External writer-extendible options: pricing and applications

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ABSTRACT

This article aims to examine a new type of exotic option, namely the external writer-extendible option. Compared to traditional options, such option has two characteristics: first, its maturity is extended by the optionwriter for a given period as soon as the option is not in-the-money at the initial maturity date, without any additional premium payment from the optionholder; second, once extended, the initial underlying asset is replaced by a new underlying asset until the extended maturity date. We derive closed-form valuation formulas for this kind of European-style call and put options, and show that firms may use these instruments in their warrant issues as well as in their risk management.

Keywords: Valuation of exotic options; extendible options; writer-extendible options; multi-asset options; external writer-extendible options.

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1. Introduction

To make stock issues or bond issues more attractive, warrants, namely traditional call options on the stock of the issuing firm, are typically attached to stocks or bonds to be issued[1]. To make warrant issues still more attractive, additional clauses, such as the "extension clause", are added to warrant contracts. Originally, the extension clause was designed to protect investors against a stock market crisis[2]. For example, some warrants have the possibility to be extended if the benchmark index decreases more than 15% within the last month before the maturity date. Such extension clauses constitute effectively a sort of "anti-crash" instrument, as they work only when stock markets fall within a short time. Compared to a traditional warrant, an extendible warrant has at least two advantages. First, by extending the maturity of the contract, it reduces the warrantholder's non-exercise risk. Second, with a higher price than a plain-vanilla warrant, it permits the issuing firm to collect more funds.

However, the extension clause may be used in a more general context by making the extension condition less restrictive. For instance, the warrant can be extended as soon as the underling stock price falls below the strike price at the maturity date. In this way, investors are not only protected against a significant decrease of the stock price within a short time, but also against a progressive, but continual decrease along the maturity of the contract. Longstaff (1990) distinguishes the "buyer-extendible option" (or simply the "extendible option") from the "writer-extendible option". For the first one, the extension decision is made by the optionholder with the payment of an additional premium, whereas for the second one, the extension decision is made by the optionwriter when the option is at-the-money or out-of-the-money at the maturity date, without any additional premium payment from the optionholder.

On the basis of extendible options, optionholder's non-exercise risk can be reduced even more by replacing the initial underlying asset by another one over the extended period. In order that such replacement makes sense, the new underlying asset should not be perfectly correlated with the initial one. For example, within the framework of a warrant, the new underlying asset could be the stock price of a parent or subsidiary of the initial underlying stock. In this way, we are led to the conception of a new type of option, entitled "external extendible options". In fact, as traditional extendible options (Longstaff, 1990), the maturity date of these options can be postponed for a specified period. Unlike traditional extendible options, the underlying asset of these options is replaced by another one over the extended period. In the sense that two or more underlying assets are involved in the option contract, external extendible options are also similar to multi-asset options (Margrabe, 1978; Stulz, 1982; Johnson, 1987).

This article aims to study external writer-extendible options, henceforth called "external extendible options". The mechanism of such options is as follows: at the initial maturity date, the option is exercised as a plain-vanilla European option if it is in-the-money; if not, it is extended as a new European option with an extended maturity date and a new underlying asset. In Section 2, the closed-form valuation formulas are derived within the framework of the model of Black & Scholes for this type of European options, and on this basis, analytical properties are analyzed. In Section 3, two examples are presented as corporate applications of these options. The fourth and last section summarizes the main results obtained and presents concluding remarks.

2. Pricing external extendible options

2.1. Valuation framework

Option valuation is made in the framework of the model of Black and Scholes (1973), generalized by Harrison and Kreps (1979), Harrison and Pliska (1981). We assume that the prices of the initial underlying asset, S_1, and of the second underlying asset, S_2, are two basic assets whose prices follow a joint geometric Brownian motion (GBM):

$$\frac{dS_1(t)}{S_1(t)} = (\mu_1 - \delta_1)dt + \sigma_1 d\tilde{W}_1(t) \tag{1a}$$

$$\frac{dS_2(t)}{S_2(t)} = (\mu_2 - \delta_2)dt + \sigma_2 d\tilde{W}_2(t) \tag{1b}$$

Where t is time, $\tilde{W}_1$ and $\tilde{W}_2$ are two standard Wiener processes, and ρ_{12} is the instantaneous correlation coefficient between $\tilde{W}_1$ and $\tilde{W}_2$, with $d\tilde{W}_1(t) \times d\tilde{W}_2(t) = \rho_{12} dt$. In Equations (1a) and (1b), μ_1 and μ_2 can be interpreted as the expected total returns (resulting from capital gain as well as dividend income) on the underlying assets, δ_1 and δ_2 as the continuous dividend yields, σ_1 and σ_2 as the instantaneous volatility rates. Furthermore, we designate the riskless interest rate as r and we assume that the valuation is made at time 0. For simplicity, $S_i(0)$ is noted as S_i.

