium, and its asymptotic properties are tested. Section IV provides an efficient algorithm for the computation of the first hitting time density of the underlying price process, which allows the comparison, in section VI, of the different specifications of the early exercise boundary discussed in section V. Section VII concludes.

II. Model Setup

The valuation of American options will be explored in the context of a stochastic intertemporal economy with continuous trading on the time-interval $[t_0, T]$, for some fixed time $T > t_0$, where uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Throughout the paper, $\mathbb{Q}$ will denote the martingale probability measure obtained when the numéraire of the economy under analysis is taken to be a money market account B_t, whose dynamics are governed by the following ordinary differential equation:

$$dB_t = rB_t dt, \tag{1}$$

where r denotes the riskless interest rate, which is assumed to be constant.

Although the alternative representation of the early exercise premium that will be proposed in Theorem 1 requires only that the underlying asset price process S_t be Markovian, the subsequent empirical analysis will be based on the following one-dimensional diffusion process:

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma(t, S)\, dW_t^{\mathbb{Q}}, \tag{2}$$

where q represents the dividend yield for the asset price, $\sigma(t, S)$ corresponds to the instantaneous volatility (per unit of time) of the asset returns and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion, initialized at zero and generating the augmented, right continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$. Nevertheless, equation (2) encompasses several well known option pricing models as special cases: for example, it corresponds to the geometric Brownian motion if $\sigma(t, S) = \sigma$ is a constant; and it yields the CEV process when

$$\sigma(t, S) = \delta S_t^{\frac{\beta}{2} - 1}, \tag{3}$$

for $\delta, \beta \in \mathbb{R}$.[2]

Hereafter, the analysis will focus on the valuation of an American option on the asset price S, with strike price K, and with maturity date T, whose time-t ($\leq T$) value will be denoted by $V_t(S, K, T; \phi)$, where $\phi = -1$ for an American call or $\phi = 1$ for an American put. Since the American option can be exercised at any time during its life, it is well known—see, for example, (Karatzas (1988), Theorem 5.4)—that its price can be represented by the *Snell envelope*:

$$V_{t_0}(S, K, T; \phi) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r[(T \wedge \tau) - t_0]} \left(\phi K - \phi S_{T \wedge \tau} \right)^+ \middle| \mathcal{F}_{t_0} \right\}, \tag{4}$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$ generated by the underlying price process and taking values in $[t_0, \infty]$.[3]

Since the underlying asset price is a diffusion and both interest rates and dividend yields are assumed to be deterministic, for each time $t \in [t_0, T]$ there exists a *critical asset price* E_t below (above) which the American put (call) price equals its intrinsic value and, therefore, early exercise should occur—see, for instance, (Carr, Jarrow, and Myneni (1992), equations 1.2 and 1.3). Consequently, the optimal policy should be to exercise the American option when the underlying asset price first touches its critical level. Representing the first passage time of the underlying asset price to its moving boundary by

$$\tau_e := \inf \{ u \geq t_0 : S_u = E_u \} \tag{5}$$

and considering that the American option is still alive at the valuation date (i.e., $\phi S_{t_0} > \phi E_{t_0}$), equation (4) can then be restated as:

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r[(T \wedge \tau_e) - t_0]} \left(\phi K - \phi S_{T \wedge \tau_e} \right)^+ \middle| \mathcal{F}_{t_0} \right\} \\ &= \mathbb{E}_{\mathbb{Q}} \left[e^{-r(\tau_e - t_0)} \phi \left(K - E_{\tau_e} \right) 1\!\!1_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T - t_0)} \mathbb{E}_{\mathbb{Q}} \left[\left(\phi K - \phi S_T \right)^+ 1\!\!1_{\{\tau_e \geq T\}} \middle| \mathcal{F}_{t_0} \right], \end{aligned} \tag{6}$$

[2] The underlying asset can be thought of as a stock, a stock index, an exchange rate, or a financial futures contract, so long as the parameter q is understood as, respectively, a dividend yield, an average dividend yield, the foreign default-free interest rate, or the domestic risk-free interest rate.

[3] $\mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t)$ denotes the expected value of the random variable X, conditional on $\mathcal{F}_t$, and computed under the equivalent martingale measure $\mathbb{Q}$. Similarly, $\mathbb{Q}(\omega | \mathcal{F}_t)$ will represent the probability of event ω, conditional on $\mathcal{F}_t$, and computed under the probability measure $\mathbb{Q}$.

where the first line of equation (6) follows from equation (5), and $\mathbb{1}_{\{A\}}$ denotes the indicator function of the set A. Note that $K \geq E_{\tau_e}$ for the American put, because the exercise boundary is limited from above by min $\left(K, \frac{r}{q}K\right)$—see, for instance, (Huang, Subrahmanyam, and Yu (1996), footnote 5). For the American call, $K \leq E_{\tau_e}$ because the early exercise boundary is limited from below by max $\left(K, \frac{r}{q}K\right)$—see, for example, (Kim and Yu (1996), p. 67).

For $\phi = 1$, equation (6) is equivalent to (Kim and Yu (1996), eq. 7) and decomposes the American put into two components. The first one corresponds to the present value of a non-deferrable (and, in general, also non-constant) rebate $(K - E_{\tau_e})$, payable at the optimal stopping time τ_e. The second component is simply the time-t_0 price of a European down-and-out put on the asset S, with strike price K, maturity date at time T, and (time-dependent) barrier levels $\{E_t, t_0 \leq t \leq T\}$. Assuming a convenient parametric specification for the barrier function E_t, it is possible to convert equation (6) into a closed-form solution. Such an approach was pursued, for instance, by Ingersoll (1998) using both constant and exponential specifications, and by Sbuelz (2004), also under a constant barrier formulation. Unfortunately, the time path $\{E_t, t_0 \leq t \leq T\}$ of critical asset prices, which is called the *exercise boundary,* is not known *ex ante* and therefore the assumption of a specific parametric form for the barrier function simply transforms equation (6) into a lower bound for the true American put option value.

The "integral representation approach" adopted by Kim and Yu (1996) also starts from equation (6). By imposing an exogenously specified *value-matching condition*, these authors are able to rewrite equation (6) only in terms of the underlying asset transition density function. In contrast, this paper proposes an alternative characterization of the American option price, which is endogenously consistent with the *value-matching condition*, although it involves knowledge of the first hitting time density function for the underlying asset price.

III. The Early Exercise Premium

Similarly to Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992), the American option price can be divided into two components: the corresponding European option price and an early exercise premium. For this purpose, and because $\mathbb{1}_{\{\tau_e \geq T\}} = 1 - \mathbb{1}_{\{\tau_e < T\}}$, equation (6) can be rewritten as:

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} \phi (K - E_{\tau_e}) \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned}$$

And, since

$$e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] := v_{t_0}(S, K, T; \phi) \tag{7}$$

can be understood (under a deterministic interest rate setting) as the time-t_0 price of the corresponding European option (with technical features identical to those of the American contract under analysis), then

$$\begin{aligned} V_{t_0}(S, K, T; \phi) &= v_{t_0}(S, K, T; \phi) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} \phi (K - E_{\tau_e}) \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned} \tag{8}$$

The last two terms on the right-hand-side of equation (8) correspond to the early exercise premium, for which a quasi-analytical solution will be proposed in the next theorem.

A. An alternative characterization

The theorem presented below provides a new characterization for the early exercise premium.

Theorem 1 *Assuming that the underlying asset price process S_t is Markovian and that the interest rate r is constant, the time-t_0 value of an American option $V_{t_0}(S, K, T; \phi)$*

on the asset price S, with strike price K, and with maturity date T can be decomposed into the corresponding European option price $v_{t_0}(S,K,T;\phi)$ and the early exercise premium $eep_{t_0}(S,K,T;\phi)$, i.e.,

$$V_{t_0}(S,K,T;\phi) = v_{t_0}(S,K,T;\phi) + eep_{t_0}(S,K,T;\phi), \tag{9}$$

with

$$eep_{t_0}(S,K,T;\phi) := \int_{t_0}^{T} e^{-r(u-t_0)} \left[\phi(K-E_u) - v_u(E,K,T;\phi)\right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right), \tag{10}$$

where $\mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_e, as defined by equation (5), $\phi = -1$ for an American call and $\phi = 1$ for an American put.

