Pricing Derivatives with Barriers in a Stochastic Interest Rate Environment

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Abstract

This paper develops a general valuation approach to price barrier options when the term structure of interest rates is stochastic. These products' barriers may be constant or stochastic, in particular we examine the case of discounted barriers (at the instantaneous interest rate). So, in practice, we extend Rubinstein and Reiner (1991), who give closed-form formulas for pricing barrier options in a Black and Scholes context, to the case of a Vasicek modeling of interest rates. We are therefore in the situation of pricing barrier options semi-explicitly or explicitly (depending on the shape of the barrier) with stochastic Vasicek interest rates. The model is illustrated with a specific contract, an up and out call with rebate, hence a typical barrier option. This example is merely here to show how any standard barrier option can be priced in such a context. The validity of the approximation is analyzed and the sensitivity to the barrier level and to discretization schemes are also derived.

Keywords: Change of Numéraire, Vasicek model, Barrier Option, Markovian Approximation

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Introduction

In this article, we focus our analysis on the pricing of financial contracts with barriers in a stochastic interest rate environment. The applications of barrier options are multiple and go far beyond the study of derivative products. Barrier options are building blocks of diverse fields such as investment choice theory, the study of the capital structure of the firm (see the standard reference of Black and Cox (1976) for instance, or the interesting contribution of François and Morellec (2004)), or life insurance (see for instance Grosen and Jorgensen (2002)). Recall that these contracts payoffs depend on whether or not the price of their underlying assets cross a barrier from above or from below. They are the essential part of the standard structured products that are guaranteeing the maximum of a capital and the performance of a financial index.

Barrier options have been studied in great detail for a long time. Under the assumption of a unique and constant interest rate, closed-form solutions were given by Merton (1974) for down and out calls, then by Rubinstein and Reiner (1991) for vanilla barrier options. Other contributions include the works of Geman and Yor (1996) and Pelsser (2000) who priced double barrier options, and the innovative article of Chesney, Jeanblanc and Yor (1997) who introduced Parisian barrier options. The payoff of the latter contracts depends on the time spent above or below the barrier. Later on, Linetsky (1999) pioneered step options. In all these papers, the standard Black and Scholes framework is the starting point and in particular the risk-free interest rate is assumed constant. For short term contracts, a constant term structure of interest rates can be considered reasonable; yet, for medium or long term notes this assumption cannot hold.

The study of exotic barrier options in the context of stochastic interest rates is a rather difficult problem. Although it arises frequently in practice, it is usually solved in the financial industry by means of Monte-Carlo simulations or partial differential equations. This article is, as far as we know, the first in finance to take into account a stochastic term structure of interest rates to price barrier options by means of closed or semi-closed form formulas in continuous time. To do so, our framework considers a type of Markovian approximation due to Fortet (1943) and used by Longstaff and Schwartz (1995) to value risky debt. Collin-Dufresne and Goldstein (2001) generalized Fortet's approximation to the case of two-dimensional Markov processes. As suggested by these authors, we use their extension to price exotic barrier options. In the actuarial field, Bernard, Le Courtois and Quittard-Pinon (2005) priced successfully life insurance contracts owning many covenants in a similar stochastic context. Our paper goes beyond the article of these last authors to show how the extended

Fortet's method can be used in the finance realm for barrier options.

This article is organized as follows. In the first section, we show how standard barrier options can be priced with semi-closed-form formulas, when the interest process is stochastic and of the Vasicek type. This section is therefore the direct extension of the work of Rubinstein and Reiner (1991). The second section illustrates our approach with a particular exotic contract, the shark option, which is in fact a barrier option with rebate. This section also develops a subsetting where it is possible to reduce the semi-closed form formulas to closed-form formulas, while keeping the randomness of the underlying interest rate process. The last section applies our results in the context of a numerical analysis.

1 Standard Barrier Options in a Vasicek Model

Let us start by considering a financial market with a primary asset, say a stock S, on which a barrier option is written. The underlying asset price is assumed to follow a geometric Brownian motion. The interest rate model is a Vasicek one, in particular the instantaneous interest rate r enjoys the Markovian property. The uncertainty is modeled by a filtered space $(\Omega, F, \{F_t\}_{t\geqslant 0}, \Pi)$ where Ω is the usual fundamental space, $\{F_t\}_{t\geqslant 0}$ is the filtration generated by the Brownian motions, and Π is the historical probability measure. Trading takes place continuously and the prices of all assets follow correlated diffusions. In particular, the interest rate process is correlated to the stock process, or put differently, the economy is driven by two correlated Brownian motions. The market is complete and frictionless, and Q denotes the risk-neutral probability.

Because standard barrier options can be *up* or *down*, *in* or *out*, *call* or *put* options, there are eight types of such options. For the sake of brevity, we will only price in this section call options (the put option formulas can be obtained straightforwardly from parity relationships), that is to say *up* and *out*, *up* and *in*, *down* and *out*, and *down* and *in* barrier call options.

Denoting by T the maturity of the options, by K their strike, and by H their barrier level, one can write the following arbitrage-free pricing formulas for the *up* and *out*, and *up* and *in* call options:

$$\begin{cases} C^{uo} &= E_Q\left(e^{-\int_0^T r_s ds}\left(S_T-K\right)^+ \mathbb{1}_{S_{\max}\leqslant H}\right) \\ C^{ui} &= E_Q\left(e^{-\int_0^T r_s ds}\left(S_T-K\right)^+ \mathbb{1}_{S_{\max}> H}\right) \end{cases} \tag{1}$$

As concerns the *down* and *out*, and *down* and *in* calls, they admit the following valuation formulas:

$$\left\{\begin{array}{ll} C^{do} & = E_Q\left(e^{-\int\limits_0^T r_s ds}\left(S_T-K\right)^+ \mathbb{1}_{S_{\min}\geqslant H}\right) \\ C^{di} & = E_Q\left(e^{-\int\limits_0^T r_s ds}\left(S_T-K\right)^+ \mathbb{1}_{S_{\min}<H}\right) \end{array}\right. \tag{2}$$

The goal of this section will be to show how the formulas in (1) and (2) can be priced in semi-closed form.

1.1 Pricing Framework

We will need in the coming developments to use the forward-neutral dynamics of the stock, of the default-free zero-coupons and of the stock expressed in units of default-free zero-coupon. The dynamics of the default-free zero-coupons $P(t,T)$ classically write, in the historical world, as:

$$\frac{dP(t,T)}{P(t,T)} = \lambda(t,T)dt - \sigma_P(t,T)dZ_1(t)$$

where $\lambda(t,T)$ is their expected return, $\sigma_P(t,T)$ their volatility, and Z_1 a standard Brownian motion under Π.

The option's underlying price at time t, denoted by S_t , is modeled by a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_2(t)$$

where Z_2 is a standard Brownian motion correlated with Z_1: we define the correlation coefficient ρ by $dZ_1.dZ_2 = \rho dt$.

These dynamics are given in the historical universe. Using standard results from risk-neutral analysis, we know that there exists a unique probability measure Q under which the discounted price of securities are martingales. After decorrelating the above Brownian motions, we can write under Q:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T)d\widehat{Z}_1(t)$$

and for the underlying's price:

$$\frac{dS_t}{S_t} = r_t dt + \sigma\left(\rho d\widehat{Z}_1(t) + \sqrt{1-\rho^2}d\widehat{Z}_2(t)\right)$$

where $\widehat{Z}_1$ and $\widehat{Z}_2$ are now two uncorrelated Q-Brownian motions.

Using Itō's lemma, we can express the risk-neutral dynamics of S_t and $P(t,T)$ as:

$$S_t = S_0 \exp\left(\int_0^t r_u du - \frac{1}{2}\sigma^2 t + \int_0^t \rho\sigma d\widehat{Z}_1(u) + \int_0^t \sigma\sqrt{1-\rho^2} d\widehat{Z}_2(u)\right) \quad (3)$$

and

$$P(t,T) = P(0,T)\exp\left(\int_0^t r_u du - \frac{1}{2}\int_0^t \sigma_P^2(u,T)du - \int_0^t \sigma_P(u,T)d\widehat{Z}_1(u)\right) \quad (4)$$

We now aim at writing the dynamics of S in the T-forward-neutral universe. First, we start using the martingale property of the relative price $\frac{S_t}{P(t,T)}$, which reads:

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)} \exp\left(\begin{array}{c} \int_0^t (\sigma_P(u,T)+\rho\sigma)\, dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^T(u) \\ -\frac{1}{2}\int_0^t \left((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du \end{array}\right) \quad (5)$$

and we readily set:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(\begin{array}{c} -\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))\, dZ_1^T(u) \\ +\frac{1}{2}\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))^2\, du \end{array}\right)$$

where Z_1^T and Z_2^T are two uncorrelated Q_T-Brownian motions, defined by the two following relationships: $dZ_1^T(t) = d\widehat{Z}_1(t) + \sigma_P(t,T)dt$ and $dZ_2^T(t) = d\widehat{Z}_2(t)$.