2.2. External extendible calls

2.2.1. Analytical pricing formula

We designate the value at time t of an external extendible call option as WECO(S_1(t), S_2(t), K_1, T_1 – t, K_2, T_2 – t), where T_1 and T_2 represent the initial and the extended maturity dates, K_1 and K_2 represent the initial and the new strike prices, and S_1 and S_2 represent the initial and the new underlying assets. At time T_1, if $S_1(T_1) > K_1$, then the call option is exercised as a plain-vanilla call option; on the contrary, if $S_1(T_1) \leq K_1$, then the call option is extended by the optionwriter and no additional amounts are paid by the optionholder. The payoff of the option at T_1 is:

$$WECO(S_1(T_1), S_2(T_1), K_1, 0, K_2, T_2 - T_1) = \begin{cases} S_1(T_1) - K_1 & \text{if } S_1(T_1) > K_1 \\ C(S_2(T_1), K_2, T_2 - T_1) & \text{if } S_1(T_1) \leq K_1 \end{cases} \tag{2}$$

Where $C(S_2(T_1), K_2, T_2 - T_1)$ represents the value of a traditional European-style call option with an underlying asset price $S_2(T_1)$, a strike price K_2, and a maturity $T_2 - T_1$. The right-hand term of Equation (2) can be transformed as follows:

$$[S_1(T_1) - K] \times 1_{S_1(T_1) > K} + C(S_2(T_1), K_2, T_2 - T_1) \times 1_{S_1(T_1) \leq K} \tag{3}$$

Where $1_{\text{Condition}}$ is an indicator function, with $1_{Condition} = \begin{cases} 1 & \text{if the condition is met} \\ 0 & \text{if not} \end{cases}$. In Equation (3), the first component is the payoff at the maturity date of a traditional call option with an underlying asset price $S_1(T_1)$, a strike price K_1, and a maturity date T_1, whereas the second component is the payoff of the extension clause. Thus, the value of the external extendible call option is the sum of the value of the traditional European-style call option and that of the extension clause. As the Black and Scholes formula derives the value of a traditional European-style call option, the valuation problem consists in valuing the extension clause. According to the valuation theory of contingent securities, the value at time t (with t $\in$ [0,

T_1]) of the extension clause, designated as WECC(S_1(t), S_2(t), K_1, T_1 – t, K_2, T_2 – t), can be written as:

$$WECC(S_1(t),S_2(t),K_1,T_1-t,K_2,T_2-t)=e^{-r(T_1-t)}E_{P*}\left[C(S_2(T_1),K_2,T_2-T_1)1_{S_1(T_1)\le K}\right]$$

Where E_{P*}[.] is the mathematical expectation with the risk-neutral probability P*. As we assume that the valuation is made at time 0, we have:

$$WECC(S_1,S_2,K_1,T_1,K_2,T_2)=e^{-rT_1}E_{P*}\left[C(S_2(T_1),K_2,T_2-T_1)1_{S_1(T_1)\le K}\right] \quad (4)$$

To get the closed-form formula from Equation (4), the mathematical development could be realized in three steps.

1) In the first step, the value at T_1 of the extended call option, $C(S_2(T_1),K_2,T_2-T_1)$, can be written with its closed-form expression thanks to the Black & Scholes formula. Namely, we have:

$$C(S_2(T_1),K_2,T_2-T_1)$$

$$=S_2(T_1)e^{-\delta_2(T_2-T_1)}N(Z_{S_2(T_1)})-K_2e^{-r(T_2-T_1)}N(Z_{S_2(T_1)}-\sigma_2\sqrt{T_2-T_1}) \quad (5)$$

Where N(a) is the cumulative probability of the standard normal density in the interval [–∞, a], with $N(a)=\int_{-\infty}^{a}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}du$, and $Z_{S_2(T_1)}=\frac{ln\left[\frac{S_2(T_1)}{K_2}\right]+\left(r-\delta_2+\frac{\sigma_2{}^2}{2}\right)(T_2-T_1)}{\sigma_2\sqrt{T_2-T_1}}$. As a result, the valuation problem consists in calculating the mathematical expectation of a function of S_1(T) and S_2(T).

2) In the second step, S_1(T) and S_2(T) can be written as a function of their respective initial levels, S_1 and S_2:

$$S_1(T_1)=S_1e^{(r-\delta_1-\sigma_1{}^2/2)T_1+\sigma_1\sqrt{T_1}\varepsilon_1} \quad (6a)$$

$$S_2(T_1) = S_2 e^{(r-\delta_2-\sigma_2^2/2)T_1+\sigma_2\sqrt{T_1}\varepsilon_2} \tag{6b}$$

Where (ε_1, ε_2) follows a standard bivariate normal distribution with a correlation coefficient ρ_{12}, which is the same as the correlation coefficient between $\tilde{W}_1$ and $\tilde{W}_2$.