Proof. Noting that the only random variable contained in the second term on the right-hand-side of equation (8) is the first passage time, then

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e-t_0)}\phi(K-E_{\tau_e}) \mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] \\ &= \int_{t_0}^{T} e^{-r(u-t_0)}\phi(K-E_u)\,\mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right). \end{aligned} \tag{11}$$

Concerning the third term on the right-hand-side of equation (8), it is necessary to consider the joint density of the two random variables involved: the first passage time τ_e and the terminal asset price S_T. Hence,

$$\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}} (\phi K - \phi S)^+ \,\mathbb{Q}\left(S_T \in dS, \tau_e < T \middle| \mathcal{F}_{t_0}\right), \tag{12}$$

where the integration can be restricted to the domain $\mathbb{R}_+$ if, for example, the geometric Brownian motion assumption is imposed. Because the underlying asset price is assumed to be a Markov process, the joint density contained in equation (12) is simply the convolution between the density of the first passage time τ_e and the transition probability density function of the terminal asset price S_T:

$$\mathbb{Q}\left(S_T \in dS, \tau_e < T \middle| \mathcal{F}_{t_0}\right) = \int_{t_0}^{T} \mathbb{Q}\left(S_T \in dS \middle| S_u = E_u\right) \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right). \tag{13}$$

Therefore, combining equations (12) and (13),

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\left(\phi K-\phi S_T\right)^{+}\mathbb{1}_{\{\tau_e<T\}}\middle|\,\mathcal{F}_{t_0}\right]\\
&=\int_{t_0}^{T}\left[\int_{\mathbb{R}}\left(\phi K-\phi S\right)^{+}\mathbb{Q}\left(S_T\in dS\middle|\,S_u=E_u\right)\right]\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_{t_0}\right)\\
&=\int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}}\left[\left(\phi K-\phi S_T\right)^{+}\middle|\,S_u=E_u\right]\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_{t_0}\right).
\end{aligned}\tag{14}$$

Moreover, considering equation (7), the expectation contained in the right-hand-side of equation (14) can be expressed in terms of a European option price:

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\phi K-\phi S_T\right)^{+}\mathbb{1}_{\{\tau_e<T\}}\middle|\,\mathcal{F}_{t_0}\right]=\int_{t_0}^{T}e^{r(T-u)}v_u\left(E,K,T;\phi\right)\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_{t_0}\right).\tag{15}$$

Finally, combining equations (8), (11) and (15), the early exercise representation (10) follows. ∎

Under the usual geometric Brownian motion assumption, equation (10) yields a closed-form solution to the early exercise premium (modulo to the specification of the first passage time density), because the term $v_u\left(E,K,T;\phi\right)$ can be computed using the Merton (1973) formulae. The same reasoning applies to the CEV model since, in this case, European option prices can be computed through the analytical solutions provided by Emanuel and MacBeth (1982), or by Schroder (1989), for $\beta>2$ or $\beta<2$, respectively. Note, however, that the proof of Theorem 1 relies only on the much weaker assumption of a Markovian asset price. That is, the early exercise representation (10) is still valid for other asset price processes beyond the general class represented by the stochastic differential equation (2).

The representation offered by Theorem 1 is also amenable to an intuitive interpretation. As shown in Proposition 1,

$$\lim_{S\to E_u}V_u\left(S,K,T;\phi\right):=V_u\left(E,K,T;\phi\right)=\phi\left(K-E_u\right).$$

Hence, equation (10) can be rewritten as

$$\begin{aligned}
&eep_{t_0}\left(S,K,T;\phi\right)\\
&=\int_{t_0}^{T}e^{-r(u-t_0)}\left[V_u\left(E,K,T;\phi\right)-v_u\left(E,K,T;\phi\right)\right]\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_{t_0}\right).
\end{aligned}$$

Using equation (9), today's early exercise premium can now be easily understood as the discounted expectation of the early exercise premium stopped at the first passage time:[4]

$$eep_{t_0}(S,K,T;\phi) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t_0)} eep_{\tau_e}(E,K,T;\phi)\, \mathbb{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0}\right]. \tag{16}$$

That is, the discounted and stopped early exercise premium is, as expected, a martingale under measure $\mathbb{Q}$.[5]

Such an interpretation is substantially different from the one implicit in the characterization of the American option already offered by Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), Kim and Yu (1996), and Detemple and Tian (2002). For all these authors, the early exercise premium corresponds to the compensation that the option holder would require (in the *stopping region*) in order to postpone exercise until the maturity date. Under the geometric Brownian motion assumption, and for some early exercise boundary specifications—see, for example, Ju (1998)—it is possible to obtain closed-form solutions for such early exercise representation. However, for more general underlying diffusion price processes, as the ones proposed by Kim and Yu (1996), and Detemple and Tian (2002), it is necessary to solve numerically and recursively a set of exogenously specified *value-matching* implicit integral equations, which can be too time-consuming for practical purposes. To improve efficiency, Huang, Subrahmanyam, and Yu (1996) calculate only option values based on a few points on an approximation to the exercise boundary, and then use Richardson extrapolation. Such accelerated recursive scheme is very fast but not very accurate, especially for medium- and long-term options—see, for example, (Ju (1998), Tables 1 and 2).

Alternatively, the new characterization offered by Theorem 1 can be efficiently applied for any early exercise boundary specification, and under any Markovian (and diffusion)

[4] It is well known that the discounted price process of an American option is a supermartingale under the risk-neutral measure. Nevertheless, such relative price process behaves as a martingale during any period of time in which it is not optimal to exercise the option. Therefore, the same result obtains until the first passage time to the exercise boundary.

[5] Alternatively and as suggested by an anonymous referee, the right-hand-side of equation (10) is simply the expected value of the cash flow that arises from liquidating (at the first passage time to the exercise boundary) a static portfolio that includes a long position on an American option and a short position on the corresponding European contract.

underlying price process, which constitutes an innovation with respect to the representations of the early exercise premium already offered in the literature.

B. Asymptotic properties

Before implementing Theorem 1 and in order to investigate its limits, the asymptotic properties of the early exercise representation (10) are first explored.

Proposition 1 *Under the assumptions of Theorem 1, the early exercise premium and the American option value satisfy the following boundary conditions for $t \leq T$:*

$$\lim_{r \downarrow 0} eep_t \left(S, K, T; 1 \right) = 0, \tag{17}$$

$$V_T \left(S, K, T; \phi \right) = \left(\phi K - \phi S_T \right)^+ , \tag{18}$$

$$\lim_{S \uparrow \infty} V_t \left(S, K, T; 1 \right) = 0, \tag{19}$$

$$\lim_{S \downarrow 0} V_t \left(S, K, T; -1 \right) = 0, \tag{20}$$

and

$$\lim_{S \to E_t} V_t \left(S, K, T; \phi \right) = \phi \left(K - E_t \right), \tag{21}$$

where $\phi = -1$ for an American call or $\phi = 1$ for an American put.

Proof. See Appendix A. ∎

Once the general diffusion process (2) is adopted, the usual parabolic partial differential equation follows for the price of the American option.

Proposition 2 *Under the diffusion process (2), the American option value function given by Theorem 1 satisfies, for $\phi S_t > \phi E_t$ and $t \leq T$, the partial differential equation*

$$\mathcal{L} V_t \left(S, K, T; \phi \right) = 0, \tag{22}$$

where $\mathcal{L}$ *is the parabolic operator*

$$\mathcal{L} := \frac{\sigma(t,S)^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r-q) S \frac{\partial}{\partial S} - r + \frac{\partial}{\partial t}, \tag{23}$$

$\phi = -1$ *for an American call and* $\phi = 1$ *for an American put.*

Proof. See Appendix B. ∎

The relevance of Propositions 1 and 2 emerges from the fact that the American option price is, under the stochastic differential equation (2), the unique solution of the initial value problem represented by the partial differential equation (22) and by the boundary conditions (18) through (21). Moreover, according to equation (21), and contrary to the characterization offered by Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), Kim and Yu (1996), and Detemple and Tian (2002), the American option representation contained in Theorem 1 is automatically consistent with the so-called *value-matching condition.* Hence, the proposed early exercise premium formulae is, also in this sense, more general than the alternative representations already available in the literature.[6]

Next proposition shows that the American option representation contained in Theorem 1 converges to the appropriate perpetual limit. This result contrasts with the characterization offered by Carr, Jarrow, and Myneni (1992) or Kim and Yu (1996), and can be relevant for the pricing of long-term option contracts. Explicit pricing solutions are also given for both the Merton (1973) and the CEV models, which will be used in the subsequent empirical analysis. The latter result constitutes an innovation with respect to the previous literature.

[6] It is well known, at least since the analysis of McKean (1965), that in order to uniquely determine both the American option value and the exercise boundary, the initial value problem represented by equations (18) through (22) must be transformed into a larger free boundary problem through the inclusion of an additional *high contact condition.* Unfortunately and as with all previous early exercise representations, the general solution proposed in Theorem 1 is not automatically consistent with this last *smooth fit condition.*

Proposition 3 *Under the geometric Brownian motion assumption, that is for $\sigma(t,S)=\sigma$ in equation (2), the American option value function given by Theorem 1 converges, in the limit, to the perpetual formulae given by McKean (1965) or Merton (1973), i.e.*

$$\lim_{T\uparrow\infty} V_t(S,K,T;\phi)=\phi(K-E_\infty)\left(\frac{E_\infty}{S_t}\right)^{\gamma(\phi)}, \tag{24}$$

where $\phi S_t>\phi E_\infty$, E_∞ denotes the constant exercise boundary,

$$\gamma(\phi):=\frac{r-q-\frac{\sigma^2}{2}+\phi\sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2+2\sigma^2 r}}{\sigma^2}, \tag{25}$$

$\phi=-1$ for an American call and $\phi=1$ for an American put.