Finally, we can obtain the forward-neutral expression of S_t that is going to be used in the remainder of this paper:

$$S_t = \frac{S_0}{P(0,t)} \exp\left(\begin{array}{c} \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2}\right) du \\ + \int_0^t \ (\sigma_P(u,t)+\rho\sigma) dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^T(u) \end{array}\right) \quad (6)$$

or equivalently:

$$\ln(S_t) = \ln\left(\frac{S_0}{P(0,t)}\right) + \left(\begin{array}{c} \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t)+\rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2}\right) du \\ + \int_0^t \ (\sigma_P(u,t)+\rho\sigma) dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^T(u) \end{array}\right)$$

Hence, under Q_T, the underlying price is lognormal, and $\ln(S)$ is a Gaussian process. Denoting it by l, we can also remark that:

$$dl_t = \left(r_t - \frac{\sigma^2}{2} - \sigma\,\rho\,\sigma_P(t,T)\right) dt + \sigma\,\rho\, dZ_1^T(t) + \sigma\sqrt{1-\rho^2}\, dZ_2^T(t) \quad (7)$$

We will also need the following moments: M_t, V_t and $Cov\,(v,t)$, $v \leqslant t$, which respectively denote the mean, variance and auto-covariance of the underlying. Their generic expressions are:

$$\begin{cases} M_t & = \ln\left(\frac{S_0}{P(0,t)}\right) + \int_0^t \left(-\sigma_P\,(u,T)\,(\sigma_P\,(u,t) + \rho\sigma) + \frac{\sigma_P^2(u,t)-\sigma^2}{2}\right) du \\ V_t & = \int_0^t \left(\sigma^2 + \sigma_P^2\,(u,t) + 2\rho\sigma\sigma_P\,(u,t)\right) du \\ Cov\ (v,t) & = \int_0^v \left(\sigma^2 + \rho\sigma\,(\sigma_P\,(u,t) + \sigma_P\,(u,v)) + \sigma_P\,(u,v)\,\sigma_P\,(u,t)\right) du \end{cases}$$

Furthermore, using standard probabilistic results on bidimensional Gaussian vectors, we know that the conditional law of $\ln(S_t)$ given $(\ln(S_v) = \ln(H))$, where $\ln(H) = h$ is an arbitrary given level, is normal and possesses the following mean $\widehat{M}$ and variance $\widehat{V}$:

$$\begin{cases} \widehat{M}\,(v,t) & = M_t + \frac{Cov\,(v,t)}{V_v}\,(\ln(H) - M_v) \\ \widehat{V}\,(v,t) & = V_t - \frac{Cov^2\,(v,t)}{V_v} \end{cases}$$

Standard computations enable computing explicitly the above moments in the two cases of linear and exponential volatility structures. The results for M, V and Cov are given in appendix II (from them, one obtains straightforwardly the expressions for $\widehat{M}$ and $\widehat{V}$) in the case of an exponential structure of volatility which corresponds to the Vasicek model.

1.2 Semi-Closed Form formulas

We can now start deriving the quasi-closed expressions of the arbitrage-free formulas (1) and (2) of barrier call options. We start with the *up* and *in* and the *up* and *out* options.

Pricing *up* call options

To begin with, we can reexpress the formula of the *up* and *in* option in (1) in the forward-neutral universe:

$$C^{ui} = P(0,T)E_{Q_T}\left((S_T - K)^+\,\mathbb{1}_{S_{\max}>H}\right)$$

This can alternatively be written as:

$$\frac{C^{ui}}{P(0,T)} = E_{Q_T}\left(S_T\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H} - K\mathbb{1}_{S_T>K}\mathbb{1}_{S_{\max}>H}\right)$$

or as:

$$\frac{C^{ui}}{P(0,T)} = E_{Q_T}\left(S_T \mathbb{1}_{S_T>K}\,\mathbb{1}_{S_{\max}>H}\right) - K\,Q_T\left(S_T > K,\ S_{\max} > H\right)$$

Finally, the *up* and *in* call option price C^{ui} is given by:

$$\frac{C^{ui}}{P(0,T)} = \mathcal{A} - K\,\mathcal{B} \tag{8}$$

where

$$\begin{cases} \mathcal{A} & = E_{Q_T}\left(S_T\,\mathbb{1}_{S_T>K}\,\mathbb{1}_{S_{\max}>H}\ \right) \\ \mathcal{B} & = Q_T\left(S_T > K,\ S_{\max} > H\right) \end{cases}$$

Note the following problem that appears in the computation of $\mathcal{A}$ and $\mathcal{B}$: the explicit expression of the law of $S_{\max}$, and *a fortiori* of the joint law of $S_{\max}$ and S_T, is not known. The event $\{S_{\max} > H\}$ is indeed equivalent to the first passage time of the process S through the barrier level H occurring before the maturity T of the option. Let us denote by γ^u this first passage time ("u" for an "up" barrier). One readily has $\{S_{\max} > H\} = \{\gamma^u \leqslant T\}$. We do not know the explicit joint distribution of γ^u and r_{γ^u}; yet, a discretized version of it can be obtained using the recursive argument of Collin-Dufresne and Goldstein (2001). In Appendix I, we expose this algorithm, titled the extended Fortet's method, along a new and clean presentation (relying in particular on distributions and not on densities).

The distribution function of the random vector (r_{γ^u}, γ^u) at time t under the T-forward-neutral measure Q_T is unknown, as previously said. We approximate it by discretizing along the time and interest rate dimensions. The interval $[0,T]$ is subdivided into n_T subperiods of length $\delta_t = T/n_T$, and the interest rate is subdivided between $r_{\min}$ and $r_{\max}$ into n_r intervals of length $\delta_r = \frac{r_{\max}-r_{\min}}{n_r}$. Finally, we denote by $t_j = j\delta_t$ and $r_i = r_{\min} + i\delta_r$ the discretized values of time and interest rate. Next, denote also by:

$$q^u(i,j) = \mathrm{Q}_T\left(r_{\gamma^u} \in [r_i, r_{i+1}],\ \gamma^u \in [t_j, t_{j+1}]\right)$$

the discretized version of the first-passage time distribution. One obtains the semi-closed-form formulas for $\mathcal{A}$ and $\mathcal{B}$, according as:

$$\begin{cases} \mathcal{A} & \approx \sum\limits_{j=0}^{n_T}\sum\limits_{i=0}^{n_r}\sum\limits_{k=0}^{n_r} \delta_r f_r(r_k\,|\,r_i)\,\kappa\Big(\widehat{\mu}_{t_j,T};\,\widehat{\Sigma}_{t_j,T};\,K\Big)\ q^u(i,j) \\ \mathcal{B} & \approx \sum\limits_{j=0}^{n_T}\sum\limits_{i=0}^{n_r}\sum\limits_{k=0}^{n_r} \delta_r f_r(r_k\,|\,r_i)\,\mathcal{N}\left(\frac{\ln(K)-\widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q^u(i,j) \end{cases} \tag{9}$$

where M_T and V_T are the moments of $\ln(S_T)$, and where κ is defined, for a Gaussian random variable X following the law $\mathcal{N}(m,\sigma^2)$, by:

$$\kappa(m;\sigma;a)\ =\ E\left(e^X \mathbb{1}_{e^X>a}\right)\ =\ \exp\left(m + \frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{m+\sigma^2-\ln(a)}{\sigma}\right)$$

and where f_r is the transition density of r (whose conditional moments are denoted by m and v):

$$f_{r_t}(\, r \mid r_s) = \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{(r-m)^2}{2v}}$$

and where $\widehat{\mu}_{s,T} = \mu(\, r_T,\, l_s,\, r_s\,)$ and $\widehat{\Sigma}^2_{s,T} = \Sigma^2(\, r_T,\, l_s,\, r_s\,)$ are the first two centered moments of the law of l_T conditional on $\mathcal{F}_s$ and given r_T. Their expressions are given in Appendix II.