3) In the third and last step, $C(S_2(T_1), K_2, T_2 - T_1)$, $S_1(T)$, and $S_2(T)$ are replaced by their expressions written in Equations (5), (6a), and (6b), respectively. Then, the right-hand term of Equation (4) can be transformed into a function of bivariate normal distributions. These developments lead to (for more details, see Appendix):

$$WECC(S_1, S_2, K_1, T_1, K_2, T_2)$$

$$= S_2 e^{-\delta_2 T_2} N_2(-x_1 + (\sigma_1 - \rho_{12}\sigma_2)\sqrt{T_1}, x_2; -\rho\rho_{12})$$

$$-K_2 e^{-rT_2} N_2(-x_1 + \sigma_1\sqrt{T_1}, x_2 - \sigma_2\sqrt{T_2}; -\rho\rho_{12}) \tag{7}$$

Where $N_2(a, b ; \rho*)$ is the cumulative probability of the standard bivariate normal density with correlation coefficient $\rho*$ for the rectangular region $[-\infty, a] \times [-\infty, b]$,

with $N_2(a,b;\rho*) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{exp\left[-\frac{u^2 - 2\rho* uv + v^2}{2(1-\rho*^2)}\right]}{2\pi\sqrt{1-\rho*^2}} dvdu$, $x_1 = \frac{ln\left(\frac{S_1}{K_1}\right) + \left(r - \delta_1 + \frac{\sigma_1^2}{2}\right)T_1}{\sigma_1\sqrt{T_1}}$,

$x_2 = \frac{ln\left(\frac{S_2}{K_2}\right) + \left(r - \delta_2 + \frac{\sigma_2^2}{2}\right)T_2}{\sigma_2\sqrt{T_2}}$, and $\rho = \sqrt{\frac{T_1}{T_2}}$.

In Equation (7), the first component represents the present value of the average price of the second underlying asset when the option is extended and exercised at the extended maturity date, whereas the second one represents the present value of the second strike price in the same case. The value of the extension clause can be delimited by some rational bounds. For example, it can never be negative, but is smaller than $C(S_2, K_2, T_2)$, the value of the traditional European-style call option with the second underlying asset, the second strike price, and the extended maturity date. In other words, the value of the external extended call option is bigger than that of the initial traditional call option, but smaller than the sum of the initial traditional call option and the extended call option.

The extension clause has a certain number of interesting special cases. For example, if the prices of the two underlying assets are perfectly correlated (i.e., $\rho_{12} = 1$) and the two strike prices are the same (i.e., $K_1 = K_2 = K$), then the closed-form formula derived by Longstaff (1990) can be obtained. If $S_2 \to 0$ and $K_2 \to \infty$, then the value of the extension clause tends towards zero, as the probability of the extension approaches zero. If $S_1 \to 0$ and $K_1 \to \infty$, then the value of the extension clause tends towards $C(S_2, K_2, T_2)$, as the probability of the extension approaches one. If the prices of the two underlying assets are independent (i.e., $\rho_{12} = 0$), then the value of the extension clause is equal to the multiplication of the value of the extended call option and the probability of the extension.

2.2.2. Analytical properties

In deriving comparative statics, we focus especially on the differences between the classical extension clause (i.e., without changing the underlying asset over the extended period) and the external extension clause (i.e., with a new underlying asset over the extended period). It is also noteworthy that the computation of the cumulative probability of the standard bivariate normal density is based on Drezner's algorithm (Drezner, 1978).

As indicated in Longstaff (1990), the classical extension clause is not always a monotone increasing function of S_1, the initial underlying asset price. In fact, an increase in S_1 reduces the probability of the extension at T_1, and on the other hand, increases the value of the extended option at T_2. This is no longer the case with the external extension clause, which is a monotone decreasing function of S_1 (cf. Figure 1), as an increase in S_1 reduces only the probability of the extension, without having any effect on the extended option whose underlying asset is no more S_1. This is effectively the most important advantage that external extendible call options have compared to classical extendible options: even if S_1 tends towards zero, as soon as S_2 does not approach zero, the value of the option tends towards the value of the extended option, rather than zero.

(Insert Figure 1 here)

As for the classical extension clause, the external extension clause is not always a monotone function of σ_1 – the volatility of the initial underlying asset price. In fact, a high volatility may lead to a high level of S_1, which leads to a low probability of the extension, as well as a low level of S_1, which leads to a high probability of the extension. As a result, σ_1 may have a positive or negative effect on the value of the extension clause.

As for the classical extension clause, an increase in T_1 has an indeterminate effect on the value of the external extension clause (cf. Figure 2). In fact, on the one hand, a longer duration of T_1 gives more chance for the initial underlying asset to exceed the initial strike price, which reduces the probability of the extension. On the other hand, a longer duration of T_1 gives more chance for the new underlying asset to exceed the new strike price, which increases the value of the extended option. As a result, T_1 may have a positive or negative effect on the value of the extension clause.

(Insert Figure 2 here)

As for a traditional European-style call option, the value of the external extension clause is a monotone increasing function of S_2 (cf. Figure 3), σ_2 and T_2, but a monotone decreasing function of K_2 and δ_2. However, its sensitivity to these variables is lower than that for the traditional option, as the probability of the extension is smaller than one.

(Insert Figure 3 here)

As shown in Figure 4, the value of the external extension clause is a monotone decreasing function of ρ_{12} – the correlation coefficient between the two underlying asset prices. In fact, what finally accounts for the extension clause is the value of the extended option at its maturity date. For this reason, analysis should be concentrated on the case when S_2 exceeds K_2 at the extended maturity date: if S_1 and S_2 are positively correlated (i.e., $\rho_{12} > 0$), when S_2 rises, S_1 also tends to rise, which reduces the probability of the extension, and so the value of the extension clause; on the contrary, if S_1 and S_2 are negatively correlated (i.e., $\rho_{12} < 0$), when S_2 rises, S_1 rather tends to decrease, which increases the probability of the extension, and so the value of the extension clause. As a result, a positive correlation between S_1 and S_2 is more favorable for the extension of the initial option and for the exercise of the extended option. When $\rho_{12} = +1$, the value of the external extension clause reaches its minimum, which is also the value of the extension clause of a traditional extendible option. On the contrary, when $\rho_{12} = -1$, it reaches its maximum.

