A Market-Based Framework for Bankruptcy Prediction

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Abstract

We estimate probabilities of bankruptcy for 5,784 industrial firms in the period 1988-2002 in a model where common equity is viewed as a down-and-out barrier option on the firm's assets. Asset values and volatilities as well as firm-specific bankruptcy barriers are simultaneously backed out from the prices of traded equity. Implied barriers are significantly positive and monotonic in the firm's leverage and asset volatility. Our default probabilities display better calibration and discriminatory power than the ones inferred in a standard Black and Scholes (1973)/Merton (1974) and KMV frameworks. However, accounting-based measures such as Altman Z- and Z"-scores outperform structural models in one-year-ahead bankruptcy predictions, but lose relevance as the forecast horizon is extended.

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I. Introduction

The purpose of this paper is twofold. First, we estimate default probabilities for close to 6,000 industrial firms in the period 1988-2002, refining the estimation procedure within a structural model. We argue for the inclusion of a firm-specific early bankruptcy barrier so as to reflect the nature of many bankruptcy codes, jurisdictions and covenants, which allow bondholders to extract value (or to force liquidation/reorganization) when some trigger event occurs. The inclusion of such a barrier is also necessary from an investment policy point of view to account for bounded managerial risk choices. The second contribution of our paper is methodological: how are we to judge the quality of default probabilities coming out of different models, since we do not observe "true" probabilities of default? We link the two contributions by comparing the discriminatory power and calibration of the barrier option model with that of other structural and accounting-based measures.

In their seminal work, Black and Scholes (1973) and Merton (1974) offer the insight that the we can see the common stock of a firm as a standard call option on the underlying assets of the firm: while shareholders have in effect sold the corporation to their creditors, they hold the option of reacquiring it by paying back the face value (plus interest) of their debt obligations. Alternatively, using the put/call parity we can see shareholders as holding the firm's underlying assets (bought after borrowing money from creditors) as well as a put option with strike price equal to the face value of the debt. If the assets of the firm are worth less than the face value of the debt at maturity, shareholders can walk away without repaying their debt obligations, in effect "selling" the firm to bondholders for the face value of the debt. In turn, bondholders hold a portfolio consisting of riskless debt and a short put option on the firm's assets.[1] However, as Galai and Masulis (1976) first point out, this option analogy suffers from the fact that the value of a standard call option is strictly increasing in the volatility of the underlying assets. Hence, a shareholder-aligned manager choosing between two different projects would "invest in the project of higher variance. Moreover, it is even possible that a more profitable investment project will be rejected in favor of a project with a higher variance of percentage returns", thereby transferring wealth from bondholders to shareholders. This asset-substitution problem has been much developed in the agency literature, starting with Jensen and Meckling (1976). A shareholder-aligned manager who acts to maximize shareholder wealth (as opposed to firm value) might even invest in a negative NPV investment solely to increase the volatility of the firm's assets. Indeed, if common stock were literally a standard option on the firm's assets, shareholder-aligned manager should take infinite risk.

On the other hand, the option analogy is very appealing, especially in cases where a more "classical" valuation model fails. For instance, a discounted dividend analysis cannot be expected to properly value a firm that does not yet distribute dividends, especially if it is anyone's guess when that firm will start paying out dividends to its shareholders (and how large the first dividend will be). Moreover, since equities are traded daily, a valuation method based on equity prices (provided financial markets are somewhat efficient) produces asset values that are

[1] This last analogy may explain Longstaff and Schwartz's (1995) and Duffee's (1998) findings that credit premia tend to decrease as Treasury rates increase. Indeed, if the difference between corporate debt and riskless debt is a put option, this difference will decrease as interest rates rise since the value of a put option is decreasing in the riskfree interest rate. See Merton (1974) for a full analysis of the corporate yield spread.

much more up-to-date and reliable than those we would back out from infrequently updated accounting data. This last observation laid the foundation for KMV Corporation's approach to estimating market values of assets and to pricing risky debt (see for example Crosbie and Bohn (2002) or Vasicek (2001)).

Such an equity-based valuation may well lead to better default predictions. For instance, Hillegeist, Keating, Cram, and Lundstedt (2004) report that probabilities of bankruptcy backed out from a Black-Scholes-Merton structural model are up to fourteen times more informative than the ones inferred from accounting-based statistics such as the Altman (1968) Z-score. But appealing as this framework is, it does not provide any rationale for observed managerial (bounded) risk choices. Moreover, probabilities of default (PDs) coming from this framework are miscalibrated, leading KMV to depart from the assumption of normality used in option pricing and to recalibrate predicted PDs using a proprietary database of historical defaults.

A model is needed that keeps the option analogy but is also consistent with the empirically observed bounded risk choices of shareholder-aligned managers. We need a counterforce that pulls the price of common stock (seen as an option) down when risk becomes too large. Such a requirement is satisfied if we consider the firm's common stock to be a European down-and-out (knock-out) barrier option. A European down-and-out barrier call (resp. put) option is a derivative contract giving its holder the right to acquire (resp. sell) an underlying asset at a prespecified date (maturity date) for a prespecified price (strike), provided the underlying asset has not crossed a prespecified bound between the time the option was written and the maturity date.[2] Following Black and Cox (1976), Leland and Toft (1996), Chesney and Gibson-Asner (1999), and Brockman and Turtle (2003), we see shareholders as having in effect sold the firm to their creditors; however, shareholders hold the option to reacquire the firm if they repay the debt, provided the assets of the firm have not sunk in the meantime below a critical level. If the assets of the firm do reach this lower bound at any time between the date at which the option is written (money is borrowed) and the maturity date of the option (debt's repayment due date), bankruptcy is triggered, followed by the firm's takeover by bondholders and the shareholders' loss of any claim they had on the firm's cash flows (early bankruptcy). This framework accommodates salient features of different bankruptcy codes, from shareholder-friendly environments like the US (a rebate can be introduced to model violations of the Absolute Priority Rule (APR)) to stricter environments like the UK, where corporations are sold piecemeal with proceeds going to bondholders, or Germany (zero rebate and large barrier levels to model the German overindebtedness-triggered bankruptcy). Of course, a firm can also go bankrupt even though the value of its assets never crosses the barrier level before debt maturity, if it fails to pay back its debt obligations at maturity (late bankruptcy).

This approach also implies that bondholders own a portfolio consisting of riskless debt, a short put option on the firm, and a long down-and-in call option on the firm's assets. Brockman and Turtle (2003) (in the remainder of the paper: BT) uncover empirical evidence that "firm-specific barriers are non-zero across firms and that these implied barriers provide meaningful information about firms in the cross-section." This firm-specific barrier level is robust to different test specifications, and leads to better bankruptcy predictions than the Altman (1968) Z-score.

[2] An additional feature of some barrier options is the rebate, i.e., a fixed (prespecified) amount paid to the option buyer upon hitting the barrier to "compensate" him/her for the loss of the option.

They conclude that the barrier option framework significantly improves on the standard Black-Scholes/Merton (in the remainder of this paper: BSM) option approach.

However, BT's model presents two major shortcomings. First, they assume that the market value of debt can be proxied for by the book value of debt (thus implying that debt is riskless[3]). In doing so, they force a structure on the data that biases their implied barrier towards the amount of debt outstanding (and the volatility of the asset process downwards). They indeed report implied barriers uniformly larger than the amount of debt outstanding, which is suspicious (it would imply negative yield spreads or even negative yields). Moreover, the use of book values of debt will certainly not capture *changes* in risk and discount rates over time. Secondly, they ignore in the computation of (risk-neutral) probabilities of bankruptcy the possibility that, even if the underlying asset value of the firm never crosses the barrier level, the firm may nevertheless be unable to pay back its debt obligations at maturity. This second shortcoming is a direct consequence of the first one. If the barrier level B is above the debt level (as is the case in BT's paper), bankruptcy can only happen when the firm's asset value hits the barrier (if the firm's asset value is lower than its debt obligations F at maturity, it must have crossed $B>F$ at some earlier time). But such a high barrier leads to an overstatement of the default probability (compounded by the fact that the actual- or physical - drift of the asset process is usually larger than the riskfree rate), which may explain why BT can "foresee" 85% of bankruptcies happening within a year. They do not report their type I error, i.e., firms that according to their predictions should have defaulted but did not, nor offer a tradeoff analysis between Type I and Type II errors. As a consequence, this statistic of 85% may not be very meaningful. A thorough comparative analysis of default models should discuss how different models fare both in terms of ranking (a good model should assign larger PDs to firms that actually default) and in terms of calibration (how closely predicted PDs approximate true historical default frequencies). Accuracy can be a pretty biased statistic for the quality of a model.

In this paper, we address these issues by simultaneously backing out from equity prices, in a barrier option framework, the market value and the volatility of assets, as well as the firm-specific default barrier implicitly priced by the market. We find that such a barrier is indeed significantly different from zero, but not nearly as large as the one reported by BT. In particular, default barriers are generally lower than the firm's debt obligations. Our methodology also yields default probabilities that outperform the ones inferred from a BSM and KMV model, both in terms of ranking (we predict greater probabilities of bankruptcy for firms that actually default later on) and calibration (less underestimation of the PDs in the low range, and less overestimation for highly levered and volatile firms).

However, probabilities of default coming from all structural models seem to be miscalibrated, which explains KMV's departure from the normality assumption and their non-parametric recalibration. For that reason, we recalibrate predicted probabilities of default using maximum likelihood in a logistic regression framework. Even after this parametric recalibration, probabilities of default inferred from BSM or KMV approaches are still dominated by the ones computed in a barrier option framework.

[3] If the coupon rate is close to the demanded yield, the market value of debt is close to its book value. But if one assumes that the debt is in the form of a zero-coupon bond (as BT, and most structural models, do), proxying for the market value of debt using its book value seriously biases debt valuation upwards.

But our most surprising result is that accounting-based measures such as the Altman (1968, 1993) Z- and Z”-scores outperform structural models for one-year-ahead bankruptcy predictions. Brockman and Turtle (2003) and Hillegeist *et al.* (2004) obtained the opposite results because they used book values of debt and risk-neutral probabilities.[4] However, as the forecast horizon extends beyond one year, forward-looking measures based on equity prices surpass accounting-based scores. These results are robust to both in-sample and out-of-sample recalibrations. Empirical implications for risk management are derived and discussed. In particular, we recommend combining accounting-based measures and structural models (for instance in a logit model) to achieve an optimal term structure of default predictions.

The paper is organized as follows: Section II presents an overview of the studies that form the background for our paper and examines the theoretical framework that motivates our empirical tests. In Section III, we expose our methodology for backing out implied market values of assets, asset volatilities, and bankruptcy barriers when the equity of the firm is modelled as a down-and-out call option on the underlying assets of the firm. We also present our first empirical results and compare them with alternative frameworks. In Section IV, we use these implied firm characteristics to predict bankruptcy and compare our predictions to those of other structural models and to alternative measures such as Altman Z- and Z”-scores. In particular, we discuss appropriate methods for the evaluation of predicted default probabilities and analyze the advantages and drawbacks of accounting-based measures vs. structural models. Section V concludes. All proofs are relegated to the Appendix.

II. Theoretical Framework

A. The KMV Methodology[5]

KMV uses the Black-Scholes (1973)/Merton (1974) analogy by first converting the debt structure of the firm into an equivalent zero-coupon bond with a one-year maturity, for a total promised repayment (interest plus principal) of F. Although KMV claims that its methodology can accommodate different classes of debt (including convertible debt), its publications are silent on how this equivalent amount F is distilled from more complex capital structures. In any case, it serves as the exercise price that shareholders have to pay at maturity (one year in the case of a year-ahead prediction) to reacquire the assets of the firm. Since KMV also predicts two- and five-year-ahead probabilities of bankruptcy, we will more generally denote by $\tau=T-t$ the maturity of the option (i.e., the maturity of the equivalent debt contract). The firm’s assets are assumed to follow a geometric Brownian motion with constant relative drift μ_A, payout ratio δ, and volatility σ_A: $dV_s = (\mu_A - \delta)V_s dt + \sigma_A V_s dW(s)$, where V_s denotes the asset value as of time s and $\{W(s)\}$ is a standard Wiener process. The option value at time t, E_t, is known (equity is traded on a daily basis) and so is the (nonstochastic) riskfree rate r. However, two inputs remain unknown: the underlying asset

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[4] Hillegeist et al. (2004) try to infer a drift for the asset value process, but their estimates cannot be consistent due to the telescoping property. Moreover, whenever their estimation procedure yields a negative value for the drift, they replace it, in a rather *ad hoc* manner, with the riskfree rate.

[5] This section is based on the description of KMV’s methodology by two of its vice presidents, Crosbie and Bohn (2002).

value at time t (V_t) and the volatility of the asset process (σ_A). These two unknowns are backed out using two equations. The first one is the BSM equation for the option (i.e., equity) price E_t, and the second one comes from Itô's lemma, linking the equity's relative volatility σ_E, which can be estimated from historical equity quotes, to the relative volatility of assets σ_A. The equations are the following:

$$E_t = V_t e^{-\delta(T-t)}\Phi(d_1) - e^{-r(T-t)}F\Phi(d_1 - \sigma_A\sqrt{T-t}) , \tag{1}$$

$$\text{and } \sigma_E = \frac{V_t}{E_t}\Delta\sigma_A , \tag{2}$$

where $d_1 = \dfrac{ln(V_t / F) + (r - \delta + \sigma_A^2 / 2)(T-t)}{\sigma\sqrt{T-t}}$, $\Delta = \dfrac{\partial E_t}{\partial V_t}$ is the hedge ratio, equal to $e^{-\delta(T-t)}\Phi(d_1)$, Φ is the normal cumulative distribution function, and all other parameters have been previously defined.

Once V_t and σ_A are known, we can compute the risk-neutral probability of bankruptcy, which is equal to $\Pr[V_T \le F \mid V_t] = \Pr\left[\ln(V_T) \le \ln(F) \mid V_t\right] = 1 - \Phi\left(d_1 - \sigma_A\sqrt{T-t}\right)$ since $ln(V_T)$ is normally distributed with mean $\ln(V_t) + (r - \delta - \sigma_A^2 / 2)(T-t)$ and variance $\sigma_A^2(T-t)$ under the risk-neutral probability measure. At this point, KMV publications note that this methodology clearly understates probabilities of bankruptcies, resulting in a predicted rating of "Aaa or better" for 75% of the firms in their sample. They argue that "the normal distribution is a very poor choice to define the probability of default [since] the default point is in reality also a random variable [due to] firms often adjust[ing] their liabilities as they near default" (Crosbie and Bohn (2002)). For that reason, KMV departs here from the assumption of normality, which would be to imply the default probabilities within the model. They rather compute a standardized "distance-to-default" (DD), i.e., the number of standard deviations away from default that the firm is expected to be at time T. It is easily shown that under the risk-neutral measure, $DD = \left[ln(V_t / F) + (r - \delta - \sigma_A^2 / 2)(T-t)\right] / \left(\sigma_A\sqrt{T-t}\right)$. This DD is used as a key summary statistic for the credit quality of the obligor in question and is assumed to contain all default-relevant information (i.e., leverage and asset volatility). It is used in table lookup/univariate scoring models to compare with the default experience of other firms that had the same distance to default using the Moody's KMV proprietary defaults database. This yields the Expected Default Frequency (EDF), the frequency with which firms with the same distance-to-default actually defaulted. This EDF "has much wider tails than the normal distribution. For example, a distance-to-default of four standard deviations maps to a default rate of around 100 bp [i.e., 1%]. The equivalent probability from the normal distribution is essentially zero" (Crosbie and Bohn (2002)).

In other words, KMV merely uses the contingent claims approach (assuming normality of returns) to back out V_t and σ_A from the Black-Scholes/Merton equations and to compute the key indicator for credit quality (the DD), but then leaves the model framework to imply default likelihoods from historical data. This inconsistency is justified by the large improvement in predicting power over traditional rating models, which use potentially stale accounting data, and over a standard BSM approach without remapping, which would yield unrealistically low probabilities of default. In particular, KMV claims that using forward-looking equity prices helped them foresee the Monsanto and Enron bankruptcies several months ahead.

This dichotomy is still an embarrassment for the model builder. If the distributional assumptions are not to be taken seriously, it is difficult to see how to interpret the firm values and asset volatilities identified using them. If they are to be taken seriously, DD's are indeed cardinal measures, whose rejection by the data implies rejection either of the distribution (normal returns) or of the model – equity as a standard option on the firm's assets. We follow the last argument and consider equity as a down-and-out call (DOC) option on the assets of the firm. The firm goes bankrupt not only if it cannot pay back its debt obligations at maturity, but also if its net asset value falls through a prespecified barrier at any time before maturity, raising the probability of default without discarding the normality assumption. Finally, backing out a firm-specific barrier enables us to make firm-specific default predictions. When KMV maps a DD to a default probability using a proprietary database, it compares a specific case (say, a small bank) to all firms with a comparable DD, thus discarding much of the firm-specific information. However, one may argue that a better default prediction would be carried out by comparing this firm's DD to the DD of banks, of financial firms, or of small firms. More generally, even putting aside the aforementioned theoretical inconsistency, the KMV methodology cannot be carried out without their large proprietary database.

It is worth noting that a barrier option framework is implicitly used by Huang and Huang (2002), who back out market values of assets and asset volatilities from market data to test different structural models with early bankruptcy, including the ones by Longstaff and Schwartz (1995) with an exogenous barrier and stochastic interest rate; Leland and Toft (1996) with an endogenously derived optimal barrier but constant interest rate; Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) with strategic defaults; and Collin-Dufresne and Goldstein (2001) with mean-reverting leverage ratios. However, their focus is the explanation of corporate debt yield spreads, and they calibrate the different models to actual default frequencies and equity premia *per rating category*. In particular, they assume that the barrier B is the same for all firms with the same rating, thus not using the discriminatory power of firm-specific barriers. To the best of our knowledge, Brockman and Turtle (2003) offer the first study that backs out firm-characteristic barriers and uses them to predict bankruptcy. Unfortunately, it presents serious weaknesses and limitations.

B. Bankruptcy Barriers and Actual Defaults

Brockman and Turtle (2003) model the equity of a corporation as a down-and-out call (DOC) option on the assets of the firm. Their goal is to estimate the barrier level implied in traded equity prices. To do so, they assume that the market value of assets (MVA) is equal to the market value of equity plus the book value of debt. They estimate the volatility of the MVA using quarterly percentage changes in those asset values over 10 years and assume an average debt maturity of 10 years (their results do not depend on this maturity assumption). They back out implied barriers that are positive and significantly different from zero. In their 10-year sample period between 1989 and 1998, the implied firm-specific barrier is on average equal to 69.2% of the firm's asset value, compared with an average debt proportion equal to 44.7% of asset value. Moreover, this implied barrier (as a fraction of asset value) is larger than the leverage ratio for all firms with a debt-to-asset ratio lower than 90% and arguably for quite a few firms with even larger leverage (see their Table 2, Panel D).[6] They conclude that the standard option approach overestimates the value of equity and underestimates the value of debt.

It is possible for the implied bankruptcy bound to be larger than the firm's liabilities. This could be the case if shareholders decide to strategically enter into bankruptcy proceedings so as to break unfavorable contracts and protect themselves from litigation, or because they believe they can extract more value from other stakeholders after emerging from bankruptcy as an ongoing concern. Alternatively, it could be the case that short-term liquidity constraints prevent the firm from meeting its obligations even though the firm's asset value is still larger than its total liabilities (positive net worth). However, in such a case, shareholders will most likely be successful in renegotiating their debt. Finally, a barrier level B larger than the firm's obligations F is applicable to the case of banks (with early closure rules when bank capital falls below a threshold equal to 2% of assets).[7] But we doubt that such a large implied barrier pertains for industrial firms in general, the subject of BT's and our study, particularly if there is no preset mechanism to force liquidation of a firm that is not yet insolvent.