The above development of $\mathcal{A}$ can be justified as follows. Start with the expression:

$$\mathcal{A} = E_{Q_T}\left[S_T \mathbb{1}_{\ln(S_T)>\ln(K)} \mathbb{1}_{\gamma^u \leqslant T}\ \right]$$

which can be simplified according as:

$$\mathcal{A} = E_{Q_T}\left[e^{l_T} \mathbb{1}_{l_T>\ln(K)} \mathbb{1}_{\gamma^u \leqslant T}\ \right]$$

Using conditioning operators, we develop this formula as:

$$\mathcal{A} = \int_0^T \int_{-\infty}^{+\infty} E_{\mathrm{Q}_T}\left[e^{l_T} \mathbb{1}_{l_T>\ln(K)} \mid r_{\gamma^u} = r,\, \gamma^u = s\right] \mathrm{Q}_T\left(r_{\gamma^u} \in dr,\, \gamma^u \in ds\right)$$

Because the process (l_T, r_T) is a Gaussian vector, the conditional law of l_T given r_T is normal. Using this conditional distribution and the transition density f_r of an Ornstein-Uhlenbeck process, together with $E^s_{\mathrm{Q}_T}$ the expectation operator condition on the available information $\mathcal{F}_s$, we can write:

$$\mathcal{A} = \int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dr' f_{r_T|r_s=r}(r')\, E^s_{\mathrm{Q}_T}\left[e^{l_T} \mathbb{1}_{l_T>\ln(K)} \mid r_T = r'\right] \mathrm{Q}_T\left(r_{\gamma^u} \in dr,\, \gamma^u \in ds\right)$$

This expectation can be rewritten as:

$$\mathcal{A} = \int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dr' f_{r_T|r_s=r}(r')\ \kappa\Big(\widehat{\mu}_{s,T};\, \widehat{\Sigma}_{s,T};\, K\Big)\, \mathrm{Q}_T\left(r_{\gamma^u} \in dr,\, \gamma^u \in ds\right)$$

and, finally, one obtains the discretized approximation of $\mathcal{A}$:

$$\mathcal{A} \approx \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_{r_T|r_{t_j}=r_i}(r_k)\ \kappa\Big(\widehat{\mu}_{t_j,T};\, \widehat{\Sigma}_{t_j,T};\, K\Big)\, q^u(i,j)$$

This explains the discretized approximation of $\mathcal{A}$ in formula (9). The one of $\mathcal{B}$ can be obtained in a similar way. Therefore, one has all the necessary elements to compute the *up* and *in* barrier call options formula (8). As mentioned above, the terms $q^u(i,j)$ can be computed using the methodology in Appendix I.

Now, to price an *up* and *out* call, it is sufficient to use the following parity relationship:

$$C^{uo} = P(0,T)E_{\mathrm{Q}_T}\left((S_T - K)^+\right) - C^{ui}$$

noting that:

$$P(0,T)E_{\mathrm{Q}_T}\left((S_T - K)^+\right) = P(0,T)\left(\kappa(M_T, S_T, K) - K\,\mathcal{N}\left(\frac{\ln(K) - M_T}{\sqrt{V_T}}\right)\right) \tag{10}$$

where all the moments and symbols are the same as defined before. Let us now come to the pricing of *down* barrier call options.

Pricing *down* call options

We will sketch the main ideas and formulas in this paragraph; clearly, all the derivations are analogical to the ones of the previous paragraphs. We start with the pricing of *down* and *in* call options. Their valuation formula in (2) can be reexpressed in the forward-neutral universe as:

$$C^{di} = P(0,T)E_{\mathrm{Q}_T}\left((S_T - K)^+ \mathbb{1}_{S_{\min} < H}\right)$$

Next, we denote by γ^d the first passage time by S of a *down* barrier H. Defining $S_{\min}$ on $[0,T]$, one has: $\{S_{\min} < H\} = \left\{\gamma^d \leqslant T\right\}$. By analogy with (8), we write:

$$\frac{C^{di}}{P(0,T)} = \mathcal{C} - K\,\mathcal{D} \tag{11}$$

where:

$$\begin{cases} \mathcal{C} & = E_{Q_T}\left(S_T \mathbb{1}_{S_T > K}\, \mathbb{1}_{\gamma^d \leqslant T}\right) \\ \mathcal{D} & = Q_T\left(S_T > K, \;,\; \gamma^d \leqslant T\right) \end{cases}$$

and where these formulas can be discretized in semi-closed form as:

$$\begin{cases} \mathcal{C} & = \sum\limits_{j=0}^{n_T}\sum\limits_{i=0}^{n_r}\sum\limits_{k=0}^{n_r} \delta_r f_r(r_k \,|\, r_i)\, \kappa\Big(\widehat{\mu}_{t_j,T};\, \widehat{\Sigma}_{t_j,T};\, K\Big)\; q^d(i,j) \\ \mathcal{D} & = \sum\limits_{j=0}^{n_T}\sum\limits_{i=0}^{n_r}\sum\limits_{k=0}^{n_r} \delta_r f_r(r_k \,|\, r_i) \mathcal{N}\left(\frac{\ln(K) - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q^d(i,j) \end{cases} \tag{12}$$

As concerns the *down* and *out* call, its pricing follows readily from the following parity relationship:

$$C^{do} = P(0,T)E_{\mathrm{Q}_T}\left((S_T - K)^+\right) - C^{di}$$

where the first term is given by Equation (10).

To conclude this section, we have constructed semi-closed-form expressions for standard barrier options under a Vasicek model for the interest rate dynamics. The term “semi” in “semi-closed form” refers to the fact that the $q^u(i,j)$ and $q^d(i,j)$ factors are only the discretized version of the first passage time distribution. In practice, and as the final section we show, these semi-closed form formulas can be computed extremely quickly. The next section shows how this methodology can be used to price a particular exotic contract.

2 Pricing a Structured Barrier Option

Our aim will now be to shed some light on the use of the above method to price some exotic contracts. We start defining the “shark” option, which was introduced a couple of years ago by the Equity desk of an international bank.

2.1 The Shark Index Option

In its most basic form, a shark option is an option whose holder is entitled to receive a rebate at expiry if the underlying index hits a barrier and a European payoff otherwise. The latter depends on the value of the underlying index at expiry and may take the form of a European call or a functional of it, as the following developments will show. The underlying index may be a financial asset, an interest rate, an exchange rate or an equity index. In full generality, it is correlated to the interest rates. Here, we assume that payments are always settled at expiry (ranging typically from one to five years for these options). The presence of a barrier decreases the premium, compared to vanilla options. The barrier can be hit from below or from above and can be a knock-in or a knock-out one. It may also be constant, deterministic or stochastic.

For the sake of clarity, we shall consider from now on a special kind of shark option. Yet, the reader should keep in mind that our method can be applied to value many other similar products. Let us describe more precisely our contract: it is a medium-term structured note, having a one to five year maturity, guaranteeing the investor (purchaser of the shark) 100% of his capital, and linearly linked to an Equity Index. However, this link is cut as soon as the growth rate of the index is equal or greater than $\alpha\%$ during the shark's life, in which case the investor receives $\beta\%$ of his initial investment at the end. In formal terms, the investor receives at expiry time T:

$$\begin{cases} M.\left(1+R_T\right) & \text{if } S_{\max} \leqslant (1+\alpha)\, S_0 \\ M.\beta & \text{otherwise} \end{cases} \tag{13}$$

where $R_T = \frac{(S_T - S_0)^+}{S_0}$, M is a notional amount, S_t is the index at time t or the underlying shark's price at time t, and $S_{\max}$ is the maximum of S before the shark's maturity, that is, over $[0, T]$.

As concerns α and β, they respectively describe the barrier level and the value of the rebate, and we let them satisfy $\alpha > 0$ and $0 < \beta < 1 + \alpha$. We call this structured product a standard shark and, without loss of generality, we assume $M = 1$ for the sake of simplicity. In our example and for our numerical analysis in section 3: $\alpha = 0.35$ and $\beta = 1.1$. We denote by H the barrier level:

$$H = (1 + \alpha) S_0$$

The payoff at maturity (assuming $M = 1$) then writes:

$$(1 + R_T)\, \mathbb{1}_{S_{\max} \leqslant H} + \beta\, \mathbb{1}_{S_{\max} > H} \tag{14}$$

In fact, one has $1 + R_T = 1 + \frac{(S_T - S_0)^+}{S_0}$. This allows rewriting the payoff as:

$$1 + \frac{1}{S_0}\, (S_T - S_0)^+\, \mathbb{1}_{S_{\max} \leqslant H} + (\beta - 1)\, \mathbb{1}_{S_{\max} > H} \tag{15}$$

Technically, a shark option is merely an *up* and *out* barrier call option with rebate. Indeed, $(S_T - S_0)^+\, \mathbb{1}_{S_{\max} \leqslant H}$ is the payoff of an *up* and *out* call on the underlying S, with a strike price $K = S_0$, and a barrier H.

