Options on Energy Portfolios in an HJM Framework

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Abstract

This paper implements a pricing procedure for commodity options, taking into account the stochastic nature of spot prices, currency exchange rates and convenience yields. Interest rate risk is also mentioned, but is not included in the numerical implementation. The analytic framework is heavily inspired by Miltersen&Schwartz(1998); however, while they allow the interest rate to be stochastic they do not consider exchange rate risk. Furthermore, our work differs from theirs by extending the analysis to derivatives, where the underlying energy commodity is allowed to be either a portfolio of single-delivery contracts or a flow forward. The numerical implementation is an adaptation of the stochastic duration approach in Munk(1999), and we find that this numerical method produces results quite close to the "true" prices found through Monte Carlo simulation.

Key words: Stochastic exchange rates, forward and futures prices, convenience yields, options, portfolios of single-delivery contracts, flow forwards, stochastic duration.

1 Introduction

It is a delicate matter to trade spot products and financial derivatives in energy markets. Opposite to bond and stock markets, the underlying assets are *real* products, and a significant part of the demand for them represents a *real* need for the products, which can only be substituted away with some difficulties; or, in some cases, only in a prohibitively costly manner. This is particularly true in the spot market, where the demand is almost always met, but where the spot price processes can be quite different from the spot price processes conventionally used in the pricing of derivatives. This pattern of real demand is also the main reason for the existence of the well-known convenience yield in energy markets.

Convenience yields exist in most commodity markets which are not primarily driven by speculation, cf. Brennan (1991). A popular and widely used definition of net convenience yield of a commodity is given in Brennan (1991) as

> "the flow of services that accrues to the holder of a physical commodity, but *not* to the owner of a contract for *future* delivery − measured per unit time and unit of the commodity."

In order to explain what convenience yield implies for the pricing and hedging of energy derivatives, convenience yield is often compared to (continuously paid) dividends on common stocks. This is a good first explanation. However, the dynamics of convenience yields in energy markets is typically very different from the modest dynamics of stock dividends, where an assumption about a constant continuously paid dividend rate or discretely paid fixed amounts is often sufficient to make a derivatives pricing model work well. Opposite to this, modeling of convenience yields on energy commodities requires a more sophisticated approach, because empirical evidence clearly states that convenience yields are stochastic. Furthermore, it shows the existence of a significant *term structure* of convenience yields. This calls for a modeling approach in the spirit of Heath, Jarrow, and Morton (1992), abbreviated HJM, where the spot price and the initial term structure of futures prices are taken as given. Based on this the drift of the convenience yield under an equivalent martingale measure is derived.

Obviously, the spot prices of energy has to be modeled as a stochastic process too. Furthermore, the introduction of a stochastic exchange rate is important for energy investors in almost any other country than the US, the prime example being crude oil and oil products that are almost exclusively traded in US dollars. Unless otherwise stated it is understood that energy commodities are internationally traded commodities with identical exchange rate adjusted spot prices in different markets. This is no restriction whenever the different currencies are used solely as units of account. But whenever the use of different currencies are used to compare spot prices in geographically separated areas it is no longer obvious that the exchange rate adjusted spot prices are identical. Transportation costs and delivery time (e.g. oil and related products), physical constraints in transmission capacity or the entire lack of such capacity (e.g. natural gas, electricity) and even politically determined flow constraints (e.g. oil export tariffs from Russia) are frictions that may cause markets to separate such that the exchange rate adjusted prices can no longer be assumed identical. However, such frictions due to the *real* nature of the commodities are not modeled within this paper. On the other hand, even with frictionless conversions of spot prices across currencies the analogous conversion of forward and futures prices is not entirely straightforward. This also goes for the relation between the convenience yields across currencies. We derive these relations in this paper.

The electricity market is the prime example where some of the basic assumptions and properties underlying conventional derivatives pricing models do not apply. This is due to the lack of storeability outside the hydropower based markets. Hence, neither the conventional hedging techniques for the pricing of

financial derivatives nor the ability to learn about an appropriate equivalent martingale measure from the spot market apply. Similarly, it is meaningless to interpret the convenience yield as a flow of services to an investor from holding an inventory. In this case one can, of course, always give up the usual interpretation of the convenience yield and *define* it in a backwards manner such that the usual no arbitrage based pricing relations are valid. On the other hand, electricity is also an example of a real good that can be transported instantly over long distances, provided the relevant transmission lines have been established.

This paper implements a pricing procedure for commodity options, taking into account the stochastic nature of spot prices, currency exchange rates and convenience yields. Interest rate risk is also mentioned, but is not included in the numerical implementation. The analytic framework is heavily inspired by Miltersen and Schwartz (1998); however, while they allow the interest rate to be stochastic they do not consider exchange rate risk. Furthermore, our work differs from theirs by extending the analysis to derivatives, where the underlying energy commodity is allowed to be either a portfolio of single-delivery contracts or a flow forward.

A flow forward is an agreement to deliver a commodity continuously through some specified period as opposed to the standard single-delivery energy commodity like crude oil or cracked oil products, where the delivery of the entire bulk takes place at a single point in time. The relevant energy commodities with flow feature are electricity and natural gas. Our approach to cope with the flow feature is to view the contract as an indivisible portfolio of periodical (weekly or daily) single-delivery forwards. This is analogous to a coupon bond, in which the cash flows occur at a number of times. We apply variants of the technique developed in Munk (1999) for pricing options on coupon bonds in multi-factor models. This technique replaces the underlying bond by a pure discount bond with the same stochastic duration and provides an approximate pricing method; however, Monte Carlo simulation results show that this approximation method is indeed quite accurate.

The paper is organized as follows. Section 2 outlines the set-up and section 3 describes our basic model for energy options, where the underlying is allowed either to be a portfolio of single-delivery futures or forwards with a flow feature (flow forwards). Potential applications to two different energy markets is discussed in section 4. Section 5 describes our results from comparing the approximate pricing method with the results from a Monte Carlo simulation which produces "close to theoretically correct" prices that are not obtainable in analytical form. Concluding remarks end the paper in section 6.

2 The set-up of the general model

A similar description of the model setup used here can be found in Miltersen and Schwartz (1998). However, considering only the one country/currency case they do not incorporate a stochastic exchange rate in their model.

2.1 The stochastic processes

The current time is 0 and we assume a finite investment horizon is $\overline{T}$. The relevant filtered probability space upon which the objective probability measure $\mathbf{P}$ works is denoted $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{\{0 \leq t \leq \overline{T}\}}, \mathbf{P}\right)$. This probability space carries a d-dimensional Wiener process W_u, and we assume that the filtration is the one generated by this Wiener process: $\mathscr{F}_t = \sigma\left(W_u \mid 0 \leq u \leq t\right)$.

We assume that the trading of energy commodities − spot and forward/futures − takes place in another currency domiciled in another country than the one used in the country where the energy derivatives in question are being priced and traded. The former country is denoted the *foreign* country and the latter the *domestic* country. Thus, the situation is seen from the point of view of the domestic country. All foreign variables are written with superscript * as commonly used in two country models in economic literature to identify the foreign country. The domestic spot price S_t and the foreign spot price S_t^* are assumed to be directly related through the spot exchange rate x_t, i.e. $S_t = x_t S_t^*$. Hence, as mentioned in the introduction, we disregard frictions of *real* nature in the spot markets, which otherwise would call for yet another source of risk to be modeled.

The following stochastic processes are defined on the probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{\{0 \leq t \leq \overline{T}\}}, \mathbf{P} \right)$:

- the spot price processes of the underlying energy commodity, S_t and S_t^*

- the spot interest rate processes, r_t and r_t^*, and the families of zero-coupon bond prices, $P(t,T)$ and $P^*(t,T)$

- the spot exchange rate process x_t

- the futures price processes, $F(t,T)$ and $F^*(t,T)$

- the forward price processes, $G(t,T)$ and $G^*(t,T)$

- the forward convenience yield processes, $\delta(t,s)$ and $\delta^*(t,s)$

- the futures convenience yield processes, $\varepsilon(t,s)$ and $\varepsilon^*(t,s)$

For technical reasons all processes are required to be $\mathscr{H}_2(\mathbf{P})$ processes, meaning that all the stochastic integrals w.r.t. the Wiener process are not only well-defined, but are $\mathbf{P}$-continuous square integrable martingales[1].