This doubt is fueled by our intuition that bondholders will find it extremely difficult to enforce in a court of law a covenant that forces the firm into bankruptcy if the value of the underlying assets V_s crosses a barrier level B greater than the face value of debt F at any time s before debt maturity. In that case, when $F \leq V_s \leq B$, bondholders push the firm into bankruptcy, while its assets are still more valuable than the amount of money owed. The firm could then sell its assets, repay the debt and some money will still be left for shareholders, unless liquidation costs are prohibitive. But goodwill and the ongoing concern value (intangibles such as unrealized growth opportunities) would be lost forever.

Leland (1994) and Leland and Toft (1996) argue that as long as the value of the gamble for resurrection is larger than the cash needed to keep the firm alive (which certainly is the case if the asset value is above the amount owed but below the barrier level $B>F$), shareholders will pay coupons out of their own pockets if necessary to keep the firm alive. Even when the barrier level is chosen so as to maximize firm and equity values, "the endogenous bankruptcy-triggering asset value will typically be less than the principal value of debt. Hence the firm will continue to operate despite having negative net worth." Their simulations yield optimal barrier levels around 30% of asset values for a wide range of leverage, risk, and debt maturity. Crosbie and Bohn (2002) note that historically, even when the asset value reaches the technical default point, i.e., the sum of total liabilities, most firms do not default immediately. They rather tap into credit lines and other cash-generating mechanisms to service their debt. For that reason, KMV sets the default point to the sum of total short-term liabilities plus half of long-term liabilities (and this is the F, in equation (1), that KMV uses). In the barrier option framework, it means that B should be lower than F by (at least) half the amount of long-term debt obligations.

[6] BT report an average barrier level for each leverage decile that is greater than the upper bound of that leverage decile. Only the last leverage decile (debt-to-assets ratio between 0.9 and 1) displays an average barrier-to-assets ratio of 96.57%, which might be smaller than the average leverage in that decile.

[7] We thank Linda Allen for pointing this out.

Similarly, Huang and Huang (2002) note that in some models "default is assumed to occur when the firm value falls to the bond face value. This assumption, however, may not be very reasonable given that bondholders on average recover about 51% of the face value given default; after all, it is difficult to believe that 49% of the firm value is lost due to bankruptcy costs or the violation of absolute priority rule." For that reason, they carry out their empirical tests with a default boundary equal to 60% of the bond's face value.

Other papers go even further to argue that, since the net worth of a corporation can be significantly negative without that company being liquidated, even crossing a barrier level below the total amount of liabilities is not enough to trigger actual default. Helwege (1999) documents cases of firms emerging from bankruptcy as going concerns after remaining in distress (not making any coupon payments) for up to seven years without being formally liquidated. Literature built on this evidence argues that a firm should be considered formally bankrupt only after its asset value process has spent a certain time below the barrier. François and Morellec (2003) consider that equity is best modelled as a Parisian option (the firm is not liquidated unless a certain *uninterrupted* amount of time is spent below the barrier), while Moraux (2002) argues for a Parasian option framework (instead of resetting the clock to zero every time the asset value rises above the barrier, the model considers a firm bankrupt after its asset value has spent a certain *cumulative* time below the barrier). More recently, Galai, Raviv, and Wiener (2005) generalize the two approaches and allow for the distress clock to give more weight to more recent visits below the barrier and for the severity of distress (how far below the barrier the asset process dips) to play a role.[8] This research is based on the realization that negative net worth (i.e., the asset value dropping below the firm's debt obligations) is not sufficient to trigger automatic default, making a default-triggering barrier level *larger* then these debt obligations even more unrealistic.

Moreover, if the barrier is larger than the amount owed to creditors, defaultable bonds cannot experience any loss at all at default if bankruptcy costs (or violations of the APR) are not explicitly modelled since bondholders would force liquidation precisely when they start suffering a loss: corporate debt is uniformly riskless in BT's framework, and should not command any yield spread (except, maybe, for firms with a leverage ratio greater than 90%). Yield spreads (and even yields in some cases) should actually be negative, if bondholders have a positive probability of taking over a firm worth $B>F$, i.e., get more than their dues! BT, aware of this problem, state that a "barrier level set above the exercise price of the option (overcollateralization) […] is more likely to trigger a loan recall or some other punitive action than forced bankruptcy proceedings." They nevertheless use implied barriers strictly larger than the firm's liabilities to predict bankruptcies.

As a consequence, their bankruptcy predictions suffer from a potentially serious problem. In their framework, default is defined solely as crossing the barrier before the debt maturity (if the firm's asset value is lower than F at maturity, it must have crossed $B>F$ at some earlier time). This leads to an overstatement of the probability of bankruptcy, and may explain why BT can "foresee" 85% of bankruptcies happening within a year. However, if the prior for the majority class (surviving firms) is very different from the prior for the minority class (defaulting firms)

[8] Although Parisian options admit a closed-form pricing formula (see Chesney, Jeanblanc-Piqué, and Yor (1995)), these models are very difficult to test empirically since these formulas are very sensitive to small variations in parameters, resulting in non-convergence of algorithms. Moreover, a numerical procedure to invert a Laplace transform is needed.

and uncertainty is large, accuracy (percentage of correctly forecasted bankruptcies and survivals) becomes irrelevant. As Stein (2002) points out, since actual default rates are of the order of 1% to 5% depending on the horizon considered, a model that predicts that all firms survive should achieve an accuracy between 95 and 99%, although such a model is totally irrelevant for risk management purposes. BT report 56 delistings within a year. Since their sample consists of 7,787 firm-years, we infer that the one-year-ahead prior of survival is equal to 1-56/7,787=99.28% in their sample. Hence, an accuracy (after the recalibration implemented in the logistic regression) of nearly 99.3% would be expected. We will turn to this issue in detail in Section IV.

BT's extremely large barriers (implying that debt is riskless) are due to their use of the book value of debt to proxy for the market value of debt (thus *starting* with the implicit assumption that debt is riskless – at best – given their assumption that debt is in the form of a zero-coupon bond). When the market value of assets is set equal to the book value of debt plus the market value of equity, a structure is forced onto the model. As the underlying asset value reaches the barrier level B, the equity (a down-and-out call option) is worthless. Hence the market value of assets (equal to B as the barrier is hit) is equal at that point to the market value of debt. If the market value of debt is assumed to be equal to the book value of debt F, *the barrier level B is forced to be equal to this book value of debt F*. Since B and F are assumed constant in the barrier option framework, B is not only forced to be equal to F at bankruptcy, but at any point in time. The only reason BT find barrier levels close to – but not equal to – leverage levels is noise in data.

Actually, if equity is not quite worthless when the asset value reaches the barrier (due to violations of the APR and/or shareholders' strategic default in the hope of emerging from bankruptcy proceedings as an ongoing concern), the implied barrier (i.e., value of assets at default) will be forced to be equal to F plus the expected residual equity value, which probably explains why BT back out barriers larger than F. Another explanation for BT's results is put forward by Wong and Choi (2006). They show that the fair value of a barrier option will be equal to its intrinsic value (which is what BT implicitly assume when using the face value of debt as a proxy for its market value) only if the barrier is larger than the strike price. They find results very close to those of BT, without even looking at real data, by merely equating the price of the option to its intrinsic value (see their Table 1). Finally, book values of debt will certainly not reflect *changes* in risk and discount rates over time.

If, however, we allow the market values of debt to be less than its book value, the problem vanishes: as the asset value process reaches the barrier, the equity (DOC) becomes worthless, forcing the *market* value of debt to be equal to the barrier level. Since the market value of debt at that point is less than its book value (Altman and Kishore (1996) find an average recovery rate on defaulted bonds equal to 41%, while Moody's report an average recovery rate of 51.3% on senior unsecured bonds), the barrier level has to be lower than the face value of debt.

In such a framework where the barrier level B is lower than the strike price F, the estimated probability of bankruptcy should take two possible events into consideration. First, the firm's asset value might cross the barrier before the option (i.e., debt) maturity. This event is summarized in BT's equation (3), which is the probability of the first passage time being less than T, the debt maturity. Second, bankruptcy can occur if the firm's asset value is insufficient to meet the debt obligation, even though the barrier has never been hit (late bankruptcy). It is the latter event that cannot, by construction, happen in BT's framework.[9] We refer the reader to Figure 1 for a synopsis of the different cases of bankruptcy. It is also worth noting that the relevant probability of bankruptcy is the physical one (a function of μ_A, the actual asset drift), not the risk-neutral one specified by BT. Since the asset drift μ_A is generally larger than the riskfree rate, this actual probability of default will typically be lower than the risk-neutral one. As noted by Leland and Toft (1996), who argue for using the actual asset value drift rather than the riskfree rate, a small difference in the drift can result in major differences in the probability of default.

Finally, since the book value of debt does not fluctuate much, it is much less volatile than the volatility of the market value of debt, and BT's study underestimates the volatility of asset values (which they approximate with the sum of the book value of debt and market value of equity). In the words of Crosbie and Bohn (2002), "asset volatility is related to, but different from, equity volatility. A firm's leverage has the effect of magnifying its underlying asset volatility. As a result, industries with low asset volatilities (for example, banking) tend to take on larger amounts of leverage while industries with high asset volatility (for example, computer software) tend to take on less. As a consequence of these compensatory differences in leverage, equity volatility is far less differentiated by industry and asset size than is asset volatility."

We now address these issues by adopting the KMV approach of backing out asset values and volatilities from equity market data, while at the same time keeping the DOC framework and estimating implied barriers as well.

III. Common Stock as a Down-And-Out Barrier Option

A. Model and Assumptions

We model the corporation as an entity fully financed with a share of equity and a single zero-coupon bond, both of which are traded on a perfect financial market (no arbitrage opportunities, no taxes or transaction costs, perfect asset divisibility, and continuous trading). Since the sum of the bond and the stock value is equal to the firm's asset value in all states of the world, we can consider the firm's asset value as a traded security. Uncertainty is represented by an underlying probability space $(\Omega,\mathcal{F},P)$ that is assumed to be complete. The probability measure P represents the common probability beliefs held by all investors or traders. The arrival of information is represented by an augmented filtration $\{\mathcal{F}_s\}$, $s \in [t,T]$. Under the measure P, the firm's asset value process follows a lognormal diffusion, that is $dV(s) = (\mu_A - \delta)V(s)ds + \sigma_A V(s)dW(s)$, where μ_A is the instantaneous drift of the asset return process, σ_A its instantaneous dispersion or volatility coefficient, δ the constant fraction of value paid to security holders (both shareholders and creditors), and $dW(s)$ the increment of a standard Wiener process defined relative to the filtration $\{\mathcal{F}_s\}$. We assume moreover the existence of a default-free money market account M

[9] We could do away with the second bankruptcy event (bankruptcy at maturity) by modeling the barrier as time dependent, for instance $B(s)=Fe^{-k(T-s)}$ for any time s between t and T where k is an arbitrarily chosen decay factor. In that case, $B(T)=F$, and bankruptcy at maturity amounts to having crossed the barrier at some previous time. This specification, introduced by Black and Cox (1976), further stresses that a realistic barrier level is lower than the amount of debt owed by the firm.

appreciating at a constant continuous interest rate r, i.e. $dM(s)=rM(s)ds$.[10] Markets are therefore dynamically complete. As a result, there exists a unique martingale measure that makes the firm's asset value process a martingale and claims written on these assets can be priced by replication.

The firm's debt, issued before or at time t, has the form of a pure discount bond of promised payment F that matures at time T. At that time, the firm is required to pay debt claims, if possible. Since the asset value is lognormally distributed, any amount of outstanding debt is risky. We also assume that bond covenants prevent the (shareholder-aligned) manager from selling assets between the time the bond was issued and the time the debt is due (T). If allowed to do so, the shareholder-aligned manager would "strip" the firm, i.e., distribute a liquidating dividend to shareholders and default strategically, leaving an empty shell for bondholders to take over.[11] Covenants specify that if the assets of the firm drop below a certain prespecified boundary (barrier) $B<V_t$ at any time s between the present time t and debt maturity T, bankruptcy is triggered: creditors take away the firm from shareholders and either reorganize the firm or liquidate it. In any event, previous shareholders have lost any claim they may have had on the cash flows of the firm. The same happens at maturity if shareholders are unable to honor their debt obligations. Thus, we can see equity as a down-and-out call option on the firm's assets, with a barrier level of B, strike price of F and a maturity of $\tau=T-t$. If the asset value at time T is not as large as the amount of debt outstanding F, shareholders can walk away (limited liability), provided they are still in control of the company, i.e., the asset level has never crossed B from above (in which case bondholders have already taken over the firm). See Figure 1 for a synopsis of the different cases of bankruptcy.

A few comments on those assumptions are necessary. We assume that the debt is in the form of a zero-coupon bond because in the case of coupon-bearing debt, shareholders might find it of interest to default right before making a coupon payment. Hence, servicing the debt can be seen as acquiring the option to keep the option (common stock) alive with the next coupon payment: the stock has now to be seen as a compound option (see Geske (1978)); see also Mella-Barral and Perraudin (1997), Pan (1999), and Fan and Sundaresan (2000) for a strategic approach to debt service, where bargaining between equityholders and debtholders can lead to lesser debt service than promised, but without formal default.

Since there is no interest payment until maturity, bankruptcy cannot be defined in our model as a lack of cash to service the debt. Instead, bankruptcy occurs either if the asset value is insufficient to cover debt obligations (bankruptcy at maturity) or if the firm's assets cross a lower bound prespecified in the debt contract (early bankruptcy). There is a clear example of such a case where a sufficient drop in the (market) value of the firm's assets triggers bankruptcy: in December 2001, Enron's debt was downrated to junk, following a long and consistent

[10] Note that this money market account is not even necessary since two linearly independent securities (the firm's stock and bond) are already traded in the market and $\{W_s\}$ is the only source of risk (and hence we can always produce a synthetic riskless instrument), but it makes the modeling much easier. Longstaff and Schwartz (1995) and Kim, Ramaswamy, and Sundaresan (1993) show that introducing a stochastic default-free interest rate process has virtually no effect on credit spreads, while rendering the analysis significantly more complex.

[11] This assumption was already implied in the law of motion of the firm's assets, where the payout ratio δ was assumed constant. In particular, sudden increases in δ triggered by a liquidation of assets are ruled out.

drop in the stock price. As pointed out by Gillan and Martin (2004), "earnings hedges engineered by Enron executives inextricably linked Enron's credit rating to its own stock price." Since Enron's debt included the covenant that a debt rating below Baa3 gave creditors the right to force the firm into bankruptcy, the company had to file a Chapter 11 procedure;[12] see also Bhanot (2003) for a structural rating model where rating agencies downrate firms following a substantial drop in asset value. Alternatively, B can be seen as the asset value at which the firm has to borrow more money to continue operating but cannot because of, say, prohibitive informational costs (for instance, it may be the minimum asset value necessary to satisfy the firm's obligations towards its employees, suppliers, or other debtors and nobody is willing to put more money in the firm at that point, as was the case in the Swissair and Sabena bankruptcies). In the first case, B is a quantity arbitrarily specified by shareholders in bond covenants, and which determines the price paid by bondholders for a debt of a given face value F. In the second case, B is not explicitly specified, but rather corresponds to the general market conditions for the industry in which the firm operates. It can then be seen as a "reservation value" below which creditors are not willing to wait any longer and take the risk of being repaid even less.[13] In any case, since the asset value is stochastic, the distance-to-default is stochastic as well, and modeling B as a random quantity (as in Nielsen, Saa-Requejo, and Santa-Clara (1993)) would unduly complicate our framework and thereby render it impossible to back out parameters.[14]

Note that default can lead to restructuring in or outside bankruptcy proceedings or to liquidation. We abstract from detailed considerations about the actual process followed, at least as long as we are considering the equity side, and will assume that regardless of the procedure, equity has become worthless (APR). We also abstain from considering bankruptcy costs and assume that in case of bankruptcy, bondholders recover the full remaining asset value. As emphasized by Black and Cox (1976), bankruptcy costs are unlikely to alter the qualitative results of a structural model: "We are considering bankruptcy as simply a transfer of the entire ownership of the firm to bondholders. The physical activities of the firm need not be affected. The bondholders may not want to actively run the company, but probably the stockholders did not want [to] either. The bondholders could retain the old managers or hire new ones, or they could refinance the firm and sell all or part of their holdings. Certain legal costs may be involved in the act of bankruptcy, but if contracts are carefully specified in the first place with an eye towards minimizing these costs, then their importance may be significantly reduced."[15]

It is worth noting that the barrier option framework could accommodate departures from the APR. An alternative specification for barrier options allows for a rebate, a fixed payment made to the option holder at the time he/she is stripped of the option. In a shareholder-friendly bankruptcy code (such as the American one), equity

[12] Stulz (2004) also stresses that Enron had "credit rating triggers" that required payments to be made on complicated derivatives positions (or these positions to be closed) if the firm's rating fell below a prespecified trigger level. These payments were so large ($3.9 billion) that the firm could no longer survive once the triggers were activated.

[13] Alternatively, if asset sales were allowed to fund bond-related payments, B could be the minimum value of assets below which even a total asset sale would not supply the minimum amount bondholders would be willing to accept in a renegotiation of their claim.

[14] But since our empirical study backs out an implied barrier for every firm-year, we *de facto* allow B to vary over time for a given firm.

[15] Andrade and Kaplan (1998) find bankruptcy costs ranging from 10 to 20% of pre-distressed firm value.

can therefore be modeled as a DOC with a positive rebate. For instance, Eberhart, Moore, and Roenfeldt (1990), Weiss (1990), Franks and Torous (1989, 1994), and Layish (2001) document that in 75% of all U.S. bankruptcies, debt structures are renegotiated so as to allow some deviation from absolute priority, resulting in positive payments to equity holders even if bondholders are not fully paid. For instance, Franks and Torous (1989, 1994) report that in 41 Chapter 11 bankruptcies, junior claimants extracted \$878 million that should have been received by senior claimants and that common stockholders received a third of those \$878 million, although they had no valid claim on any of it. They also report a similar pattern in 47 workouts (in those cases, the deviation from the APR were –3.54% for bank debt, but +9.51% for equity). In a stricter bankruptcy code (such as in the U.K., where corporations are sold piecemeal with proceeds going to bondholders, or such as the German one, where firms have a legal obligation to report overindebtedness[16]), the rebate would be set to zero. The barrier option framework can thus accommodate quite a variety of bankruptcy codes. To keep the model empirically tractable, we will abstain from considering rebates, and assume that the APR holds (the limitation here is not the data, but the number of equations needed to back out firm-specific market values of assets, volatilities, implied barriers, *and* rebates).

Appendix 1 is a reminder on the pricing of a down-and out call option. The formula below assumes that the barrier level B is below the amount of total liabilities F. As mentioned earlier, this is dictated by the application of the DOC framework to corporate liabilities. As the asset value crosses the barrier, equity becomes worthless, so that the asset value is equal to the market value of debt, which, at this distress point, has to be lower than the book value of debt (see the evidence in Altman and Kishore (1996)). Besides reasons cited previously (the near impossibility of enforcing in a court of law a covenant pushing the firm into bankruptcy while its asset value is still greater than the value of its liabilities, or shareholders realizing that it is in their interest to supply more capital to keep alive the gamble for resurrection), the most compelling reason for considering the case where $B<F$ is that since we do not model deviations from the APR nor bankruptcy costs, defaultable bonds would not experience any loss at all at default in our model if the barrier was larger than the amount owed to creditors. We also provide in Appendix 1 the pricing formula for the case where $B>F$, which in a few cases will turn out to be useful for backing out the implied barrier.

In the framework where $B<F$, the price of the firm's common stock can then be expressed as:

$$E_t = V_t e^{-\delta\tau}\Phi\left(d_1\right) - Fe^{-r\tau}\Phi\left(d_1 - \sigma_A\sqrt{\tau}\right) - \left[V_t e^{-\delta\tau}\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma^2}+1}\Phi\left(d_1^B\right) - Fe^{-r\tau}\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma^2}-1}\Phi\left(d_1^B - \sigma_A\sqrt{\tau}\right)\right] \tag{3}$$

with $d_1 = \dfrac{\ln(V_t/F) + (r-\delta+\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}$ and $d_1^B = \dfrac{\ln\left[B^2/(V_tF)\right] + (r-\delta+\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}$. Note that the value of the common stock can be decomposed into two terms: i) the value of a standard call option on the underlying assets of the firm, minus ii) the ex-ante loss in equity value corresponding to the option bondholders have to "pull the plug" early (terms in square brackets).