Denoting by r the risk-free interest rate, and using the fundamental result of arbitrage pricing theory, and the expression of the final payoff (14), we can express the shark's option equilibrium price at time 0 as:

$$C(0, T) = E_Q \left(e^{-\int_0^T r_s ds} \left(1 + \frac{1}{S_0}\, (S_T - S_0)^+\, \mathbb{1}_{S_{\max} \leqslant H} + (\beta - 1)\, \mathbb{1}_{S_{\max} > H} \right) \right) \tag{16}$$

Coming now to the practical valuation of our barrier product, we set ourselves in the forward-neutral world where the underlying follows (6). One readily obtains using this latter world:

$$\frac{C(0, T)}{P(0, T)} = \left(1 + \frac{1}{S_0}\, C^{uo}(S_T,\, K = S_0,\, \text{Barrier } H) + (\beta - 1) \mathrm{Q}_T \left(S_{\max} > H \right) \right) \tag{17}$$

The only term that cannot be computed using the first section is $\mathrm{Q}_T \left(S_{\max} > H \right)$. We denote it by $\mathcal{E}$ and this is in fact $\mathrm{Q}_T \left(\gamma^u \leqslant T \right)$. Using the approximation of the distribution of γ^u (see Appendix I), one obtains:

$$\mathcal{E} = \mathrm{Q}_T \left(\gamma^u \leqslant T \right) \approx \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} q^u(i, j)$$

Indeed, to obtain this formula, one should start writing:

$$\mathcal{E} = \int_0^T \int_{-\infty}^{+\infty} \mathrm{Q}_T \left(r_{\gamma^u} \in dr,\, \gamma^u \in ds \right)$$

and then discretize along time and interest rate, and introduce the $q^u(i,j)$ terms.

Using this discretized version of the first-passage time distribution, one can obtain the following formula for the shark contract value when the barrier is constant:

$$C(0,T) = P(0,T) + \frac{1}{S_0}\, C^{uo}(S_T, K = S_0, H) + (\beta - 1)\, P(0,T)\, \mathcal{E}$$

which computes straightforwardly using our previous results. We shall now concentrate on the particular case where the barrier is slightly modified in terms of a zero-coupon bond: this case is particularly interesting because fully closed-form formulas can be obtained.

2.2 Discounted Barrier Options

In this subsection, we take into account the effect of discounting the barrier. At first look, such a structured product seems difficult to value fully explicitly. In fact, we show below that this is the contrary and that the pricing problem can be solved in closed-form. We assume that the frontier is given by a discounted constant barrier. Formally, K being a constant, the barrier is a stochastic process $(D_t)_{t\in[0,T]}$ such that:

$$D_t = K P(t,T) \tag{18}$$

where, only in this section, this expression replaces in the contract covenant the barrier $H = (1+\alpha)\, S_0$. The shark's formula then becomes:

$$C(0,T) \;=\; P(0,T)\, E_{Q_T}\left[(1+R_T)\, \mathbb{1}_{\{\forall t\in[0,T],\ S_t \leqslant D_t\}} + \beta \mathbb{1}_{\{\exists t\in[0,T],\ S_t > D_t\}}\right]$$

In fact, the factor $1 + R_T = 1 + \frac{(S_T - S_0)^+}{S_0}$ can also be written as:

$$1 + R_T = \mathbb{1}_{\{S_T < S_0\}} + \frac{S_T}{S_0}\mathbb{1}_{\{S_T > S_0\}} \tag{19}$$

which allows to write together with equation (18):

$$\begin{aligned} C(0,T) \;=\; & \beta\ P(0,T)\ Q_T\left(\sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) >\ K\right) \\ & +\ P(0,T)\ Q_T\left(S_T < S_0,\ \sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) \leqslant\ K\right) \\ & +\ E_Q\left[e^{-\int_0^T r_s ds}\ \frac{S_T}{S_0}\mathbb{1}_{\left\{S_T \geqslant S_0,\ \sup\limits_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right)\leqslant\ K\right\}}\right] \end{aligned}$$

Notice that the third term is expressed under the risk-neutral probability Q. To simplify the following developments, we divide the Shark contract into a sum of three expressions according as:

$$C[0,T] = P(0,T)\,[\beta E_1 + E_2] + E_3$$

where the three sub-contributions to the contract can be defined as:

$$\begin{cases} E_1 &= Q_T\left(\sup\limits_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) > K\right) \\ \\ E_2 &= Q_T\left(S_T < S_0,\ \sup\limits_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right) \leqslant K\right) \\ \\ E_3 &= E_Q\left[e^{-\int_0^T r_s ds}\ \frac{S_T}{S_0}\mathbb{1}_{\left\{S_T \geqslant S_0,\ \sup\limits_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right)\leqslant K\right\}}\right] \end{cases} \tag{20}$$

Then, against all expectations, one can obtain the following proposition:

Proposition 2.1. *The three components of a shark contract, when the barrier is proportional to a zero-coupon bond and under a Vasicek term structure of interest rates, can be written in closed-form as follows:*

$$\begin{cases} E_1 &= \mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ \\ E_2 &= \mathcal{N}\left(\frac{\ln(P(0,T)) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right) + \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ \\ E_3 &= \mathcal{N}\left(\frac{\ln\left(\frac{KP(0,T)}{S_0}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ & -\mathcal{N}\left(\frac{\ln(P(0,T)) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \end{cases} \tag{21}$$

where $\tau(T) = \int_0^T \left[(\sigma_P(u,t) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right] du$ *and* $\mathcal{N}$ *is the cumulative standard normal distribution function.*

The proof of this proposition can be found in Appendix III. To sum up, we have obtained a closed-form formula for the shark option in the case of a stochastic barrier defined as in (18). Moreover, this closed-form formula is very simple and has the same computational efficiency as the one we would obtain with a constant term structure of interest rates (see Rubinstein and Reiner (1991) for the pricing of barrier options in a Black and Scholes context).

Unfortunately, the simplicity of the above result does not hold when the barrier is merely a constant one, as exposed in the beginning of this article: semi-closed form formulas are then in order. In the coming section, we shall compare the two main contracts (defined respectively with a discounted and a constant barrier) ; a full sensitivity analysis of these products will be presented.

3 Numerical Analysis

One of the main goals of this article being to develop a new methodology to study barrier products in the presence of stochastic interest rates, we start by checking its accuracy by comparing the results it provides to the ones obtained by means of Monte-Carlo simulations. By doing so, we show that the extended Fortet's method does indeed work correctly, and that it is much faster than the Monte-Carlo method.

Secondly, and from subsection 3.3 on, we shall concentrate on the analysis of the shark option, which is the core product example of our study. We compare the prices and sensitivities of these contracts written either with a stochastically discounted barrier, e.g. $(1+\alpha)S_0\,P(t,T)$, or with a constant barrier, e.g. $(1+\alpha)S_0$. Amongst the sensitivities studied here are the ones computed with respect to the barrier level, to the underlying index's volatility or to its correlation with the interest rates. Let us start by giving the values of the parameters involved in our numerical analysis.

3.1 Parameters

In Table 1, we give some values for the general parameters useful for the coming option valuations. Some of them will vary later on, and this shall be indicated in due time.

M	S_0	σ	T	α	β	ρ
1	100	20%	1	0.35	1.1	0.3

Table 1: Data

We briefly recall the meanings of the above coefficients. The nominal of the contract, M, is set to one for the sake of simplicity. S_0 stands for the initial value of the Equity Index. σ is the underlying's volatility and is set to 20%. The contract's maturity, T, is equal to 1 year. As concerns the maximum yield, in other words the factor governing the level of the barrier, it is given by $1 + \alpha = 1.35$. The barrier level is indeed given by $H = (1 + \alpha)S_0 = 135$. The rebate's percentage is equal to $\beta = 110\%$. ρ is the correlation coefficient between the Index process and the instantaneous interest rate process r.

We made our study with an exponential structure of the volatility of the interest rates, specified by the two parameters a and ν. The values chosen for the interest rate process parameters are given in Table 2.

a	ν	r_0	θ
0.46	0.007	0.015	0.05

Table 2: Interest Rate Process

r_0 and θ are necessary to specify the initial term structure of interest rates. In the particular calibration subsetting chosen here, this is equivalent to knowing the Government yield curve.

3.2 Fortet's Methodology

We start by pricing a shark option when the main parameters are defined as in Table 1 and 2. Table 3 displays numerical estimations of the option, done implementing the extended Fortet's method.

Extended Fortet	Shark Price	Time
$n_T = 100,\ n_r = 30$	1.0092	13 s
$n_T = 100,\ n_r = 50$	1.0175	40 s
$n_T = 200,\ n_r = 50$	1.0172	2 min
$n_T = 400,\ n_r = 50$	1.0168	8 min

Table 3: Shark Option Values

Table 4 gives numerical results obtained with Monte-Carlo simulations. To obtain accurate results, when using the Monte-Carlo method to price path-dependent derivatives, it is well known that one needs to choose a thin discretization time step and to simulate a lot of sample paths. In this context, N refers to the number of paths, and δt is the time step. One of the first conclusions stemming from the analysis of Table 4 is that setting a time step small enough is of critical importance to the fairness of the evaluation. If not, a discretization bias shifts the value of the contract, whatever the number of simulations. Note that an alternative to increasing the number of time steps is to use a method correcting for the bias induced by the hitting probabilities between two time steps; see for instance the paper of Andersen and Brotherton-Ratcliffe (1996).