In accordance with standard assumptions in the literature we assume the existence of the usual equivalent martingale measures, $\mathbf{Q}$ in the domestic country with the bank account $\beta_t \equiv \exp(\int_0^t r_s ds)$ as the numeraire and $\mathbf{Q}^*$ in the foreign country with the bank account $\beta_t^* \equiv \exp(\int_0^t r_s^* ds)$ as the numeraire.

The continuously compounded *forward interest rates*, $f(t,s)$ and $f^*(t,s)$, are defined by

$$P(t,T) = \mathrm{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} \mid \mathscr{F}_t] = e^{-\int_t^T f(t,s)ds} \tag{2.1}$$

$$P^*(t,T) = \mathrm{E}^{\mathbf{Q}^*}[e^{-\int_t^T r_s^* ds} \mid \mathscr{F}_t] = e^{-\int_t^T f^*(t,s)ds} \tag{2.2}$$

The continuously compounded *forward convenience yields*, $\delta(t,s)$ and $\delta^*(t,s)$, are defined in a similar manner by the forward prices as

$$G(t,T) = \frac{S_t}{P(t,T)} e^{-\int_t^T \delta(t,s)ds} = S_t e^{\int_t^T (f(t,s)-\delta(t,s))ds} \tag{2.3}$$

$$G^*(t,T) = \frac{S_t^*}{P^*(t,T)} e^{-\int_t^T \delta^*(t,s)ds} = S_t^* e^{\int_t^T (f^*(t,s)-\delta^*(t,s))ds} \tag{2.4}$$

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[1] $\mathscr{H}_2(\mathbf{P})$ is the space of progressively measurable real processes such that $\mathrm{E}^{\mathbf{P}}[\int_0^t Y_s^2 ds] < \infty \ \forall t \in [0, \overline{T}]$.

while continuously compounded *futures convenience yields*, $\varepsilon(t,s)$ and $\varepsilon^*(t,s)$, are defined by the futures prices as

$$F(t,T) = \frac{S_t}{P(t,T)} e^{-\int_t^T \varepsilon(t,s)ds} = S_t e^{\int_t^T (f(t,s)-\varepsilon(t,s))ds} \tag{2.5}$$

$$F^*(t,T) = \frac{S_t^*}{P^*(t,T)} e^{-\int_t^T \varepsilon^*(t,s)ds} = S_t^* e^{\int_t^T (f^*(t,s)-\varepsilon(t,s)^*)ds} \tag{2.6}$$

The standard boundary conditions, $f(t,t) = r_t$ for all t and $G(t,t) = F(t,t) = S_t$ for all t, apply.

The economic interpretation of the continuously compounded forward convenience yields is that they represent a "non-pecuniary dividend yield", cf. Brennan (1991), as mentioned in the introduction. There is no similar easy interpretation of the futures convenience yields, because of the continuous resettlement payment structure for futures contracts. However, when interest rates are deterministic the forward prices and the futures prices are identical; hence, the two convenience yield curves in each currency are necessarily also identical. The difference arises solely from the interaction of the continuous resettlement feature with the stochastic variation in the interest rate process. It is a pure definition, although a very useful one. Due to the martingale property of the futures prices $F(t,T)$ w.r.t. to the **Q** measure it is easier to develop models for the stochastic dynamics of futures prices than for similar forward prices.

2.2 The dynamics

The model is based on the dynamics for the spot price of the underlying energy commodity, the term structure of futures convenience yields and the term structure of forward interest rates. This choice is explained by the experience from the HJM analysis, which shows that it is appropriate to initiate the building of the model for zero-coupon bond prices from the dynamics of the forward interest rates. Similarly, the futures prices are most appropriately modeled from the dynamics of the futures convenience yields. Because the objective of the analysis is to price derivatives we start the specification of the model directly under the equivalent **Q** measure.

The spot commodity price is assumed to follow the SDE

$$S_t = S_0 + \int_0^t S_u (r_u - \varepsilon(u,u)) \, du + \int_0^t S_u \sigma_S(u) \cdot dW_u^{\mathbf{Q}} \tag{2.7}$$

with the usual convenience yield adjusted drift. The spot exchange rate is assumed to follow the SDE

$$x_t = x_0 + \int_0^t x_u (r_u - r_u^*) \, du + \int_0^t x_u \sigma_x(u) \cdot dW_u^{\mathbf{Q}} \tag{2.8}$$

where the drift is given by the usual uncovered parity relation under Q.

The family of SDEs for the continuously compounded futures convenience yields is given by

$$\varepsilon(t,s) = \varepsilon(0,s) + \int_0^t \mu_\varepsilon(u,s)du + \int_0^t \sigma_\varepsilon(u,s) \cdot dW_u^{\mathbf{Q}} \tag{2.9}$$

and the family of SDEs for the continuously compounded forward interest rates is given by

$$f(t,s) = f(0,s) + \int_0^t \mu_f(u,s)du + \int_0^t \sigma_f(u,s) \cdot dW_u^{\mathbf{Q}} \tag{2.10}$$

Any possible correlation between the three processes is specified through the diffusion terms. At this

point of the analysis the drift and diffusion terms are not specified further, except for the fact that they must fulfill certain regularity conditions in order to ensure the existence of strong solutions to the SDEs. So, in the general case, stochastic/state dependent diffusion terms and correlation between the processes is possible.

Under the **Q** measure the drift terms of the continuously compounded forward interest rates as well as the convenience yields are completely determined by the diffusion terms. This is one of the main lessons from the HJM methodology. More specifically, from Heath, Jarrow, and Morton (1992) we have

$$\mu_f(t,T) = \sigma_f(t,T) \cdot \int_t^T \sigma_f(t,v)dv \tag{2.11}$$

and in Miltersen and Schwartz (1998), Appendix A, it is shown that the drift of any of the continuously compounded futures convenience yield processes under **Q** is given by

$$\mu_\varepsilon(t,T) = \sigma_f(t,T) \cdot \int_t^T \sigma_f(t,v)dv + \Big(\sigma_f(t,T) - \sigma_\varepsilon(t,T)\Big) \cdot \Big(\sigma_S(t) + \int_t^T (\sigma_f(t,v) - \sigma_\varepsilon(t,v))\Big)dv \tag{2.12}$$

2.3   Cross-currency relations

Intuitively, forward convenience yields, being rates analogous to dividend yields, should be identical in the foreign and the domestic country. This is proven as the following Theorem 1. However, the futures dividend yields processes, reflecting a possible correlation pattern between the price of the underlying asset and the interest rate process, will in general be different processes.

We apply the notation $E_t[X]$ to denote the conditional expected value of X given the σ-algebra $\mathscr{F}_t$. Similarly, we denote the Radon-Nikodym derivative, relative to the σ-algebra $\mathscr{F}_t$, between two equivalent probability measures P and Q as $\frac{dQ}{dP} \mid_{\mathscr{F}_t}$.