(Insert Figure 4 here)

2.3. External extendible puts

2.3.1. Analytical pricing formula

We designate the value at time t of an external extendible put option as WEPO(S_1(t), S_2(t),

K_1, $T_1 - t$, K_2, $T_2 - t$). At time T_1, if $S_1(T_1) < K_1$, then the put option is exercised as a plain-vanilla put option; on the contrary, if $S_1(T_1) \geq K_1$, then the put option is extended by the optionwriter and no additional amounts are paid by the optionholder. The payoff of the option at time T_1 is:

$$WEPO(S_1(T_1), S_2(T_1), K_1, 0, K_2, T_2 - T_1) = \begin{cases} K_1 - S_1(T_1) & \text{if } S_1(T_1) < K_1 \\ P(S_2(T_1), K_2, T_2 - T_1) & \text{if } S_1(T_1) \geq K_1 \end{cases} \tag{8}$$

Where $P(S_2(T_1), K_2, T_2 - T_1)$ represents the value of a traditional European-style put option with an underlying asset price $S_2(T_1)$, a strike price K_2, and a maturity $T_2 - T_1$. We designate the value at t of the extension clause as WECP($S_1(t)$, $S_2(t)$, K_1, $T_1 - t$, K_2, $T_2 - t$). Proceeding as before, the value of the extension clause can be derived as follows:

$$\begin{aligned} & WECP(S_1, S_2, K_1, T_1, K_2, T_2) \\ & = -S_2 e^{-\delta_2 T_2} N_2(x_1 - (\sigma_1 - \rho_{12}\sigma_2)\sqrt{T_1}, -x_2; -\rho\rho_{12}) \\ & + K_2 e^{-rT_2} N_2(x_1 - \sigma_1\sqrt{T_1}, -(x_2 - \sigma_2\sqrt{T_2}); -\rho\rho_{12}) \end{aligned} \tag{9}$$

As for the call option, the value of the extension clause for the put option should be superior or equal to zero, but inferior to $P(S_2, K_2, T_2)$, the value of the traditional European-style put option with the second underlying asset, the second strike price, and the extended maturity date. It also has a certain number of interesting special cases. For example, if the prices of the two underlying assets are perfectly correlated (i.e., $\rho_{12} = 1$) and the two strike prices are the same (i.e., $K_1 = K_2 = K$), then the closed-form valuation formula derived by Longstaff (1990) can be obtained. If $S_2 \to \infty$ and $K_2 \to 0$, then the value of the extension clause tends towards zero, as the probability of the extension approaches zero. If $S_1 \to \infty$ and $K_1 \to 0$, then the value of the extension clause tends towards $P(S_2, K_2, T_2)$, as the extension probability approaches one. If the prices of the two underlying assets are independent (i.e., $\rho_{12} = 0$), then the value of the extension clause is equal to the multiplication of the value of the extended put option and the probability of the extension.

2.3.2. Analytical properties

As for call options, similar properties can be derived for the extension clause for external extendible put options. More precisely, unlike the extension clause of traditional extendible put options, the value of the external extension clause is a monotone increasing function of S_1, but may increase or decrease with σ_1 and T_1. As a traditional put option, the value of the external extension clause is a monotone increasing function of K_2, δ_2, σ_2, and T_2, and is a monotone decreasing function of S_2. As for call options, the value of the external extension clause for put options decreases with ρ_{12}, the correlation coefficient between S_1 and S_2.

3. Examples of external extendible options

External extendible options can be used by firms in different contexts. In this section, two applications are presented, one for corporate warrants and the other for corporate risk management.

3.1. Corporate warrants

Corporate warrants constitute an important financing tool for firms (Smith, 1977). To make warrants even more attractive for investors, the external extension clause may be added to traditional warrants. In fact, to protect warrantholders against a general fall that is common for the whole stock market, the extension clause can be introduced as a first measure. For example, the CAC40 index fell 17% in 1994, whereas it rose 0.5% in 1995 and 25% in 1996.

In such a context, the extension, for the duration of one year, of a warrant expiring in 1995 gives warrantholders an additional period to exercise their options. Furthermore, to protect warrantholders against a fall that is rather specific to the stock price of the issuing firm, it may be useful to change the underlying stock price over the extended period. For example, though the CAC40 index rose 20% within the first semester of 1996, the real estate sector fell 0.81% and the financial sector rose only 0.56%. In this case, for investors holding warrants issued by a bank, it is more interesting to change the initial underlying stock by a new stock that is in an industrial sector, such as automobiles, petroleum, or pharmaceuticals.