Monte-Carlo	Shark Price	Time
$N = 10^6,\ \delta t = 1/12$	1.0375	2 min
$N = 10^6,\ \delta t = 1/300$	1.0344	40 min
$N = 10^6,\ \delta t = 1/10\,000$	1.0336	2h30

Table 4: Shark Option Values

The main conclusion that can be drawn from the analysis of Table 4 is that the extended Fortet's method proves much faster than the Monte-Carlo one. The extended Fortet's method gives three digits of precision in ten minutes of computation time, which if not instantaneous is yet extremely efficient, considering that we are doing the numerical valuation of a path-dependent contract under a stochastic term structure of interest rates.

Furthermore, we observe a rather good convergence for the option values with the extended Fortet's method, whilst the Monte-Carlo method is very slow to converge. Indeed, and not surprisingly, this path-dependent problem requires a very thin time discretization and many sample paths. Hence to obtain a sufficient precision with Monte-Carlo, it would be necessary to launch simulations lasting many hours, which is unacceptable for practical use. Finally note that the implementation of both methods has been done making an extensive use of Matlab's vectorization tools, on a $3\,GHz$ computer.

3.3 Comparison of Contracts

We want to compare the two types of contracts described in section 2.1 and 2.2 (shark contracts with respectively a constant barrier H and a discounted barrier $KP(t,T)$). To enable an efficient comparison of both contracts, we first set $H = K = S_0(1+\alpha)$ (identical levels at contract maturity) and then $K = (1+\alpha)S_0$ and $H = (1+\alpha)S_0P(0,T)$ (identical levels at inception of the contracts). The results of our computations are given in Table 5, where the hitting probability and the contract value $C(0,T)$ are displayed.

Values	Discounted Barrier $(1+\alpha)S_0P(t,T)$	Constant Barrier $H_1 = (1+\alpha)S_0$	Constant Barrier $H_2 = (1+\alpha)S_0P(0,T)$
Hitting Prob.	0.144	0.133	0.161
$C(0,T)$	1.033	1.013	1.006

Table 5: Comparison of Several Contracts

When the underlying touches the barrier, the final payoff is worth β (see Formula (13) where M is set to one); this is to be compared to $1+\alpha$, the payoff obtained when the barrier remained unattained. Clearly, the crossing upward of the barrier by the index is not advantageous to the optionholder (because $\beta < 1 + \alpha$). Then, it should be noticed that the constant barrier $H_2 = (1 + \alpha)S_0 P(0, T)$ is inferior to the constant barrier $H_1 = (1 + \alpha)S_0$. The probability to hit H_2 is higher than the probability to hit H_1 (0.161 instead of 0.133). The contract built with H_2 is therefore less interesting and its value is smaller (1.006 instead of 1.013), compared to the one built with H_1. The main point here is that, whatever the choice of constant barrier, the contract with a discounted barrier is always the most expensive and interesting one - though the hitting probability of the stochastic barrier does not fall down.

3.4 Sensitivity to the Barrier Level

Let us now come to the numerical study of the sensitivities to the barrier level. To do this, we plot the probability of hitting the barrier, in Figure 1, and $C(0, T)$, the contract's value, in Figure 2 with respect to the barrier level (defined respectively by $H = (1 + \alpha)S_0$ and $HP(t, T)$). We keep the parameter values from Table 1, except α which ranges between 0.1 and 1 (in correspondence to H which ranges between 110 and 200).

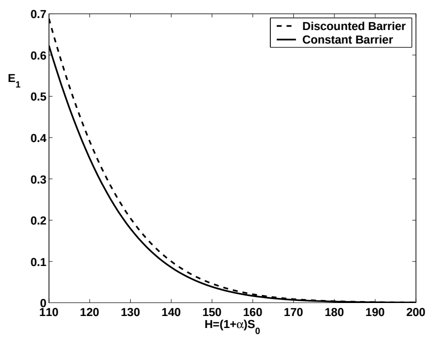

Figure 1: Hitting Prob. w.r.t. H

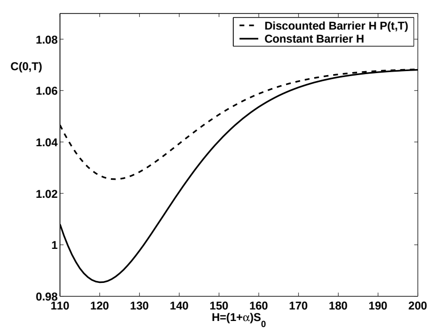

Figure 2: $C(0, T)$ w.r.t. H

The interpretation of Figure 1 is straightforward: as the barrier increases, the probability that it be hit sharply diminishes. When the barrier value is high enough (say 170), the probability to reach it is nearly null.

Despite the gross appearance of Figure 2, the influence of the barrier level on the price is indeed quite small. In particular, the contract's price shows a relative variation of less than 3% when the barrier goes from 110 to 200 (assuming $\beta = 1.1 < 1 + \alpha$). The explanation of this phenomenon obtains directly from a P&L analysis. At expiry time T, the investor gets back his initial investment, whether the barrier has been reached or not, and this payment mostly determines the price of the contract. Obviously, for a knock-out option without rebate, we would observe a stronger influence of the barrier on the price.

Let us now come to a finer description of Figure 2. One can observe that the price of the option is decreasing with respect to the barrier for low levels of the barrier. This comes from the fact that for $\beta = 1.1$, the rebate is quite important. Choosing a rebate $\beta = 0.3$ and *ceteris paribus*, one would obtain the graph displayed in Figure 3.

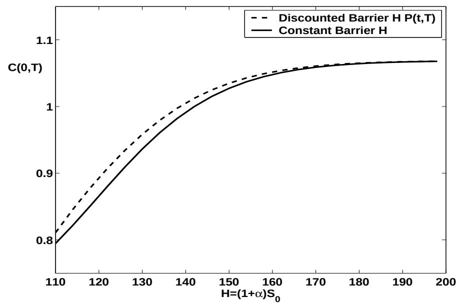

Figure 3: $C(0,T)$ w.r.t. H with $\beta = 30\%$

Now, how can we explain the weird behavior of the shark price when the barrier varies between 110 and 120 in Figure 2? In general, it is advantageous not to hit the barrier; yet, in the presence of a high rebate, say when $1+\alpha \approx \beta$, the probability to get a yield strictly superior to β is equal to the joint probability that the following events occur: $S_{\max} < (1+\alpha)S_0$ and $S_T > \beta S_0$. As this joint probability is very weak, it is in the interest of the optionholder that the barrier be hit, in order to ensure a return at least equal to β. To conclude on this particular situation, when the barrier level is increased, the probability to reach it is diminished, and the contract becomes less interesting, which explains the decrease of its price.

3.5 Sensitivity to the Index Volatility σ

Let us now come to a brief study of our product's sensitivity with respect to the underlying index volatility.

In Figure 4, we represent the hitting probability with respect to σ, the volatility of the Equity Index. All parameters are chosen as in subsection 3.1, except σ which ranges between 1% and 80%. The conclusion of this study is straightforward: for both contracts, when the volatility of the underlying increases, the hitting time probability is increased accordingly. This is indeed a standard fea-

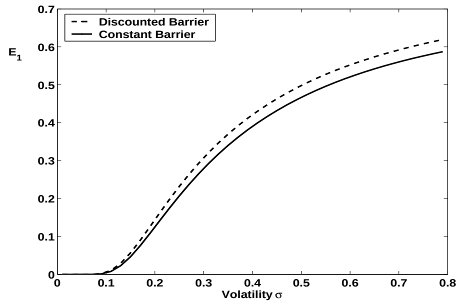

Figure 4: Hitting Probability w.r.t. σ

ture that we recover here and which is common to all barrier derivatives.

3.6 Sensitivity to the Correlation ρ

Figures 5 and 6 plot respectively the probability to hit the barrier (before the contract maturity) and the contract value with respect to ρ, the correlation coefficient between the Equity Index and the interest rate. We let the correlation ρ vary between -0.8 and 0.8 in both graphs.