Theorem 1 *Forward convenience yields as defined in equations (2.3) and (2.4) are identical. When the futures price processes and the forward price processes are identical the futures convenience yields, as defined in equations (2.5) and (2.6), are identical and identical to the forward convenience yields.*

When the futures price processes and the forward price processes differ the futures convenience yields will only be identical by coincidence. ∎

Proof *First, consider a situation where an arbitrageur enters into two opposite forward positions, one in domestic currency and another in foreign currency. Each of these contracts have zero present value, hence the portfolio of opposite positions also has zero present value. The final settlement payment, expressed in domestic currency, is*

$$x_T G^*(t,T) - G(t,T) \tag{2.13}$$

The zero present value of this position leads to the relations

$$E_t^Q \left[e^{-\int_t^T r_s ds} (x_T G^*(t,T) - G(t,T)) \right] = 0 \qquad \Leftrightarrow \tag{2.14}$$

$$E_t^Q \left[x_T e^{-\int_t^T r_s ds} \right] G^*(t,T) = G(t,T)P(t,T) \quad \Leftrightarrow \tag{2.15}$$

$$x_t P^*(t,T)G^*(t,T) = G(t,T)P(t,T) \quad \Leftrightarrow \tag{2.16}$$

$$x_t S_t^* e^{-\int_t^T \delta^*(t,s)ds} = S_t e^{-\int_t^T \delta(t,s)ds} \quad \Leftrightarrow \tag{2.17}$$

$$\int_t^T \delta^*(t,s)ds = \int_t^T \delta(t,s)ds \tag{2.18}$$

Since this holds for all T we conclude that $\delta(t,T) = \delta^*(t,T)$ for all T. Alternatively stated, the domestic forward price is the foreign forward price converted at the forward exchange rate:

$$G(t,T) = \left(x_t \frac{P^*(t,T)}{P(t,T)} \right) G^*(t,T) \tag{2.19}$$

For futures contracts recall the well known fact that the futures price $F(t,T)$ is a Q-martingale and, analogously, that the futures price $F^*(t,T)$ is a Q^*-martingale. The Radon-Nikodym derivative, cf. Andreasen (1995), is

$$\frac{dQ}{dQ^*} \bigg|_{\mathscr{F}_t} = \frac{x_t}{x_T} e^{\int_t^T (r_s - r_s^*)ds} \tag{2.20}$$

Identical futures convenience yields is equivalent to having (2.16) fulfilled for the futures prices, i.e.

$$x_t P^*(t,T) F^*(t,T) = P(t,T) F(t,T) \tag{2.21}$$

This is equivalent to

$$E_t^{Q^*} \left[x_t P^*(t,T) \frac{S_T}{x_T} \right] = E_t^Q \left[P^*(t,T) e^{\int_t^T (r_s^* - r_s)ds} S_T \right] = E_t^Q [P(t,T) S_T] \tag{2.22}$$

This will only be fulfilled by coincidence. If forward and futures prices only differ in one currency, e.g. because the interest rate is only stochastic in one country, it is clearly impossible to have (2.21) fulfilled.

As another example, consider the widely used specification of the exchange rate process as a process with a constant or deterministic drift; i.e. $r_s - r_s^*$ is deterministic, cf. (2.8). In that case the fulfillment of (2.22) boils down to whether the exchange rate process $\frac{x_T}{x_t}$ is (Q-)uncorrelated with the domestic discount factor:

$$P^*(t,T) = E_t^{Q^*} \left[e^{-\int_t^T r_s^* ds} \right] = E_t^{Q^*} \left[e^{-\int_t^T (r_s^* - r_s)ds} e^{-\int_t^T r_s ds} \right]$$

$$= e^{-\int_t^T (r_s^* - r_s)ds} E_t^Q \left[\frac{x_T}{x_t} e^{\int_t^T (r_s^* - r_s)ds} e^{-\int_t^T r_s ds} \right]$$

$$= e^{-\int_t^T (r_s^* - r_s)ds} P(t,T) + Cov_t^Q \left(\frac{x_T}{x_t}, e^{-\int_t^T r_s ds} \right) \tag{2.23}$$

As usual in this area it is difficult to have an intuitive feeling for such covariance properties under Q; but under the physical measure, at least, it is unlikely that this covariance term vanishes. ∎

Remark 1 *While the relation between the forward prices is both well known in the literature and very simple to derive, the relation between the futures prices has not to our knowledge been spelled out previously.* ∎

3 Energy options

3.1 Options on a single-delivery futures contract

Our first objective is to price a strike-K call option on one single-delivery futures contract at date 0. The option expires at date t and the price of the futures contract with expiry at date $T > t$ is given by $F(t,T)$.
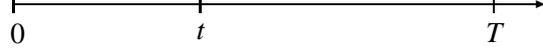
Figure 1: *Time line showing the time of option pricing (current time 0), the option expiration time t and the expiration time T of the underlying futures contract.*

The pricing procedure involves a standard change of probability measure from $\mathbf{Q}$ to $\mathbf{Q}^t$, the **t-forward measure** with the price process for the t-maturity zero-coupon bond as numeraire process. We denote the price of the option by $C^F(0;t,T,K)$. By standard change of numeraire calculations we get the following pricing equation:

$$
\begin{aligned}
C^F(0;t,T,K) &= \mathrm{E}_0^{\mathbf{Q}}\left[\frac{[F(t,T)-K]^+}{\beta(t)}\right] = \mathrm{E}_0^{\mathbf{Q}}\left[\frac{(F(t,T)-K)\mathbf{1}_{\{F(t,T)\geq K\}}}{\beta(t)}\right] \\
&= \mathrm{E}_0^{\mathbf{Q}}\left[\frac{F(t,T)\mathbf{1}_{\{F(t,T)\geq K\}}}{\beta(t)}\right] - K\mathrm{E}_0^{\mathbf{Q}}\left[\frac{\mathbf{1}_{\{F(t,T)\geq K\}}}{\beta(t)}\right] \\
&= P(0,t)\mathrm{E}_0^{\mathbf{Q}^t}\left[F(t,T)\mathbf{1}_{\{F(t,T)\geq K\}}\right] - P(0,t)K\mathrm{E}_0^{\mathbf{Q}^t}\left[\mathbf{1}_{\{F(t,T)\geq K\}}\right]
\end{aligned}
\tag{3.1}
$$

This pricing equation holds in any case. But, in order to arrive at an operational closed-form Black-Scholes like expression for the option price, we restrict ourselves to the Gaussian case and assume that the diffusion terms are *deterministic* functions. The problem then is to work out standard calculations with truncated lognormal distributions. From Appendix A in Miltersen and Schwartz (1998) we know that the dynamics for the futures price process, which is a martingale under $\mathbf{Q}$, is given by

$$
dF(u,T) = F(u,T)\widetilde{s}_F(u,T) \cdot dW_u^{\mathbf{Q}}
\tag{3.2}
$$

where

$$
\widetilde{s}_F(u,T) = \sigma_S(u) + \int_u^T (\sigma_f(u,s) - \sigma_\varepsilon(u,s))ds = \sigma_{S^*}(u) + \sigma_x(u) + \int_u^T (\sigma_f(u,s) - \sigma_\varepsilon(u,s))ds
\tag{3.3}
$$

The dynamics for the zero-coupon bond expiring at date t is given by

$$
dP(u,t) = P(u,t)r_u du - P(u,t)s_P(u,t)dW_u^{\mathbf{Q}}
\tag{3.4}
$$

where

$$
s_P(u,t) = \int_u^t \sigma_f(u,s)ds
\tag{3.5}
$$

The dynamics of the futures price process under $\mathbf{Q}^t$ is found by a standard transformation, cf. e.g. Proposition 24.5 from Björk (2004):

$$
dF(u,T) = F(u,T)\left(-\widetilde{s}_F(u,T) \cdot s_P(u,t)\right)du + F(u,T)\widetilde{s}_F(u,T) \cdot dW_u^{\mathbf{Q}^t}
\tag{3.6}
$$

implying that

$$
F(u,T) = F(0,T)e^{-\int_0^u \widetilde{s}_F(v,T) \cdot s_P(v,t)dv - \frac{1}{2}\int_0^u \|\widetilde{s}_F(v,T)\|^2 dv + \int_0^u \widetilde{s}_F(v,T) \cdot dW_v^{\mathbf{Q}^t}}
\tag{3.7}
$$