Under the CEV model and for $r\neq q$, the perpetual American option price is equal to

$$\begin{aligned}\lim_{T\uparrow\infty} V_t(S,K,T;\phi) &= \phi(K-E_\infty)\left(\frac{S_t}{E_\infty}\right)^{\eta(\phi)}\exp\{\eta(\phi)[x(S_t)-x(E_\infty)]\}\\ &\quad \frac{M_{\phi(\beta-2)}\left[\eta(\phi)+(-1)^{\eta(\phi)}\alpha,\frac{\beta-1-2\eta(\phi)}{\beta-2};(-1)^{\eta(\phi)}x(S_t)\right]}{M_{\phi(\beta-2)}\left[\eta(\phi)+(-1)^{\eta(\phi)}\alpha,\frac{\beta-1-2\eta(\phi)}{\beta-2};(-1)^{\eta(\phi)}x(E_\infty)\right]},\end{aligned} \tag{26}$$

where

$$\eta(\phi):=\begin{cases}\mathbb{1}_{\{r>q,\beta<2\}}\Leftarrow\phi=1\\ 1-\mathbb{1}_{\{r>q,\beta>2\}}\Leftarrow\phi=-1\end{cases}, \tag{27}$$

$$\alpha:=\frac{r}{(\beta-2)(r-q)}, \tag{28}$$

$$x(S):=\frac{2(r-q)}{\delta^2(\beta-2)}S^{2-\beta}, \tag{29}$$

and

$$M_\lambda(a,b;z):=\begin{cases}M(a,b;z)\Leftarrow\lambda>0\\ U(a,b;z)\Leftarrow\lambda<0\end{cases}, \tag{30}$$

with $M(a,b;z)$ and $U(a,b;z)$ representing the confluent hypergeometric Kummer's functions.[7] *For $r=q$,*

$$\lim_{T\uparrow\infty} V_t(S,K,T;\phi)=\phi(K-E_\infty)\sqrt{\frac{S_t}{E_\infty}}\frac{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(S_t)\sqrt{2r}\right]}{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(E_\infty)\sqrt{2r}\right]}, \tag{31}$$

[7] As defined by (Abramowitz and Stegun (1972), equations 13.1.2 and 13.1.3).

where

$$\varepsilon(S) := \frac{2S^{1-\frac{\beta}{2}}}{\delta|\beta-2|}, \tag{32}$$

and

$$I_{\nu;\lambda}(z) := \begin{cases} I_\nu(z) \Leftarrow \lambda > 0 \\ K_\nu(z) \Leftarrow \lambda < 0 \end{cases}, \tag{33}$$

with $I_\nu(z)$ and $K_\nu(z)$ representing the modified Bessel functions.[8]

Proof. See Appendix C. ∎

IV. The First Passage Time Density

To implement the new American option value representation offered by Theorem 1, it is necessary to compute the first passage time density of the underlying asset price to the moving exercise boundary.

Following (Buonocore, Nobile, and Ricciardi (1987), eq. 2.7), a Fortet (1943)-type integral equation can be obtained for the optimal stopping time density under consideration. Notably, such non-linear integral equation involves only the transition density function of the underlying asset price. This result, contained in the next proposition, is valid for any Markovian underlying price process and for any continuous representation of the exercise boundary.

Proposition 4 *Under the assumptions of Theorem 1 and considering that the optimal exercise boundary is a continuous function of time, the first passage time density of the underlying asset price to the moving exercise boundary is the implicit solution of the following non-linear integral equation:*

$$\int_{t_0}^{u} \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| S_v = E_v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right), \tag{34}$$

[8] See, for instance, (Abramowitz and Stegun (1972), p. 375).

for $\phi S_{t_0} > \phi E_{t_0}$, where $u \in [t_0, T]$, and with $\phi = -1$ for an American call or $\phi = 1$ for an American put.

Proof. Assuming that the exercise boundary is continuous on $[t_0, u]$ and that $\phi S_{t_0} > \phi E_{t_0}$, while using definition (5), the distribution function of the optimal stopping time can be written as:[9]

$$\begin{aligned}\mathbb{Q}\left(\tau_e \leq u \middle| \mathcal{F}_{t_0}\right) &= \mathbb{Q}\left\{\inf_{t_0 \leq v \leq u}\left[\phi\left(S_v - E_v\right)\right] \leq 0, \phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right\} \\ &\quad + \mathbb{Q}\left\{\inf_{t_0 \leq v < u}\left[\phi\left(S_v - E_v\right)\right] \leq 0, \phi S_u > \phi E_u \middle| \mathcal{F}_{t_0}\right\}.\end{aligned}$$

Since $\mathbb{Q}\left\{\inf_{t_0 \leq v \leq u}\left[\phi\left(S_v - E_v\right)\right] \leq 0, \phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right\} = \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right)$ and because the underlying price process is assumed to be Markovian,

$$\begin{aligned}\mathbb{Q}\left(\tau_e \leq u \middle| \mathcal{F}_{t_0}\right) &= \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0}\right) \\ &\quad + \int_{t_0}^{u} \mathbb{Q}\left(\phi S_u > \phi E_u \middle| S_v = E_v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right).\end{aligned} \tag{35}$$

Finally, considering that $\mathbb{Q}\left(\tau_e \leq u \middle| \mathcal{F}_{t_0}\right) = \int_{t_0}^{u} \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right)$, equation (34) follows immediately from equation (35). ∎

Theorem 1 and Proposition 4 show that an explicit solution for the European option and knowledge of the transition density function of the underlying price process are the only requirements for the analytical valuation of the American contract. Hence, the proposed pricing methodology can be fruitfully applied to many other Markovian pricing systems besides the one-dimensional case covered by equation (2). Nevertheless and for the sake of brevity, the extension to alternative Markovian diffusion processes is left for future research.

Proposition 4 can be specialized easily for the Merton (1973) and the CEV models, which will be used in the numerical analysis to be presented in section VI. For $\sigma(t, S) = \sigma$, the underlying price process—as given by equation (2)—becomes lognormally distributed, and equation (34) can be restated as

$$\int_{t_0}^{u} \Phi\left(\phi \frac{E_v^z - E_u^z}{\sqrt{u - v}}\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = \Phi\left(-\phi \frac{E_u^z}{\sqrt{u - t_0}}\right), \tag{36}$$

[9] Notice that $\inf_{t_0 \leq v < u}\left[-\left(S_v - E_v\right)\right] = -\sup_{t_0 \leq v < u}\left(S_v - E_v\right)$.

with

$$E_v^z := \frac{\ln\left(\frac{S_{t_0}}{E_v}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(v - t_0)}{\sigma}, \tag{37}$$

and where $\Phi(\cdot)$ represents the cumulative density function of the univariate standard normal distribution. Equation (36) is consistent with (Park and Schuurmann (1976), Theorem 1) and similar to the integral equation used by (Longstaff and Schwartz (1995), eq. A6). For $\sigma(t,S) = \delta S_t^{\frac{\beta}{2}-1}$, it is well known—see, for example, (Schroder (1989), eq. 1) for $\beta < 2$, or (Emanuel and MacBeth (1982), eq. 7) for $\beta > 2$—that

$$\mathbb{Q}\left(S_u \leq E_u \middle| S_v = E_v\right) = \begin{cases} Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa E_u^{2-\beta}\right)}\left(2\kappa E_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right) \Leftarrow \beta < 2 \\ Q_{\chi^2\left(2+\frac{2}{\beta-2}, 2\kappa E_v^{2-\beta} e^{(2-\beta)(r-q)(u-v)}\right)}\left(2\kappa E_u^{2-\beta}\right) \Leftarrow \beta > 2 \end{cases}, \tag{38}$$

with

$$\kappa := \frac{2(r-q)}{(2-\beta)\,\delta^2\left[e^{(2-\beta)(r-q)(u-v)} - 1\right]}, \tag{39}$$

and where $Q_{\chi^2(a,b)}(x)$ represents the complementary distribution function of a non-central chi-square law with a degrees of freedom and non-centrality parameter b. Combining equations (34) and (38), a non-linear integral equation follows immediately for the optimal stopping time density under the CEV model.

Except for such crude critical asset price specifications as, for example, the constant and exponential functional forms used by Ingersoll (1998) under the geometric Brownian motion assumption, the optimal stopping time density is not known in closed-form. Following Kuan and Webber (2003), the next proposition shows that such first passage time density can be efficiently computed, for any exercise boundary specification, through the standard partition method proposed by Park and Schuurmann (1976).