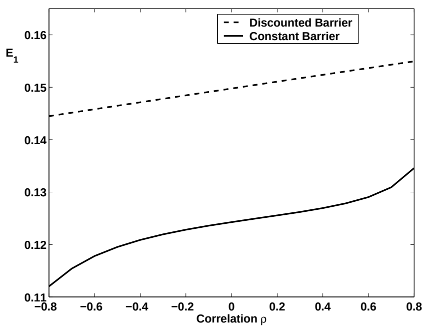

Figure 5: Hitting Prob. w.r.t. ρ

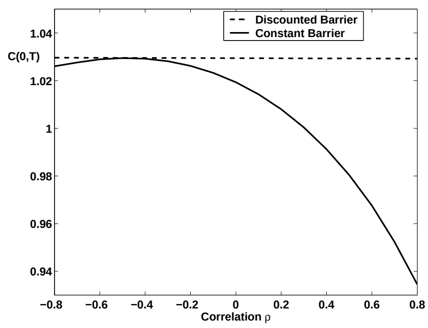

Figure 6: $C(0,T)$ w.r.t. ρ

Let us first consider the case of a discounted barrier. In this particular situation, the hitting time probability and contract price are nearly insensitive to a change in the correlation. On the contrary, when the barrier is constant, the shark's price is a remarkably decreasing function of the correlation. Indeed, one of the advantages of imposing a stochastic barrier appears here: it can help cancel the impact of the randomness of interest rates on derivative prices.

Conclusion

This article develops a general methodology useful for pricing barrier options in a Vasicek framework. When the derivative's barrier is a discounted one, we show that it is possible to obtain closed-form formulas to price it, using time change techniques. When the barrier is constant, quasi-closed-form formulas can be found. These latter formulas can be computed using the extended Fortet's method, exposed within a new and clean apparel in the first appendix of this text, and whose first implementation dates back to Collin-Dufresne and Goldstein (2001) in their seminal structural model of the firm. What we do is indeed obtaining general formulas that extend the ones of Rubinstein and Reiner (1991) for pricing barrier options when the driving risk-free interest rate is a Vasicek process. We illustrate our approach on a particular exotic derivatives, the shark index, which is indeed a type of up and out barrier option with rebate. Concluding this paper, a numerical analysis on shark options gives a practical illustration of the method, and shows how quickly it works.

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APPENDIX I

The extended Fortet's Method.

Let assume that one initially observes $\ln(A_0) = l_0 > \ln(H) = h$. The process l_t is continuous. If at time t, the process $l_t = \ell < h$ then the barrier has been hit and the *down* condition is realized. We denote by γ^d this first hitting time. It verifies $\gamma^d \in [0, t]$. Thanks to this remark, one has:

$$\begin{aligned}&\mathrm{Q}_T\left(l_t \in [\ell, \ell + d\ell[, r_t \in [r, r + dr[\mid l_0, r_0\right)\\ &= \int_0^t \int_{-\infty}^{+\infty} \mathrm{Q}_T\left(l_t \in [\ell, \ell + d\ell[, r_t \in [r, r + dr[\mid l_s = h, r_s = r'\right) \mathrm{Q}_T\begin{pmatrix} r_{\gamma^d} \in [r', r' + dr'[\\ \gamma^d \in [s, s + ds[\end{pmatrix}\end{aligned}$$

Let integrate the previous equation with respect to ℓ between $-\infty$ and h. After inverting the two integrals, one obtains:

$$\begin{aligned}&\mathrm{Q}_T\left(l_t \leqslant h, r_t \in [r, r + dr[\mid l_0, r_0\right)\\ &= \int_0^t \int_{-\infty}^{+\infty} \mathrm{Q}_T\left(l_t \leqslant h, r_t \in [r, r + dr[\mid l_s = h, r_s = r'\right) \mathrm{Q}_T\begin{pmatrix} r_{\gamma^d} \in [r', r' + dr'[\\ \gamma^d \in [s, s + ds[\end{pmatrix}\end{aligned} \tag{22}$$

To simplify the notations, we define respectively Φ and Ψ by:

$$\begin{cases} \Phi(r, t)\, dr &= \mathrm{Q}_T\left(l_t \leqslant h, r_t \in [r, r + dr[\mid l_0, r_0\right) \\ \Psi(r, t, r', s)\, dr &= \mathrm{Q}_T\left(l_t \leqslant h, r_t \in [r, r + dr[\mid l_s = h, r_s = r'\right) \end{cases}$$

Under these assumptions, Φ and Ψ could be expressed as closed-form formulas. The previous equation (22) becomes:

$$\Phi(r, t) = \int_{s \in [0,t]} \int_{r' \in \mathbb{R}} \Psi(r, t, r', s)\, \mathrm{Q}_T\left(r_{\gamma^d} \in [r', r' + dr'[, \; \gamma^d \in [s, s + ds[\right) \tag{23}$$

As the distribution function of γ^d is unknown, we approximate it. Discretizing along the time and interest rate, with n_T discretization steps along the time $t_0 = 0, t_1, ..., t_{n_T} = T$ and n_r along the interest rate. One has $r_1 = r_{\min}, ..., r_{n_r} = r_{\max}$ where $r_{\min}$ and $r_{\max}$ are chosen such as the probability that r takes values outside the interval $[r_{\min}, r_{\max}]$ is negligible. We denote by $q^d(i, j)$:

$$q^d(i, j) = \mathrm{Q}_T\left(r_{\gamma^d} \in [r_i, r_{i+1}], \gamma^d \in [t_j, t_{j+1}]\right)$$

Then, formula (23) could be written as:

$$\Phi(\, r_i,\, t_j\,) = \sum_{v=0}^{j} \sum_{u=0}^{n_r} \Psi(\, r_i,\, t_j,\, r_u,\, t_v\,)\, q^d(u,v)$$

If $j = 0$, the previous expression becomes:

$$\Phi(\, r_i,\, t_0\,) = \sum_{u=0}^{n_r} \Psi(\, r_i,\, t_0,\, r_u,\, t_0\,)\, q^d(u,0)$$

We then obtain the following expression: $q^d(i,0) = \mathrm{Q}_T\left(r_{\gamma^d} \in [r_i, r_{i+1}], \gamma^d \in [t_0, t_1]\right)$. Noting that $\Psi(\, r_i,\, t_0,\, r_u,\, t_0\,) = \mathbb{1}_{\{r_i = r_u\}}$, one readily has:

$$q^d(i,0) \;=\; \Phi(\, r_i,\, t_0\,)$$

The quantities $q^d(\, i,\, j\,)$ can be computed by means of a recursive algorithm. First, the quantities $q^d(\, i,\, 0\,)$ are computed for every i thanks to the above expression. From them the quantities $q^d(\, i,\, j\,)$ for $j \geq 1$ are recursively obtained.

$$\begin{aligned}
\Phi(\, r_i,\, t_j\,) &= \sum_{v=0}^{j} \sum_{u=0}^{n_r} q^d(\, u,\, v\,)\; \Psi(\, r_i,\, t_j,\, r_u,\, t_v\,) \\
&= \sum_{u=0}^{n_r} q^d(u,j)\; \Psi(\, r_i,\, t_j,\, r_u,\, t_j\,) + \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q^d(\, u,\, v\,)\; \Psi(\, r_i,\, t_j,\, r_u,\, t_v\,)
\end{aligned}$$

Thanks to $\Psi(\, r_i,\, t_j,\, r_u,\, t_j\,) = \mathbb{1}_{\{r_i = r_u\}}$, one has:

$$q^d(i,j) = \Phi(\, r_i,\, t_j\,) - \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q^d(\, u,\, v\,)\; \Psi(\, r_i,\, t_j,\, r_u,\, t_v\,) \tag{24}$$

To sum up, we have now, with formula (24) the possibility to compute the terms $q^d(i,\, j)$, which give us the approximated distribution function of γ^d we are looking for because we have closed-form expressions for $\Phi(\, r,\, t\,)$ and $\Psi(\, r,\, t,\, r',\, s\,)$:

$$\begin{cases}
\Phi(\, r,\, t\,)\, dr & = \mathrm{Q}_T\, (\, l_t \leqslant h, r_t \in dr \mid l_0,\, r_0) \\[2ex]
\Psi(\, r,\, t,\, r',\, s)\, dr & = \mathrm{Q}_T\, (\, l_t \leqslant h, r_t \in dr \mid l_s = h, r_s = r')
\end{cases}$$

Note that $X = (l, r)$ is a Gaussian Markov process whose dynamics are given by:

$$dX_t = d \left[\begin{array}{c} l_t \\ r_t \end{array} \right] = \left[\begin{array}{c} r_t - r_g - \frac{\sigma^2}{2} - \sigma\rho\sigma_P(t,T) \\ a\left(\theta - \frac{\nu}{a}\sigma_P(t,T) - r_t\right) \end{array} \right] dt + \left[\begin{array}{cc} \sigma\rho & \sigma\sqrt{1-\rho^2} \\ \nu & 0 \end{array} \right] \cdot \left[\begin{array}{c} dZ_1^T \\ dZ_2^T \end{array} \right].$$