Compared to a traditional extendible warrant, an external extendible warrant has at least three advantages. First, by changing the underlying asset over the extended period, it reduces warrantholders' non-exercise risk, which is not only due to the systematic risk related to the whole stock market, but also due to the specific risk related to the issuing firm. Second, with a higher price than a traditional extendible warrant, it permits the issuing firm to collect more funds. It is noteworthy that the underlying asset substitution over the extended period is particularly interesting for small firms, which are still at their start-up stage. In fact, even though the growth of these firms needs to be financed, their high risk dissuades investors from buying their stocks or warrants. In case these firms experience difficulties, the substitution of their stocks by those of their mother firms[3] may reassure investors, and such reassurance may facilitate firms' financing operation. The third and last advantage is that, for existing stockholders of the issuing firm, the dilution effect is inferior or equal to that resulting from a traditional extendible warrant. In fact, when the warrant is exercised at its initial maturity date, both of these two warrants lead to dilution and the dilution effect is the same; when the warrant is extended, but the extended warrant is not exercised at the extended maturity, neither of these warrants leads to dilution; when the warrant is extended, and the extended warrant is exercised at the extended maturity, the traditional extendible warrant leads to dilution for the stock of the issuing firm, whereas the external extendible one does not, as in the second case the stock of the issuing firm is no more involved during the extended period[4].

3.2. Risk management

Industrial firms are exposed to the risk resulting from fluctuation in both raw material prices and product prices. Let us take the example of a firm whose activity consists in transforming crude oil into a refined product. Assume that the firm plans to buy a certain quantity of crude oil at a future date T_1, and that the refined product will be sold on the market at another future date T_2, with $T_2 > T_1 > 0$. The firm is exposed to two risks at two different dates, namely the increase in the buying price of crude oil at T_1, and/or the decrease in the selling price of the refined product at T_2. Assume that T_1 and T_2 are so close that crude oil price at T_1 and the refined product price at T_2 are significantly correlated, for example, with a correlation coefficient that is higher than 0.80.

To hedge the firm's risks, one classical solution is to buy two traditional options, namely a call option on crude oil price with T_1 as maturity and a put option on the refined product price with T_2 as maturity. However, this solution over-hedges the firm's risks insofar as, in most cases, the firm cannot exercise both of these two options. In fact, if crude oil price increases at T_1, then the refined product price tends to increase at T_2. In this case, the firm can exercise its call option on crude oil price at T_1, but is not able to exercise its put option on the refined product price at T_2. On the contrary, if crude oil price decreases at T_1, then the refined product price tends to decrease at T_2. In this case, the firm will not be able to exercise its call option on crude oil price at T_1, but can exercise its put option on the refined product price at T_2.

The firm needs a solution that is more appropriate to its situation. In fact, what it needs is an option that works as a traditional call option on crude oil price with T_1 as maturity if the call option is in-the-money at T_1; as soon as the call option is not in-the-money at T_1, it is transformed into a traditional put option on the refined product price with T_2 as maturity. The payoff at T_1 of the option can be written as:

$$max[0,S_1(T_1)-K]+P(S_2(T_1),K_2,T_2-T_1)\times 1_{S_1(T_1)\leq K} \tag{10}$$

The last formula can be transformed as follows:

$$max[0,S_1(T_1)-K]+P(S_2(T_1),K_2,T_2-T_1)-P(S_2(T_1),K_2,T_2-T_1)\times 1_{S_1(T_1)>K} \tag{11}$$

In Formula (11), the first component represents the payoff at T_1 of the traditional call option on S_1 with T_1 as maturity, the second component represents the value at T_1 of a traditional put option on S_2 with T_2 as maturity, and the third component represents the value of the extension clause of an external extendible put option. This means that the new solution is the same as the classical one, except that the extension clause of an external extendible put option is sold in addition. In fact, if the call option on crude oil price is in-the-money at its maturity T_1, then the firm exercises its call option and sells at the same time its put option, as it needs no more to be protected against the decrease of the refined product price at T_2; on the contrary, if the call option on crude oil price is at-the-money or out-of-the-money at its maturity T_1, then the firm, not being able to exercise its call option, holds on to its put option, as it needs to be protected against the decrease of the refined product price at T_2. The price difference between the classical hedging solution and the new one is the value of the external extension clause. According to Formulas (9) and (11), the closed-form expression of the value of the new option contract can be written as:

$$C(S_1,K_1,T_1) - S_2 e^{-\delta_2 T_2} N_2(-x_1 + (\sigma_1 - \rho_{12}\sigma_2)\sqrt{T_1}, +x_2; -\rho\rho_{12})$$
$$+K_2 e^{-rT_2} N_2(-(x_1 - \sigma_1\sqrt{T_1}), +x_2 - \sigma_2\sqrt{T_2}; -\rho\rho_{12}) \quad (12)$$

4. Discussions

In this article, a new type of exotic options, called “external writer-extendible options”, is designed. These options have two characteristics compared to traditional ones. First, their maturity is extended for a given period by the optionwriter without any additional payment from the optionholder as soon as the option is not in-the-money at the initial maturity date. Second, once extended, their initial underlying asset is replaced by a second underlying asset until the extended maturity date. Such options enable optionsholders to reduce their non-exercise risk due to a “general” decline related to the whole stock market as well as a fall that is rather “specific” to the initial underlying stock. Within the framework of the model of Black and Scholes (1973), the closed-form valuation formulas have been derived for European call options as well as for European put options. It has also been shown that such options can be used by firms in their warrant issues and in their risk management.