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For notational simplicity define Σ_{FP} and $\widetilde{\Sigma}_F^2$, respectively, as

$$\Sigma_{FP} = \int_0^t \widetilde{s}_F(v,T) \cdot s_P(v,t)dv, \qquad \widetilde{\Sigma}_F^2 = \int_0^t \| \widetilde{s}_F(v,T) \|^2 dv \tag{3.8}$$

Now we are able to compute the pricing formula (3.1):

$$\begin{aligned}
\mathrm{E}_0^{\mathbf{Q}^t}\left[1_{\{F(t,T)\geq K\}} \right] &= \mathbf{Q}^t\left(F(0,T)e^{-\Sigma_{FP}-\frac{1}{2}\widetilde{\Sigma}_F^2 + \int_0^t \widetilde{s}_F(v,T)\cdot dW_v^{\mathbf{Q}^t}} \geq K \right) \\
&= \mathbf{Q}^t\left(-\int_0^t \widetilde{s}_F(v,T)\cdot dW_v^{\mathbf{Q}^t} \leq \log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} - \frac{1}{2}\widetilde{\Sigma}_F^2 \right) \\
&= \Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} - \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\Big)
\end{aligned} \tag{3.9}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Analogously, by standard calculations the first term is

$$P(0,t)\mathrm{E}_0^{\mathbf{Q}^t}\left[F(t,T)\mathbf{1}_{F(t,T)\geq K} \right] = P(0,t)F(0,T)\mathrm{E}_0^{\mathbf{Q}^t}\left[\frac{F(t,T)}{F(0,T)}\mathbf{1}_{\{F(t,T)\geq K\}} \right] =$$

$$P(0,t)F(0,T)\mathrm{E}_0^{\mathbf{Q}^t}\left[e^{-\Sigma_{FP}-\frac{1}{2}\widetilde{\Sigma}_F^2 + \int_0^t \widetilde{s}_F(v,T)\cdot dW_v^{\mathbf{Q}^t}}\mathbf{1}_{\{F(t,T)\geq K\}} \right] =$$

$$P(0,t)F(0,T)e^{-\Sigma_{FP}}\Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} + \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\Big) \tag{3.10}$$

Collecting terms the final result is

$$\begin{aligned}
C^F(0;t,T,K) &= P(0,t)F(0,T)e^{-\Sigma_{FP}}\Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} + \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\Big) \\
&\quad - P(0,t)K\Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} - \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\Big)
\end{aligned} \tag{3.11}$$

This provides a closed-form expression for the price at date 0 of a strike-K call with maturity date $t > 0$ written on a futures contract with maturity date $T \geq t$. In Miltersen and Schwartz (1998), Appendix B, the authors derive the same price equation for a similar call option, but in a different way without explicitly introducing the change of measure from $\mathbf{Q}$ to $\mathbf{Q}^t$. The procedure used here is considerably shorter and simpler.

An almost identical analysis would show that the price $P^F(0;t,K)$ at date of a similar strike-K put option with maturity date t on a futures contract with maturity date $T \geq t$ is given by

$$\begin{aligned}
P^F(0;t,K) &= P(0,t)K\Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} - \frac{1}{2}\widetilde{\Sigma}_F^2}{-\sqrt{\widetilde{\Sigma}_F^2}}\Big) \\
&\quad - P(0,t)F(0,T)e^{-\Sigma_{FP}}\Phi\Big(\frac{\log\left(\frac{F(0,T)}{K}\right) - \Sigma_{FP} + \frac{1}{2}\widetilde{\Sigma}_F^2}{-\sqrt{\widetilde{\Sigma}_F^2}}\Big)
\end{aligned} \tag{3.12}$$

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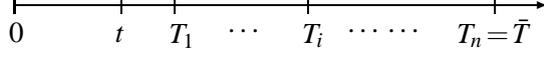
Figure 2: *Time line showing the time of option pricing (current time) t_0, the option expiration time t and the expiration times $\{T_i\}_{\{1 \le i \le n\}}$ of the underlying single-delivery futures contracts.*

Relevant sensitivity parameters, i.e. the *Greeks* like delta, gamma etc., can now be found by standard methods. Observe from (3.7) that the process $F(u,T)e^{\int_0^u \widetilde{s}_F(v,T) \cdot s_P(v,t)dv}$ is a Q^t-martingale:

$$F(u,T)e^{\int_0^u \widetilde{s}_F(v,T) \cdot s_P(v,t)dv} = F(0,T)e^{-\frac{1}{2}\int_0^u \|\widetilde{s}_F(v,T)\|^2 dv + \int_0^u \widetilde{s}_F(v,T) \cdot dW_v^{\mathbf{Q}^t}} \tag{3.13}$$

Hence, the result in (3.1) can be restated as

$$C^F(0;t,T,K) = P(0,t)F(0,T)e^{-\Sigma_{FP}}\mathrm{E}_0^{\mathbf{Q}^T}\left[\mathbf{1}_{\{F(t,T) \ge K\}}\right] - P(0,t)K\mathrm{E}_0^{\mathbf{Q}^t}\left[\mathbf{1}_{\{F(t,T) \ge K\}}\right] \tag{3.14}$$

where the **T−futures measure** Q^T is defined from Q^t by

$$\left.\frac{dQ^T}{dQ^t}\right|_{\mathscr{F}_0} = e^{-\frac{1}{2}\int_0^t \|\widetilde{s}_F(v,T)\|^2 dv + \int_0^t \widetilde{s}_F(v,T) \cdot dW_v^{\mathbf{Q}^t}} \tag{3.15}$$

If interest rates are assumed to be deterministic, i.e. $\sigma_f(u,s) = 0$ for all (u,s), the call price (3.11) simplifies to

$$C^F(0;t,T,K) = P(0,t)\left(F(0,T)\Phi\left(\frac{\log\left(\frac{F(0,T)}{K}\right) + \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\right) - K\Phi\left(\frac{\log\left(\frac{F(0,T)}{K}\right) - \frac{1}{2}\widetilde{\Sigma}_F^2}{\sqrt{\widetilde{\Sigma}_F^2}}\right)\right) \tag{3.16}$$

3.2 Options on a portfolio of futures

The pricing technique for options on single-delivery futures from the previous subsection makes it possible to price an option on a portfolio of futures contracts. Define the price of the whole contract with expiry at date $\overline{T}$ as

$$H(t) \equiv \sum_{i=1}^n \alpha_i F(t,T_i) \tag{3.17}$$

where $t \le T_i \le T_{i+1} \le \overline{T}$ for all $i = 1, \ldots, n-1$ and $T_n = \overline{T}$. The α_i's are numbers that indicate how many single-delivery contracts the investor holds for every delivery date. In the case, where the α_i's are all equal and the T_i's equidistant, it means that the portfolio is a contract for constant delivery throughout the period $[T_1,T_n]$. Figure 2 illustrates the location of the relevant points in time. For notational simplicity we omit the explicit mentioning of the time arguments T_i and the volume arguments α_i in the specification of H. Denote the price of the option by $C^H(0;t,K)$. Analogous to the pricing relation developed above

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we have that

$$
\begin{aligned}
C^H(0;t,K) &= \mathrm{E}_0^{\mathbf{Q}}\left[\frac{[H(t)-K]^+}{\beta(t)}\right] = \mathrm{E}_0^{\mathbf{Q}}\left[\frac{(H(t)-K)\mathbf{1}_{\{H(t)\geq K\}}}{\beta(t)}\right] \\
&= \mathrm{E}_0^{\mathbf{Q}}\left[\frac{H(t)\mathbf{1}_{\{H(t)\geq K\}}}{\beta(t)}\right] - K\mathrm{E}_0^{\mathbf{Q}}\left[\frac{\mathbf{1}_{\{H(t)\geq K\}}}{\beta(t)}\right] \\
&= P(0,t)\mathrm{E}_0^{\mathbf{Q}^t}\left[H(t)\mathbf{1}_{\{H(t)\geq K\}}\right] - P(0,t)K\mathrm{E}_0^{\mathbf{Q}^t}\left[\mathbf{1}_{\{H(t)\geq K\}}\right] \\
&= P(0,t)\sum_{i=1}^{n}\alpha_i F(0,T_i)\mathrm{E}_0^{\mathbf{Q}^{T_i}}\left[\mathbf{1}_{\{H(t)\geq K\}}\right] - P(0,t)K\mathrm{E}_0^{\mathbf{Q}^t}\left[\mathbf{1}_{\{H(t)\geq K\}}\right] \quad (3.18)
\end{aligned}
$$

So in principle we have found the price of the option. In order to make this pricing equation applicable in practice, it is necessary to find the probabilities of the event $\{H(t) \geq K\}$ under the $\mathbf{Q}^t$ and $\mathbf{Q}^{T_i}$ measures. However, it is not obvious how to find these, because $H(t)$ is a sum of log-normally distributed variables, so when $n \geq 2$ it is not log-normally distributed itself.