Denote by f_{l_t, r_t} the density function of (l_t, r_t) under Q_T. Thanks to conditional results, one obtains:

$$f_{l_t, r_t}(\ell,\, r) \;=\; f_{r_t}(r)\; f_{l_t | r_t}(\ell)$$

$\mathcal{F}_0$ and $\mathcal{F}_s$ represent the available information at time 0 and s. Using the Markov property of (l_t, r_t), conditioning by $\mathcal{F}_s$ is like conditioning by (l_s, r_s). One then obtains Ψ and Φ:

$$\begin{cases} \Phi(r, t) & = f_{r_t}(r|\,\mathcal{F}_0)\ \int_{-\infty}^{h} f_{l_t|r_t}(\ell \mid \mathcal{F}_0)\,d\ell \\ \\ \Psi(r, t, r', s) & = f_{r_t}(r|\,\mathcal{F}_s)\ \int_{-\infty}^{h} f_{l_t|r_t}(\ell \mid \mathcal{F}_s)\,d\ell \end{cases}$$

As the process (l_t, r_t) is Gaussian, the conditional law of $l_t|r_t$ knowing the available information at time s is Gaussian. We denote the conditional moments by $\mu(r_t, l_s, r_s)$ and $\Sigma^2(r_t, l_s, r_s)$.

$$\begin{cases} \mu(r_t, l_s, r_s) & = E_{Q_T}\,[l_t \mid \mathcal{F}_s] + \frac{\mathrm{Cov}(l_t,\, r_t \mid \mathcal{F}_s)}{V\mathrm{ar}[r_t \mid \mathcal{F}_s]}\,(r_t - E_{Q_T}\,[r_t \mid \mathcal{F}_s]) \\ \\ \Sigma^2(r_t, l_s, r_s) & = V\mathrm{ar}\,[l_t \mid \mathcal{F}_s] - \frac{\mathrm{Cov}(l_t,\, r_t \mid \mathcal{F}_s)^2}{V\mathrm{ar}[r_t \mid \mathcal{F}_s]} \end{cases}$$

The above moments are computed in Appendix II. Let $\mathcal{N}$ be the normal standard distribution function. We then obtain:

$$\begin{cases} \Phi(r, t) & = f_{r_t}(r \mid r_0)\ \mathcal{N}\left(\frac{h-\mu(r, l_0, r_0)}{\sqrt{\Sigma^2(r, l_0, r_0)}}\right) \\ \\ \Psi(r, t, r', s) & = f_{r_t}(r \mid r_s = r')\ \mathcal{N}\left(\frac{h-\mu(r, l_s=h, r')}{\sqrt{\Sigma^2(r, l_s=h, r')}}\right) \end{cases}$$

where f_r is the transition density of r. Recall that:

$$f_{r_t}(r \mid r_s) = \frac{1}{\sqrt{2\pi v}}\ e^{-\frac{(r-m)^2}{2v}}$$

where $m = E[r_t|r_s]$ and $v = V\mathrm{ar}[r_t|r_s]$ (given in Appendix II).

Remark: The *up* case.

The *up* case is in fact the case when $l_0 < h$. We define as γ^u the first hitting time of the process l_t to the barrier's level $\ln(H) = h$. The proof is exactly the same as in the *down* case. Thus, one obtains the following formulas for the approximate density of (r_{γ^u}, γ^u) (similar to formula (24)):

$$\begin{cases} q^u(i, 0) & = \Phi^u(r_i, t_0) \\ q^u(i, j) & = \Phi^u(r_i, t_j) - \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q^u(u, v)\ \Psi^u(r_i, t_j, r_u, t_v) \end{cases} \tag{25}$$

where

$$\begin{cases} \Phi^u(r, t) & = f_{r_t}(r \mid r_0)\ \mathcal{N}\left(\frac{\mu(r, l_0, r_0)-h}{\sqrt{\Sigma^2(r, l_0, r_0)}}\right) \\ \\ \Psi^u(r, t, r', s) & = f_{r_t}(r \mid r_s = r')\ \mathcal{N}\left(\frac{\mu(r, l_s=h, r')-h}{\sqrt{\Sigma^2(r, l_s=h, r')}}\right) \end{cases}$$

APPENDIX II

Moments of the Processes r_t and l_t.

We work under the forward-neutral measure Q_T. We compute in this appendix the moments of the instantaneous interest rate r and those of l associated with the index process. We choose to do the study with the exponential structure of volatility. With $\nu > 0$ and $a > 0$, the volatility structure can be written as follows:

$$\sigma_P(t,T) = \frac{\nu}{a}\left(1 - e^{-a(T-t)}\right)$$

Define B_a by:

$$B_a(u) = \frac{1}{a}\left(1 - e^{-au}\right)$$

Under the forward-neutral measure, the interest rate process r follows the dynamics given by:

$$dr_t = a\left(\theta_t - r_t\right)dt + \nu dZ_1^T(t)$$

where $\theta_t = \theta - \frac{\nu^2}{a}B_a(T-t)$. Thanks to Itō's lemma and an integration by parts, one obtains:

$$r_t = e^{-at}\left(r_u e^{au} + \int_u^t \theta_s e^{as}ds + \nu\int_u^t e^{as}dZ_1^T(s)\right).$$

In this particular case, the instantaneous interest rate r is an Ornstein-Uhlenbeck process under the forward-neutral probability Q_T. The zero-coupon bond maturing at T satisfies the relationship:

$$P(t,T) = e^{-B_a(T-t)r_t - \eta(T-t)} \tag{26}$$

where:

$$\eta(u) = \left(\theta - \frac{\nu^2}{2a^2}\right)(u - B_a(u)) + \frac{\nu^2}{4a}(B_a(u))^2$$

Conditional moments of the process r

r is a gaussian process with the following conditional moments (with $s < t$):

$$\begin{cases} E_{Q_T}[\, r_t \mid r_u \,] & = e^{-a(t-u)}r_u + \left(\theta a - \frac{\nu^2}{a}\right)B_a(t-u) + \frac{\nu^2}{a}e^{-a(T-t)}B_{2a}(t-u) \\ Var_{Q_T}[\, r_t \mid r_u \,] & = \nu^2 B_{2a}(t-u) \\ \mathrm{Cov}_{Q_T}(r_s, r_t \mid r_u) & = \frac{\nu^2}{2a}\, e^{-a(s+t)}\left(e^{2as} - e^{2au}\right) = \nu^2 e^{-a(t-s)}B_{2a}(s-u) \end{cases}$$

Conditional moments of the process l

Integrating the dynamics (7) of the process l under Q_T between u and t, one has:

$$l_t = l_u + \int_u^t r_s ds - \left(\frac{\sigma^2}{2} + \frac{\sigma\rho\nu}{a}\right)(t-u) + \sigma\rho\nu \int_u^t e^{-a(T-s)} ds + \sigma\rho \int_u^t dZ_1^T(s) + \sigma\sqrt{1-\rho^2}\int_u^t dZ_2^T(s)$$

Now remark that the integral $\int_u^t r_s ds$ is also a gaussian process: whose conditional moments are given by the following formulas:

$$\begin{cases} E_{Q_T}\left[\ \int_u^t r_s ds \mid \mathcal{F}_u\ \right] = r_u B_a(t-u) + \int_u^t e^{-as} \int_u^s e^{ax}\theta_x dx ds \\ Var_{Q_T}\left[\ \int_u^t r_s ds \mid \mathcal{F}_u\ \right] = \frac{\nu^2}{a^2}\left(t-u+B_{2a}(t-u) - 2B_a(t-u)\right) \\ \mathrm{Cov}_{Q_T}\left(\int_u^t r_v dv, \int_u^t dZ_1^T(s) \mid \mathcal{F}_u\ \right) = \frac{\nu}{a}\left(t-u-B_a(t-u)\right) \end{cases}$$