[16] An example of such a reporting of overindebtedness is the case of Metallgesellschaft in January 1994, following large losses in energy futures (see Mello and Parsons (1995)).

The second equation we will need to back out asset values and implied barriers comes from Itô's lemma:

$$\sigma_E = \frac{V_t}{E_t} \Delta_{Barrier} \sigma_A , \tag{4}$$

where $\Delta_{Barrier}$ denotes $\partial E_t / \partial V_t$, and for which a formula is given in Appendix 1. We chose this structural method over the maximum likelihood (ML) approach advocated by Duan (1994) since he only proves consistency and asymptotic normality of the volatility estimators in the classical (BSM) framework. It is unclear whether volatility estimators in a barrier option framework are as reliable. Moreover, Ericsson and Reneby (2005) estimate (in their Table 3b) the bias of the structural approach (compared to the ML approach) to be on the order of 0.1% in the BSM framework when equity volatility is estimated over a period of 250 days and is close to 40% (which corresponds to our empirical approach and results). Hence, we consider that the potential bias incurred does not warrant using estimators with unknown statistical properties; see also Bensoussan, Crouhy, and Galai (1995) and Hull, Nelken, and White (2004).

We also provide in this Appendix the derivative of the stock price with respect to the asset volatility σ_A (*vega* of the option). Contrary to the *vega* of a standard call, it is not always positive.[17] If the barrier B is close enough to the sum of total liabilities F, $\partial E_t / \partial \sigma_A$ may even be negative for all σ_A (and the more so the smaller δ and the larger τ[18]): any risky investment makes the probability of going bankrupt too large and shareholders will shy away from any risky project. In this case, far from witnessing an asset substitution problem à la Jensen and Meckling (1976), we might observe a risk-avoidance problem à la John and Brito (2000): a shareholder-aligned manager, afraid of losing growth options privy only to her, may shy away from risk and undertake projects with suboptimal risk levels.[19] The first simulation in Figure 2 illustrates this point: when T-t=10 years, r=5%, δ=3%, and B=30% (approximately our sample averages as discussed in the next section), the equity value is decreasing in σ_A for low values of leverage. As the firm becomes more and more levered and the probability of "late" bankruptcy looms larger, the "classical" risk-shifting intuition dominates again (equity value is increasing in σ_A). When using BT's parameters (an average barrier level equal to 69% of the firm's asset value and a payout ratio of zero), the equity value is decreasing in σ_A for leverages up to 50%. More generally, the DOC framework ensures that managerial risk choices are bounded, even when the manager is fully shareholder-aligned.

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17 For instance, if a corporation with current market value of assets of \$100 million and a payout ratio δ=0 has issued zero-coupon debt with a face value of \$50 million repayable in 15 years, and the per-annum standard deviation of the asset return process is 40%, the riskfree rate is 5% per annum, and bondholders have the right to take the firm over if the market value of assets drop below \$30 million, the derivative of the stock price with respect to σ_A is equal to –4.02: shareholders are better off with *less* risky investments. For a more complete set of comparative statics in the DOC framework, see also Reisz (2001).

18 In our framework, long-term debt prices suffer more from increases in risk than do short-term debt prices. The DOC framework thus lends another explanation for why riskier firms tend to issue shorter-term debt; see Barclay and Smith (1995), Stohs and Mauer (1996), and Guedes and Opler (1996).

19 Of course, risk-shifting (or risk-avoidance) becomes an issue if the manager switches her risk choices *after* debt has been issued. If the equity of the firm is commonly seen as a DOC option all along (implying an optimal risk σ_A^*), the manager has now an incentive, *after issuing debt*, to increase the payout ratio δ to not only leak more cash flows to shareholders, but also to increase the optimal level of risk σ_A^*, which is increasing in δ.

B. Implied Market Values of Assets and Barrier Levels

Our sample consists of industrial corporations (SIC codes between 2000 and 5999) between 1988 and 2002. The accounting data comes from COMPUSTAT, while share price data, SIC codes, and delisting codes used to identify bankruptcies come from CRSP (following Dichev (1998) and BT, we used CRSP's delisting codes 400 and 550-585, which indicate bankruptcy, liquidation, or poor performance). Historical Treasury rates come from the United States Department of the Treasury (www.ustreas.gov).

We start with a sample consisting of 60,110 firm-years representing 7,180 unique industrial firms. For each firm-year, we first estimate the daily volatility of log returns on equity over the elapsed year, using actual trading quotes or the average of the bid and ask if the stock was not traded on a particular day. We multiply this volatility by $\sqrt{251}$ to get an annualized equity volatility σ_E. Other statistics such as payout and leverage ratios require us to estimate the market value of the firm's assets, and this is the issue to which we now turn.

For each firm-year, we assume in a first step that the asset volatility σ_A and the barrier B remain constant between times t-1 and t, and solve a system of four nonlinear equations (equations (3) and (4) using relevant data as of times t-1 and t) for four unknowns: the market value of assets at times t-1 and t, V_{t-1} and V_t, the asset volatility σ_A and the implied barrier B. Although the barrier option framework assumes that σ_A and B are constant over the life of the firm, we allow for the estimation of a different value of σ_A and B for each firm-year (since a firm's leverage will experience cyclical variation, so should the exogenous bankruptcy barrier, as argued by Collin-Dufresne and Goldstein (2001).) The restriction that σ_A and B have to be constant over two consecutive years is merely imposed to have no more than four unknowns to be backed out from a system of four equations. However, the backed-out volatility σ_A and barrier B will be assigned to the firm as of time t (their values for firm-year t-1 will result from solving the system of equations for years t-1 and t-2, so as to avoid a hindsight bias, which would result from assigning the backed out σ_A and B to year t-1). Similarly, we keep only the value of the assets at time t, V_t (V_{t-1} comes from solving equations for times t-1 and t-2).[20] Following BT, we assume that the firm's liabilities are due in 10 years (the implied barrier does not differ much when we let debt maturity vary between 5 and 20 years). Sophisticated multi-dimensional equation solvers fail on this task due to the very flat slope of the DOC formula in some of the arguments. For that reason, we used a mix of bisection search (the DOC option is monotonic in V_t and B)[21] and full grid search. For each of the variables, we set the following possible range of values over which we search for a solution: the barrier is bounded by zero and two times the firm's liabilities,[22] the market value of assets is bounded by the market value of equity and the market value of equity plus twice the firm's liabilities, and σ_A is bounded between $0.1*\sigma_E$ *and* σ_E. When we conduct a grid search, we impose a granularity of 100 steps. Due to the rather coarse grid (our programs still take more than 72 hours on 10 co-processors to yield results), we never find an

[20] The reader may be concerned that equations for times t-1 and t and for t and $t+1$ may yield very different values of V_t or σ_A at time t. However, this is not the case: these backed-out values are remarkably stable.

[21] Due to the flatness of the slope at certain points, a Newton-Raphson-type search does not always converge.

[22] Of course, we also require this barrier to be less than the implied market value of assets since, in the opposite case, the firm would already be defunct. As our program iterates through barrier levels that are larger than the firm's liabilities, it switches to the alternative DOC and volatility formulae to be found in the Appendix.

actual solution that zeroes out our four equations simultaneously, but only one that minimizes the sum of absolute errors $abs(E_t - DOC_t)/E_t + abs(\sigma_E - \sigma_{DOC})/\sigma_E$, where E_t is the actual equity value at time t, DOC_t the value of a down-and-out barrier call (equation (3)) with the relevant parameters at that time, σ_E the actual equity volatility at time t and σ_{DOC} the volatility as given by equation (4). We keep only those observations for which the errors were less than 5 percent in each year, both in terms of equity price and equity volatility. This is the case for 33,037 firm-years (representing 5,784 unique firms). The rejection of observations with large errors may introduce a sample bias, so we compare the descriptive statistics of both samples (60,110 and 33,037 firm-years) in terms of firm characteristics as well as overall occurrences of default. The percentage of defaulted firms is only slightly larger for the original sample: Of the 7,180 (5,784) firms in the original (remaining) sample, 1,008 (799) go bankrupt at some point, corresponding to a cumulative default rate of 14.04% (13.81%). In terms of firm characteristics, both samples show very similar summary statistics and we conclude that a sample bias is unlikely.

Table I displays summary statistics for our final sample. The average market value of assets (in the remainder: MVA) is $1.267 billion, but the median is only $135 million: a few very large firms in our sample drive the average up. The median payout ratio is 2.24%, while the median volatility of the asset process is 38%. This number is larger than the 23% reported by Leland (2002) in his summary of structural models of corporate debt, but may be due to our larger sample, that is likely to include a larger number of smaller – and riskier – firms than previous studies. Moreover, our remaining sample contains more bankruptcies than previous studies, and firms that go bankrupt typically display a very large volatility of equity (and thus volatility of implied asset value) before bankruptcy. More generally, the increased volatility in the equity market at the end of the 1990's and beginning of the 2000's yields larger implied asset volatilities for our sample. We find a median leverage ratio of 39%, consistent with the average leverage ratio for Baa-rated firms as reported in Moody's (1999). We express the leverage ratio as book value of liabilities divided by the implied MVA. Since the latter is significantly lower than the sum of market value of equity and book value of liabilities, the denominator used in previous studies, our leverage ratios are by construction larger than the ones reported in studies so far. Our barrier estimate is on average equal to 30.53% (here, as well as in the remainder of the paper, we will always express the implied barrier as a percentage of the implied MVA),[23] and its median is equal to 27.58%, which is large enough to conclude that it is significantly different from zero (p-value< 0.00001) and that the barrier option framework seems to fit equity prices better than a standard BSM approach (barrier of zero). These values are also very close to the optimal barrier levels endogenously derived in Leland and Toft (1996), who report values around 30%. However, our results also point at the fact that BT may suffer from a very large bias in their barrier estimation (their average barrier is 69% of MVA), caused by their use of book value of debt in the estimation of the MVA (and its volatility). Given a certain value of equity, the larger the MVA, the larger the implied barrier should be, *ceteris paribus*, since, for a given MVA, a larger barrier will provide bondholders with more protection and lower the value of equity. The large barrier levels uncovered by BT may thus be mainly due to the inflated MVA's they use. Another striking difference is that we find an implied barrier larger than the leverage (implying that the debt is riskfree) in only 53 cases (0.16% of our sample), while BT find such large barrier levels in most cases (as mentioned before, they report an average barrier

[23] This average corresponds to a barrier level equal to 66.16% of the book value of debt obligations (median of 62.36%), consistent with the 60% assumed by Huang and Huang (2002) and with the empirical evidence of Davydenko (2005), who finds a "median market value of debt plus equity immediately before default [equal to] 63.3% of the debt."

level – as a fraction of the MVA – for all but the last leverage decile that is greater than the upper bound of the leverage decile).

To get a better idea of the empirical differences between the various models mentioned, we show in Panel B of Table I the implied market value of assets in i) the DOC framework; ii) the standard BSM model (simultaneously backing out V_t and σ_A using equations (1) and (2)); iii) the KMV framework, assuming that the strike price of the standard call option is equal to the firm's current liabilities plus one half its long-term liabilities;[24] and iv) in the BT framework, summing the market value of equity and the book value of liabilities. For the first three models, we assume that the liabilities of the firm have a maturity of 10 years.

The MVA implied from a standard BSM framework is on average 3% lower than the one backed out in a DOC framework (with a median of \$130.2 million vs. \$135.3 million). This is not surprising: for a given MVA, the option (equity) is decreasing in the barrier but increasing in the volatility of assets. Equivalently, for a given value of equity, the value of the underlying assets will be lower when the barrier is lower (in the BSM model, the implied barrier is zero by definition) and when σ_A is larger (the median implied σ_A for BSM is 45% compared to 38% in the DOC framework). This difference, equal to \$5 million on average, is the increase in the market value of debt that bondholders enjoy thanks to more protection (the right to pull the plug early). The KMV-implied MVA is slightly lower (median of \$128.8 million, and a firm value on average 5% lower than in the DOC framework). Since the strike price of the option is lower than in the standard BSM approach by half the amount of long-term debt (but the empirical volatility σ_A is similar), the option (equity) price must be larger, for a given level of MVA, than in the BSM framework. Equivalently, for a given value of equity, the value of the underlying assets will be lower. The MVA's backed out in these structural frameworks are all much smaller than the market value of equity plus the book value of liabilities, used by BT to proxy for the MVA (median of \$182.2 million). This reflects the extent to which BT overestimates the market value of corporate debt. As a consequence, the barrier levels backed out by BT are even larger than they appear, since they are equal on average to 69% of an inflated MVA. Indeed, since the median BT "market" value of assets (\$182.2 million) is 35% larger than the median MVA implied in a DOC framework (\$135.3 million), the BT average barrier is roughly equal to 1.35*0.69=93.2% of the MVA implied in the DOC framework: the median firm in BT is by all accounts already on the verge of bankruptcy.

In Table II, we present implied barriers as a function of the firm's leverage ratio (Panel A) and asset volatility (Panel B). As expected, the implied barrier is increasing in the firm's leverage ratio (as the firm has more debt outstanding, bondholders will require more protection), with the notable exception of firms with a leverage ratio greater than 1.5. These heavily levered firms (4.83% of our sample) are firms on the verge of default and are probably governed by idiosyncratic factors that are harder to uncover. The implied barrier is also decreasing in the volatility of the assets σ_A, consistent with Leland and Toft's (1996) empirical prediction. In both cases, a Kruskal Wallis test strongly rejects the null hypothesis of random ranks across leverage and volatility bins. Scheffe comparisons confirm that the average implied barriers are significantly different across leverage and volatility bins,

[24] Since the equity (option) value is increasing in both V_t and σ_A in the BSM and KMV frameworks, we use Matlab's fminsearch function to back these two quantities out. We again require the relative error, both in terms of equity price and equity volatility, to be no larger than 5%.

with exception of the largest four leverage and largest two volatility categories, whose average implied barrier is not statistically different.

To confirm our summary statistics, we regress the implied barrier on leverage and σ_A. The coefficient estimate on σ_A is significantly negative (-0.5817 with a p-value less than 0.0001) and the coefficient estimate on leverage is significantly positive (0.1052 with a p-value of less than 0.0001), with an R^2 of 0.429 for the regression (since the correlation between leverage and implied asset volatility is equal to –0.195, one needs not be too worried about multicollinearity). Controlling for further factors in the regression such as firm size, age, or cash-richness (EBIT scaled by sales or book value of assets) does not alter our results. We may have expected a *positive* relation between barrier and asset volatility (more risky firms are more closely monitored by outsiders). However, the negative relation we uncover may not be causal, but due to the negative relation between leverage and asset volatility: lenders provide the most leverage to the safest companies, i.e., to companies with the most tangible – and least volatile – assets, but also require more protection the larger the leverage.[25]

IV. Predicting Default in a Barrier Option Framework

A. Estimating Default Probabilities

In the existing literature (e.g., Crosbie and Bohn (2002), BT, or Hillegeist et al. (2004)), a one- (three-, five-, ten-) year-ahead probability of default is computed assuming that the equity is a one- (three-, five-, ten-) year option on the firm's underlying assets. The probability of default is then equal to $1-\Phi\left(d_1-\sigma_A\sqrt{\tau}\right)$ where τ is the default horizon considered. The only difference is whether, after τ years, the "strike price" of the option is the total amount of liabilities (BSM) or the sum of current liabilities plus one half of long-term liabilities (KMV), which in turn changes the implied MVA and σ_A as of time *t*. However, such a methodology obviously yields miscalibrated probabilities since it totally overestimates true default probabilities (the firm needs only pay back its current liabilities within a year to avoid bankruptcy). More generally, it is unrealistic to assume that the firm has a life of three or five years and back out MVA and σ_A again, thus implicitly assuming that the whole debt structure is due within three or five years.

In this section, we always assume that the equity is a ten-year option on the firm's underlying assets for backing out MVA and σ_A. We then compare how different models (BSM, KMV, and the DOC frameworks) perform when trying to predict one-, three-, five-, and ten-year-ahead bankruptcies, modifying the BSM and KMV models so as to allow for early bankruptcy. More precisely, the one- (three-, five-) year-ahead probability of default is redefined as the probability of not meeting, after one (three, five) years the obligations due within that time frame (several firm-years are lost due to Compustat's incomplete listing of the maturity breakdown of liabilities). This

[25] We do not report average or median barrier levels on a year by year basis, since they do not fluctutate much. Suffice it to say that they are slightly larger in 1998 and 2002: investors seem to react to major shocks on financial indices by pricing equity with a larger implied barrier.

probability is equal to $1-\Phi\left(d_1-\sigma_A\sqrt{\tau}\right)$, where τ is the horizon considered and the strike price F is equal to the sum of liabilities due within τ years (for the KMV model: the sum of current liabilities plus half of the remaining liabilities due within τ years). As for the ten-year-ahead bankruptcy prediction, we assume that all liabilities (for the KMV model: current liabilities plus half of total long-term debt) are due within ten years.

In the DOC framework, early bankruptcy is defined intuitively as the asset value crossing the barrier at any time before the horizon considered. The probability is shown in the Appendix to be equal to

$$P(t^* \leq T \mid \mathrm{F}_t) = \Phi\left(\frac{\ln(B/V_t)-(\mu-\delta-\sigma^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)+(B/V_t)^{2(\mu-\delta)/\sigma_A^2-1}\Phi\left(\frac{\ln(B/V_t)+(\mu-\delta-\sigma^2/2)\tau}{\sigma_A\sqrt{\tau}}\right) \quad (5)$$

where t^* is the first passage time of the asset value at the bankruptcy bound B and τ=T-t is the forecast horizon considered. Note that this probability is increasing in the horizon considered, since it is a cumulative probability of reaching bankruptcy before T. In case $B<F$ (99.84% of all cases, as discussed in the previous section), we have to add the probability of not being able to meet the one- (three-, five-, ten-) year obligations, even though the firm's asset value never crossed the bankruptcy bound. This probability is given by

$$\begin{aligned}P(B \leq V_T < F, \min_{t\leq s\leq T} V_s \geq B \mid \mathrm{F}_t) &= \Phi\left(\frac{\ln\left(V_t/B\right)+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)-\Phi\left(\frac{\ln\left(V_t/F\right)+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)\\ &-(B/V_t)^{2(\mu-\delta)/\sigma_A^2-1}\left\{\Phi\left(\frac{\ln\left(B/V_t\right)+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)-\Phi\left(\frac{\ln\left(B^2/(FV_t)\right)+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)\right\},\end{aligned} \quad (6)$$

as shown in the Appendix. Of course, if $B>F$, only equation (5) applies, since if the firm cannot meet $F<B$, it must have crossed B at some prior time. This may often be the case when we consider a one- (three-, five-) year horizon since the barrier is backed out assuming a ten-year option and is typically equal to 30% of the firm's market value of assets, which may be much larger than the sum of liabilities due within one, (three, five) years, although it is less than the sum of total liabilities in 99.84% of the cases.