Jamshidian (1989) and Longstaff (1993) show how to cope with this problem and obtain a closed-form solution when pricing European coupon bond options in the Vasiček and CIR *one-factor* interest rate models, respectively. The Jamshidian approach can be extended to all one-factor models, where the relative price volatility $\sigma(t,T)$ at time t of a zero-coupon bond with maturity date T can be factored as $\sigma(t,T) = (g(T) - g(t))h(t)$, where g and h satisfy certain regularity conditions. This is known as the condition under which the process for the short term rate of interest is Markovian, cf. Carverhill (1994).[2] The basic methodology rests on the perception of the *coupon bond* as a *portfolio of zero-coupon bonds*.

When it comes to bond option pricing in *multi-factor* models, several models offer a closed-form solution for the price of European options on zero-coupon bonds. All multi-factor exponential affine term structure models offer closed-form solutions for zero-coupon bond option prices. However, no closed-form solutions are known for European coupon bond option prices in multi-factor models. In order to cope with this problem, Munk (1999) suggested a fast and precise numerical method. The idea is to approximate the coupon bond option price by a multiple of the price of a European option on a zero-coupon bond with a time to maturity equal to the stochastic duration of the coupon bond.

We shall apply this approach when pricing options on portfolios of futures and look into the details in the next section. A single-delivery futures is analogous to a zero-coupon bond; hence, a portfolio of single-delivery futures is analogous to a coupon bond. We approximate the option price by the price of a European option on a single-delivery futures with a time to maturity equal to the so-called stochastic duration of the futures portfolio.

An alternative approach to pricing European coupon-bond options has been proposed in Singleton and Umantsev (2002) for the case of affine term structure models, i.e. where bond prices are affine functions of the state variables.[3] The idea is to approximate the option's optimal exercise boundary by a hyperplane in $\mathbb{R}^m$, where m equals the number of factors in the model. This approximation makes it possible to calculate the probability of the bond being in the money at the option expiration date under the relevant probability measures. This reduces to calculating the sum of $n+1$ probabilities of a zero-coupon bond being in the money, where n equals the number of remaining bond payments after the option's expiration date. Knowledge of the conditional characteristic function for the state variables and the

---

[2]More precisely, g is a $\mathscr{C}^1(\mathbb{R}, \mathbb{R}_+)$ function and h is a $\mathscr{C}(\mathbb{R}, \mathbb{R}_+)$ function.

[3]This is not the only alternative approach to Munk (1999) to be found in the literature. Other approaches are suggested in e.g. Collin-Dufresne and Goldstein (2002) and, more recently, in Schrager and Pelsser (2006); however, we have not tried to compare our approach with the suggestions in these papers.

fact that the zero-coupon bond prices are exponential affine functions of the state variables is sufficient to derive closed form solutions, cf. e.g. section 2 in Singleton and Umantsev (2002). Longstaff and Schwartz (1992) is a two state variable example of this. All relevant option sensitivity parameters can in principle be calculated along the same lines.

Examples provided by Singleton and Umantsev (2002) indicate that this approach is more precise than the stochastic duration approach and that the pricing error is independent of the moneyness of the option. However, this precision comes at a cost. The approach is much more complicated to implement, and the calculations are more time consuming. Calculating the price of a bond option with only two remaining payments at option expiry takes almost six times longer than using the stochastic duration approach according to an example provided by Singleton and Umantsev (1.42 seconds compared to 0.24 seconds). As the computing time increases linearly in the number of remaining payments, which in our case corresponds to delivery times, and can be much higher than the number of coupon payments usually found in the bond market, one can easily image a situation where this approach spends unacceptable long time to perform the calculation compared to the stochastic duration approach. Thus, despite its theoretical appeal and the higher degree of precision, we shall not deal further with this approach in the paper.

3.2.1 Pricing the option by approximation

Given that the pricing is considerably easier for a single-delivery futures one could consider different ways of approximating the dynamics of the futures portfolio by the dynamics of a single-delivery futures with maturity T. This could be in a myopic sense by replicating the infinitesimal volatility term of the portfolio, but it could equally well be an attempt to replicate some of the distributional characteristics of the portfolio at the expiration date t of the option.

We shall stick to the nomenclature in Munk (1999) and denote the relative diffusion coefficients of the futures price process as $s_{F_j}(u,T)$, $j = 1,\ldots,d$, for factor sensitivities. So the j'th factor sensitivity is a relative futures price diffusion coefficient with respect to one single source of uncertainty, namely the j'th component of the d-dimensional process $(W_u^{\mathbf{Q}})$. Note that in most cases when the text reads $s_{F_j}(u,T)$ it is understood that it is for an arbitrary $j \in \{1,\ldots,d\}$. We rewrite the dynamics of the single-delivery futures price in a way that emphasizes the dependency of a d-dimensional Wiener process:

$$dF(u,T) = F(u,T)s_F(u,T) \cdot dW_u^{\mathbf{Q}} = F(u,T)\sum_{j=1}^{d} s_{F_j}(u,T)dW_j^{\mathbf{Q}}(u) \tag{3.19}$$

where $s_F(u,T) = (s_{F_1}(u,T),\cdots,s_{F_d}(u,T))^T$ and $W_u^{\mathbf{Q}} = (W_1^{\mathbf{Q}}(u),\cdots,W_d^{\mathbf{Q}}(u))^T$ with $(\cdot,\cdots,\cdot)^T$ denoting the transpose of a row vector. Similarly the dynamics for the portfolio position is given by

$$
\begin{aligned}
dH(u) &= \sum_{i=1}^{n} \alpha_i dF(u,T_i) \\
&= \sum_{i=1}^{n} \alpha_i F(u,T_i) \sum_{j=1}^{d} s_{F_j}(u,T_i)dW_j^{\mathbf{Q}}(u) \\
&= H(u)\Big(\sum_{i=1}^{n} w(u,T_i)\sum_{j=1}^{d} s_{F_j}(u,T_i)dW_j^{\mathbf{Q}}(u)\Big) \\
&= H(u)\Big(\sum_{i=1}^{n}\sum_{j=1}^{d} w(u,T_i)s_{F_j}(u,T_i)dW_j^{\mathbf{Q}}(u)\Big)
\end{aligned}
\tag{3.20}
$$

where $w(u, T_i)$ are portfolio weights given by $w(u, T_i) = \frac{\alpha_i F(u, T_i)}{\sum_{i=1}^n \alpha_i F(u, T_i)} = \frac{\alpha_i F(u, T_i)}{H(u)}$.