Proposition 5 *Under the assumptions of Proposition 4, and dividing the time-interval* $[t_0, T]$ *into* N *sub-intervals of (equal) size* $h := \frac{T-t_0}{N}$, *then*

$$\begin{aligned} eep_{t_0}(S,K,T;\phi) \; &= \; \sum_{i=1}^{N} \left\{ \phi\left[K - E_{t_0+\frac{(2i-1)h}{2}}\right] - v_{t_0+\frac{(2i-1)h}{2}}(E,K,T;\phi) \right\} \\ &\quad e^{-r\frac{(2i-1)h}{2}} \left[\mathbb{Q}\left(\tau_e = t_0 + ih\right) - \mathbb{Q}\left(\tau_e = t_0 + (i-1)h\right)\right], \end{aligned} \tag{40}$$

where $\phi = -1$ for an American call or $\phi = 1$ for an American put. The probabilities $\mathbb{Q}(\tau_e = t_0 + ih)$ are obtained from the following recurrence relation:

$$\begin{aligned}
(41) \qquad & \mathbb{Q}(\tau_e = t_0 + ih) \\
= \; & \mathbb{Q}(\tau_e = t_0 + (i-1)h) + \left\{ F_\phi \left[E_{t_0+ih}; E_{t_0+\frac{(2i-1)h}{2}} \right] \right\}^{-1} \\
& \left\{ F_\phi \left(E_{t_0+ih}; S_{t_0} \right) - \sum_{j=1}^{i-1} F_\phi \left[E_{t_0+ih}; E_{t_0+\frac{(2j-1)h}{2}} \right] \right. \\
& \left. \left[\mathbb{Q}(\tau_e = t_0 + jh) - \mathbb{Q}(\tau_e = t_0 + (j-1)h) \right] \right\},
\end{aligned}$$

for $i = 1, \ldots, N$, where $\mathbb{Q}(\tau_e = t_0) = 0$, and with

$$(42) \qquad F_\phi \left(E_u; S_{t_0} \right) := \mathbb{Q}\left(\phi S_u \leq \phi E_u \middle| \mathcal{F}_{t_0} \right)$$

representing the risk-neutral cumulative density function, for $\phi = 1$, or the complementary distribution function, for $\phi = -1$, of the underlying price process.

Proof. See Appendix D. ∎

V. Specification of the Exercise Boundary

The pricing solution offered by Theorem 1 depends on the specification adopted for the exercise boundary $\{E_t, t_0 \leq t \leq T\}$. Although such an optimal exercise policy is not known *ex ante* (i.e., before the solution of the pricing problem), its main characteristics have already been established in the literature: *i*) The exercise boundary is a continuous function of time—see, for instance, (Jacka (1991), Propositions 2.2.4 and 2.2.5); *ii*) E_t is a non-decreasing function of time t for the American put, but non-increasing for the American call contract—see (Jacka (1991), Proposition 2.2.2); *iii*) the exercise boundary is limited by $E_T = \min\left(\phi K, \phi \frac{r}{q} K\right)$—as stated in Van Moerbeke (1976); and *iv*) $\lim_{t \uparrow \infty} E_t = E_\infty$, where E_∞ represents the (constant) critical asset price for the perpetual American case.

As described by (Ingersoll (1998), p. 89), in order to price an American option, it is necessary to choose a parametric family $\mathcal{E}$ of exercise policies $E_t(\underline{\theta})$, where each policy is characterized by an n-dimensional vector of parameters $\underline{\theta} \in \mathbb{R}^n$. Then, the early exercise value (as given by equation (10)) is expressed as a function of $\underline{\theta}$ and maximized with respect to the parameters. Since the chosen family $\mathcal{E}$ may not contain the optimal exercise boundary, the resulting American option price constitutes a lower bound for the true option value.

Of course, the more general the specification adopted for the exercise boundary, the smaller the approximation error associated with the American price estimate should be. Moreover, the chosen parametric family should at least satisfy requirements (i)–(iv) described at the start of this section. However, the parametric families already proposed in the literature have been chosen not for their generality but because they provide fast analytical pricing solutions. In order to measure the accuracy improvement provided by more general families of exercise policies, section VI will consider the following parametric specifications:

1. Constant exercise boundary:

   (43) $$E_t\left(\underline{\theta}\right) = \theta_1,\ \theta_1 > 0.$$

   This is the simplest specification one can adopt and has already been used by Ingersoll (1998) and Sbuelz (2004), under the geometric Brownian motion assumption. Although it yields a closed-form solution for equation (10), such an exercise boundary cannot simultaneously satisfy previously stated requirements (iii) and (iv).

2. Exponential family:

   (44) $$E_t\left(\underline{\theta}\right) = \theta_1 e^{\theta_2 (T-t)},\ \theta_1 > 0, \phi\theta_2 < 0.$$

   This specification, already proposed by Ingersoll (1998) for the geometric Brownian motion process, also yields an analytical solution for equation (10), but again cannot simultaneously satisfy requirements (iii) and (iv).

3. Exponential-constant family:

   (45) $$E_t\left(\underline{\theta}\right) = \theta_1 + e^{\theta_2 (T-t)},\ \phi\theta_2 < 0.$$

This new parameterization corresponds to a simple modification of equation (44) and has never been proposed in the literature. Nevertheless, section VI will show that it can produce smaller pricing errors than equation (44) for the same number of parameters.

4. Polynomial family:

$$E_t\left(\underline{\theta}\right) = \sum_{i=1}^{n} \theta_i \left(T - t\right)^{i-1}. \tag{46}$$

Because the exercise boundary is assumed to be continuous and defined on the closed interval $[t_0, T]$, the Weierstrass approximation theorem implies that E_t can be uniformly approximated, for any desired accuracy level, by the polynomial (46). By increasing the degree of the polynomial (and therefore, the number of parameters to be estimated), this new class of exercise policies allows the pricing error to be arbitrarily reduced. Section VI will reveal that with only five parameters (that is, a polynomial of degree 4) it is possible to obtain smaller pricing errors than with many alternative specifications already proposed in the literature.

5. CJM family:

$$E_t\left(\underline{\theta}\right) = \min\left(\phi K, \phi \frac{r}{q} K\right) e^{-\theta_1\sqrt{T-t}} + E_\infty\left(1 - e^{-\theta_1\sqrt{T-t}}\right),\ \theta_1 \geq 0. \tag{47}$$

Equation (47) corresponds to an exponentially weighted average between the terminal bound and the perpetual limit of the exercise boundary, and fulfills all of requirements (i)–(iv). Such a specification was proposed by (Carr, Jarrow, and Myneni (1992), p. 93), but has never been tested since it does not yield an analytical solution for the American option price. The next section will show that, with only one parameter, the magnitude of pricing errors produced by this specification is similar to that associated with the best parameterizations already available in the literature.

VI. Numerical Results

In order to test both the accuracy and efficiency of the pricing solutions proposed in Theorem 1 and the influence of the exercise boundary specification on the early exercise value, all the parametric families described in section V will be compared for different constellations of the pricing model coefficients contained in equation (2), and under two special cases: the geometric Brownian motion and the CEV processes. For this purpose, the maximization of the early exercise value (with respect to the parameters defining the exercise policy) is implemented through Powell's method, as described in (Press, Flannery, Teukolsky, and Vetterling (1994), section 10.5).[10]

To enhance the efficiency of the proposed valuation method, the parameters defining the exercise policy are first estimated by discretizing both Theorem 1 and Proposition 4 using only $N = 2^4$ time-steps. Then, and based on such an approximation for the optimal exercise boundary, the early exercise premium is computed from Proposition 5 using $N = 2^8$ time steps. The crude discretization adopted in the optimization stage should not compromise the accuracy of the pricing formulae proposed because, as noted by (Ju (1998), p. 642) in the context of the Merton (1973) model, a detailed description of the early exercise boundary is not necessary to generate accurate American option values.

Table 1 compares, in terms of both accuracy and efficiency, the valuation of short maturity American put options under different specifications of the exercise boundary, and by using the option parameters contained in (Broadie and Detemple (1996), Table 1), and (Ju (1998), Table 1) for the Black and Scholes (1973) model. Accuracy is measured by the average absolute percentage error (over the 20 contracts considered) of each valuation approach and with respect to the exact American option price. This proxy of the "true" American put value (fourth column) is computed through the binomial tree model with $15,000$ time steps, as

[10] This method requires evaluations only of the function to be maximized and therefore is faster than a *conjugate gradient* or a *quasi-Newton* algorithm. Nevertheless, it is always possible to use a more robust optimization method that also requires evaluations of the derivatives of the function to be maximized, because the derivatives of the first passage time density can be computed through a recurrence relation similar to equation (41). Details are available upon request.

suggested by (Broadie and Detemple (1996), p. 1222). Efficiency, that is, the computational speed of each valuation method, is evaluated by the total CPU time (expressed in seconds) spent to value the whole set of contracts considered. All computations were made with Pascal programs running on an Intel Pentium 4 2.80 GHz processor under a Linux operating system.

Insert Table 1 about here.

To get an idea of the magnitude of the early exercise value associated with each American option contract, the third column of Table 1 shows the price of the corresponding European put contract, which is computed using the Merton (1973) formulae. The American put prices produced by the analytical pricing solutions associated with the constant and exponential boundary specifications (fifth and sixth columns), as given by equations (43) and (44), respectively, are obtained from (Ingersoll (1998), sections 4 and 5). For comparison purposes, the last three columns of Table 1 contain the American put prices generated by the full (with $2,000$ time steps)[11] and the 10-point accelerated recursive methods of Huang, Subrahmanyam, and Yu (1996), and by the three-point multipiece exponential function method proposed by Ju (1998). The choice of the multipiece exponential approximation as a benchmark for the best pricing methods already proposed in the literature, under the geometric Brownian motion assumption, follows from (Ju (1998), Tables 3 and 5): it is faster than the Carr (1998) approach (for the same accuracy level) and much more accurate for hedging purposes, than the lower and upper bound approximation of Broadie and Detemple (1996).