This second probability may not be increasing in τ. It will actually be decreasing in τ for large enough horizons, since assets grow on average at the riskless rate r under the risk-neutral probability measure, and at an even greater rate μ under the actual probability measure. As a consequence, the probability of the assets being between B and F at a certain horizon decreases with time. The total probability of default (PD) is given by the sum,

$$P(default) = 1-\Phi\left(\frac{\ln(V_t/F)+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right)+(B/V_t)^{2(\mu-\delta)/\sigma_A^2-1}\Phi\left(\frac{\ln(B^2/(V_tF))+(\mu-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right), \quad (7)$$

which may not be increasing in τ, if $B<F$ and is given by equation (5) if $B>F$. Figure 3 shows simulations of the probability of bankruptcy under parameters close to the average backed out values (r=5%, σ_A=40%, δ=3% and B=30% of asset value). The most striking fact is that even for firms with very low leverage, the DOC framework yields non-negligible probabilities of default, which the BSM framework fails to do. This is due to the last term on the RHS of equation (7) above, which is added to the BSM PD (the first two terms on the RHS). In a companion paper, we distinguish between the effects of both events of default (early and "late" bankruptcy) on corporate debt yields. We find there that the DOC framework with empirically backed out asset values and volatilities seems well-equipped to address the two major shortcomings of structural models, i.e., their inability to explain yield spreads for firms with low financial leverage and for debt with short maturity (see Jones, Mason, and Rosenfeld (1984) and

Eom, Helwege, and Huang (2004) for example). Eom, Helwege, and Huang (2004) find that a barrier option model does not quite resolve the problem of underestimation of spreads in the low end and their overestimation for risky, highly levered firms. However, they only consider early default. In Figure 4, we offer a glimpse of the difference in predicted yield spreads depending on whether we use the probability of early bankruptcy (equation (5) above, as used, for instance, by Eom, Helwege, and Huang (2004) or Giesecke (2004)), or the total probability of bankruptcy (equation (7)), which is the correct one to use. In particular, including the probability of the firm not being able to pay back its obligations at maturity even though it has never crossed the bankruptcy bound not only predicts larger yields but also seems, most importantly, to enable us to predict positive spreads at short maturities.[26] It is interesting to note, from the first graph of Figure 4, that yield spreads in the DOC framework could, in theory, become negative: as the barrier level B approaches F and σ_A increases sharply, the probability of early takeover by bondholders approaches one, with no loss given default and an expected hitting time tending to zero. Taking this case even further, if we were to accept BT's estimates of B larger than F (under the assumption, of course, that the DOC framework is the correct one), we should observe negative yield spreads, or even negative yields (in Figure 4, we keep B below F). Indeed, a misspecified DOC framework allows bondholders to receive $B>F$ (i.e., more than their dues) upon hitting the barrier, resulting in negative yields.

Bankruptcy should be predicted using actual probabilities, not risk-neutral ones. For that purpose, we need to estimate the actual drift μ of the asset process.[27] Huang and Huang (2002) find asset risk premia ranging from 4.86% to 6.46% (see their Table 2). Leland (2002) considers that "estimates of expected returns and premia require very long data series and are suspect in an environment where risks are changing through time." For that reason, he calculates μ by adding 4% to the riskfree rate. Consistent with Leland (2002), we found that estimates of μ were unstable and very dependent on the equity market performance in a given year. In particular, we found a significant number of negative estimates for μ in our test period, which is not consistent with expected returns.[28] However, we do not feel comfortable adding a constant 4% to the riskfree rate across all firms, thereby losing some discriminatory power (probabilities of default would still differ across firms due to cross-sectional differences in MVA and σ_A). We rather calculate firm-specific μ's as the sum of the relevant riskfree rate (depending on the

[26] With this expression, we mean that predicted spreads become positive for shorter maturities in the DOC framework using the total probability of bankruptcy (7) than in the BSM framework or in the DOC framework where default is defined solely as early passage (equation (5)), the framework tested by Eom, Helwege, and Huang (2003). For instance, bond spreads are already positive for three-month notes in our framework, whereas bankruptcy defined solely as early passage would start predicting positive spreads for nine-month notes. However, even with default defined as early and/or late bankruptcy, we cannot generate spreads for very short terms (i.e., less than three months). The inability of structural models with complete information to predict very short spreads has been established theoretically by Giesecke (2001).

[27] Avellaneda and Zhu (2001) are faced with the same problem when distinguishing between physical and risk-neutral distances-to-default.

[28] A shareholder-aligned manager may choose a project with a negative expected return for the sheer sake of increasing the firm's volatility, but this may not be optimal in a barrier option framework, and would certainly not be the case for the whole portfolio of projects.

bankruptcy horizon considered) and Huang and Huang's (2002) stable estimate of the asset market price of risk $(\mu-r)/\sigma_A$ of 0.15, multiplied by the firm-specific asset risk σ_A backed out for the most recent year.[29]

Finally, we require the end of the forecast horizon to still be within our sample, i.e., before December 2002. For that reason, ten-year-ahead predictions can be made only up to December 1992. We do this to avoid biasing our calibration (see next section). For instance, imagine that a firm goes bankrupt in 1998 and we are sitting in December 1994 making predictions, to be compared to actual defaults (denoted with a one) or non-defaults (denoted with a zero). The one- and three-year-ahead actual bankruptcy codes will be zero, while the five-year-ahead code will be one (the firm actually defaults within five years). But if we also enter a one for the ten-year-ahead code (the firm actually defaults within ten years from 1994), we are biasing our results towards predicting defaults, since we cannot enter a zero for all the firms that have not yet defaulted by December 2002. Hence, as the forecast horizon increases, the number of firm-years we use decreases (we have 11,008 firm-years for our ten-year-ahead predictions, less than half our sample size for the one-year-ahead predictions).

We now turn to the issue of how to judge different models based on the PDs they yield, since "true" probabilities of bankruptcy are not observable.

B. Empirical Evaluation of Default Probabilities

To evaluate the performance of a default model, the predicted default probabilities it yields must be compared with actual default occurrences. A measure commonly used (by BT in particular) is accuracy. If the model predicts a probability of default greater (lower) than a certain threshold (e.g., 0.5) and an actual default occurs (does not occur), a "hit" is recorded. In other cases, a "miss" is recorded. Accuracy is defined as the relative frequency of "hits" in the sample.

However, accuracy is a poor measure of how well the probabilities of default generated by a model approximate true default probabilities. First, significant amounts of information are ignored since any probability predictions below (above) 0.5 are treated as equal whereas predicted default probability of 0.001 and 0.499 should lead the researcher to very different conclusions. In particular, if the costs of misclassification are vastly different across firms (small vs. large potential losses) and across type I and type II errors (predicting default when the firm actually survives versus predicting survival when it actually defaults), an accuracy measure discards too much information by ignoring the severity of the potential defaults, and no fixed cutoff (in particular not 0.5) can be justified across firms. Choosing a cutoff different from 0.5 does not solve the problem. For instance, if a cutoff of 0.025 is chosen, i.e., any firm with an estimated PD greater than 0.025 is predicted to default, a prediction error is recorded if a firm with an estimated PD of 5% does not go bankrupt, even though the firm was expected to survive with probability 95%. In the words of Hillegeist *et al.* (2004), "the 5% [PD] estimate could reasonably be considered a successful prediction of solvency, rather than an unsuccessful prediction of bankruptcy." Secondly, for

[29] Adding a constant 4% asset risk premium to the riskfree rate yields slightly lower discriminatory power, but does not change the relative performance of the different models considered.

tasks with high uncertainty and very unbalanced priors (e.g., a 99.3% one-year-ahead survival rate), most models will achieve an accuracy equal to the percentage of the majority class. For instance, if we were to predict that all firms were to survive over the next year with probability, say, 0.95 or even 1, we would be wrong in 0.7% of the cases and achieve an accuracy of 99.3%. However, predicting survival or default with a constant probability across all firms is of no use for risk management purposes.

Another shortcoming of the accuracy measure is that it leads to a dichotomous decision (the firm is predicted to default if the estimated PD is above a certain cutoff). However, a loan officer in a bank (or practically any other decision-maker) usually makes a more continuous decision choice — for instance at what rate to lend money to a corporation. While this rate varies with the estimated PD, much of the rate's variance depends on the relative costs of type I and type II errors. For instance, the costs of allowing a failing firm to continue (think Enron) will typically be much larger than the costs of denying credit to a healthy firm. Since accuracy (concordance) does not allow for these continuous choices, we consider it a poor measure to evaluate the performance of a default model. Stein (2002) presents a number of alternative measures to evaluate default probabilities that focus on two nearly orthogonal, but complementary, performance criteria: the *discriminatory power* and the *calibration*.

The discriminatory power captures to what extent the model can discriminate firms that are more likely to default from firms that are less likely to default. In a perfect model, all firms that actually default are assigned a greater probability of default than any surviving firm. More generally, a model that has a larger percentage of true defaults under firms with larger predicted probabilities is considered more powerful than a model with a lower percentage. Discriminatory power thus measures how well a model *ranks* firms. A loan officer who has to choose an interest rate on a continuous scale will typically opt for the most discriminatory model.

The calibration of a model assesses whether the predicted probabilities indeed correspond to actual default frequencies. Power analysis methods only judge the ranking of the predictions but are invariant under monotonic transformation of the probabilities that leave the order unchanged. In particular, a model that only predicts default probabilities between 0.9 and 0.99 may have a very high discriminatory power (firms are very well ranked), but the predictions clearly depart from actual default frequencies. The degree of calibration is therefore an additional performance criterion that is independent of power. Following Stein (2002), it is generally easy to recalibrate a powerful model to reflect expected default frequencies, whereas improvements in model power are very hard to achieve. Thus, the model builder will choose the model with high discriminatory power (for which the calibration can easily be corrected) over the weak, well-calibrated model.

Accuracy is a function of both calibration and power but gives meaningful comparisons only if the predictions are both well-calibrated and powerful, which is unlikely to be the case for default probabilities. If the model is badly calibrated, the accuracy can be as low as the percent of the minority class (defaults) even if the model perfectly discriminates. For instance, in the case of one-year-ahead predictions, a model could perfectly rank defaulting and non-defaulting firms but yield an accuracy of only 0.7% if all probabilities of default are larger than 0.5 (in which cases the model would record 99.3% of misses). On the other hand, a model with no discriminatory power (assigning *any* constant probability of default lower than 0.5 for all firms) will still have an accuracy as high as the percent of the majority class (non-defaults; in our example: 99.3%). *Only if the accuracy is larger than the percentage of the majority class (e.g., 99.3%) can we conclude that the model has some discriminatory power*, which is rarely the case for default predictions due to the large inherent uncertainty and the very high percentage of non-defaults. Hence, accuracy is not the right measure for the relative performance of different models yielding default probabilities.

B.1. Discriminatory Power

The discriminatory power of a model denotes its ability to rank observations by increasing probabilities of default. A perfectly discriminating model would assign all defaulting firms larger probabilities of default than non-defaulting firms. A receiver operating characteristic (ROC) curve as shown in Figure 5 reflects the quality of a ranking. The curve is constructed by varying the cutoff that maps estimated PDs to class predictions (i.e., predicting default [one] if the probability is larger than the cutoff and survival [zero] otherwise) from zero to one (for accuracy, this cutoff is usually fixed at 0.5). For every cutoff, the ROC shows the true positive rate (percentage of true defaults that the model correctly classifies as defaults) on the y-axis as a function of the corresponding false positive rate (percentage of non-defaults that are mistakenly classified as defaults) on the x-axis. In particular, it tells the researcher, for *any* type I error rate (how many false positives are incurred), what fraction of actually defaulting firms are screened out of the loan market (and hence the type II error rate – how many of the true defaults the model could not detect). The ROC of a constant or entirely random prediction corresponds to the main diagonal whereas a perfect model will have a ROC that goes straight up from (0,0) to (0,1) and then across to (1,1). Given two models, the one with better ranking will display a ROC that is further to the top left than the other. The area under the curve (AUC) is commonly used as a summary statistic for the quality of a ranking and has a well-defined statistical interpretation: it represents the probability that the predicted PD for a randomly chosen defaulting firm is greater than the predicted PD for a randomly chosen surviving firm over all firms in the sample. A model with perfect ranking has an AUC of one (all defaulting firms are assigned a larger probability of bankruptcy than any surviving firm) whereas a model with constant or random predictions has an AUC of 0.5. It can be shown that the AUC is identical to the Mann-Whitney U statistic, the Wilcoxon-Mann-Whitney statistic, and the concordance index for binary outcomes, and it is arithmetically related to Somer's D rank correlation and the Gini coefficient (see Hand, 1997).[30] All ranking measures have a number of other interesting properties, including invariance under monotonically increasing transformations of the probabilities, applicability to any continuous score (not only probabilities), and independence of the class prior (contrary to accuracy).

Table III compares the AUC scores of the probabilities inferred from the DOC, BSM, and KMV structural models, and of the Altman (1968, 1993) Z- and Z"-scores for one-, three-, five-, and ten-year-ahead default predictions. The Altman Z- and Z"-scores differ from the three structural models in the sense that they do not depend on a particular forecast horizon. We include in our tests the Z"-score (developed by Altman (1993) mainly to predict bankruptcy for private firms, since it uses the book value of equity, as opposed to the Z-score that uses

[30] There is also a one-to-one relation between the ROC plot and the Cumulative Accuracy Profile (CAP) plot, preferred by Vassalou and Xing (2004).

market value of equity) because Altman (1993) considers it a better predictor of bankruptcy for non-manufacturing industrial firms in general.

The DOC framework outperforms both the BSM and the KMV frameworks in its ranking of all but the five-year-ahead predictions, and most strongly so in the one-year prediction (arguably the most important one). In the five-year-ahead default prediction, the only case where KMV outperforms the DOC framework, the latter still has a better accuracy: the much better likelihood, indicating a better calibration, more than offsets a slightly worse ranking. It is worth noting that the BSM framework is dominated by the KMV predictions for all but the ten-year-ahead predictions. However, KMV only predicts defaults up to five years ahead, and an application of the KMV framework to ten-year-ahead predictions is of academic interest only. For all relevant purposes, our data supports KMV's claim that a strike price equal to current liabilities plus half long-term debt (as opposed to total liabilities) yields a higher discriminatory power. Altman Z- and Z"-score outperform all structural models in the one-year-ahead prediction (with the former performing slightly better), but fade in longer-term predictions. In particular, these scores largely dominate structural models in the low false-positive region, which is the most important one, since no decision-maker is likely to accept more than a 5% Type I error rate. If anything, the AUC statistic (that measures the area below the whole curve) is biased against Altman's scores. However, as the forecast horizon is extended beyond one year, the Z"-score outperforms its Z counterpart, which has a worse performance than a constant probability of default for ten-year-ahead predictions (AUC of 0.416<0.5). This is consistent with our intuition that the performance of (backward-looking) accounting-based measures relative to (forward-looking) market-based measures should decrease as the forecast horizon is extended. They will be most relevant for short-term predictions, while data implied from a (relatively) efficient market – reflecting its participants' expectations of future firm performance – will become more and more relevant as the forecast horizon is extended. Figure 5 presents two ROC plots, for the one-year-ahead and for the three-year-ahead default predictions (we do not plot the performance of Altman Z"-score, since it is extremely close to the Z-score and unnecessarily garbles the figure).

The former results are difficult to interpret from the point of view of statistical significance. While there has been work on significance tests for differences in ROC and AUC by Cumming, Grace, and Phillips (1999), their approach assumes that the probability estimates coming from different models are bivariate normally distributed. This assumption is strongly violated in the case of default probabilities. The vast majority of probability estimates is close to zero, while some large probabilities (greater than 0.5) can be observed, and the distribution of the probabilities is therefore highly skewed. Rosset (2004) derives theoretical statistical properties of the AUC and evaluates significance testing of parametric approaches. His evaluation of methods of AUC comparisons strongly suggests that empirical bootstrapping, rather then assumption-based parametric approaches, provides the most reliable results. Following his conclusions, we assess the significance of performance differences between pairs of default models using empirical bootstrapping. 100 samples are drawn with replacement from the original set of firms and the AUC is calculated for each model on each sample. Differences in model performance are considered significant if zero is either below the 5th percentile or above the 95th percentile of the empirical distribution of the bootstrapped AUC differences.

The DOC framework significantly outperforms the BSM framework across the 1, 3, and 5-year-ahead predictions. DOC dominates KMV only in one-year-ahead predictions (arguably the most important one) and outperforms both Altman scores for all long-term predictions (3, 5, and 10-year horizon). In the one-year-ahead predictions Altman Z significantly outperforms all three structural models but is significantly dominated by all structural models in longer-term predictions. Altman Z" is competitive with Altman Z in the one-year-ahead and outperforms Altman Z in all longer-term predictions (3, 5, and 10 years).

B.2. Calibration

In terms of calibration, the DOC framework consistently outperforms both the BSM and the KMV analyses (no likelihood is available for Altman Z- and Z" scores since they are not probabilities). However, structural models seem in general to fare poorly in terms of calibration: they all underestimate the probability of default for relatively safe firms (PD<0.5%) and overestimate it for riskier firms (PD>0.5%). This is consistent with Eom, Helwege, and Huang's (2004) results on corporate-Treasury yield spreads. They indeed find that structural models "severely overstate the credit risk of firms with high leverage or volatility and yet suffer from a spread underprediction with safer bonds." We find that the DOC framework outperforms in that respect other structural models by having less underestimation in the low range and less overestimation in the high range. We illustrate this in Figure 6 (calibration plot), where the DOC curve is closer to the diagonal than the BSM and KMV curves.

Likelihood is very sensitive to extreme errors (predicting a PD close to zero for firms that actually go bankrupt or close to one for firms that actually survive). The BSM (KMV, resp., DOC) framework predicts a one-year ahead PD of zero (i.e., 10^{-7} or less) for 11,966 (8,794, resp., 3,859) out of 25,582 firm-years (note here how the early bankruptcy bound drastically reduces the number of predicted PDs of zero in the DOC framework). However, 30 (14, resp., 0) of these predicted sure survivors actually go bankrupt. At the other end of the spectrum, the BSM (KMV, resp., DOC) framework predicts a one-year-ahead PD larger than 50% for 514 (1,838, resp., 79) out of 25,582 firm-years. Only 12 (42, resp., 6) of these likely defaulters actually go bankrupt. These poor results at the ends of the spectrum hurt BSM's likelihood much more than KMV's, resulting in a larger log-likelihood (lower absolute value) for the latter for all but the ten-year-ahead prediction (which is of academic interest in KMV's case anyway). Both structural models severely underperform the DOC framework in terms of calibration, for all prediction horizons.

These results make it evident that all structural models need recalibration. Fortunately, recalibrating the data to fit actual default frequencies is easier than increasing the discriminatory power of a model. Recalibration can be done in a non-parametric way (as is done by KMV when they leave the contingent claim framework and remap computed distances-to-default to actual historical default frequencies) or assuming a functional relation (e.g., logistic regression) and maximizing likelihood. Since we do not have access to KMV's large proprietary database, we recalibrate our probabilities of default using logistic regression. This distribution is used (rather than, say, the normal distribution) since it places much more weight on tails, which are critical in the estimation of PDs. With this methodology, we can also derive PDs from Altman Z- and Z"-scores, thus including them in the comparison of calibration across models.

The logistic model assumes that the maximum likelihood probability of default is equal to

$$\Pr(\mathbf{1}\{Default\}=1)=\Lambda(\boldsymbol{\beta}'\mathbf{x})=e^{\boldsymbol{\beta}'\mathbf{x}}/(1+e^{\boldsymbol{\beta}'\mathbf{x}})\,, \tag{8}$$

where **1***{Default}* is an indicator function equal to one if default occurs and zero otherwise, **x** is a set of firm characteristics, and **β** is a vector of parameters to be estimated.[31] In our case of recalibration, the firm characteristics are summarized in the probability of default p estimated within the structural model (BSM, KMV, or DOC). However, independent variables in the form of probabilities are not consistent with the logit model. Thus, rather than using these probabilities directly as independent variables, we first estimate an unobserved "firm characteristic" $\hat{x}$ using the inverse logistic function, i.e., $\hat{x}=\ln(p/(1-p))$. We then estimate β by maximum likelihood using $\hat{x}$ as independent variable and actual defaults (zero or one) as dependent variable. Since $\hat{x}$ approaches negative (positive) infinity as the predicted PD approaches zero (one), we winsorize the sample so that the minimum (maximum) PD estimated within a given framework equals 10^{-7} ($1\text{-}10^{-7}$). This will result in a minimum (maximum) $\hat{x}$ equal to –16.118 (16.118).[32] To be consistent, we also winsorize the $\hat{x}$'s coming from accounting-based scores using the same cutoff values.