The idea behind the approximation of the option price with the portfolio as underlying is to replace the true portfolio of futures with an appropriately chosen single-delivery future. This approximation is given by

$$C^H(0;t,K) \approx \varphi C^F(0;t,T,\frac{K}{\varphi}) \tag{3.21}$$

where $T \in [t, \overline{T}]$ is some chosen maturity of an approximating single-delivery futures and the scaling factor φ is given by

$$\varphi = \frac{H(0)}{F(0,T)} \tag{3.22}$$

The approximation for a similar put option on the futures portfolio is given by

$$P^H(0;t,K) \approx \varphi P^F(0;t,T,\frac{K}{\varphi}) \tag{3.23}$$

For an arbitrary maturity date T of the approximating single-delivery futures the approximation error $\varepsilon^H(T)$ in the call case can be evaluated under the T-futures measure $\mathbf{Q}^T$ with $F(u,T)$ as the numeraire process. The case of a put option is similar. Recall that all gain processes in the economy as well as futures price processes relative to the numeraire process $F(u,T)$ are $\mathbf{Q}^T$-martingales. Then

$$\varepsilon^H(T) = F(0,T)\mathrm{E}_0^{\mathbf{Q}^T}\left[\frac{\max\{H(t)-K,0\}}{F(t,T)}\right] - \frac{H(0)}{F(0,T)}F(0,T)\mathrm{E}_0^{\mathbf{Q}^T}\left[\frac{\max\{F(t,T)-K\frac{F(0,T)}{H(0)},0\}}{F(t,T)}\right]$$

$$= F(0,T)\mathrm{E}_0^{\mathbf{Q}^T}\left[\max\{\frac{H(t)}{F(t,T)}-\frac{K}{F(t,T)},0\} - \max\{\frac{H(0)}{F(0,T)}-\frac{K}{F(t,T)},0\}\right]$$

where

$$\mathrm{E}_0^{\mathbf{Q}^T}\left[\frac{H(t)}{F(t,T)}\right] = \frac{H(0)}{F(0,T)}, \quad \mathrm{E}_0^{\mathbf{Q}^T}\left[\frac{K}{F(t,T)}\right] = \frac{K}{F(0,T)} \tag{3.24}$$

For a call option deeply in the money, both maximum terms are given by the first (non-zero) argument by a probability close to one, so by equation (3.24) we see that the difference between the two terms and hence the approximation error $\varepsilon^H(T)$ is close to zero. For call options deeply out of the money, both maximum terms are zero with a probability close to one, so similarly $\varepsilon^H(T)$ is close to zero.

Only when one of the maximum terms yields zero and the other one non-zero $\varepsilon^H(T)$ is affected more substantially. This is the case, when $\frac{K}{F(t,T)}$ is between $\frac{H(t)}{F(t,T)}$ and $\frac{H(0)}{F(0,T)}$ at time t. Therefore, to keep the error $\varepsilon^H(T)$ on a minimal level, T should be chosen such that $\frac{H(t)}{F(t,T)}$ stays close to $\frac{H(0)}{F(0,T)}$.

3.2.2 Stochastic duration

For a futures portfolio we define the *stochastic duration* $\delta^H(0)$ at time 0 as the time to maturity of the single-delivery futures with the same relative price volatility as the portfolio. I.e., $\delta^H(0)$ is defined by the equation

$$\sum_{j=1}^d s_{F_j}(0, \delta^H(0))^2 = \sum_{j=1}^d \left(\sum_{i=1}^n w(0,T_i)s_{F_j}(0,T_i)\right)^2 \tag{3.25}$$

The usual concept of duration in fixed income analysis is a risk measure gauging the entire relative price reaction to a change in the yield − or in more general models the entire relative price reaction

to a change in some common exogenous variable. The present definition relates only to the volatility term. However, it shares a basic property with the usual duration measure; the definition implies that the stochastic duration of a single-delivery futures equals its time to maturity just as the duration of a zero-coupon bond equals its time to maturity.

By the very definition of the stochastic duration, choosing $T = \delta^H(0)$ assures that the relative changes in $F(\cdot, T)$ and $H(\cdot)$ at time 0 are close to each other over the next infinitesimal time interval. Hence, the approximation is a myopic one, and whether they are actually equal depends on the size of the drift of the two price processes under $\mathbf{Q}^T$. On the other hand, in many models it is an "easy to compute" measure and, as will be shown subsequently, it has a reasonably good fit when compared to option prices found by Monte Carlo simulation.

An alternative choice[4] is to measure the cumulative variance of the portfolio over the entire remaining lifetime of the option, i.e. over the interval $[0, t]$, and find a representative single futures that has the same accumulated variance over this interval. Based on these considerations we define an alternative stochastic duration $\delta_A^H(0)$ for the portfolio from the following (approximate) equality:

$$\int_0^t \sum_{j=1}^d s_{F_j}(u, \delta_A^H(0))^2 du = \int_0^t \sum_{j=1}^d \left(\sum_{i=1}^n w(0, T_i) s_{F_j}(u, T_i) \right)^2 du \tag{3.26}$$

$$\approx E^Q \left[\int_0^t \sum_{j=1}^d \left(\sum_{i=1}^n w(u, T_i) s_{F_j}(u, T_i) \right)^2 du \mid \mathscr{F}_t \right] \tag{3.27}$$

In section 5.1 we report on Monte Carlo simulations and compare the pricing errors with this measure to the myopic stochastic duration.

3.2.3 Basic properties of stochastic duration

We omit most details in this section; in most cases they can be found in section 1.2 in Munk (1999) in the case of coupon bonds. The stochastic duration measure has the following properties, most of them obvious or very simply proven:

- If the factor sensitivities $s_{F_j}(0, T)$ are either positive and increasing in T or negative and decreasing in T, then $\sum_{j=1}^d s_{F_j}(0, T)^2$ is increasing in T. If the factor sensitivities $s_{F_j}(0, T)$ are either positive and decreasing or negative and increasing in T, then $\sum_{j=1}^d s_{F_j}(0, T)^2$ is decreasing in T.

- If the factor sensitivities $s_{F_j}(0, T)$ are either all increasing or all decreasing in T for all j and at least one of them is strictly increasing respectively decreasing, then a unique solution $\delta^H(0)$ to equation (3.25) exists in the interval $T_1 \leq \delta^H(0) \leq T_n$.

There is also a parallel to the well-known Fisher-Weil duration applied in the analysis of coupon bonds. The Fisher-Weil duration is a weighted average of the times to maturity of the bond's payments, weighted with the portfolio weights of the individual payments when calculated in accordance with the existing zero-coupon term structure. In the context of futures the duration measure is similar, but we shall call it the Fisher-Weil futures duration.

---

[4]Yet another candidate was suggested in Munk (1999), namely to incorporate explicitly the drift and look for $T' = \operatorname{argmin}_T \operatorname{Var}_0^{\mathbf{Q}^T} \left(\left\{ \frac{\text{drift+diffusion of } F(u,T)}{F(u,T)} - \frac{\text{drift+diffusion of } F(u,\overline{T})}{F(u,\overline{T})} \right\} \Big|_{u=0} \right)$. As Munk (1999) argues, T' and $\delta^H(0)$ will be equal for all one-factor models, whereas they may differ in multi-factor models. However, according to the numerical results reported in Munk (1999) for two-factor models, including a two-factor HJM model, this did not improve the pricing accuracy.

Definition 1 *The Fisher-Weil futures duration δ_{FW}^H is defined by*

$$\delta_{FW}^H = \frac{\sum_{i=1}^n \alpha_i F(0,T_i)(T_i)}{\sum_{i=1}^n \alpha_i F(0,T_i)} = \sum_{i=1}^n w(0,T_i)(T_i) = \sum_{i=1}^n w(0,T_i)T_i \qquad (3.28)$$

where $w(0,T_i) = \frac{\alpha_i F(0,T_i)}{\sum_{i=1}^n \alpha_i F(0,T_i)}$ are positive weights and $\sum_{i=1}^n w(0,T_i) = 1$. ∎

The following proposition deals with a situation, where the stochastic duration and the Fisher-Weil futures duration are identical. This is the case for the special class of models with so-called linear factor sensitivities, i.e. when

$$s_{F_j}(u,T) = (T-u)b_j(u) \quad \forall j \qquad (3.29)$$

for some smooth maturity-independent functions b_j for all j.