All the other early exercise boundary approximations (i.e., from the seventh to the tenth columns of Table 1) are implemented through Proposition 5. For the exponential-constant (seventh column) and polynomial (of degree 4 and 5, on the eighth and ninth columns, respectively) boundary specifications, the parameter corresponding to the constant term in equations (45) and (46) is initialized at the Barone-Adesi and Whaley (1987) estimate (and

[11] As suggested by (Detemple and Tian (2002), p. 924).

at zero, for the other parameters). For the CJM exercise boundary approximation, the initial guess for the single parameter involved in equation (47) is also set at zero.

Table 2 presents the same information, but for long maturity option contracts, and yields results similar to the ones contained in Table 1 as a consequence of the asymptotic property described in Proposition 3. In general, one may conclude that the fastest approximations (in terms of CPU time) are the constant, the exponential, and the three-point multipiece exponential specifications, as well as the accelerated recursive method of Huang, Subrahmanyam, and Yu (1996): they all possess computational times below 0.2 seconds for the range of all contracts under consideration. However, the pricing errors generated by the constant and the exponential parameterizations can be significant. For instance, in Table 1, the average mispricing of the constant parameterization equals 41 basis points. Moreover, and as shown by Table 2, the accuracy of the 10-point recursive scheme deteriorates as the option maturity increases: its mean absolute percentage pricing error is now about four times the error associated with the full recursive scheme, whereas in Table 1 both errors had the same order of magnitude.

As expected, the pricing errors produced by the specifications described in section V are negative because any approximation of the optimal exercise policy can yield only a lower bound for the true American put price. The only exceptions correspond to the approximation suggested by Ju (1998), for which the pricing errors are consistently positive, and to the 10-point recursive method, when valuing long-term options. This behavior might be explained by the non-uniform convergence of the Richardson extrapolation employed.

Insert Table 2 about here.

With the same number of parameters as the already known exponential approximation, the new exponential-constant parameterization can yield pricing errors about three times smaller, as shown in Tables 1 and 2. Even more interestingly, the CJM approximation suggested by Carr, Jarrow, and Myneni (1992) and tested here possesses an accuracy similar to the three-point multipiece exponential approach: the average absolute pricing errors are between two and three basis points. This result is relevant since the CJM approximation satisfies all the requirements described in section V for the early exercise boundary specification.

Tables 1 and 2 also show that the implementation of a polynomial approximation is able to achieve smaller pricing errors than the Ju (1998) approach. The Huang, Subrahmanyam, and Yu (1996) full recursive method yields an even higher precision level, but at the expense of a prohibitive computational effort. Overall, taking into consideration both accuracy and efficiency, the best pricing methodology, under the geometric Brownian motion assumption, is still the multipiece exponential approach of Ju (1998). Even though such parameterization does not obey the requirements enunciated in section V, it seems to be flexible enough to capture the behavior of the critical asset prices. Nevertheless, the disparity of pricing errors contained in Tables 1 and 2 shows that the early exercise premium depends largely on the specification adopted for the early exercise boundary.

Insert Table 3 about here.

Tables 3 and 4 repeat the analysis contained in Tables 1 and 2 for the same parameter values, but under the CEV model. Table 3 assumes $\beta = 3$ (> 2) and prices American put contracts with a time-to-maturity of three years, while Table 4 considers a square root process with $\beta = 1$ (< 2) and American call options with a time-to-maturity of five years.[12] Parameter δ is computed from equation (3) by imposing the same instantaneous volatility as in Tables 1 and 2.

The proxy of the exact American option price (fourth column) is now computed through the Crank-Nicolson finite difference method with $15,000$ time intervals and $10,000$ space steps. Besides the early exercise boundary specifications described in section V, Tables 3

[12] The test of the proposed pricing solution for American call options is restricted to the CEV process because, under the Merton (1973) model, the pricing accuracy for American calls would be perfectly correlated with the results already obtained in Tables 1 and 2 for American puts since both contracts can be linked through the parity relation derived by McDonald and Schroder (1998).

and 4 also contain the full recursive scheme (eleventh column), as suggested by (Detemple and Tian (2002), Proposition 3), and a 10-point accelerated recursive approach (last column), along the lines of (Kim and Yu (1996), subsection 3.4).[13]

To implement Proposition 4, using equation (38), the non-central chi-square cumulative density function is computed from routine "cumchn", which is contained in the Fortran library of Brown, Lovato, and Russell (1997). This routine is based on (Abramowitz and Stegun (1972), eq. 26.4.25), and is found to be more precise than the algorithm offered by Schroder (1989) or the *Wiener germ* approximations proposed by Penev and Raykov (1997), especially for large values of the non-centrality parameter or of the upper integration limit.

Insert Table 4 about here.

As before, the constant specification generates excessively large (absolute) pricing errors and the new exponential-constant parameterization yields an accuracy higher than the exponential specification for American put contracts (see Table 3). In contrast, Table 4 shows that the exponential boundary is more accurate for American call contracts than the new formulation given by equation (45). Under the CEV model, the CJM approximation presents an excellent performance even though the pricing errors are now affected by the approximation employed to evaluate the non-central chi-square distribution function, as well as by the root-finding routine used to extract the optimal constant exercise boundary E_∞ from equations (26) and (31).

In terms of accuracy, the Detemple and Tian (2002) approach constitutes the best pricing method for the CEV model. However, this approach is based on the full recursive method (with $2,000$ time steps) of Huang, Subrahmanyam, and Yu (1996), which is very time consuming—six times slower than the exact Crank-Nicolson implicit finite-difference

[13]The trinomial approach developed by Boyle and Tian (1999) for the valuation of barrier and lookback options under the CEV model (for $0 \leq \beta < 2$) can also be used to price American standard calls and puts. However, the numerical experiments run have shown that the adopted Crank-Nicolson scheme possesses better convergence properties.

scheme. The accelerated recursive scheme of Kim and Yu (1996) is much more efficient but can also be very inaccurate for medium- and long-term options. The last column of Table 4 shows a mean absolute percentage error of about 28 basis points. On the contrary, Tables 1 through 4 show that the accuracy of the pricing methodology proposed in Theorem 1 is not affected by the time-to-maturity of the option contract under valuation. Moreover, for almost all the parameterizations tested (with the single exception of the polynomial specification), the computational time of the proposed pricing methodology corresponds to less than one second per contract.

Insert Table 5 about here.

Under the CEV model, the best trade-off between accuracy and efficiency is given by the polynomial approximations presented in Tables 3 and 4, since their accuracy can always be improved by increasing their degree. Table 5 applies different polynomial specifications to a random sample of $1,250$ American put options, where all the option parameters, with the exception of β and δ, are extracted from the same uniform distributions as in (Ju (1998), Table 3).[14] With a six-degree polynomial it is possible to obtain an average absolute percentage error (computed against the Crank-Nicolson solution) of only 1.5 basis points and a maximum absolute percentage error of about 9 basis points, which corresponds to a higher accuracy than that associated with the 10-point accelerated recursive scheme.

In summary, the numerical results presented in Tables 3, 4 and 5 configure the implementation of Theorem 1 through a polynomial specification of the early exercise boundary as the best pricing alternative, under the CEV model, for medium- and long-term American option contracts.

[14] From the uniform distribution adopted for the instantaneous volatility, the parameter δ is obtained from equation (3). Parameter β is assumed to possess a uniform distribution between 0 and 4.0. The scenario $\beta < 0$ is ignored because it would imply unrealistic economic properties for the CEV process; namely, bankruptcy would be attainable for sufficiently negative values of β (which is implausible, for instance, when considering options on stock indices), and underlying asset price volatility would explode as the spot price tends to the origin.

VII. Conclusions

The main theoretical contribution of this paper consists in deriving an alternative characterization of the early exercise premium, which is valid for any Markovian representation of the underlying asset price and for any parameterization of the exercise boundary. Moreover, the proposed characterization is shown to be automatically consistent with the *value-matching condition* and to possess appropriate asymptotic properties.

The intuitive representation offered by Theorem 1 is simply based on the observation that the discounted and stopped early exercise premium must be a martingale under the risk-neutral measure. Additionally, the Markov property ensures analytical tractability since it enables the decomposition of the joint density between the first hitting time and the underlying asset price through the convolution of their marginal densities.

To test the proposed pricing methodology and to highlight its generality, several parameterizations of the exercise boundary were compared under the geometric Brownian motion assumption and for the CEV process. For both option pricing models, the single-parameter approximation suggested by Carr, Jarrow, and Myneni (1992) was shown to be extremely accurate. Moreover, the continuity of the early exercise boundary allows the pricing errors to be arbitrarily reduced through a polynomial specification, which can be easily accommodated by the proposed methodology.

Under the Merton (1973) model, the multipiece exponential approach of Ju (1998) offers the best compromise between accuracy and efficiency. However, under the CEV model, Theorem 1 provides the best pricing alternative for medium- and long-term American options. Whereas the early exercise premium formula proposed in equation (10) involves only a single time-integral, the representations offered by Kim and Yu (1996) or Detemple and Tian (2002) pose a more demanding two-dimensional integration problem (with respect to time and to the underlying asset price). Moreover, although Theorem 1 requires the numerical evaluation of the first passage time density (which is shown to be easily recovered from the transition density function), the formulas offered by Kim and Yu (1996), and Detemple and Tian (2002) rely on the numerical and recursive solution of a set of (exogenously specified) value-matching (or high-contact) implicit integral equations, which are too time-consuming for practical purposes. And, even though such a recursive scheme can be accelerated through Richardson extrapolation, the pricing methodology proposed by Huang, Subrahmanyam, and Yu (1996) yields inaccurate results for medium- and long-term options.