A model can be recalibrated using the full data set (in-sample recalibration) or, more appropriately, using a subset of historical data (training sample) to estimate β and evaluating the predictive performance of the resulting probabilities on the remaining, unused, most recent data (out-of-sample recalibration). The former evaluates the degree of miscalibration in probabilities inferred from structural models. The latter is an unbiased evaluation of how well the various models will be able to predict *future* defaults, which by definition are out-of-sample. If there is substantial cyclical variation in default rates (and this certainly was the case as defaults became much more numerous towards the end of the 90's and beginning of the 00's), the out-of-sample method is the *only* valid one. For the out-of-sample evaluation, we use the two most recent forecast years (i.e., 2001-*n* and 2002-*n*, where *n* is the forecast horizon) as test period (and report the AUC and likelihood for these two years) and all previous years (1988 to 2000-*n*) as estimation window. Since recalibration (whether in- or out-of-sample) is a monotonic transformation, the AUC statistic (quality of the ranking) will remain unchanged for a given sample. The AUC we report for out-of-sample recalibration is different only because the sample is smaller.

When evaluating PDs from Altman Z- and Z"-scores, we use those scores directly, with the original weightings as in Altman (1968, 1993) as independent variables $\hat{x}$, as opposed to re-estimating optimal weightings of the four or five accounting-based ratios. We do this following Hillegeist *et al.*'s (2004) surprising result (confirmed in our data) that the original Z-score (with weights estimated by multivariate discriminant analysis on a small sample containing 33 bankrupt firms in the period 1946-1965) outperforms the five variables it contains (i.e., a Z-score

[31] Shumway (2001) advocates including a variable that measures the secular variation in the underlying or baseline risk of bankruptcy to avoid biases when firms appear multiple times in a sample. PDs implied from structural models trivially account for that baseline risk if the general performance of the stock market is (negatively) correlated to the number of bankruptcies in the economy. For instance, in a recession, as the latter increases, the MVA implied from equity prices will decrease, thus reducing the distance-to-default and increasing the probability of bankruptcy for all firms, before and after recalibration. Our model thus takes into account the effect of macroeconomic variables on PDs through their effect on the stock market.

[32] The likelihood reported in Table 3 (i.e., on raw, non-recalibrated, data) also uses this winsorization.

with re-estimated coefficients using current data) when predicting defaults over the recent past, in spite of the Bankruptcy Reform Act of 1978 and the consequent substantial changes in the legal environment.

The likelihood of the PDs coming from Altman Z- and Z"-scores should be compared only to the likelihood of recalibrated structural PDs, and not to the likelihood of the original structural PDs, for the following reason: In order to get PDs from Altman scores, logistic regression maximizes likelihood using actual defaults (zeros and ones), whereas PDs inferred from structural models (before being recalibrated) do not take into account actual default data. Finally, since our samples for Altman scores and structural models contain a different number of observations, the correct number to compare across models is the average loglikelihood. Since all our recalibrations use only one explanatory variable (inferred from an accounting-based score or PD estimated within a structural model), there is no need to correct our results for different numbers of variables à la Vuong (1989).

After recalibration, Altman Z-score still outperforms all structural models in the one-year-ahead prediction, both in terms of ranking and calibration, especially in out-of-sample recalibration. On the other hand, the Z"-score is outperformed by all other models in out-of-sample ranking, although it is still relatively well-calibrated. The Z-score achieves better results than its Z" counterpart because it uses the market value of equity (hence its structural "flavor"), while the Z"-score only takes the book value of equity into consideration. As pointed out by Duffie and Lando (2001), in a world where outsiders cannot perfectly observe the true underlying asset value (for instance, because financial statements do not fully reflect the true asset value), additional information about the likelihood of default can be derived from any variable that is correlated with the underlying market value of assets (e.g., market value of equity). This may explain why forward-looking equity-based measures perform better than accounting-based measures as the forecast horizon is increased to three, five, and ten years, with the DOC framework outperforming all other models (including the KMV framework in the five-year-ahead prediction after out-of-sample recalibration).

Our results unambiguously indicate that accounting-based measures dominate structural models for short-term default prediction, whereas the latter are more reliable for medium- and long-term prediction. The performance of accounting-based scores is best illustrated in Figure 7 (reproduced from Saunders and Allen (2002)). It clearly shows that by January 2000 the (quarterly updated) Altman Z"-score already implied a worse rating for Enron, in spite of its fudged accounting, than the (monthly updated) KMV EDF. The latter "caught up" only in May 2001. A possible reason for the better short-term predicting of accounting-based measures is their ability to pick up information about the probability of the firm violating accounting-based debt covenants, as modeled by Core and Schrand (1999). The effects of such covenants are certainly not captured in structural models, thus limiting their performance.

Structural models take over as the forecast horizon is extended for a variety of reasons. First, they incorporate the two important dimensions of default likelihood: leverage and volatility of the asset value process. The latter is totally ignored by accounting-based scores. But two firms with the same leverage will have very different distances-to-default if they exhibit different asset volatilities, and the firm with the larger asset volatility will also typically have a larger true probability of default. For that reason, even a Z-score that takes into account the market value of equity (and thus has a structural "flavor") cannot compete with structural models for medium- and long-term default prediction, although it is still competitive for short-term prediction, where asset volatility plays less of a role. Second, structural models allow the researcher to specify an arbitrary time horizon, whereas accounting-based measures are the same for a given firm, regardless of the forecast horizon. Third and finally, the very nature of accounting may explain its inability to reliably forecast defaults more than one year ahead. As noted by Hillegeist *et al.* (2004), "financial statements are formulated under the going-concern principle, which assumes that firms will not go bankrupt. Thus, their ability to accurately and reliably assess the probability of bankruptcy will be limited by design. Additionally, the conservatism principle often causes asset values to be understated relative to their market values […], [further] limit[ing] the performance of any accounting-based [bankruptcy prediction]."

Based on our results, sound risk management should incorporate both structural models and accounting-based scores to infer a maturity structure of corporate yield spreads. A simple approach to evaluating the joint performance of accounting-based scores and structural models is to estimate a logistic model using two independent variables $\hat{x}_1$ and $\hat{x}_2$, unobserved firm characteristics implied by the probability of default coming from a structural model and by Altman Z- score respectively. This approach can only be evaluated out-of-sample since adding more variables to a model will always improve its in-sample performance.[33] Indeed, combining the firm characteristics implied by Altman Z-score with the ones inferred from the DOC-PD yields a better out-of-sample performance than either of the two taken individually in one-year-ahead default predictions. The AUC (quality of ranking) for the one-year-ahead prediction based on the combination is equal to 0.8274, as opposed to 0.8063 and 0.7865 for the Altman Z-score and the DOC-implied probability of default respectively. Unreported likelihood tests confirm that adding the Z-score to the unobserved firm characteristics implied by the PD coming from a structural model (or vice versa) increases the predictive performance of the latter in one-year-ahead – but not in longer-term – default predictions.[34]

Figure 5 gives an intuitive explanation for our results: since the one-year-ahead receiver operating characteristic curves for the Altman Z-score and the DOC intersect, combining both is tantamount to taking the convex hull of both curves.

V. Conclusion

In this paper, we model the stock price of a levered corporation as a down-and-out barrier option (as opposed to a standard call option) on the underlying assets of the firm with the barrier being a bankruptcy trigger exogenously specified in the bond covenants or a level of asset value below which nobody is willing to further invest in the firm. In so doing, we successfully explain on the theoretical side why shareholder-aligned managers would stop short of increasing the risk of the firm's operations without bound. On the empirical side, we simultaneously back out from

[33] We can also use an Akaike (1974) Information Criterion or a Schwarz (1978) Bayesian Information Criterion (BIC) to penalize models using more variables (for a comprehensive treatment of the issue in a logit framework, see Vuong, 1989). However, such a method only addresses the issue of an increased likelihood, but does not correct for the improved ranking.

[34] See also Dionne *et al.* (2006), who document that adding PDs coming from a structural model to accounting variables not only helps improve default prediction for Canadian firms, but also adds a large amount of dynamic information to explain the evolution of default probabilities over the course of a year.

equity prices the market value and the volatility of assets, as well as a firm-specific default barrier implicitly priced by the market. We find that the implicit barrier is on average equal to 30% of the firm's market value of assets (significantly different from zero) and increases (decreases) with leverage (asset volatility), consistent with the predictions of Leland and Toft (1996). This methodology enables us to outperform both the standard Black-Scholes/Merton and the KMV approaches when predicting bankruptcies one, three, five, and ten years ahead, both in terms of ranking and in terms of calibration. However, our most striking result may be that Altman Z- and Z”-scores outperform all structural models, both in terms of ranking and calibration, in the one-year-ahead default prediction, although they fare poorly as the forecast horizon is extended.

Probabilities of default coming from all three structural models tend to underestimate PDs for safe firms and overestimate PDs for highly levered and volatile firms, although modeling bondholders' right to pull the plug early drastically reduces the extent of this miscalibration. Our results confirm that PDs coming from structural models must be recalibrated to fit actual default history more closely, and partly explain KMV's decision to depart from the contingent claim framework and to use actual historical default data to map distances-to-default into default probabilities. We use logistic regression for recalibration and to obtain default probabilities from Altman Z- and Z”-scores. In particular, we advocate out-of-sample recalibration to better evaluate how well models will predict *future* defaults. Such monotonic transformations of the predicted probabilities of default do not change the relative rankings across models, and confirm the domination of Altman Z- and Z”-scores for short-term default prediction, and of the DOC framework for medium- and long-term default prediction.

Obviously, (backward-looking) accounting-based measures are most relevant for short-term bankruptcy prediction, while (forward-looking) market-based structural models are best suited for medium-and long-term default predictions. Of the latter, a barrier option framework seems to be most relevant. However, we carry out but one kind of recalibration (logistic regression). It is quite possible that other kinds of recalibration, in particular a non-parametric recalibration using proprietary data, as conducted by KMV, could redeem structural models and establish their superiority over accounting-based measures even in short-term default predictions. In the absence of such a proprietary default database, we find that combining in a logistic regression an accounting-based measure such as the Z-score with a probability of default inferred from a structural model (preferably one modeling the bondholders' right to pull the plug early) further improves short-term default predictions. On the other hand, longer-term default predictions are not enhanced by the addition of an accounting score to the PD estimated within a structural model.

Future research should extend our generalized market methodology to the pricing of corporate bonds. Extant research documents that structural models cannot explain the extent of corporate yield spreads over Treasury (see for instance Jones, Mason, and Rosenfeld (1984) and Eom, Helwege, and Huang (2003)). Kealhofer (2000) attributes this failure to faulty default probabilities and uncovers evidence that most of the variation in yield spreads over Treasury is due to variation in the probability of default (captured by KMV's EDFs), as opposed to variation in the demanded risk premium for a given level of default risk. He is further convinced that “using empirically validated default probabilities (EDFs), the option framework provides a good explanation of typical corporate debt prices” (see also Vasicek (2001) for evidence that trading bonds based on EDFs leads to abnormal profits). We introduce in this article a new, more powerful methodology– simultaneously backing out from traded equity prices the firm's market value of assets and volatility, as well as a firm-specific bankruptcy barrier implicitly priced by the market. Since the backed-out market values of debt in our paper (i.e., the difference between the implied market values of assets and the market values of equity) are significantly lower than the book values, we believe that our methodology could avoid the underestimation of corporate debt yields typical of a BSM framework, especially since a barrier option approach does not yield negligible PDs for firms with low leverage. Moreover, the DOC model's better calibration may solve the underestimation of spreads in the low end and their overestimation for risky, highly levered firms documented by Eom, Helwege, and Huang (2003), for the same reasons that it predicts defaults better than other structural models. Using recalibrated PDs may enable us to further fit actual bond prices. Of course, to avoid the temptation of data snooping, one should use out-of-sample recalibration, i.e., infer from a training sample the logistic regression coefficients to be used to determine equilibrium yield spreads on new bond issues. Another avenue of research is to generalize our methodology to imply, from equity and possibly bond market data, the extent of bankruptcy costs as a function of (true) pre-distress asset value.

Appendix

1. Pricing a Down-and-Out Barrier Option

Here is a quick reminder of the pricing of a down-and-out call option when the barrier B is strictly smaller than the strike price F and the starting value of the assets V_t, i.e., $F > B$ and $V_0 > B$ (the relationship between V_0 and F is not important). The reader who is familiar with the proof can skip this Appendix.

- Lemma A1: Let $X(s) = \sigma W(s)$, where $\{W(s)\}$ is a standard Brownian motion. Then for $\beta<0$ and $x\geq\beta$,

$$\frac{\partial}{\partial x} P(X(t) < x, \min_{s\leq t} X(s) < \beta) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(2\beta-x)^2}{2\sigma^2 t}}$$

Proof:

$$P(X(t) > x, \min_{s\leq t} X(s) < \beta) = P(X(t) < 2\beta - x, \min_{s\leq t} X(s) < \beta) = P(X(t) < 2\beta - x)$$

where the first equality results from the reflection principle and the second from the fact that $x\geq\beta$. Now, since

$$P(X(t) < x, \min_{s\leq t} X(s) < \beta) = P(\min_{s\leq t} X(s) < \beta) - P(X(t) > x, \min_{s\leq t} X(s) < \beta),$$

$$\frac{\partial}{\partial x} P(X(t) < x, \min_{s\leq t} X(s) < \beta) = -\frac{\partial}{\partial x} P(X(t) < 2\beta - x)$$

$$= -\frac{\partial}{\partial x} \int_{-\infty}^{2\beta-x} \frac{e^{-\frac{u^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} du = -\frac{\partial}{\partial x} \Phi\left(\frac{2\beta - x}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}} \varphi\left(\frac{2\beta - x}{\sigma\sqrt{t}}\right)$$

where Φ and φ denote the standard normal cumulative probability function and density function respectively. □

- Lemma A2:

Let $\{Y(t)\}$ be a Brownian motion with mean αt (with $\alpha\neq0$) and variance $\sigma^2 t$ (in the sequel: a (α,σ^2)-BM), then, for $\beta<0$ and $y\geq\beta$,

$$\frac{\partial}{\partial y} P(Y(t) < y, \min_{0\leq s\leq t} Y(s) < \beta) = e^{2\alpha\beta/\sigma^2} \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(y-2\beta-\alpha t)^2}{2\sigma^2 t}} \equiv g_t(y,\beta)$$

Proof: Let $\hat{W}(t) = W(t) - \frac{\alpha}{\sigma} t$; then, by Girsanov's theorem, $\{\hat{W}(t)\}$ is a standard Brownian motion on $[0,T]$ under the probability measure $\hat{P}$ defined by $\frac{d\hat{P}}{dP} = \exp\left\{\frac{\alpha}{\sigma} W(T) - \frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2 T\right\} \equiv Z(T)$. Hence $\sigma W(t) = \sigma\hat{W}(t) + \alpha t$ is a (α,σ^2)-BM under $\hat{P}$. Let $\underline{M}_t = \min_{0\leq s\leq t} X(s)$, where $\{X(t)\}$ is, as above, equal to $\sigma\{W(t)\}$, i.e., a $(0,\sigma^2)$-BM under the probability measure P. Then

$$\hat{P}\left(\sigma W(t) \leq y, \underline{M}_t \leq \beta\right) = \hat{E}\left[\mathbf{1}\{\sigma W(t) \leq y, \underline{M}_t \leq \beta\}\right] = E\left[Z(t)\mathbf{1}\{\sigma W(t) \leq y, \underline{M}_t \leq \beta\}\right]$$

$$= E\left[\exp\left(\frac{\alpha}{\sigma^2} X(t) - \frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2 t\right)\mathbf{1}\{X(t) \leq y, \underline{M}_t \leq \beta\}\right] = \int_{-\infty}^{y} e^{\frac{\alpha}{\sigma^2}x - \frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2 t} P(X(t) \in dx, \underline{M}_t \leq \beta)$$

where **1** is the indicator function, i.e. ***1***{*A*}=*1* if event $A \subset \Omega$ happens and *0* otherwise. This in turn is equal to, by Lemma A1, $\int_{-\infty}^{y} e^{\frac{\alpha}{\sigma^2}x-\frac{1}{2}\left(\frac{\alpha}{\sigma}\right)^2 t} \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(2\beta-x)}{2\sigma^2 t}} dx$. Differentiating with respect to *y* gives, after completing the square, the desired result. □

We assume here that the assets of the firm follow a lognormal diffusion, i.e., $dV_s = (\mu_A - \delta)V_s dt + \sigma_A V_s dW(s)$. We can rewrite this as $dV_s = (r-\delta)V_s dt + \sigma_A V_s d\tilde{W}(s)$, where $\tilde{W}(s) \equiv W(s) + [(\mu - r)/\sigma_A]s$ is, by Girsanov's theorem, a standard BM under the probability measure $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = \exp\left\{-[(\mu-r)/\sigma_A]W(T) - [(\mu-r)/\sigma_A]^2 T/2\right\}$. Hence $V_T = V_t e^{(r-\delta-\sigma_A^2/2)(T-t)+\sigma_A(\tilde{W}(T)-\tilde{W}(t))} \equiv V_t e^{Y(t,T)}$ where $\{Y(t,T)\}$ is a $\tilde{P}$-Brownian motion with drift $(r-\delta-\sigma_A^2/2)(T-t) \equiv \alpha(T-t) \equiv \alpha\tau$ and variance $\sigma_A^2\tau$ (we denote by τ the maturity *T-t* of the debt contract). In complete markets, the down-and-out option is replicable and its time-*t* price can be written as

$$
\begin{aligned}
& e^{-r\tau}\tilde{E}\left[\left(V_T - F\right)^+ \left\{1 - \mathbf{1}\left(\min_{t\le s\le T} V(s) \le B\right)\right\} \mid \mathcal{F}_t\right] \\
& = e^{-r\tau}\tilde{E}\left[\left(V_T - F\right)^+ \mid \mathcal{F}_t\right] - e^{-r\tau}\tilde{E}\left[\left(V_T - F\right)^+ \mathbf{1}\left(\min_{t\le s\le T} V(s) \le B\right) \mid \mathcal{F}_t\right] \qquad \text{(A1)}
\end{aligned}
$$

where $\tilde{E}[\cdot]$ refers to the expectation under the measure $\tilde{P}$ and $(k)^+$ denotes $\max(k,0)$. The first term of expression (A1) is the price of a standard (Black-Scholes) call option and the second of a down-and-in call option. Moreover, since the only stochastic term in *V(t)* is *Y(t,T)* as emphasized above, the conditioning on $\mathcal{F}_t$ can be dropped. We will now concentrate on the second term. Note that $\min_{t\le s\le T} V(s) \le B \Leftrightarrow \min_{t\le s\le T} Y(s) \le \beta$ where $\beta = \ln(B/V_t) < 0$. Then, by Lemma A2,[35] the time-*t* price of the down-and-in call option is:

$$
\begin{aligned}
& e^{-r\tau} \int_{\ln(F/V_t)}^{\infty} \left(V_t e^y - F\right) \tilde{P}(Y_T \in dy, \min_{t\le s\le T} Y(s) \le \beta) \\
& = e^{-r\tau} V_t \int_{\ln(F/V_t)}^{\infty} e^y g_\tau(y,\beta)dy - e^{-r\tau} F \int_{\ln(F/V_t)}^{\infty} g_\tau(y,\beta)dy \equiv e^{-r\tau}\left[V_t I_1 - F I_2\right].
\end{aligned}
$$

Now,

$$
\begin{aligned}
I_2 &= \exp\left(2\alpha\beta/\sigma_A^2\right) \int_{\ln\left(\frac{F}{V_t}\right)}^{\infty} \frac{1}{\sigma_A\sqrt{2\pi\tau}} \exp\left(-\frac{\left[y-2\beta-\alpha\tau\right]^2}{2\sigma_A^2\tau}\right) dy = \exp\left(2\alpha\beta/\sigma_A^2\right)\Phi\left(\frac{2\beta+\alpha\tau-\ln(F/V_t)}{\sigma_A\sqrt{\tau}}\right) \\
&= \left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1} \Phi\left(\frac{\ln\left[B^2/(V_t F)\right] + (r-\delta-\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}\right), \text{ and}
\end{aligned}
$$

[35] The time homogeneity of Brownian motions ensures that the density of *Y(T)* given $\mathcal{F}_t$ is the same as the density of *Y(T-t)* given $\mathcal{F}_0$. This is true whether we look at the simple density of *Y*, *f(y)*, or the joint density of *Y* and its minimum (i.e., $g_\tau(y,\beta)$).

$$I_1 = \exp\left(2\alpha\beta/\sigma_A^2\right) \int_{\ln(F/V_t)}^{\infty} \frac{e^y}{\sigma_A\sqrt{2\pi\tau}} \exp\left(-\frac{\left[y-2\beta-\alpha\tau\right]^2}{2\sigma_A^2\tau}\right) dy \overset{v=(y-2\beta)/\sigma_A\sqrt{\tau}}{=}$$

$$\exp\left(\frac{2\alpha\beta}{\sigma_A^2}+2\beta\right) \int_{\frac{\ln(F/V_t)-2\beta}{\sigma_A\sqrt{\tau}}}^{\infty} \frac{e^{v\sigma_A\sqrt{\tau}}}{\sqrt{2\pi}} \exp\left(-\left[v-\alpha\sqrt{\tau}/\sigma_A\right]^2/2\right) dv = e^{(r-\delta)\tau}\left(B/V_t\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1} \Phi\left(\frac{\ln\left(B^2/V_tF\right)+(r-\delta+\sigma_A^2/2)\tau}{\sigma\sqrt{\tau}}\right)$$

after completing the square. Putting everything together, we get the formula (3) in the text.