Proposition 1 *If the factor sensitivities $s_{F_j}(0,T)$ are linear for all j, then the stochastic duration $\delta^H(0)$ is uniquely determined and identical with the Fisher-Weil futures duration $\delta_{FW}^H(0)$.* ∎

Proof *Linear factor sensitivities imply $\sum_{i=1}^n w(0,T_i)s_{F_j}(0,T_i) = s_{F_j}\left(0,\sum_{i=1}^n w(0,T_i)T_i\right)$ for all j. Hence, $\sum_{j=1}^d \left(\sum_{i=1}^n w(0,T_i)s_{F_j}(0,T_i)\right)^2 = \sum_{j=1}^d \left(s_{F_j}(0,\sum_{i=1}^n w(0,T_i)T_i)\right)^2$.*

By the definition of the stochastic duration δ^H and the Fisher-Weil futures duration δ_{FW}^H the LHS of the last expression equals $\sum_{j=1}^d s_{F_j}(0,\delta^H(0))^2$, while the RHS is equal to $\sum_{j=1}^d \left(s_{F_j}(0,\delta_{FW}^H(0))\right)^2$. Furthermore, the LHS is an increasing function in the variable $\delta^H(0)$, which is to be determined; the partial derivative is $2\sum_{j=1}^d \delta^H(0)b_j(0)^2$ and hence positive. Consequently $\delta^H(0) = \delta_{FW}^H(0)$. ∎

Thus, in the special case where all factor sensitivities are linear, the simple Fisher-Weil futures duration is indeed a valid risk measure for the futures portfolio $F(0,\overline{T})$ and is identical to the risk measure implied by the stochastic duration. Conversely, for one-factor models where the factor sensitivity has the same sign for all maturities the implication in the proof of proposition 1 turns into a biimplication and the converse to the proposition is true: If the stochastic duration and the Fisher-Weil futures duration are equal, then the factor sensitivity is linear. For multi-factor models it is not so straightforward and the reverse of proposition 1 is not true in general. Stronger assumptions on the factor sensitivity are needed for this.

3.3 Options on flow forward contracts

3.3.1 Pricing flow forward contracts

Flow forward contracts can be regarded as a finite sum of n (properly discounted) single-delivery forwards. The discounting is done in order to take into consideration when the forward payments are actually made. If the flow forward is a contract for $\frac{1}{n}$ unit of the energy commodity per sub-period, i.e. one unit over the whole period, and the whole forward payment is done upfront, then the value of the flow forwards is simply $J(t) \equiv \frac{1}{n}\sum_{i=1}^n G(0,T_i)$.

Inspired by the payment structure at the Nordic power forward market, where a flow forward contract is settled through daily payments, we investigate the case where the forward price $J(t)$ of the entire flow contract is paid sequentially as a constant fraction of $J(t)/n$ during the delivery period $[T_1,T_n]$. By a

simple discounting argument one gets that

$$\frac{J(t)}{n} = \sum_{i=1}^{n} \left(\frac{e^{-\int_t^{T_i} f(t,s)ds}}{\sum_{i=1}^{n} e^{-\int_t^{T_i} f(t,s)ds}} \right) \frac{G(t,T_i)}{n} \equiv \sum_{i=1}^{n} \gamma_i G(t,T_i) \tag{3.30}$$

So the forward price $J(t)$ may be interpreted as a weighted average of single-delivery forward prices over the delivery period $[T_1, T_n]$, where the weights are determined by a discounting reflecting the term structure of interest rates and the difference in delivery dates of the single-delivery contracts.

In the following we will limit ourselves to the case of deterministic interest rates. This implies that the weights γ_i are deterministic, enabling us to define the stochastic duration of a flow forward in the following section. We will furthermore motivate this choice in section 4, where we present two concretely specified examples of models using our numerical approach.

3.3.2 The stochastic duration of a flow forward

Given that the γ_i's are deterministic throughout the period of interest $[0,t]$ we define the stochastic duration $\delta^J(0)$ of a flow forward at time 0 similar to the stochastic duration for a futures portfolio; i.e. as the time to maturity of the single-delivery forward with the same relative price volatility as the flow forward. This means that $\delta^J(0)$ is defined by the equation

$$\sum_{j=1}^{d} s_{F_j}(0, \delta^J(0))^2 = \sum_{j=1}^{d} \left(\sum_{i=1}^{n} v(0,T_i) s_{F_j}(0,T_i) \right)^2 \tag{3.31}$$

where

$$v(0,T_i) = \frac{\gamma_i G(0,T_i)}{\sum_{i=1}^{n} \gamma_i G(0,T_i)} = \frac{\gamma_i G(0,T_i)}{J(0)} \tag{3.32}$$

We see that the only difference between the two stochastic durations δ^J and δ^H comes via differences in the portfolio weights − the v's and the w's, respectively. Consequently, the properties of δ^J are as described for δ^H in section 3.2.3.

3.3.3 Pricing the flow option

The price approximation for the flow forward option follows the same line as for the option on a futures portfolio. Let $C^G(0;t,T,K)$ be the price at time 0 of a strike-K European call option maturing at time t on a single-delivery forward maturing at time T. And let $C^J(0;t,K)$ be the price at time 0 of a strike-K European call option maturing at time t on a flow forward maturing at time $\overline{T}$ and with $J(0) = \sum_{i=1}^{n} \gamma_i G(0,T_i)$ denoting the price of the flow forward at time 0, The approximation is then given by

$$C^J(0;t,K) \approx \rho C^G(0;t,\delta^G(0), \frac{K}{\rho}) \tag{3.33}$$

where $\delta^G(0)$ is the stochastic duration at time 0 of the flow forward and the factor ρ is given by

$$\rho = \frac{J(0)}{G(0,\delta^G(0))} \tag{3.34}$$

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The approximation for the price of a similar put option is given by

$$P^J(0;t,K) \approx \rho P^G(0;t,\delta^G(0),\frac{K}{\rho}) \tag{3.35}$$

Remark 2 *A portfolio of flow forwards does not principally differ from one single flow forward in the sense that both simply can be regarded as a sum (or indivisible portfolio) of single-delivery forwards. Thus, the stochastic duration for a portfolio of flow forwards is defined in exactly the same way as for one flow forward. Similarly, the pricing formula for options written on a portfolio of flow forwards is identical to the pricing formula for a single flow forward.* ▌

4   Application to crude oil and electricity markets

We consider two different markets. The crude oil market and the electricity market. In the crude oil market oil is delivered in the normal way for commodities, i.e. as discrete separated bulks. This is opposed to the electricity market where the commodity is delivered as a continuous flow. So, the electricity market is exactly an example where the flow feature of the forward contracts becomes relevant. The natural gas market has the same flow feature as the electricity market, and the demand pattern for the two commodities exhibit a similar degree of seasonal variation, so one could expect that a model suited for the electricity market would be suited for the natural gas market as well. However, this is not pursued further within the limitations of this paper.

4.1   The crude oil market

The model is inspired by the three-factor model used in Schwartz (1997) and re-used in Miltersen and Schwartz (1998) in a different setting. Our model has three factors as well, but the difference from the original model is that we model a stochastic exchange rate instead of stochastic interest rates. In order not to expand the number of factors we believe that the exchange risk is a more important risk factor than the interest rate risk in the oil market. Moreover, in the formulation of the diffusion term for the convenience yield we have added a constant $\eta \geq 0$ to an exponential function, which ensures that the convenience yield volatility will converge towards η and not necessarily towards zero for the maturity date converging towards infinity, cf. the specification of σ_ε below. Specifically, we assume the following diffusion terms in our three-factor Gaussian model.