Since the analytical pricing of American options under the geometric Brownian motion process is already well established through the randomization approach of Carr (1998) or the multipiece exponential boundary approximation of Ju (1998), the characterization proposed in Theorem 1 can be more fruitfully applied under alternative (but Markovian) stochastic processes for the underlying asset price, as exemplified, in this paper, by the CEV model. For this purpose to be accomplished in an efficient way, it is required only that the selected price process provides a viable valuation method for European options and for its transition density function. This should be the case for multivariate Markovian models accommodating stochastic volatility and/or stochastic interest rates, for which the recursive scheme of Kim and Yu (1996) cannot be applied. Nevertheless, given space constraints, both extensions are left for further research.

Appendix A. Proof of Proposition 1

Concerning the boundary condition (17), since

$$\begin{aligned}\lim_{r\downarrow 0} E_T &= \min\left(K, \lim_{r\downarrow 0}\frac{r}{q}K\right)\\ &= 0,\end{aligned}$$

and because the exercise boundary $\{E_u, t \leq u \leq T\}$ is a non-decreasing function of u for an American put, then

$$\lim_{r\downarrow 0} E_u = 0,\ \forall u \in [t, T]\,. \tag{A-1}$$

Combining equations (10) and (A-1),

$$\lim_{r\downarrow 0} eep_t\left(S, K, T; 1\right) = \int_t^T \left[K - \lim_{r\downarrow 0} v_u\left(0, K, T; 1\right)\right] \lim_{r\downarrow 0} \mathbb{Q}\left(\tau_e \in du\middle|\, \mathcal{F}_t\right). \tag{A-2}$$

Finally, since $\left[e^{-r(T-u)}K - S_u\right]^+ \leq v_u\left(S, K, T; 1\right) \leq e^{-r(T-u)}K$ follows from straightforward no-arbitrage arguments, then $\lim_{r\downarrow 0} v_u\left(0, K, T; 1\right) = K$ and equation (A-2) can be rewritten as

$$\begin{aligned}\lim_{r\downarrow 0} eep_t\left(S, K, T; 1\right) &= \int_t^T (K - K) \lim_{r\downarrow 0} \mathbb{Q}\left(\tau_e \in du\middle|\, \mathcal{F}_t\right)\\ &= 0.\end{aligned}$$

The terminal condition (18) follows immediately from equation (9) because $v_T\left(S, K, T; \phi\right) = \left(\phi K - \phi S_T\right)^+$ and $eep_T\left(S, K, T; \phi\right) = 0$.

Concerning the boundary condition (19), because $\lim_{S\uparrow\infty} v_t\left(S, K, T; 1\right) = 0$, equations (9) and (10) imply that:

$$\begin{aligned}&\lim_{S\uparrow\infty} V_t\left(S, K, T; 1\right)\\ &= \int_t^T e^{-r(u-t)}\left[\left(K - E_u\right) - v_u\left(E, K, T; 1\right)\right] \lim_{S\uparrow\infty} \mathbb{Q}\left(\tau_e \in du\middle|\, \mathcal{F}_t\right).\end{aligned} \tag{A-3}$$

Assuming that $\lim_{S\uparrow\infty} S_u = \infty,\ \forall u \geq t$, then

$$\begin{aligned}\lim_{S\uparrow\infty} \mathbb{Q}\left(\tau_e \in du\middle|\, \mathcal{F}_t\right) &= \lim_{S\uparrow\infty} \mathbb{Q}\left(S_u = E_u,\ \inf_{t\leq v<u}\left(S_v - E_v\right) > 0 \middle|\, \mathcal{F}_t\right)\\ &= 0,\end{aligned} \tag{A-4}$$

because the exercise boundary is independent of the current asset price and finite. Combining equations (A-3) and (A-4), the boundary condition (19) is obtained.

Similar reasoning can be applied to derive the boundary condition (20). Considering that $\lim_{S\downarrow 0} v_t(S,K,T;-1)=0$, equations (9) and (10) yield:

$$\text{(A-5)}\qquad \begin{aligned} & \lim_{S\downarrow 0} V_t(S,K,T;-1) \\ = & \int_t^T e^{-r(u-t)}\left[(E_u-K)-v_u(E,K,T;-1)\right]\lim_{S\downarrow 0}\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right). \end{aligned}$$

Since, for an American call, $E_T=\max\left(K,\frac{r}{q}K\right)>0$, it follows that the exercise boundary $\{E_u, t\leq u\leq T\}$ is also strictly positive because it is a non-increasing function of u. Therefore, the spot price S_u can never touch (from below) the critical price E_u (>0), i.e.,

$$\text{(A-6)}\qquad \lim_{S\downarrow 0}\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right)=0,$$

as long as $\lim_{S\downarrow 0} S_u=0,\ \forall u\geq t$.

Finally, the *value-matching condition* (21) is also easily derived from equations (9) and (10):

$$\text{(A-7)}\qquad \begin{aligned} & \lim_{S\to E_t} V_t(S,K,T;\phi) \\ = & \ v_t(E,K,T;\phi) \\ & +\int_t^T e^{-r(u-t)}\left[\phi(K-E_u)-v_u(E,K,T;\phi)\right]\lim_{S\to E_t}\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right). \end{aligned}$$

Since

$$\lim_{S\to E_t}\mathbb{Q}\left(\tau_e\in du|\,\mathcal{F}_{t_0}\right)=\delta(u-t),$$

where $\delta(\cdot)$ is the Dirac-delta function, equation (A-7) yields

$$\begin{aligned} \lim_{S\to E_t} V_t(S,K,T;\phi) &= v_t(E,K,T;\phi)+e^{-r(t-t)}\left[\phi(K-E_t)-v_t(E,K,T;\phi)\right] \\ &= \phi(K-E_t). \end{aligned}$$

∎

Appendix B. Proof of Proposition 2

Applying the parabolic operator $\mathcal{L}$ to equations (9) and (10), and using Leibniz's rule,

$$\begin{aligned}
\text{(B-1)} \qquad & \mathcal{L}V_t\left(S,K,T;\phi\right) \\
= \; & \mathcal{L}v_t\left(S,K,T;\phi\right) \\
& + \int_t^T re^{-r(u-t)}\left[\phi\left(K-E_u\right)-v_u\left(E,K,T;\phi\right)\right]\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_t\right) \\
& + \int_t^T e^{-r(u-t)}\left[\phi\left(K-E_u\right)-v_u\left(E,K,T;\phi\right)\right]\mathcal{L}\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_t\right) \\
& - e^{-r(t-t)}\left[\phi\left(K-E_t\right)-v_t\left(E,K,T;\phi\right)\right]\mathbb{Q}\left(\tau_e=t\middle|\,\mathcal{F}_t\right).
\end{aligned}$$

Because $\mathcal{L}v_t\left(S,K,T;\phi\right)=0$, considering that $\mathbb{Q}\left(\tau_e=t\middle|\,\mathcal{F}_t\right)=0$—since Proposition 2 assumes that $\phi S_t>\phi E_t$—and using definition (23), equation (B-1) can be simplified to

$$\begin{aligned}
\text{(B-2)} \qquad & \mathcal{L}V_t\left(S,K,T;\phi\right) \\
= \; & \int_t^T e^{-r(u-t)}\left[\phi\left(K-E_u\right)-v_u\left(E,K,T;\phi\right)\right]\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_t\right),
\end{aligned}$$

where

$$\mathcal{A}:=\frac{\sigma\left(t,S\right)^2S^2}{2}\frac{\partial^2}{\partial S^2}+\left(r-q\right)S\frac{\partial}{\partial S}$$

is the infinitesimal generator of S. Since

$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\mathbb{Q}\left(\tau_e\in du\middle|\,\mathcal{F}_t\right)=0$$

can be interpreted as a Kolmogorov backward equation, the partial differential equation (22) is obtained. ∎

Appendix C. Proof of Proposition 3

For the perpetual American option, the critical asset price is a time-invariant constant, that is, $E_u = E_\infty, \forall u \in [t, T]$. Hence, the limit of equation (9), as the option's maturity date tends to infinity, is given by

$$
\begin{aligned}
& \lim_{T\uparrow\infty} V_t\left(S, K, T; \phi\right) \\
= \; & \lim_{T\uparrow\infty} v_t\left(S, K, T; \phi\right) \\
& + \lim_{T\uparrow\infty} \int_t^T e^{-r(u-t)} \left[\phi\left(K - E_\infty\right) - v_u\left(E_\infty, K, T; \phi\right)\right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t\right).
\end{aligned}
$$

Furthermore, the fair value of a perpetual European put or call option on a dividend-paying asset is equal to zero and, consequently,

$$
\begin{aligned}
\lim_{T\uparrow\infty} V_t\left(S, K, T; \phi\right) &= \phi\left(K - E_\infty\right) \int_t^\infty e^{-r(u-t)} \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_t\right) \\
&= \phi\left(K - E_\infty\right) \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau_e - t)} \mathbb{1}_{\{\tau_e < \infty\}} \middle| \mathcal{F}_t\right],
\end{aligned}
\tag{C-1}
$$

where τ_e is the first passage time of the underlying asset price to the constant exercise boundary. Hence, equation (C-1) shows that the proposed characterization of the American option converges to the correct perpetual limit for any Markovian underlying price process.