From equation (3) in the text,

$$\Delta_{Barrier} \equiv \frac{\partial E_t}{\partial V_t} = e^{-\delta\tau}\Phi\left(d_1\right) + e^{-\delta\tau}\left(\frac{2(r-\delta)}{\sigma_A^2}\right)\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1} \Phi\left(d_1^B\right) - V_t e^{-\delta\tau}\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1} \varphi\left(d_1^B\right)\frac{\partial d_1^B}{\partial V_t}$$
$$+Fe^{-r\tau}\left\{\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1} \varphi\left(d_1^B-\sigma_A\sqrt{\tau}\right)\frac{\partial\left(d_1^B-\sigma_A\sqrt{\tau}\right)}{\partial V_t} - \left(\frac{2(r-\delta)}{\sigma_A^2}-1\right)\frac{B}{V_t^2}\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-2} \Phi\left(d_1^B-\sigma_A\sqrt{\tau}\right)\right\}.$$

Since $B^2e^{-\delta\tau}\varphi\left(d_1^B\right) = V_tFe^{-r\tau}\varphi\left(d_1^B-\sigma_A\sqrt{\tau}\right)$, the above expression simplifies to

$$\Delta_{Barrier} = e^{-\delta\tau}\Phi\left(d_1\right) + \left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1} \left\{\frac{Fe^{-r(T-t)}}{V_t}\Phi\left(d_1^B-\sigma_A\sqrt{\tau}\right) + \left(\frac{2(r-\delta)}{\sigma_A^2}\right)\left[\frac{B^2e^{-\delta\tau}}{V_t^2}\Phi\left(d_1^B\right) - \frac{Fe^{-r\tau}}{V_t}\Phi\left(d_1^B-\sigma_A\sqrt{\tau}\right)\right]\right\}.$$

It is also worth noting that

$$\frac{\partial E_t}{\partial \sigma_A} = V_te^{-\delta\tau}\sqrt{\tau}\left[\varphi\left(d_1\right) - \left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1}\varphi\left(d_1^B\right)\right] + \frac{4(r-\delta)}{\sigma_A^3}\ln\left(\frac{B}{V_t}\right)\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1}\left[\frac{B^2e^{-\delta\tau}}{V_t}\Phi(d_1^B) - Fe^{-r\tau}\Phi\left(d_1^B-\sigma_A\sqrt{\tau}\right)\right],$$

which is not positive for all values of parameters: the barrier option framework yields bounded managerial risk choices.

In the case where $B>F$, it suffices to note that if the option ends up alive, it is also in the money. Its price is thus equal to

$$e^{-r\tau}\tilde{E}\left[\left(V_T-F\right)^+ \mathbf{1}\left(\min_{t\le s\le T} V(s)>B\right) \mid \mathrm{F}_t\right] = e^{-r\tau}\tilde{E}\left[\left(V_T-F\right)\mathbf{1}\left(\min_{t\le s\le T} V(s)>B\right) \mid \mathrm{F}_t\right].$$

Now, since $\tilde{P}(Y_T\in dy, \min_{t\le s\le T} Y(s)>\beta) = \tilde{P}(Y_T\in dy) - \tilde{P}(Y_T\in dy, \min_{t\le s\le T} Y(s)\le\beta) = f_\tau(y)dy - g_\tau(y,\beta)$, where $f_\tau(y)$ is the unconditional density of $Y(\tau)$, i.e., $f_\tau(y) = \frac{1}{\sigma_A\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{y-\alpha\tau}{\sigma_A\sqrt{\tau}}\right)^2}$, the arbitrage value of equity is, using notations previously defined:

$$e^{-r\tau}\int_{\ln(B/V_t)}^{\infty}\left(V_te^y-F\right)\tilde{P}(Y_T\in dy, \min_{t\le s\le T} Y(s)>\beta) = e^{-r\tau}\left[\int_{\ln(B/V_t)}^{\infty}\left(V_te^y-F\right)f(y)dy - \int_{\ln(B/V_t)}^{\infty}\left(V_te^y-F\right)g_\tau(y,\beta)dy\right].$$

The first integral in the RHS is the price of a BSM call option when the lower bound $\ln(F/V_t)$ is substituted with $\ln(B/V_t)$, while the second integral is computed as above. The final result is:

$$E_t = V_t e^{-\delta\tau}\left[\Phi\left(d_2\right)-\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1}\Phi\left(d_2^B\right)\right]-Fe^{-r\tau}\left[\Phi\left(d_2-\sigma_A\sqrt{\tau}\right)-\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1}\Phi\left(d_2^B-\sigma_A\sqrt{\tau}\right)\right], \quad \text{(A2)}$$

where $d_2 = \dfrac{\ln(V_t/B)+(r-\delta+\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}$ and $d_2^B = \dfrac{\ln(B/V_t)+(r-\delta+\sigma_A^2/2)\tau}{\sigma_A\sqrt{\tau}}$ (note the continuity between this formula and formula (3) in the text when $F=B$). Differentiating the above result with respect to V_t yields, after some algebra,

$$\Delta_{Barrier} \equiv \frac{\partial E_t}{\partial V_t} = e^{-\delta\tau}\Phi\left(d_2\right)+\frac{(B-F)e^{-\delta\tau}}{B\sigma_A\sqrt{\tau}}\left[\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1}\varphi\left(d_2^B\right)+\varphi\left(d_2\right)\right]+e^{-\delta\tau}\left(\frac{2(r-\delta)}{\sigma_A^2}\right)\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}+1}\Phi\left(d_2^B\right)$$
$$-\frac{Fe^{-r\tau}}{V_t}\left(\frac{2(r-\delta)}{\sigma_A^2}-1\right)\left(\frac{B}{V_t}\right)^{\frac{2(r-\delta)}{\sigma_A^2}-1}\Phi\left(d_2^B-\sigma_A\sqrt{\tau}\right).$$

2. Computing the Probability of Bankruptcy

Bankruptcy is defined as:

a) either the underlying asset value crosses the barrier B from above at any time before T, or

b) conditional on having always stayed above the barrier B, the underlying asset value is greater than B but less than F, the amount of debt owed, at maturity (date T).

We derive here actual probabilities of bankruptcy, i.e., using the actual asset drift. Let t^* be the first time the asset value process crosses the bankruptcy barrier B. Using the same notation as in Appendix 1, the probability of crossing the barrier before maturity (event a) above) is equal to:

$$P(t^* \le T) = P(\min_{t\le s\le T} Y(s) \le \beta) = P(Y(T)\le\beta)+P(\min_{t\le s\le T} Y(s)\le\beta, Y(T)>\beta)$$
$$= \int_{-\infty}^{\beta} f_\tau(y)dy + \int_{\beta}^{\infty} g_\tau(y,\beta)dy = \Phi\left(\frac{\beta-\alpha\tau}{\sigma_A\sqrt{\tau}}\right)+e^{2\alpha\beta/\sigma_A^2}\Phi\left(\frac{\beta+\alpha\tau}{\sigma_A\sqrt{\tau}}\right),$$

where the expression for $g_\tau(y,\beta)$ comes from Lemma A2. It is integrated in the same way as for the computation of I_2 in the previous Appendix, replacing $\ln(F/V_t)$ by $\ln(B/V_t)$ as the lower limit of integration. Since we are interested in actual (as opposed to risk-neutral) probabilities of default, α is now equal to $(\mu-\delta-\sigma_A^2/2)$, and the probability of early passage is equal to

$$P(t^* \le T) = \Phi\left(\frac{\ln(B/V_t)-(\mu-\delta-\sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right)+(B/V_t)^{2(\mu-\delta)/\sigma_A^2-1}\Phi\left(\frac{\ln(B/V_t)+(\mu-\delta-\sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right) \quad \text{(A3)}$$

As for event b) above, it suffices to realize that

$$P(Y(T)\in dy, \min_{t\le s\le T} Y(s)\ge\beta) = P(Y(T)\in dy) - P(Y(T)\in dy, \min_{t\le s\le T} Y(s)<\beta) = f_\tau(y)dy - g_\tau(y,\beta)dy$$

to write the probability of event b) as:

$$P(B \le V_T < F, \min_{t \le s \le T} Y(s) \ge \beta) = P(B \le V_t e^{Y(\tau)} < F, \min_{t \le s \le T} Y(s) \ge \beta) = \int_{\beta}^{\ln(F/V_t)} f_\tau(y)dy - \int_{\beta}^{\ln(F/V_t)} g_\tau(y,\beta)dy$$

$$= \left\{ \int_{\beta}^{\infty} f_\tau(y)dy - \int_{\ln(F/V_t)}^{\infty} f_\tau(y)dy - \int_{\beta}^{\infty} g_\tau(y,\beta)dy + \int_{\ln(F/V_t)}^{\infty} g_\tau(y,\beta)dy \right\}.$$

Those four integrals are integrated in much the same way the integral I_2 is computed in the previous Appendix for integrals involving $g_\tau(y,\beta)$, or as $\Phi(d_2)$ is computed in the Black-Scholes case for integrals involving $f_\tau(y)$. We come to the result:

$$P(B \le V_T < F, \min_{t \le s \le T} Y(s) \ge \beta) = \Phi\left(\frac{\alpha\tau - \beta}{\sigma_A\sqrt{\tau}}\right) - \Phi\left(\frac{\ln(V_t/F) + \alpha\tau}{\sigma_A\sqrt{\tau}}\right) - e^{2\alpha\beta/\sigma_A^2}\left\{\Phi\left(\frac{\alpha\tau + \beta}{\sigma_A\sqrt{\tau}}\right) - \Phi\left(\frac{\ln(B^2/(FV_t)) + \alpha\tau}{\sigma_A\sqrt{\tau}}\right)\right\}$$

$$= \Phi\left(\frac{\ln(V_t/B) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right) - \Phi\left(\frac{\ln(V_t/F) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right)$$

$$-(B/V_t)^{2(\mu-\delta)/\sigma_A^2 - 1}\left\{\Phi\left(\frac{\ln(B/V_t) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right) - \Phi\left(\frac{\ln(B^2/(FV_t)) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right)\right\} \quad \text{(A4)}$$

Summing equations (A4) (bankruptcy at maturity of the debt contract) and (A3) (early bankruptcy), we get the formula for the probability of bankruptcy:

$$P(bankruptcy) = 1 - \Phi\left(\frac{\ln(V_t/F) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right)$$

$$+(B/V_t)^{2(\mu-\delta)/\sigma_A^2 - 1}\Phi\left(\frac{\ln(B^2/(V_tF)) + (\mu - \delta - \sigma_A^2/2)(T-t)}{\sigma_A\sqrt{T-t}}\right) \quad \text{(A5)}$$

The first two terms (RHS on the first line) correspond to the standard Black-Scholes case (probability of not finishing in the money); hence, due to the probability of crossing the barrier at any time before the maturity of the debt contract (probability (A3) above), the PD in our barrier option framework exceeds the BSM PD.

NB: The complement of the PD is the probability of the option finishing in the money. Indeed the reader will notice that this complement probability is the factor of $-Fe^{-r(T-t)}$ - with μ replaced by r -in equation (3), the pricing formula for the DOC option. This was to be expected, since being long a DOC option is equivalent to being long a down-and-out asset-or-nothing option and short F down-and-out cash-or-nothing options, the value of which is the discounted risk-neutral probability of finishing in the money (Arrow-Debreu price of the solvent states).

In the case where $B>F$, bankruptcy is defined solely as an early crossing of the barrier, and the probability of bankruptcy is given by equation (A3), which is the same as BT's equation (3). It is also the complement of the factor of $-Fe^{-r(T-t)}$ – with μ replaced by r - in the relevant pricing equation (A2).

References

Altman, Edward I., 1968. Financial ratios, discriminant analysis, and the prediction of corporate bankruptcy. Journal of Finance 4, 589-609.

Altman, Edward I., 1993. Corporate Financial Distress and Bankruptcy (second edition). Wiley Finance, New York.

Altman Edward I., and Kishore, Velore M., 1996. Almost everything you wanted to know about recoveries on defaulted bonds. Financial Analysts Journal 52, 57-64.

Anderson, R., and Sundaresan, Suresh, 1996. Design and valuation of debt contracts. Review of Financial Studies 9, 37-68.

Andrade, Gregory, and Kaplan, Steven, 1998. How costly is financial (not economic) distress? Evidence from highly levered transactions that became distressed. Journal of Finance 53, 1443-1493.

Avellaneda, Marco, and Jingyi Zhu, 2001. Modeling the distance-to-default process of a firm. *RISK Magazine*.

Barclay, Michael J., and Smith, Clifford W., 1995. The maturity structure of corporate debt. Journal of Finance 50, 609-631.

Bensoussan, A., Crouhy, Michel, and Galai, Dan, 1995. Stochastic equity volatility and the capital structure of the firm. In: Howison, S. D., Kelly, F.P., and Wilmott P. (Editors), Mathematical Models in Finance, The Royal Society/Chapman and Hall, pp. 81-92.

Bhanot, Karan, 2003. Pricing corporate bonds with rating-based covenants. Journal of Fixed Income 12, 57-64.

Black, Fisher, and Scholes, Myron, 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637-659.

Black, Fisher, and Cox, John C., 1976. Valuing corporate securities: Some effects of bond indenture provisions. Journal of Finance 31, 351-367.

Brockman, Paul, and Turtle, Harry J., 2003. A barrier option framework for corporate security valuation. Journal of Financial Economics 67, 511-529.

Chesney, Marc, Jeanblanc-Piqué, Monique , and Yor, Marc, 1995. Brownian excursions and Parisian barrier options. Advances in Applied Probability 29, 165-184.

Chesney, Marc, and Gibson-Asner, Rajna, 1999. The investment policy and the pricing of equity in a levered firm: A re-examination of the contingent claims valuation approach. European Journal of Finance 5, 95-107.

Collin-Dufresne, Pierre, and Goldstein, Robert, 2001. Do credit spreads reflect stationary leverage ratios? Journal of Finance 56, 1929-1957.

Core, J., and Schrand, C., 1999. The effect of accounting-based debt covenants on equity valuation. Journal of Accounting and Economics 27, 1-34.

Crosbie, Peter J., and Bohn, Jeffrey R., 2002. Modeling default risk. KMV Corporation, San Francisco.

Cummins, J. David, Martin F. Grace, and Richard D. Phillips, 1999. Regulatory solvency prediction in property-liability insurance: Risk-based capital, audit ratios, and cash flow simulation. Journal of Risk and Insurance 66, 417-458.

Davydenko, Sergei A., 2005. When do firms default? A study of the default boundary. Working paper, London Business School.

Dichev, I., 1998. Is the risk of bankruptcy a systematic risk? Journal of Finance 53, 1131-1147.

Dionne, Georges, Sadok Laajimi, Sofiane Mejri, and Madalina Petrescu, 2006. Estimation of the default risk of publicly traded Canadian companies. Working paper 06-05, HEC Montréal.

Duan, J.-C,. 1994. Maximum likelihood estimation using price data of the derivative contract. Mathematical Finance 4, 155-167.

Duffee, Gregory, 1998. The relation between Treasury yields and corporate bond yield spreads. Journal of Finance 53, 2225-41.

Duffie, Darrell, and Lando, David, 2001. Term structure of credit spreads with incomplete accounting information. Econometrica 69, 633-664.

Eberhart, Allan C., Moore W. T., and Roenfeldt, R. L., 1990. Security pricing and deviations from the absolute priority rule in bankruptcy proceedings. Journal of Finance 45, 1457-69.

Eom, Young Ho, Helwege, Jean, and Huang, Jing-zhi, 2004. Structural models of corporate bond pricing: An empirical analysis. Review of Financial Studies 17, 499-544.

Ericsson, Jan, and Reneby, Joel, 2005. Estimating structural bond pricing models. Journal of Business 78, 707-735.

Fan, Hua, and Sundaresan, Suresh, 2000. Debt valuation, renegotiation, and optimal dividend policy. Review of Financial Studies 13, 1057-1099.

François, Pascal, and Morellec, Erwan, 2003. Capital structure and asset prices: some effects of bankruptcy procedures. Journal of Business, forthcoming.

Franks, Julian R., and Torous, Walter N., 1989. An empirical investigation of U.S. firms in reorganization. Journal of Finance 44, 747-769.

Franks, Julian R., and Torous, Walter N., 1994. A comparison of financial recontracting in distressed exchanges and Chapter 11 reorganizations. Journal of Financial Economics 35, 349-370.

Galai, Dan, and Masulis, Ronald, 1976. The option pricing model and the risk factor of stock. Journal of Financial Economics 3, 53-81.

Galai, Dan, Raviv, Alon, and Wiener, Zvi, 2005. Liquidation triggers and the valuation of equity and debt. Unpublished working paper, Hebrew University Jerusalem.

Geske, Robert, 1978. The valuation of compound options. Journal of Financial Economics 7, 63-81.

Giesecke, Kay, 2006. Default and information. *Journal of Economic Dynamics and Control* 30, 2281-2303.

Giesecke, Kay, 2004. Credit risk modeling and valuation: An introduction. Forthcoming in David Shimko (Editor): Credit Risk: Models and Management, vol. 2, Riskbooks, London.

Gillan, Stuart L., and Martin, John D., 2004. Financial engineering, corporate governance, and the collapse of Enron. Unpublished working paper, University of Delaware.

Guedes, Jose, and Opler, Tim, 1996. The determinants of the maturity of corporate debt issues. Journal of Finance 51, 1809-33.

Hand, David, 1997. Construction of Assessment of Classification Rules. Wiley Series in Probability and Statistics, New York.

Helwege, Jean, 1999. How long do junk bonds spend in default? Journal of Finance 54, 341-357.

Hillegeist, Stephen A., Keating, Elizabeth K., Cram, Donald P., and Lundstedt, Kyle G., 2004. Assessing the probability of bankruptcy. Review of Accounting Studies 9, 5-34.

Huang, Jing-Zhi, Huang, Ming, 2002. How much of the corporate-Treasury yield spread is due to credit risk? Working paper, Graduate School of Business, Stanford University.

Hull, John, Izzy Nelken, and Alan White, 2004. Merton's model, credit risk, and volatility skews. Journal of Credit Risk 1, 1-27.

Jensen, Michael C., Meckling, William H. 1976. Theory of the firm: Managerial behavior, agency costs and ownership structure. Journal of Financial Economics 3, 305-360.

John, Kose, and Brito, Jose A., 2000. Risk-avoidance induced by risky debt. Unpublished working paper, Stern School of Business, New York University.