This enables us to obtain the following conditional moments for the process l_t when $s < t$:

$$\begin{cases} E_{Q_T}[\ l_t \mid \mathcal{F}_u\ ] = l_u - \left(r_g + \frac{\sigma^2}{2} + \frac{\sigma\rho\nu}{a} - \theta + \frac{\nu^2}{a^2}\right)(t-u) - \frac{\nu^2}{a^2}e^{-a(T-t)}B_{2a}(t-u) \\ \qquad + \left(r_u - \theta + \frac{\nu^2}{a^2} + \frac{\nu^2}{a^2}e^{-a(T-t)} + \frac{\sigma\rho\nu}{a}e^{-a(T-t)}\right)B_a(t-u) \\ Var_{Q_T}[\ l_t \mid \mathcal{F}_u\ ] = \left(\sigma^2 + \frac{\nu^2}{a^2} + 2\frac{\sigma\rho\nu}{a}\right)(t-u) - 2\left(\frac{\nu^2}{a^2} + \frac{\sigma\rho\nu}{a}\right)B_a(t-u) + \frac{\nu^2}{a^2}B_{2a}(t-u) \\ \mathrm{Cov}(l_s, l_t \mid \mathcal{F}_u\ ) = \frac{\nu^2}{a^2}e^{-a(t-s)}B_{2a}(s-u) + \left(\sigma^2 + \frac{2\sigma\rho\nu}{a} + \frac{\nu^2}{a^2}\right)(s-u) \\ \qquad - \left(\frac{\nu^2}{a^2} + \frac{\sigma\rho\nu}{a}\right)\left(e^{-a(t-s)} + 1\right)B_a(s-u) \end{cases}$$

Covariance between l_t and r_t

$$\mathrm{Cov}_{Q_T}\left(\ l_t, r_t \mid \mathcal{F}_u\ \right) = \left(\frac{\nu^2}{a} + \rho\sigma\nu\right)B_a(t-u) - \frac{\nu^2}{a}B_{2a}(t-u)$$

Moments of the first and second order for the process $l_t = \ln(S_t)$ Replacing u by 0 in the above expressions of the conditional moments of l_t, we obtain the following formulas:

$$\begin{cases} M_{exp}(t) & = \ln\left(\frac{S_0}{P(0,t)}\right) + \frac{\nu^2}{4a^3} - \left(\frac{\nu^2}{2a^2} + \frac{\rho\sigma\nu}{a} + \frac{\sigma^2}{2}\right)t - \frac{\nu^2}{4a^3}\, e^{-2at} \\ & \quad + \left(\frac{\nu^2}{2a^3} + \frac{\rho\sigma\nu}{a^2}\right) e^{-a(T-t)} - \left(\frac{\nu^2}{a^3} + \frac{\rho\sigma\nu}{a^2}\right) e^{-aT} + \frac{\nu^2}{2a^3} e^{-a(T+t)} \\ & \\ V_{\exp}(t) & = \left(\sigma^2 + \frac{\nu^2}{a^2} + \frac{2\rho\sigma\nu}{a}\right)t - \frac{3\nu^2}{2a^3} - \frac{2\rho\sigma\nu}{a^2} + \frac{2\nu(\nu+a\rho\sigma)}{a^3} e^{-at} - \frac{\nu^2}{2a^3}\, e^{-2at} \\ & \\ C_{exp}(v,t) & = -\left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{a^3}\right) + \left(\sigma^2 + \frac{2\rho\sigma\nu}{a} + \frac{\nu^2}{a^2}\right)v - \frac{\nu^2}{2a^3}\, e^{-a(t+v)} \\ & \quad + \left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{a^3}\right)(e^{-av} + e^{-at}) - \left(\frac{\rho\sigma\nu}{a^2} + \frac{\nu^2}{2a^3}\right) e^{-a(t-v)} \end{cases}$$

APPENDIX III

Proof of Proposition 2.1

We show here how one can compute the three terms (depending on the supremum of the underlying process). Our main tool is the Dubins-Schwarz theorem which says that a continuous local martingale (say M) can be represented as a Brownian motion time-changed by the quadratic variation of the continuous local martingale (say $B_{<M>}$).

Let us denote by N the stochastic integral in formula (5):

$$N_t = \int_0^t (\sigma_P(u,T) + \rho\sigma)\, dZ_1^T(u) + \int_0^t \sigma\sqrt{1-\rho^2} dZ_2^T(u)$$

Let also τ be its quadratic variation: $\tau(t) = < N >_t$. N is a martingale, with $N(0) = 0$, and its quadratic variation satisfies:

$$\tau(t) = \int_0^t \left[(\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right] du$$

Consequently, we may write (5) as:

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)} \exp\left[N_t - \frac{\tau(t)}{2}\right]$$

Computation of E_1:

Finally, the expression of E_1, the first term of (20), can be expressed as:

$$E_1 = Q_T\left(\sup_{t\in[0,T]}\left\{-\frac{\tau(t)}{2} + N_t\right\} > \ln\left(\frac{KP(0,T)}{S_0}\right)\right)$$

Using the Dubins-Schwarz theorem, N is a τ time-changed Q_T-Brownian motion B. This readily yields:

$$E_1 = Q_T\left(\sup_{\tau\in[\tau(0),\tau(T)]}\left\{-\frac{\tau}{2}+B_\tau\right\} > \ln\left(\frac{KP(0,T)}{S_0}\right)\right)$$

Then, armed with the law of the supremum of an arithmetic Brownian motion (see for instance the third chapter of Jeanblanc, Yor, Chesney (2005)), we can obtain the closed-form formula:

$$E_1 = \mathcal{N}\left(\frac{-\ln\left(\frac{KP(0,T)}{S_0}\right)-\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right)+\frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{-\ln\left(\frac{KP(0,T)}{S_0}\right)+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right)$$

Computation of E_2:

To compute E_2, we start noting that:

$$E_2 = Q_T\left(-\frac{\tau(T)}{2}+B_{\tau(T)} < \ln\left(P(0,T)\right),\ \sup_{\tau\in[\tau(0),\tau(T)]}\{-\frac{\tau}{2}+B_\tau\} \leqslant \ln\left(\frac{KP(0,T)}{S_0}\right)\right)$$

Here, the problem is solved using the joint law of an arithmetic Brownian motion and its supremum (see the same reference as above). This yields directly:

$$E_2 = \mathcal{N}\left(\frac{\ln\left(P(0,T)\right)+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right)-\frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right)+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right)$$

Computation of E_3:

To compute E_3 we recall from equation (3) that

$$\exp\left(-\int_0^T r_u du\right)\frac{S_T}{S_0} = \exp\left(-\frac{\sigma^2 T}{2}+\int_0^T \sigma\rho d\widehat{Z}_1(u)+\int_0^T \sigma\sqrt{1-\rho^2}d\widehat{Z}_2(u)\right)$$

Using Girsanov's Theorem, we know that $\widetilde{Z}_1(u) = \widehat{Z}_1(u)-\sigma\rho u$ and $\widetilde{Z}_2(u) = \widehat{Z}_2(u)-\sigma\sqrt{1-\rho^2}u$ are two standard Brownian motions under the appropriate measure $\widetilde{Q}$ built with the Radon-Nikodym density process:

$$\frac{d\widetilde{Q}}{dQ} = \exp\left(-\frac{\sigma^2 T}{2}+\int_0^T \sigma\rho d\widehat{Z}_1(u)+\int_0^T \sigma\sqrt{1-\rho^2}d\widehat{Z}_2(u)\right)$$

After changing the measure, one obtains:

$$E_3 = \widetilde{Q}\left(S_T \geqslant S_0,\ \sup_{0\leqslant t\leqslant T}\left(\frac{S_t}{P(t,T)}\right)\leqslant K\right)$$

We need the expressions of $\frac{S_t}{P(t,T)}$ under $\widetilde{Q}$. After changing probability measure in the dynamics (3) and (4) of S_t and $P(t,T)$, we can write:

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)} \exp\left(\frac{\widetilde{\tau}_t}{2} + H_t\right)$$

where $H_t = \int_0^t (\sigma_P(u,T) + \rho\sigma)\, d\widetilde{Z}_1(u) + \int_0^t \sigma\sqrt{1-\rho^2} d\widetilde{Z}_2(u)$ and $\widetilde{\tau}_t =< H >_t$.

Then, one obtains:

$$E_3 = \widetilde{Q}\left(\frac{\widetilde{\tau}}{2} + \widetilde{B}_{\widetilde{\tau}} \geqslant \ln\left(P(0,T)\right), \sup_{\widetilde{\tau}\in[\widetilde{\tau}(0),\widetilde{\tau}(T)]} \left(\frac{\widetilde{\tau}}{2} + \widetilde{B}_{\widetilde{\tau}}\right) \leqslant \ln\left(\frac{KP(0,T)}{S_0}\right)\right)$$

where $\widetilde{B}$ is a standard $\widetilde{Q}$-Brownian motion. Using the same classical results as for E_2 and noting that $\widetilde{\tau}_t =< N >_t = \tau_t$, one finally obtains:

$$\begin{aligned} E_3 = {} & \mathcal{N}\left(\frac{\ln\left(\frac{KP(0,T)}{S_0}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ & - \mathcal{N}\left(\frac{\ln\left(P(0,T)\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{KP(0,T)}{S_0}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right) - \frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right). \end{aligned}$$