Appendix: Pricing the extension clause of the external writer-extendible call option

This appendix aims at deriving the analytical pricing formula at time 0 of the extension clause of an external extendible call option. The payoff at T_1 of the extension clause is:

$$WECC(S_1,S_2,K_1,T_1,K_2,T_2) = e^{-rT_1} E_{P*}\left[C(S_2(T_1),K_2,T_2-T_1)1_{S_1(T_1)<K}\right] \quad (4)$$

where $E_{P*}[.]$ is the mathematical expectation with the risk-neutral probability P*, and

$$C(S_2(T_1),K_2,T_2-T_1)$$

$$= S_2(T_1)e^{-\delta_2(T_2-T_1)}N(Z_{S_2(T_1)}) - K_2 e^{-r(T_2-T_1)}N(Z_{S_2(T_1)} - \sigma_2\sqrt{T_2-T_1}) \quad (5)$$

where

$$Z_{S_2(T_1)} = \frac{ln\left[\frac{S_2(T_1)}{K_2}\right] + \left(r - \delta_2 + \frac{\sigma_2{}^2}{2}\right)(T_2 - T_1)}{\sigma_2\sqrt{T_2-T_1}}$$

From Equations (4) and (5), we have:

$$WECC(S_1,S_2,K_1,T_1,K_2,T_2) = WECC(1) - WECC(2) \quad (A1)$$

where

$$WECC(1) = e^{-rT_1} E_{P*}\left\lfloor S_2(T_1)e^{-\delta_2(T_2-T_1)}N(Z_{S_2(T_1)})1_{S_1(T_1)<K}\right\rfloor \quad (A2)$$

$$WECC(2) = e^{-rT_1} E_{P*}\left\lfloor K_2 e^{-r(T_2-T_1)}N(Z_{S_2(T_1)} - \sigma_2\sqrt{T_2-T_1})1_{S_1(T_1)<K}\right\rfloor \quad (A3)$$

The pricing of the extension clause consists now in deriving the closed-form expression of WECC(1) and WECC(2). For this, we write:

$$Z_{S_2(T_1)} = \frac{x_2 + \rho(\varepsilon_2 - \sigma_2\sqrt{T_1})}{\sqrt{1-\rho^2}} \quad (A4)$$

$$Z_{S_2(T_1)} - \sigma_2\sqrt{T_2 - T_1} = \frac{x_2 - \sigma_2\sqrt{T_2} + (\rho\sigma_1\varepsilon_2/\sigma_2)}{\sqrt{1-\rho^2}} \quad \text{(A5)}$$

$$1_{S_1(T_1)<K} = 1_{\varepsilon_1 < -(x_1 - \sigma_1\sqrt{T_1})} \quad \text{(A6)}$$

A.1. Closed-form expression of WECC(1)

From Equations (A2), (A4), (A6), and (6b), we have:

$$WECC(1) = S_2 e^{-\delta_2 T_2 - \frac{\sigma_2^2 T_1}{2}} E_{P*}\left[e^{\sigma_2\sqrt{T_1}\varepsilon_2} N\left(\frac{x_2 + \rho(\varepsilon_2 - \sigma_2\sqrt{T_1})}{\sqrt{1-\rho^2}}\right) 1_{\varepsilon_1 < -(x_1 - \sigma_1\sqrt{T_1})}\right] \quad \text{(A7)}$$

As (ε_1, ε_2) follows a standard bivariate normal distribution with a correlation coefficient ρ_{12}, the last mathematical expectation can be written by using the density function of (ε_1, ε_2), which is:

$$p(u,v) = \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} \times exp\left[-\frac{u^2 - 2\rho_{12}uv + v^2}{2(1-\rho_{12}^2)}\right] \quad \text{(A8)}$$

We have:

$$WECC(1) = S_2 e^{-\delta_2 T_2 - \frac{\sigma_2^2 T_1}{2}} \times Integral1 \quad \text{(A9)}$$

where

$$Integral1 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{+\infty} e^{\sigma_2\sqrt{T_1}v} \times N\left(\frac{x_2 + \rho(v - \sigma_2\sqrt{T_1})}{\sqrt{1-\rho^2}}\right) \times \frac{exp\left[-\frac{u^2 - 2\rho_{12}uv + v^2}{2(1-\rho_{12}^2)}\right]}{2\pi\sqrt{1-\rho_{12}^2}} dvdu \quad \text{(A10)}$$

We replace the standard normal distribution function N[.] by an integral of the standard normal density function as follows:

$$N(a)=\int_{-\infty}^{a}\frac{e^{-w^2/2}}{\sqrt{2\pi}}dw \qquad \text{(A11)}$$

We have:

$$Integral1=\int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})}\int_{-\infty}^{+\infty}e^{\sigma_2\sqrt{T_1}v}\left(\int_{-\infty}^{\frac{x_2+\rho(v-\sigma_2\sqrt{T_1})}{\sqrt{1-\rho^2}}}\frac{e^{-w^2/2}}{\sqrt{2\pi}}dw\right)\times\frac{\exp\left[-\frac{u^2-2\rho_{12}uv+v^2}{2(1-\rho_{12}^2)}\right]}{2\pi\sqrt{1-\rho_{12}^2}}dvdu \qquad \text{(A11)}$$