Under the geometric Brownian motion assumption and for $\phi S_t > \phi E_\infty$, solving the stochastic differential equation (2), for $\sigma(t, S) = \sigma$, and redefining the optimal stopping time τ_e as

$$
\begin{aligned}
\tau_e &= \inf\left\{u \geq t : S_u = E_\infty\right\} \\
&= \inf\left\{u \geq t : -\frac{\phi}{\sigma}\left(r - q - \frac{\sigma^2}{2}\right)(u - t) - \phi \int_t^u dW_v^{\mathbb{Q}} = \frac{\phi}{\sigma} \ln\left(\frac{S_t}{E_\infty}\right)\right\},
\end{aligned}
$$

the dividend-adjusted (Merton (1973), p. 174) solution shown in equation (24) follows after applying (Shreve (2004), Theorem 8.3.2).

Under the CEV model, the expectation contained on the right-hand-side of equation (C-1) can easily be computed using, for instance, (Davydov and Linetsky (2001), equations 2 and 38), which yields equations (26) and (31) for $\phi = 1$. For the perpetual American call case ($\phi = -1$), equations (26) and (31) also follow from (Davydov and Linetsky (2001), equations 4 and 37), and (Abramowitz and Stegun (1972), equations 13.1.27 and 13.1.29). ∎

Appendix D. Proof of Proposition 5

Equation (40) is simply the discretization of equation (10) for the partition $t_0 < t_1 < \ldots < t_N = T$, where $t_i = t_0 + ih$ $(i = 1, \ldots, N)$, and $u = \frac{t_i + t_{i-1}}{2}$.

The representation (42) follows from the Markovian nature of the adopted pricing model. Therefore, equation (34) can be rewritten as

$$\int_{t_0}^{u} F_\phi\left(E_u; E_v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = F_\phi\left(E_u; S_{t_0}\right). \tag{D-1}$$

Applying the same discretization to equation (D-1), then

$$\begin{aligned} & F_\phi\left(E_{t_0+ih}; S_{t_0}\right) \\ = & \sum_{j=1}^{i} F_\phi\left[E_{t_0+ih}; E_{t_0+\frac{(2j-1)h}{2}}\right]\left[\mathbb{Q}\left(\tau_e = t_0 + jh\right) - \mathbb{Q}\left(\tau_e = t_0 + (j-1)h\right)\right], \end{aligned} \tag{D-2}$$

for $i = 1, \ldots, N$. Finally, solving equation (D-2) in order to the probability $\mathbb{Q}\left(\tau_e = t_0 + ih\right)$, equation (41) arises. ∎

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Table 1: Prices of American put options under the Merton (1973) model, with $S_{t_0} = \$100$ and $T - t_0 = 0.5$ years

Option parameters	Strike	Europ.	American put: Exact	Early exercise boundary specification: Const.	Exp.	ExpC.	4d Pol.	5d Pol.	CJM	HSY2000	HSY10	EXP3
	80	0.215	0.219	0.218	0.219	0.219	0.219	0.219	0.219	0.219	0.219	0.220
$r = 7\%$	90	1.345	1.386	1.376	1.385	1.386	1.386	1.386	1.386	1.386	1.386	1.387
$q = 3\%$	100	4.578	4.783	4.750	4.778	4.781	4.782	4.782	4.781	4.782	4.782	4.784
$\sigma = 20\%$	110	10.421	11.098	11.049	11.092	11.097	11.097	11.097	11.095	11.097	11.098	11.099
	120	18.302	20.000	20.000	20.000	19.999	19.999	19.999	19.996	20.000	20.002	20.000
	80	2.651	2.689	2.676	2.687	2.688	2.688	2.688	2.688	2.689	2.689	2.690
$r = 7\%$	90	5.622	5.722	5.694	5.719	5.721	5.721	5.721	5.720	5.722	5.722	5.724
$q = 3\%$	100	10.021	10.239	10.190	10.233	10.237	10.237	10.237	10.236	10.239	10.238	10.240
$\sigma = 40\%$	110	15.768	16.181	16.110	16.173	16.180	16.179	16.179	16.177	16.181	16.181	16.183
	120	22.650	23.360	23.271	23.350	23.358	23.358	23.358	23.355	23.359	23.359	23.362
	80	1.006	1.037	1.029	1.036	1.037	1.037	1.037	1.037	1.037	1.037	1.038
$r = 7\%$	90	3.004	3.123	3.098	3.120	3.122	3.122	3.122	3.122	3.123	3.123	3.125
$q = 0\%$	100	6.694	7.035	6.985	7.029	7.034	7.034	7.034	7.032	7.035	7.035	7.037
$\sigma = 30\%$	110	12.166	12.955	12.882	12.946	12.953	12.953	12.953	12.951	12.955	12.953	12.957
	120	19.155	20.717	20.650	20.710	20.716	20.716	20.716	20.713	20.717	20.718	20.719
	80	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664
$r = 3\%$	90	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495
$q = 7\%$	100	9.251	9.250	9.251	9.251	9.251	9.251	9.251	9.251	9.251	9.251	9.251
$\sigma = 30\%$	110	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798
	120	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706
Mean Absolute Percentage Error				0.407%	0.054%	0.020%	0.017%	0.015%	0.026%	0.003%	0.005%	0.023%
CPU (seconds)			451.32	0.01	0.03	2.12	8.87	10.07	1.91	3,215.35	0.17	0.08

Table 1 values American put options under the Merton (1973) model and for different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial model with 15,000 time steps. The fifth, sixth and seventh columns report the American put prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (43), (44), and (45). The eighth and ninth columns are both based on a polynomial boundary—see equation (46)—with four and five degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary of equation (47). The next two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996). The last column presents the American put prices generated by the three-point multipiece exponential method of Ju (1998).

Table 2: Prices of American put options under the Merton (1973) model, with $S_{t_0} = \$100$ and $T - t_0 = 5$ years

Option parameters	Strike	Europ.	American put									
				Early exercise boundary specification								
			Exact	Const.	Exp.	ExpC.	4d Pol.	5d Pol.	CJM	HSY2000	HSY10	EXP3
	80	2.828	3.617	3.585	3.609	3.614	3.615	3.615	3.616	3.617	3.619	3.619
$r = 7\%$	90	4.792	6.394	6.347	6.383	6.390	6.391	6.391	6.392	6.393	6.393	6.397
$q = 3\%$	100	7.382	10.305	10.247	10.292	10.301	10.302	10.302	10.303	10.304	10.300	10.309
$\sigma = 20\%$	110	10.582	15.502	15.441	15.488	15.499	15.499	15.499	15.500	15.500	15.515	15.506
	120	14.353	22.122	22.071	22.111	22.119	22.120	22.120	22.120	22.119	22.099	22.125
	80	12.224	14.456	14.353	14.434	14.449	14.449	14.450	14.449	14.454	14.463	14.461
$r = 7\%$	90	15.863	19.044	18.920	19.018	19.036	19.036	19.037	19.036	19.042	19.048	19.050
$q = 3\%$	100	19.850	24.184	24.043	24.155	24.177	24.176	24.177	24.176	24.182	24.171	24.192
$\sigma = 40\%$	110	24.144	29.845	29.690	29.813	29.838	29.837	29.838	29.836	29.842	29.825	29.854
	120	28.709	35.998	35.832	35.964	35.991	35.989	35.990	35.988	35.995	35.994	36.007
	80	5.313	7.092	7.033	7.077	7.086	7.087	7.088	7.088	7.090	7.095	7.095
$r = 7\%$	90	7.666	10.564	10.491	10.547	10.558	10.559	10.560	10.560	10.562	10.553	10.568
$q = 0\%$	100	10.458	14.881	14.797	14.861	14.876	14.876	14.877	14.876	14.879	14.878	14.886
$\sigma = 30\%$	110	13.662	20.078	19.988	20.057	20.073	20.072	20.073	20.073	20.075	20.099	20.083
	120	17.246	26.181	26.094	26.161	26.177	26.176	26.177	26.176	26.177	26.181	26.186
	80	17.504	17.610	17.606	17.610	17.610	17.610	17.610	17.610	17.610	17.609	17.609
$r = 3\%$	90	23.113	23.291	23.285	23.291	23.291	23.291	23.291	23.291	23.291	23.290	23.289
$q = 7\%$	100	29.231	29.508	29.500	29.508	29.508	29.508	29.508	29.508	29.508	29.511	29.506
$\sigma = 30\%$	110	35.769	36.179	36.166	36.178	36.178	36.178	36.178	36.178	36.178	36.182	36.176
	120	42.653	43.230	43.214	43.230	43.230	43.230	43.230	43.230	43.230	43.229	43.227
Mean Absolute Percentage Error				0.438%	0.098%	0.030%	0.027%	0.023%	0.023%	0.011%	0.040%	0.027%
CPU (seconds)			448.46	0.01	0.04	1.81	9.95	12.50	1.96	2,926.52	0.17	0.09

Table 2 values American put options under the Merton (1973) model and for different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial model with 15,000 time steps. The fifth, sixth and seventh columns report the American put prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (43), (44), and (45). The eighth and ninth columns are both based on a polynomial boundary—see equation (46)—with four and five degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary of equation (47). The next two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996). The last column presents the American put prices generated by the three-point multipiece exponential method of Ju (1998).