Jones, E. Philip, Mason, Scott P., and Rosenfeld, Eric. 1984. Contingent claims analysis of corporate capital structures: An empirical investigation. Journal of Finance 39, 611-625.

Kealhofer, Stephen, 2000. The quantification of credit risk. KMV Corporation, San Francisco.

Kim, In Joon, Ramaswamy, Krishna, and Sundaresan, Suresh, 1993. Does default risk in coupons affect the valuation of corporate bonds? A contingent claims model. Financial Management 22, 117-131.

Layish, D. N., 2001. A monitoring role for deviations from absolute priority in bankruptcy resolution. Unpublished dissertation thesis, Baruch College, City University of New York.

Leland, Hayne E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49, 987-1019.

Leland, Hayne E., 2002. Predictions of expected default frequencies in structural models of debt. Unpublished working paper, Haas School of Business, University of California at Berkeley.

Leland, Hayne E., and Toft, Klaus Bjerre, 1996. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. Journal of Finance 51, 987-1019.

Longstaff, Francis A., and Schwartz, Eduardo S., 1995. A simple approach to valuing risky fixed and floating rate debt. Journal of Finance 50, 789-819.

Mella-Barral, Pierre, and Perraudin, William 1997. Strategic debt service. Journal of Finance 52, 531-556.

Mello Antonio S., and Parsons, John E., 1995. Maturity structure of a hedge matters: Lessons from the Metallgesellschaft debacle. Journal of Applied Corporate Finance 8, 106-120.

Merton, Robert C., 1974. On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance 29, 449-470.

Moody's Investors Services, 1999. The evolving meaning of Moody's bond ratings, Special Comment.

Moraux, Franck, 2002. Valuing corporate liabilities when the default threshold is not an absorbing barrier. Working paper, Université de Rennes I.

Nielsen, Lars Tyge, Saa-Requejo, Jesus, and Santa-Clara, Pedro, 1993. Default risk and interest rate risk: the term structure of default spreads. Working paper, INSEAD.

Pan, Yonghua, 1999. Design and valuation of corporate securities with strategic debt service and asymmetric information. International Journal of Theoretical and Applied Finance 2, 201-219.

Reisz, Alexander S., 2001. Hitting the barrier, revisiting the contingent claim framework. Working paper, Center for Financial Studies, Frankfurt, Germany.

Rosset, Saharon, 2004. Model selection via the AUC. International Conference on Machine Learning (ICML-04).

Saunders, Anthony, and Allen, Linda, 2002. Credit Risk Measurement (second edition). Wiley Finance, New York.

Shumway, Tyler, 2001. Forecasting bankruptcy more accurately: A simple hazard model. Journal of Business 74, 101-124.

Stein, Roger, 2002. Benchmarking default prediction models: Pitfalls and remedies in model validation. Technical report # 020305, Moody's Investor Services.

Stohs, Mark Hoven, and Mauer, David C., 1996. The determinants of corporate debt maturity structure. Journal of Business 69, 279-312.

Stulz, René, 2004. Should we fear derivatives? Journal of Economic Perspectives, forthcoming.

Vasicek, Oldrich Alfons, 2001. EDF™ credit measure and corporate bond pricing. KMV Corporation, San Francisco.

Vassalou, Maria, and Xing, Yuhang, 2004. Default risk in equity returns. Journal of Finance 54, 831-868.

Vuong, Q., 1989. Likelihood ratio tests for model selection and non-nested hypotheses. Econometrica 57, 307-333.

Weiss, L. A., 1990. Bankruptcy resolution: direct costs and valuation of priority of claims. Journal of Financial Economics 27, 285-314.

Wong, Hoi Ying, and Tsz Wang Choi, 2006. Estimating default barriers from market information. Working Paper, Chinese University of Hong Kong.

Table I

Descriptive Statistics of the Firm Sample During the Period 1988-2002

This table presents summary statistics for our remaining sample of 33,238 non-bankrupt and 799 bankrupt firm-years (corresponding to 5,784 unique industrial firms) for which we could back out implicit bankruptcy barriers. Payout ratio denotes (dividends + interest expense + stock repurchases)/implied market value of assets (MVA); leverage denotes total liabilities/implied MVA; one- (three-, five-) year leverage denotes the liabilities due within one (three, five) years/implied MVA; implied barrier is the implied barrier as a fraction of the implied MVA or total liabilities (TL); CL denotes current liabilities; and LTD denotes long-term debt.

Panel A: General

Average interest rate	1-year	3-year	5-year	10-year
	0.0543	0.602	0.0632	0.0664

Variable	Number of observations	Average	Median	Standard deviation	Minimum	Maximum
Market value of equity (in millions)	33,037	1,142.10	108.08	7,119.31	0.1749	274,429.9
Annualized equity volatility	33,037	0.5926	0.5525	0.2974	0.0626	1.5998
Total liabilities (in millions)	33,037	390.76	49.736	1,303.80	0.0050	33,528.0
Book value of assets	33,037	701.43	119.43	2,284.45	0.1260	73,037.0
Implied market value of assets	33,037	1,267.06	135.31	6,775.52	0.2495	227,981.3
Implied (annual) asset volatility	33,037	0.4358	0.3813	0.2619	0.0346	1.5791
Payout ratio	33,037	0.0338	0.0224	0.0460	0.0000	2.3112
Leverage	33,037	0.5162	0.3900	0.4722	0.0001	14.017
Implied barrier (% of MVA)	33,037	0.3053	0.2758	0.2447	0.0000	0.9682
Implied barrier (% of TL)	33,037	0.6616	0.6236	0.3977	0.0000	1.1402
One-year leverage	25,582	0.3088	0.2233	0.3061	0.0001	11.090
Three-year leverage	20,256	0.3938	0.2863	0.3712	0.0035	12.682
Five-year leverage	16,514	0.4654	0.3492	0.4146	0.0056	13.141

Panel B: Comparison of implied market value of assets using different models

Implied MVA using …						
DOC framework	33,037	1,267.06	135.31	6,775.52	0.2495	247,282.3
Black-Scholes-Merton (strike=total liabilities)	33,037	1,225.86	130.17	7,415.97	0.2730	232,995.2
KMV: BSM framework (strike=CL+LTD/2)	33,037	1,202.68	128.77	7,382.49	0.2451	227,981.0
BT: MVA=MVE+BVD	33,037	1,532.86	182.24	7,982.90	0.3451	285,613.9
Relative errors						
(DOC-BSM)/DOC	33,037	0.0325	0.0397	0.2390	-0.5363	0.5446
(DOC-KMV)/DOC	33,037	0.0508	0.0565	0.2864	-0.3927	0.7460
(DOC-BT)/DOC	33,037	-0.2098	-0.2432	0.4772	-20.669	0.1179

Table II

Descriptive Statistics for the Implied Barrier During the Period 1988-2002

This table presents summary statistics for the implied bankruptcy barriers per leverage bin and per (implied) asset volatility bin, based on 32,238 non-bankrupt firm-years and 799 bankrupt firm-years, corresponding to 5,784 unique industrial firms.

Leverage ratio (total liabilities/implied MVA)	Number of observations	Average	Standard deviation	Student t-statistic	p-value
Panel A: Implied barrier (as a fraction of implied market value of assets) and leverage					
Leverage≤0.1	1,922	0.0240	0.0294	35.79	0.0001
0.1<Leverage≤0.2	2,991	0.0771	0.0650	64.87	0.0001
0.2<Leverage≤0.3	3,499	0.1479	0.0973	89.91	0.0001
0.3<Leverage≤0.4	3,576	0.2194	0.1265	103.72	0.0001
0.4<Leverage≤0.5	3,283	0.2833	0.1532	105.96	0.0001
0.5<Leverage≤0.6	3,099	0.3493	0.1778	109.36	0.0001
0.6<Leverage≤0.7	2,907	0.4138	0.1990	112.11	0.0001
0.7<Leverage≤0.8	2,535	0.4443	0.2255	99.20	0.0001
0.8<Leverage≤0.9	2,170	0.4683	0.2490	87.61	0.0001
0.9<Leverage≤1	1,605	0.4700	0.2506	75.14	0.0001
1.0<Leverage≤1.5	3,854	0.4770	0.2556	105.85	0.0001
1.5<Leverage	1,596	0.3493	0.2358	59.18	0.0001
Panel B: Implied barrier (as a fraction of implied market value of assets) and asset volatility					
$\sigma_A \leq 0.1$	1,205	0.8344	0.1989	145.62	0.0001
$0.1 < \sigma_A \leq 0.2$	5,113	0.7169	0.2152	238.21	0.0001
$0.2 < \sigma_A \leq 0.3$	6,153	0.5762	0.1822	248.07	0.0001
$0.3 < \sigma_A \leq 0.4$	4,884	0.4611	0.1641	196.37	0.0001
$0.4 < \sigma_A \leq 0.5$	4,171	0.3706	0.1527	156.74	0.0001
$0.5 < \sigma_A \leq 0.6$	3,588	0.2928	0.1348	130.11	0.0001
$0.6 < \sigma_A \leq 0.7$	2,685	0.2457	0.1246	102.18	0.0001
$0.7 < \sigma_A \leq 0.8$	1,906	0.1842	0.1087	73.98	0.0001
$0.8 < \sigma_A \leq 0.9$	1,265	0.1286	0.0981	46.62	0.0001
$0.9 < \sigma_A \leq 1.0$	818	0.0975	0.1003	27.80	0.0001
$1.0 < \sigma_A$	1,249	0.0629	0.0652	34.09	0.0001

Table III

Comparative Performance of Bankruptcy Predictions During the Period 1988-2002

This table presents the comparative performance of three structural models (Black-Scholes/Merton, KMV, and barrier [DOC]) and Altman Z- and Z"-scores for one-, three-, five- and ten-year-ahead default predictions for the time period 1988-2002, based on 32,238 non-bankrupt firm-years and 799 bankrupt firm-years. A *n*-year-ahead default prediction is only made for firm-years up to 2002-*n*. The reported results use actual, not risk-neutral, probabilities, with an asset drift equal to the riskfree rate plus 0.15 times the backed-out volatility of the firm's asset process. The area under the receiver operating characteristic curve measures the ability of a model to correctly rank firms, the log-likelihood estimates the goodness of fit (how well predicted probabilities of default proxy for actual default rates) and the accuracy is the percentage of correctly predicted bankruptcies and survivals. Reported log-likelihoods are based on a winsorized sample, so that the minimum (maximum) predicted probability of default is set equal to 10^{-7} ($1-10^{-7}$).
Significant differences in AUC between DOC and BSM are identified by †, between DOC and KMV by ‡, and between DOC and the better of the two Altman scores by * (the symbol is entered next to the AUC of the dominating model). Significance of the performance differences is assessed using empirical bootstrapping. 100 samples of firms are drawn with replacement and the differences between the AUCs are calculated for each sample. Differences are considered significant if the 5th percentile of the empirical distribution of differences is above zero or if the 95th percentile is below zero.

Panel A: 1 Year					
Model	**BSM**	**KMV**	**DOC**	**Altman Z-score**	**Altman Z''-score**
Number of Observations	25,582	25,582	25,582	22,462	22,462
Prior Survival Rate	0.9930	0.9930	0.9930	0.9944	0.9944
Area Under ROC	0.7170	0.7478	0.7636†‡	0.7794*	0.7769
Accuracy (Concordance)	0.9738	0.9244	0.9903	NA	NA
Log-likelihood	-9,819.17	-5,483.97	-4,486.29	NA	NA
Panel B: 3 Year					
Number of Observations	18,216	18,216	18,216	15,892	15,892
Prior Survival Rate	0.9515	0.9515	0.9515	0.9555	0.9555
Area Under ROC	0.7369	0.7597	0.7625†*	0.6890	0.7279
Accuracy (Concordance)	0.9063	0.8542	0.9092	NA	NA
Log-likelihood	-11,804.74	-10,090.41	-9,200.75	NA	NA
Panel C: 5 Year					
Number of Observations	12,977	12,977	12,977	11,206	11,206
Prior Survival Rate	0.9238	0.9238	0.9238	0.9269	0.9269
Area Under ROC	0.7480	0.7663	0.7646†*	0.6483	0.6831
Accuracy (Concordance)	0.8789	0.8321	0.8878	NA	NA
Log-likelihood	-9,289.16	-8,672.11	-7,653.99	NA	NA
Panel C: 10 Year					
Number of Observations	11,008	11,008	11,008	9,691	9,691
Prior Survival Rate	0.8394	0.8394	0.8394	0.8459	0.8459
Area Under ROC	0.6773	0.6739	0.6823*	0.4160	0.6615
Accuracy (Concordance)	0.7838	0.7508	0.7995	NA	NA
Log-likelihood	-12,617.41	-12,691.42	-12,409.79	NA	NA

Table IV

Comparative Performance of Recalibrated Bankruptcy Predictions During the Period 1988-2002

This table presents the comparative performance of three structural models (Black-Scholes/Merton, KMV, and barrier [DOC]) and Altman Z- and Z"-scores for one-, three-, five- and ten-year-ahead default predictions after recalibration for the time period 1988-2002, based on 32,238 non-bankrupt firm-years and 799 bankrupt firm-years. A n-year-ahead default prediction is only made for firm-years up to 2002-n. The reported results use actual, not risk-neutral, probabilities, with an asset drift equal to the riskfree rate plus 0.15 times the backed-out volatility of the firm's asset process. The area under the receiver operating characteristic curve (AUC) measures the ability of a model to correctly rank firms, the average log-likelihood estimates the goodness of fit (how well predicted probabilities of default proxy for actual default rates) and the accuracy is the percentage of correctly predicted bankruptcies and survivals.
In-sample recalibration uses the full data set to reestimate probabilities, while out-of-sample recalibration uses a subset of historical data to estimate β and evaluates the predictive performance of the resulting probabilities on the remaining, unused, most recent data. For the latter, we use the two most recent forecast years (i.e., 2001-n and 2002-n, where n is the forecast horizon) as test period (for which we report AUC and log-likelihood) and all previous years (1988 to 2000-n) as estimation window. Reported log-likelihoods are based on a winsorized sample, so that the minimum (maximum) predicted probability of default is set equal to 10^{-7} ($1-10^{-7}$).
Significant differences in AUC between DOC and BSM are identified by †, between DOC and KMV by ‡, and between DOC and the better of the two Altman scores by * (the symbol is entered next to the AUC of the dominating model). Significance of the performance differences is assessed using empirical bootstrapping. 100 samples of firms are drawn with replacement and the differences between the AUCs are calculated for each sample. Differences are considered significant if the 5th percentile of the empirical distribution of differences is above zero or if the 95th percentile is below zero.

Panel A: 1 Year; estimation period: 1988-1999; evaluation period: 2000-2001					
Model	**BSM**	**KMV**	**DOC**	**Altman Z-score**	**Altman Z''-score**
In-sample recalibration					
Number of Observations	25,582	25,582	25,582	22,462	22,462
Prior Survival Rate	0.9930	0.9930	0.9930	0.9944	0.9944
Area Under ROC	0.7170	0.7478	0.7636†‡	0.7794*	0.7769
Accuracy (Concordance)	0.9930	0.9930	0.9930	0.9942	0.9943
Log-likelihood	-2,033.54	-2,006.28	-1,992.04	-1,431.87	-1,461.10
Average Log-likelihood	-0.0795	-0.0784	-0.0779	-0.0637	-0.0650
Out-of-sample recalibration					
Number of Observations	2,761	2,761	2,761	2,456	2,456
Prior Survival Rate	0.9931	0.9931	0.9931	0.9939	0.9939
Area Under ROC	0.7308	0.7653	0.7865†‡	0.8063*	0.7268
Accuracy (Concordance)	0.9931	0.9931	0.9931	0.9927	0.9935
Log-likelihood	-215.88	-211.03	-208.77	-174.51	-177.81
Average Log-likelihood	-0.0782	-0.0764	-0.0756	-0.0711	-0.0724
Panel B: 3 Year; estimation period: 1988-1997; evaluation period: 1998-1999					
In-sample recalibration					
Number of Observations	18,216	18,216	18,216	15,892	15,892
Prior Survival Rate	0.9515	0.9515	0.9515	0.9555	0.9555
Area Under ROC	0.7369	0.7597	0.7625$^{\dagger *}$	0.6890	0.7279
Accuracy (Concordance)	0.9515	0.9515	0.9515	0.9554	0.9550
Log-likelihood	-6,513.08	-6,437.45	-6,421.25	-5,617.30	-5,493.12
Average Log-likelihood	-0.3575	-0.3534	-0.3525	-0.3534	-0.3457
Out-of-sample recalibration					
Number of Observations	2,592	2,592	2,592	2,267	2,267
Prior Survival Rate	0.9117	0.9117	0.9117	0.9153	0.9153
Area Under ROC	0.7380	0.7664	0.7795$^{\dagger *}$	0.7055	0.6997
Accuracy (Concordance)	0.9117	0.9117	0.9117	0.9162	0.9127
Log-likelihood	-1491.61	-1,468.16	-1,444.00	-1,361.07	-1,385.83
Average Log-likelihood	-0.5755	-0.5664	-0.5571	-0.6004	-0.6113
Panel C: 5 Year; estimation period: 1988-1995; evaluation period: 1996-1997					
In-sample recalibration					
Number of Observations	12,977	12,977	12,977	11,206	11,206

Prior Survival Rate	0.9238	0.9238	0.9238	0.9269	0.9269
Area Under ROC	0.7480	0.7663	0.7646$^{\dagger *}$	0.6483	0.6831
Accuracy (Concordance)	0.9238	0.9238	0.9238	0.9267	0.9259
Log-likelihood	-6,309.83	-6,208.11	-6,229.29	-5,697.07	-5,623.45
Average Log-likelihood	-0.4862	-0.4784	-0.4800	-0.5084	-0.5018
Out-of-sample recalibration					
Number of Observations	2,564	2,564	2,564	2,204	2,204
Prior Survival Rate	0.8768	0.8768	0.8768	0.8793	0.8793
Area Under ROC	0.7375	0.7573	0.7701$^{\dagger *}$	0.6538	0.6578
Accuracy (Concordance)	0.8768	0.8768	0.8768	0.8789	0.8784
Log-likelihood	-1,848.85	-1,841.87	-1,808.52	-1,695.48	-1,678.27
Average Log-likelihood	0.7211	-0.7184	-0.7054	-0.7693	-0.7615
Panel C: 10 Year; estimation period: 1988-1990; evaluation period: 1991-1992					
In-sample recalibration					
Number of Observations	11,008	11,008	11,008	9,691	9,691
Prior Survival Rate	0.8394	0.8394	0.8394	0.8459	0.8459
Area Under ROC	0.6773	0.6739	0.6823*	0.4160	0.6615
Accuracy (Concordance)	0.8394	0.8394	0.8394	0.8459	0.8457
Log-likelihood	-9,258.09	-9,379.98	-9,271.87	-8,327.99	-8,024.46
Average Log-likelihood	-0.8410	-0.8521	-0.8423	-0.8501	-0.8280
Out-of-sample recalibration					
Number of Observations	4,382	4,382	4,382	3,859	3,859
Prior Survival Rate	0.8505	0.8505	0.8505	0.8515	0.8515
Area Under ROC	0.6806	0.6682	0.6934$^{\ddagger *}$	0.5426	0.6373
Accuracy (Concordance)	0.8505	0.8505	0.8505	0.8515	0.8518
Log-likelihood	-3,538.96	-3,582.57	-3,524.37	-3,322.49	-3,134.59
Average Log-likelihood	-0.8076	-0.8176	-0.8043	-0.8610	-0.8123