In the last equation, we need to withdraw the variable v from the superior bound of the integral relative to w. For this, we resort to a variable change, namely $w=\frac{w'+\rho v}{\sqrt{1-\rho^2}}$, which leads to:

$$Integral1=\int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_2-\rho\sigma_2\sqrt{T_1}}\frac{exp[X]}{(2\pi)^{3/2}\sqrt{(1-\rho_{12}^2)(1-\rho^2)}}dw'\,dv\,du \qquad \text{(A12)}$$

where

$$X=\frac{\left(u-\rho_{12}\sigma_2\sqrt{T_1}\right)^2+2\rho\rho_{12}(u-\rho_{12}\sigma_2\sqrt{T_1})(w'+\rho\sigma_2\sqrt{T_1})+\left(w'+\rho\sigma_2\sqrt{T_1}\right)^2}{2(1-\rho^2\rho_{12}^2)}+\frac{\sigma_2^2T_1}{2}$$

$$-\frac{\left[\sqrt{1-\rho^2\rho_{12}^2}\times v+\frac{\rho(1-\rho_{12}^2)w'-\rho_{12}(1-\rho^2)u-(1-\rho^2)(1-\rho_{12}^2)\sigma_2\sqrt{T_1}}{\sqrt{1-\rho^2\rho_{12}^2}}\right]^2}{2\left(1-\rho_{12}^2\right)\left(1-\rho^2\right)} \qquad \text{(A13)}$$

In order to withdraw the integral relative to the variable v from Equation (A12) by the fact that $\int_{-\infty}^{+\infty}\frac{e^{-v^2/2}}{\sqrt{2\pi}}dv=N(\infty)=1$, we need to resort to the following variable change:

$$v'=-\frac{\sqrt{1-\rho^2\rho_{12}^2}\times v+\frac{\rho(1-\rho_{12}^2)w'-\rho_{12}(1-\rho^2)u-(1-\rho^2)(1-\rho_{12}^2)\sigma_2\sqrt{T_1}}{\sqrt{1-\rho^2\rho_{12}^2}}}{\sqrt{\left(1-\rho_{12}^2\right)\left(1-\rho^2\right)}} \qquad \text{(A14)}$$

This variable change leads to:

$$Integral1 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{x_2-\rho\sigma_2\sqrt{T_1}} \frac{exp[X_1]}{2\pi\sqrt{1-\rho^2\rho_{12}^2}} dw'\, du \qquad \text{(A15)}$$

where

$$X_1 = \frac{\left(u-\rho_{12}\sigma_2\sqrt{T_1}\right)^2 + 2\rho\rho_{12}(u-\rho_{12}\sigma_2\sqrt{T_1})(w'+\rho\sigma_2\sqrt{T_1}) + \left(w'+\rho\sigma_2\sqrt{T_1}\right)^2}{2(1-\rho^2\rho_{12}^2)} + \frac{\sigma_2^2 T_1}{2} \qquad \text{(A16)}$$

To transform the integral into a standard bivariate normal distribution function relative to u and w', we need to use the following variable changes:

$$u' = u - \rho_{12}\sigma_2\sqrt{T_1} \qquad \text{(A17)}$$

$$w'' = w' + \rho\sigma_2\sqrt{T_1} \qquad \text{(A18)}$$

These variable changes lead to:

$$Integral1 = e^{\frac{\sigma_2^2 T_1}{2}} \times \int_{-\infty}^{-x_1+(\sigma_1-\rho_{12}\sigma_2)\sqrt{T_1}} \int_{-\infty}^{x_2} \frac{exp\left[\frac{u'^2+2\rho\rho_{12}u'w''+w''^2}{2(1-\rho^2\rho_{12}^2)}\right]}{2\pi\sqrt{1-\rho^2\rho_{12}^2}} dw''\, du' \qquad \text{(A19)}$$

Equations (A9) and (A19) lead to:

$$WECC(1) = S_2 e^{-\delta_2 T} \times N_2(-x_1 + (\sigma_1 - \rho_{12}\sigma_2)\sqrt{T_1}, x_2; -\rho\rho_{12} \qquad \text{(A20)}$$

where

$$N_2(a,b;\rho^*) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{exp\left[\frac{u^2+2\rho^* uw+w^2}{2(1-\rho^{*2})}\right]}{2\pi\sqrt{1-\rho^{*2}}} dw\, du \qquad \text{(A21)}$$

A.2. Closed-form expression of WECC(2)

From Equations (A3), (A5), and (A6), we have:

$$WECC(2) = Ke^{-rT_2} E_{P*}\left[N\left(\frac{x_2 - \sigma_2\sqrt{T_2} + (\rho\sigma_1\varepsilon_2/\sigma_2)}{\sqrt{1-\rho^2}} \right) 1_{\varepsilon_1 < -(x_1 - \sigma_1\sqrt{T_1})} \right] \quad \text{(A22)}$$