Table 3: Prices of American put options under the CEV model, with $\beta = 3$, $S_{t_0} = \$100$ and $T - t_0 = 3$ years

Option parameters	Strike	Europ.	American put	Early exercise boundary specification							
			Exact	Const.	Exp.	ExpC.	4d Pol.	5d Pol.	CJM	DT	KY
	80	1.903	2.238	2.213	2.232	2.235	2.236	2.236	2.235	2.238	2.237
$r = 7\%$	90	4.130	4.986	4.939	4.976	4.982	4.983	4.984	4.982	4.986	4.989
$q = 3\%$	100	7.401	9.184	9.115	9.170	9.180	9.180	9.181	9.178	9.184	9.179
$\delta = 0.02$	110	11.646	14.864	14.783	14.849	14.861	14.861	14.861	14.858	14.866	14.875
	120	16.728	21.967	21.891	21.954	21.965	21.964	21.965	21.961	21.971	21.964
	80	9.404	10.475	10.385	10.458	10.469	10.470	10.471	10.466	10.472	10.469
$r = 7\%$	90	13.718	15.409	15.288	15.387	15.403	15.403	15.404	15.397	15.404	15.409
$q = 3\%$	100	18.636	21.100	20.950	21.073	21.093	21.093	21.094	21.085	21.094	21.100
$\delta = 0.04$	110	24.049	27.435	27.258	27.403	27.427	27.426	27.427	27.416	27.427	27.422
	120	29.867	34.315	34.116	34.279	34.307	34.305	34.306	34.293	34.307	34.293
	80	4.051	4.982	4.929	4.971	4.978	4.979	4.980	4.976	4.982	4.984
$r = 7\%$	90	6.934	8.668	8.588	8.652	8.663	8.664	8.665	8.659	8.667	8.671
$q = 0\%$	100	10.594	13.453	13.346	13.432	13.447	13.447	13.448	13.441	13.452	13.443
$\delta = 0.03$	110	14.943	19.256	19.130	19.233	19.251	19.251	19.252	19.242	19.256	19.261
	120	19.880	25.979	25.843	25.954	25.974	25.973	25.974	25.963	25.979	25.996
	80	11.349	11.350	11.350	11.350	11.350	11.350	11.350	11.350	11.350	11.350
$r = 3\%$	90	16.963	16.968	16.968	16.968	16.968	16.968	16.968	16.968	16.968	16.968
$q = 7\%$	100	23.348	23.363	23.362	23.363	23.363	23.363	23.363	23.363	23.364	23.364
$\delta = 0.03$	110	30.327	30.361	30.360	30.361	30.361	30.361	30.361	30.361	30.362	30.362
	120	37.759	37.827	37.825	37.826	37.827	37.827	37.827	37.826	37.828	37.827
Mean Absolute Percentage Error				0.564%	0.108%	0.033%	0.029%	0.025%	0.057%	0.012%	0.031%
CPU (seconds)			1,090.96	5.16	6.82	7.22	22.52	30.11	5.88	6,216.83	0.39

Table 3 values American put options under the CEV model and for different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps. The fifth, sixth and seventh columns report the American put prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (43), (44), and (45). The eighth and ninth columns are both based on a polynomial boundary—see equation (46)—with four and five degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary specification of equation (47). The last two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996), as suggested by Detemple and Tian (2002) and Kim and Yu (1996), respectively.

Table 4: Prices of American call options under the CEV model, with $\beta = 1$, $S_{t_0} = \$100$ and $T - t_0 = 5$ years

Option parameters	Strike	Europ.	American call								
				Early exercise boundary specification							
			Exact	Const.	Exp.	ExpC.	2d Pol.	3d Pol.	CJM	DT	KY
	80	32.989	33.082	33.080	33.082	33.081	33.082	33.082	33.082	33.082	33.083
$r = 7\%$	90	27.736	27.773	27.772	27.773	27.773	27.773	27.773	27.773	27.773	27.766
$q = 3\%$	100	23.014	23.029	23.028	23.028	23.028	23.028	23.028	23.028	23.029	23.014
$\delta = 2$	110	18.848	18.853	18.853	18.853	18.853	18.853	18.853	18.853	18.853	18.831
	120	15.238	15.240	15.240	15.240	15.240	15.240	15.240	15.240	15.240	15.212
	80	43.211	43.967	43.919	43.943	43.937	43.944	43.944	43.941	43.944	44.025
$r = 7\%$	90	39.275	39.802	39.769	39.786	39.782	39.787	39.787	39.785	39.787	39.933
$q = 3\%$	100	35.633	36.000	35.978	35.989	35.986	35.990	35.990	35.989	35.990	36.189
$\delta = 4$	110	32.272	32.528	32.512	32.520	32.518	32.520	32.520	32.520	32.520	32.757
	120	29.179	29.357	29.346	29.352	29.350	29.352	29.352	29.352	29.352	29.615
	80	49.762	49.765	49.762	49.762	49.762	49.762	49.762	49.762	49.762	49.746
$r = 7\%$	90	44.754	44.756	44.754	44.754	44.754	44.754	44.754	44.754	44.754	44.754
$q = 0\%$	100	40.093	40.094	40.093	40.093	40.093	40.093	40.093	40.093	40.093	40.107
$\delta = 3$	110	35.780	35.780	35.780	35.780	35.780	35.780	35.780	35.780	35.780	35.805
	120	31.812	31.812	31.812	31.812	31.812	31.812	31.812	31.812	31.812	31.846
	80	20.226	27.016	26.894	26.989	26.964	27.006	27.006	27.012	27.010	27.114
$r = 3\%$	90	16.711	21.880	21.756	21.853	21.825	21.871	21.871	21.877	21.876	21.963
$q = 7\%$	100	13.718	17.657	17.538	17.631	17.602	17.648	17.648	17.654	17.654	17.727
$\delta = 3$	110	11.193	14.194	14.086	14.171	14.143	14.187	14.187	14.192	14.192	14.272
	120	9.080	11.367	11.270	11.346	11.320	11.360	11.360	11.365	11.365	11.444
Mean Absolute Percentage Error				0.184%	0.045%	0.088%	0.021%	0.021%	0.014%	0.013%	0.278%
CPU (seconds)			1,048.92	6.80	9.08	8.16	11.08	12.02	6.80	5,927.45	0.51

Table 4 values American call options under the CEV model and for different specifications of the exercise boundary. The third column contains European call prices, while the exact American call values (fourth column) are based on the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps. The fifth, sixth and seventh columns report the American call prices associated with the constant, the exponential, and the exponential-constant boundary specifications, as given by equations (43), (44), and (45). The eighth and ninth columns are both based on a polynomial boundary—see equation (46)—with four and five degrees of freedom, respectively. The American call prices contained in the tenth column are obtained from the exercise boundary specification of equation (47). The last two columns implement the full (with 2,000 time steps) and the 10-point recursive methods of Huang, Subrahmanyam, and Yu (1996), as suggested by Detemple and Tian (2002) and Kim and Yu (1996), respectively.

Table 5: Accuracy of the polynomial specification for a large sample of randomly generated American puts

	Polynomial specifications					KY
	2nd degree	3rd degree	4th degree	5th degree	6th degree	
Percentage Errors						
mean	-0.0289%	-0.0223%	-0.0198%	-0.0175%	-0.0144%	0.0070%
maximum	0.0164%	0.0180%	0.0225%	0.0277%	0.0315%	0.3953%
minimum	-0.1691%	-0.1162%	-0.1071%	-0.1003%	-0.0902%	-0.0563%
99th percentile	0.0039%	0.0043%	0.0046%	0.0051%	0.0054%	0.1898%
1st percentile	-0.1308%	-0.1037%	-0.0949%	-0.0889%	-0.0758%	-0.0480%
Absolute Percentage Errors						
mean	0.0293%	0.0228%	0.0203%	0.0181%	0.0150%	0.0168%
maximum	0.1691%	0.1162%	0.1071%	0.1003%	0.0902%	0.3953%
minimum	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
99th percentile	0.1308%	0.1037%	0.0949%	0.0889%	0.0758%	0.1898%

Table 5 reports the pricing errors associated with the valuation of 1,250 randomly generated American put options, under the CEV model and through different polynomial parameterizations of the exercise boundary, as given by equation (46). For comparison purposes, the last column contains the pricing errors associated with the 10-point recursive scheme of Huang, Subrahmanyam, and Yu (1996), as suggested by Kim and Yu (1996). The strike price is always set at $100 while the other option features were generated from uniform distributions and within the following intervals: instantaneous volatility between 10% and 60%; interest rate and dividend yield between 0% and 10%; underlying spot price between $70 an $130; beta between 0 and 4.0; and time-to-maturity ranging from 0 to 3.0 years. The pricing errors produced by the alternative boundary specifications were computed against the Crank-Nicolson method with 15,000 time intervals and 10,000 space steps.