Figures

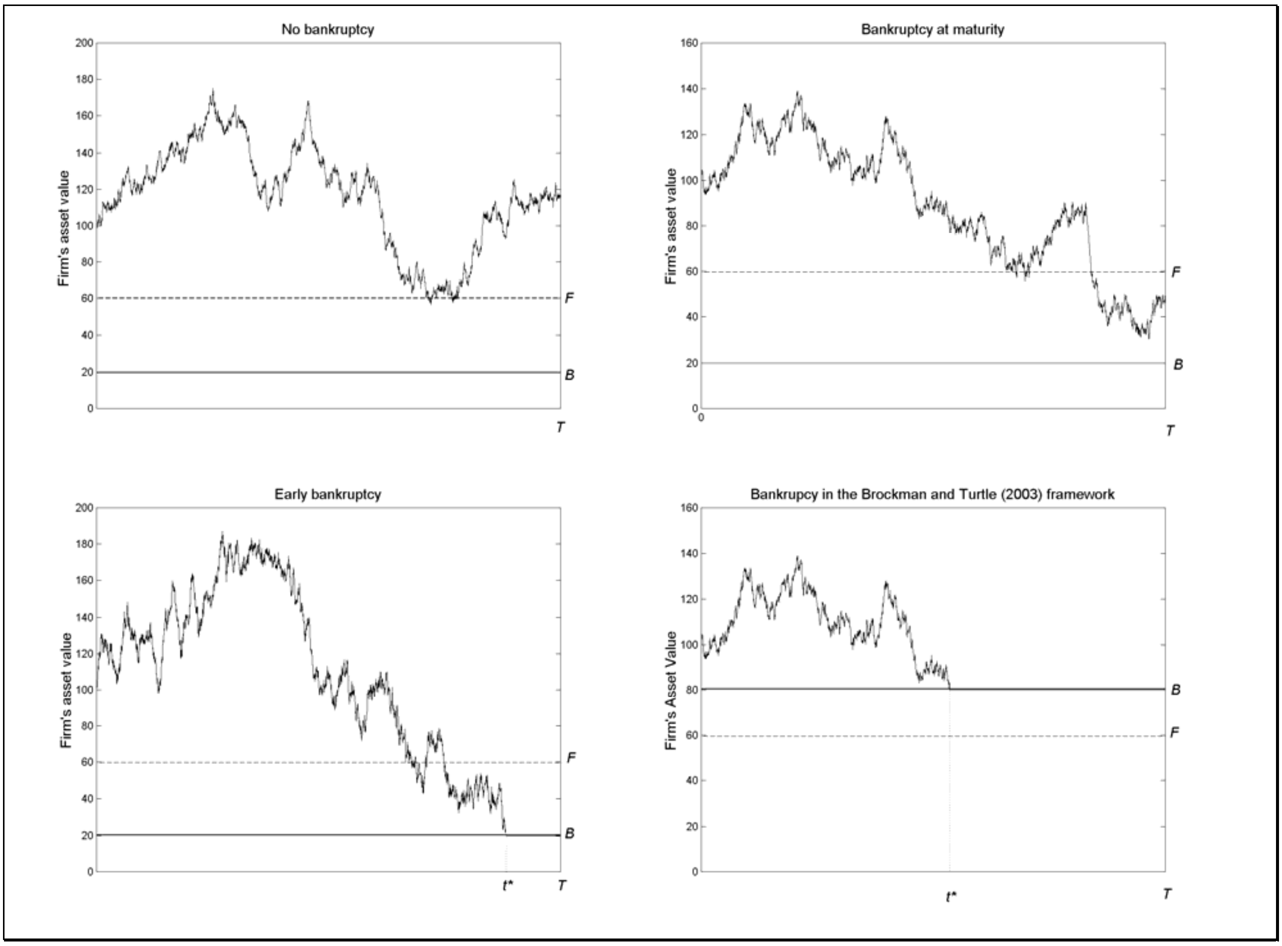

Figure 1. Asset value process and possible bankruptcy scenarios. These graphs show four possible sample paths (corresponding to a case of no bankruptcy and three different cases of bankruptcy) for an asset value process following a geometric Brownian motion with initial value of 100, relative drift of 12% per annum, book leverage of 60%, payout ratio of 3% per annum, and relative asset volatility of 40% per annum. In the Brockman and Turtle (2003) case, the barrier is assumed to be equal to 80% of the firm's starting asset value, while on the remaining graphs it is assumed equal to 20% of the firm's asset value.

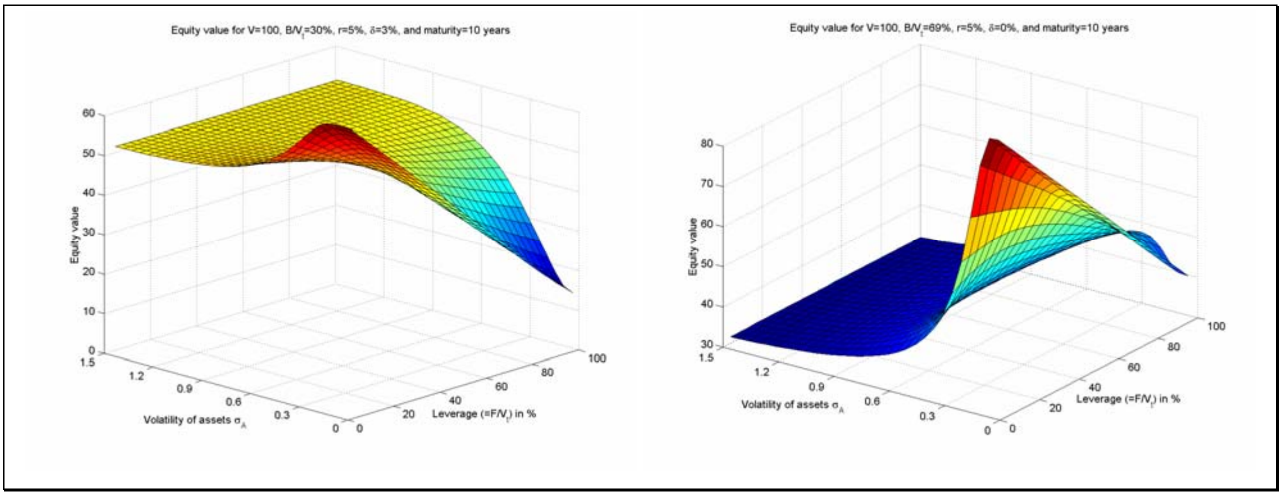

Figure 2. Equity value in the DOC framework as a function of leverage and asset volatility. These graphs show the value of equity coming out of a structural model that assume that the firm's equity is a down-and-out barrier option on the firm's assets. The equity value is indicated as a function of the leverage (expressed as book value of debt divided by market value of assets) and of the relative asset volatility. The first graph assumes a barrier level equal to 30% of the firm's asset value and a payout ratio equal to 3% (those parameters are close to our empirical backed out values. The second graph assumes a barrier level equal to 69% of the firm's asset value and a payout ratio equal to 0% (Brockman and Turtle's (2003) parameters). The riskfree rate is equal to 5% per annum for both graphs.

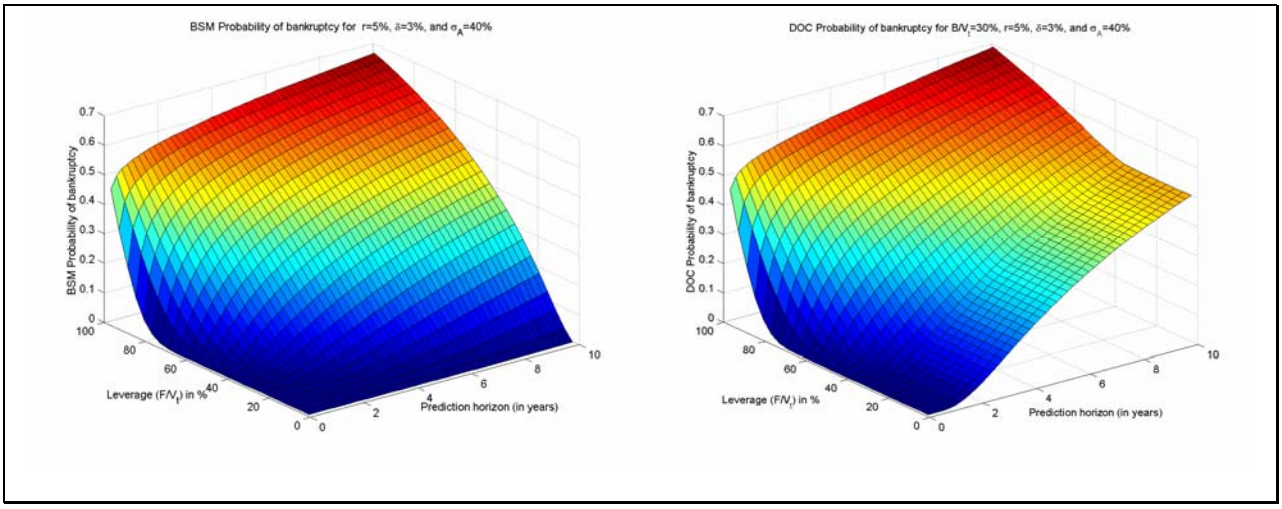

Figure 3. Estimated probability of bankruptcy in the BSM and the DOC frameworks. These graphs show the probability of bankruptcy coming out of structural models that assume that the firm's equity is a standard Black-Scholes(1973)/Merton (1974) call option and a down-and-out barrier option on the firm's assets respectively. Probabilities are indicated as a function of the prediction horizon (the assumed debt maturity) and the leverage (expressed as book value of debt divided by market value of assets). The riskfree rate is equal to 5% per annum, the firm's payout ratio is equal to 3% per annum, and the volatility of the firm's assets is equal to 40% per annum. In the case of the barrier option, the barrier is equal to 30% of the firm's asset value. These parameters are close to our empirical backed out values.

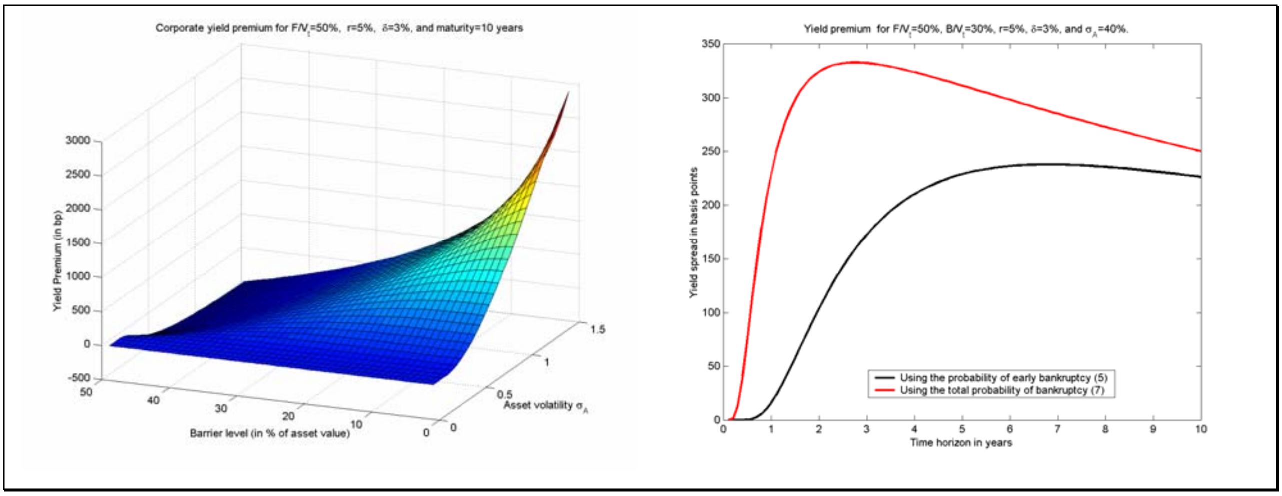

Figure 4. Corporate yield premium in the DOC framework. The first graph shows how the yield premium on corporate debt varies as a function of barrier level and asset volatility when the equity is viewed as a down-and-out barrier option on the firm's underlying assets (zero bankruptcy costs). The second graph shows, for a barrier level equal to 30% of the firm's asset value and a recovery rate of 50% of the asset value at the time of default, the yield premium when the probability of default is equal to the probability of crossing the barrier early (equation (5) in the text), or equal to the total probability of default (equation (7) in the text). For both graphs, the riskfree rate (payout ratio) is equal to 5% (3%) per annum, and the firm's leverage (expressed as book value of debt divided by market value of assets) is equal to 50%, close to our empirical backed out values.

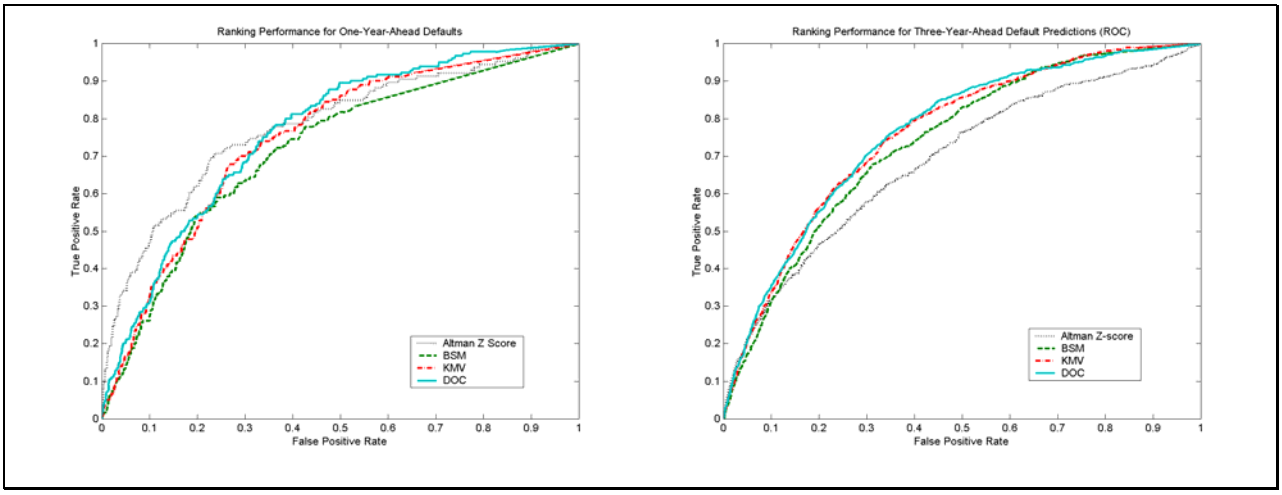

Figure 5. Receiver operating curves. These graphs show the receiver operating curves (ROC) for three structural models (Black-Scholes (1973)/Merton (1974), the down-and out-barrier option, and the KMV frameworks), as well as for the Altman (1968) Z-score. These graphs indicate what percentage of true defaults each model could detect in our sample of 5,784 industrial firms between 1988 and 2002, for an arbitrary percentage of false positives (i.e., wrongly predicted defaults) on the x-axis.

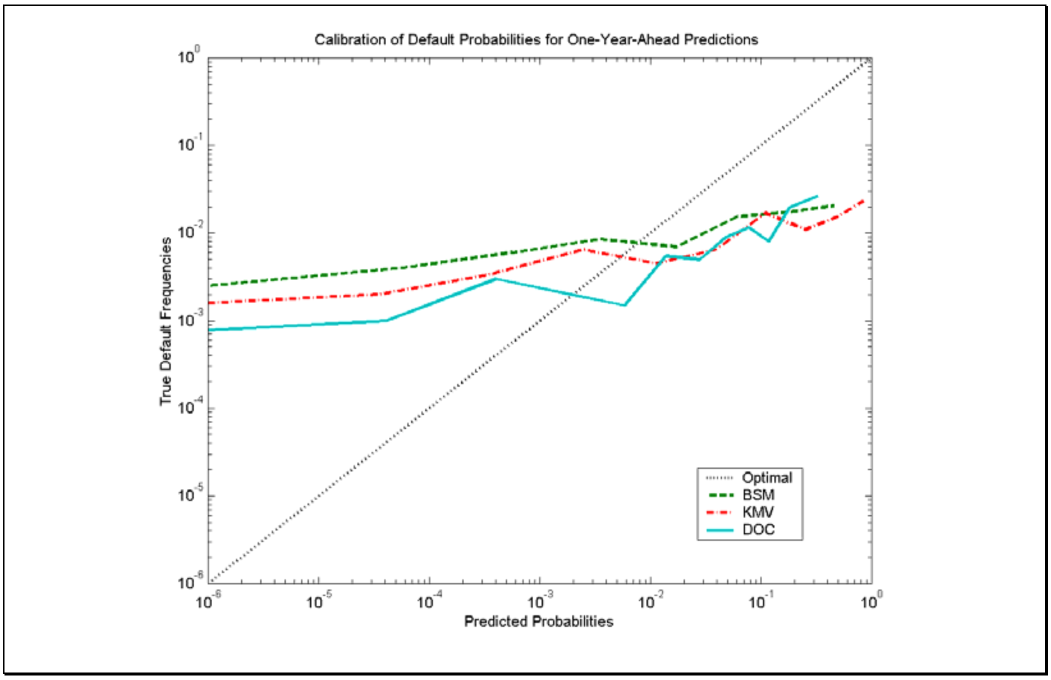

Figure 6. Calibration plot. This graph shows how well-calibrated our three structural models (Black-Scholes (1973)/Merton (1974), the down-and out-barrier option, and the KMV frameworks) are, i.e., how close the estimated probabilities of default coming out of these models are to true default rates in our sample of 5,784 industrial firms between 1988 and 2002. Since the Altman (1968) Z-score does not predict probabilities of default (before recalibration), it does not appear in this graph.

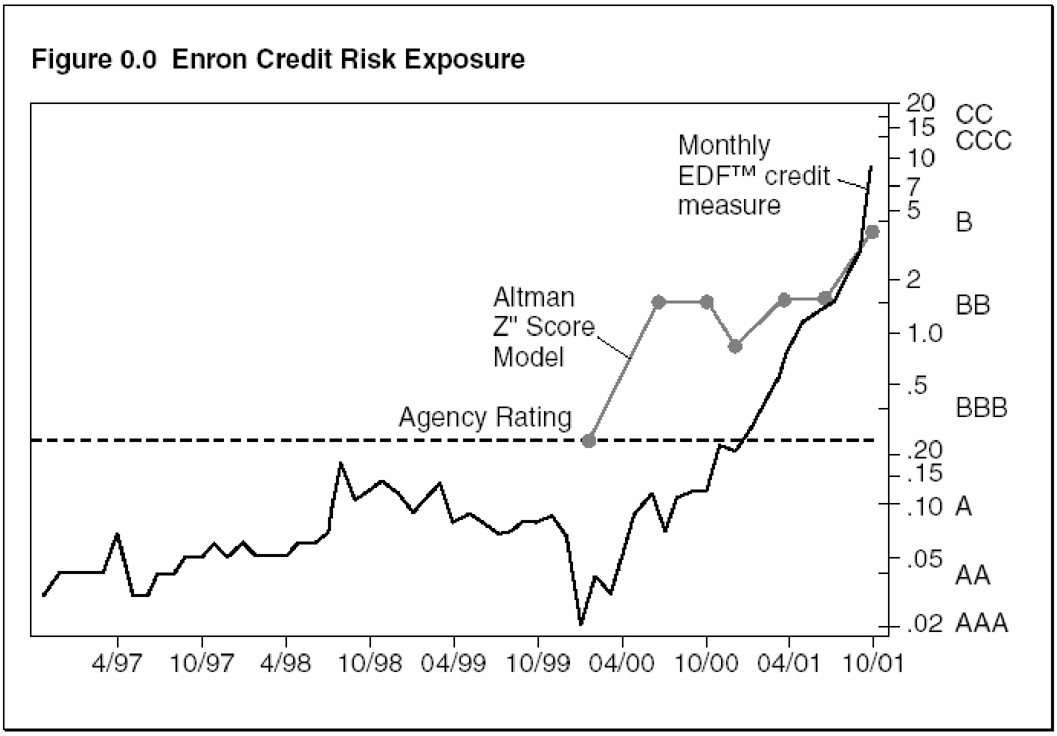

Figure 7. KMV's Expected Default Frequency and Altman's Z"-score as bankruptcy predictors: the case of Enron. This figure, reproduced from Saunders and Allen (2002), shows the deterioration in Enron's credit quality, as tracked by KMV's EDF and the Altman (1993) Z"-score.