By using the standard normal density function of (ε_1, ε_2), we have:

$$WECC(2) = Ke^{-rT_2} \times Integral2 \quad \text{(A23)}$$

where

$$Integral2 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{+\infty} N\left(\frac{x_2 - \sigma_2\sqrt{T_2} + (\rho\sigma_1 v/\sigma_2)}{\sqrt{1-\rho^2}} \right) \times \frac{exp\left[-\frac{u^2 - 2\rho_{12}uv + v^2}{2(1-\rho_{12}^2)} \right]}{2\pi\sqrt{1-\rho_{12}^2}} dvdu \quad \text{(A24)}$$

By replacing the standard normal distribution function N[.] by an integral of the standard normal density function, we have:

$$Integral2 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\frac{x_2-\sigma_2\sqrt{T_2}+(\rho\sigma_1 v/\sigma_2)}{\sqrt{1-\rho^2}}} \frac{e^{-w^2/2}}{\sqrt{2\pi}} dw \right) \times \frac{exp\left[-\frac{u^2 - 2\rho_{12}uv + v^2}{2(1-\rho_{12}^2)} \right]}{2\pi\sqrt{1-\rho_{12}^2}} dvdu \quad \text{(A25)}$$

In the last equation, we need to withdraw the variable v from the superior bound of the integral relative to w. For this, we resort to a variable change, namely, $w' = w\sqrt{1-\rho^2} - \rho v\left(\frac{\sigma_1}{\sigma_2}\right)$, which leads to:

$$Integral2 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_2-\sigma_2\sqrt{T_2}} \frac{exp[Y]}{(2\pi)^{3/2}\sqrt{(1-\rho_{12}^2)(1-\rho^2)}} dw'\, dv\, du \quad \text{(A26)}$$

where

$$Y = -\frac{u^2 + 2\rho\rho_{12}uw' + w'^2}{2(1-\rho^2\rho_{12}{}^2)} - \frac{\left[\sqrt{1-\rho^2\rho_{12}{}^2} \times v + \frac{\rho(1-\rho_{12}{}^2)w' - \rho_{12}(1-\rho^2)u}{\sqrt{1-\rho^2\rho_{12}{}^2}}\right]^2}{2\left(1-\rho_{12}{}^2\right)\left(1-\rho^2\right)} \qquad \text{(A27)}$$

In order to withdraw the integral relative to the variable v from Equation (A26) by the fact that $\int_{-\infty}^{+\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv = N(\infty) = 1$, we need to resort to the following variable change:

$$v' = -\frac{\sqrt{1-\rho^2\rho_{12}{}^2} \times v + \frac{\rho(1-\rho_{12}{}^2)w' - \rho_{12}(1-\rho^2)u}{\sqrt{1-\rho^2\rho_{12}{}^2}}}{\sqrt{\left(1-\rho_{12}{}^2\right)\left(1-\rho^2\right)}} \qquad \text{(A28)}$$

This variable change leads to:

$$Integral2 = \int_{-\infty}^{-(x_1-\sigma_1\sqrt{T_1})} \int_{-\infty}^{x_2-\rho\sigma_2\sqrt{T_1}} \frac{\exp\left[-\frac{u^2 + 2\rho\rho_{12}uw' + w'^2}{2(1-\rho^2\rho_{12}{}^2)}\right]}{2\pi\sqrt{1-\rho^2\rho_{12}{}^2}} dw'du \qquad \text{(A29)}$$

Equations (A23) and (A29) lead to:

$$WECC(2) = Ke^{-rT_2} \times N_2\left(-(x_1 - \sigma_1\sqrt{T_1}), x_2 - \rho\sigma_2\sqrt{T_1}; -\rho\rho_{12}\right) \qquad \text{(A30)}$$

Equations (A1), (A20), and (A30) lead to:

$$WECC(S_1, S_2, K_1, T_1, K_2, T_2)$$

$$= S_2 e^{-\delta_2 T_2} N_2(-x_1 + (\sigma_1 - \rho_{12}\sigma_2)\sqrt{T_1}, x_2; -\rho\rho_{12}) - K_2 e^{-rT_2} N_2(-x_1 + \sigma_1\sqrt{T_1}, x_2 - \sigma_2\sqrt{T_2}; -\rho\rho_{12}) \qquad \text{(A31)}$$

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Liste of figure captions

[1] Once issued, the warrants are detached from stocks or bonds to be traded independently.

[2] For instance, in France, holders of warrants expiring at the end of 1987 were not able to exercise their options due to the stock market crash in October. To reduce their losses, companies having issued these warrants asked the COB (or *Commission des Opérations de Bourse*) – the equivalent Security Exchange Committee in France – to give its permission to extend the warrant contracts. Such request was systematically rejected by the COB due to the fact that no extension clause had been planned in the contracts. Since then, some companies have added to their warrant contracts an extension clause in order to protect warrantholders against an eventual stock market debacle.

[3] Legally, this mechanism is possible. For example, the warrants issued in June 1996 by Northumbrian Water Group (NWG) gave to their holders the right to buy a stock of Lyonnaise des Eaux, the mother firm of NWG, rather than a stock of NWG.

[4] In other words, for an external extended warrant, the exercise of the extended warrant leads to dilution for the new underlying stock, rather than dilution for the initial underlying one.