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_x(t) = \sigma_x \begin{pmatrix} \rho_{Sx} \\ \sqrt{1-\rho_{Sx}^2} \\ 0 \end{pmatrix} \tag{4.1}$$

and

$$\sigma_\varepsilon(t,s) = \sigma_\varepsilon \left(e^{-\kappa_\varepsilon(s-t)} + \eta \right) \begin{pmatrix} \rho_{s\varepsilon} \\ \frac{\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon}}{\sqrt{1-\rho_{Sx}^2}} \\ \sqrt{1-\rho_{S\varepsilon}^2 - \frac{(\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon})^2}{1-\rho_{Sx}^2}} \end{pmatrix} \tag{4.2}$$

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Written as quadratic variation and mixed variation terms this means that the diffusion terms are given by

$$d\langle S\rangle_t = \sigma_S^2 S_t^2 dt \tag{4.3}$$

$$d\langle x\rangle_t = \sigma_x^2 x_t^2 dt \tag{4.4}$$

$$d\langle \varepsilon(\cdot,s)\rangle_t = \sigma_\varepsilon^2 \left(e^{-\kappa_\varepsilon(s-t)} + \eta\right)^2 dt \tag{4.5}$$

$$d\langle S,x\rangle_t = \rho_{Sx}\sigma_S\sigma_x S_t x_t dt \tag{4.6}$$

$$d\langle S,\varepsilon(\cdot,s)\rangle_t = \rho_{S\varepsilon}\sigma_S\sigma_\varepsilon S_t \left(e^{-\kappa_\varepsilon(s-t)} + \eta\right) dt \tag{4.7}$$

$$d\langle x,\varepsilon(\cdot,s)\rangle_t = \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon x_t \left(e^{-\kappa_\varepsilon(s-t)} + \eta\right) dt \tag{4.8}$$

Thus, using equation (3.3) one gets that

$$s_F(u,T) = \sigma_S \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sigma_x \begin{pmatrix} \rho_{Sx} \\ \sqrt{1-\rho_{Sx}^2} \\ 0 \end{pmatrix} +$$

$$\sigma_\varepsilon \left(\frac{1}{\kappa_\varepsilon}\left(e^{-\kappa_\varepsilon(T-u)} - 1\right) - \eta(T-u)\right) \begin{pmatrix} \rho_{S\varepsilon} \\ \frac{\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon}}{\sqrt{1-\rho_{Sx}^2}} \\ \sqrt{1-\rho_{S\varepsilon}^2 - \frac{(\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon})^2}{1-\rho_{Sx}^2}} \end{pmatrix} \tag{4.9}$$

and

$$\| s_F(u,T) \|^2 = \| \sigma_S(u) + \sigma_x(u) - \int_u^T \sigma_\varepsilon(u,s)ds \|^2$$

$$= \sigma_\varepsilon^2 \left(\frac{1}{\kappa_\varepsilon^2}\left(1 + e^{-2\kappa_\varepsilon(T-u)} - 2e^{-\kappa_\varepsilon(T-u)}\right) + \eta^2(T-u)^2 + \frac{2\eta}{\kappa_\varepsilon}\left(1 - e^{-\kappa_\varepsilon(T-u)}\right)(T-u)\right)$$

$$- 2\left(\rho_{S\varepsilon}\sigma_S\sigma_\varepsilon + \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\right)\left(\frac{1}{\kappa_\varepsilon}\left(1 - e^{-\kappa_\varepsilon(T-u)}\right) + \eta(T-u)\right) + \sigma_S^2 + \sigma_x^2 + 2\rho_{Sx}\sigma_S\sigma_x \tag{4.10}$$

This implies that

$$\Sigma_F^2 = \int_0^t \| s_F(u,T) \|^2 du$$

$$= \sigma_\varepsilon^2 \left\{ \frac{1}{\kappa_\varepsilon^2}\left(t + V_{2\kappa_\varepsilon}(t) - V_{2\kappa_\varepsilon}(0) - 2\left(V_{\kappa_\varepsilon}(t) - V_{\kappa_\varepsilon}(0)\right)\right)\right.$$

$$\left. - \frac{1}{3}\eta^2\left((T-t)^3 - T^3\right) + \frac{2\eta}{\kappa_\varepsilon}\left(\frac{1}{2}T^2 - \frac{1}{2}(T-t)^2 + Y_{\kappa_\varepsilon}(0) - Y_{\kappa_\varepsilon}(t)\right)\right\}$$

$$- 2\left(\rho_{S\varepsilon}\sigma_S\sigma_\varepsilon + \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon\right)\left\{\frac{1}{\kappa_\varepsilon}\left(t + V_{\kappa_\varepsilon}(0) - V_{\kappa_\varepsilon}(t)\right) - \frac{\eta}{2}\left((T-t)^2 - T^2\right)\right\}$$

$$+ \left(\sigma_S^2 + \sigma_x^2 + 2\rho_{Sx}\sigma_S\sigma_x\right)t \tag{4.11}$$

where V_a and Y_a for notational convenience are defined for a general constant a by

$$V_a(u) \equiv \frac{1}{a}e^{-a(T-u)}, \qquad Y_a(u) = \frac{1}{a^2}\left(e^{-a(T-u)} + a(T-u)e^{-a(T-u)}\right) \tag{4.12}$$

Given the eight parameters of the model, listed in Table 1, it is now possible to price single-delivery futures (or forward) European options by equations (3.16) and European options on a futures portfolio by equation (3.21). Or stated in an other way: given observed market prices on options on single-delivery futures it is possible to find approximate prices on futures portfolios via equation (3.21).

[INSERT Table 1]

4.2 The electricity market

This model is an extended version of a model introduced by Bjerksund, Rasmussen, and Stensland (2000). Their model is used as a description of the (risk neutral) dynamics for futures and forward contracts on the Norwegian electricity market. The model has been implemented in Elviz, a risk management software marketed by the Norwegian company Viz Risk Management Services AS. To replicate their model in our setting we describe it as a two-factor model with the two factors equalling the spot price and the forward convenience yield, but with perfect positive correlation between the two factors. A one-dimensional Wiener process under $\mathbf{Q}$ is the driving process. Using the notation from the original paper we can formulate the model in terms of the diffusion terms as

$$\sigma_S(t) = \left(c + \frac{a}{b}\right)\begin{pmatrix} 1 \end{pmatrix} \quad \text{and} \quad \sigma_\varepsilon(t,s) = a(s-t+b)^{-2}\begin{pmatrix} 1 \end{pmatrix} \tag{4.13}$$

This is slightly different from the way Bjerksund, Rasmussen and Stensland formulate their model. They start by explicitly stating the volatility $s_F(t,T)$ of the prise process for the futures with maturity T. But the volatility can be calculated easily from the diffusion terms above as

$$s_F(t,T) = \sigma_S(t) - \int_t^T \sigma_\varepsilon(t,s)ds = c + \frac{a}{b} - a\int_t^T (s-t+b)^{-2}ds = c + a\left(\frac{1}{T-t+b}\right) \tag{4.14}$$

For $T=t$ one gets that the volatility for $\sigma_F(t,t)$, i.e. the spot price process, is $c + \frac{a}{b}$, which indeed is how $\sigma_S(t)$ is specified above. For a general $T \geq t$ the SDE governing $(F(t,T))$under $\mathbf{Q}$ is given by

$$dF(t,T) = s_F(t,T)dW_t^{\mathbf{Q}} = \left(c + \frac{a}{T-t+b}\right)dW_t^{\mathbf{Q}}, \quad t \leq T \tag{4.15}$$

Before extending the model we state it in terms of our own notation with $\sigma_S \equiv c + \frac{a}{b}$, $\sigma_\varepsilon \equiv a$ and $b \equiv \frac{\sigma_\varepsilon}{\sigma_\varepsilon - c}$ as

$$\sigma_S(t) = \sigma_S \quad \text{and} \quad \sigma_\varepsilon(t,s) = \sigma_\varepsilon\left(s - t + \frac{\sigma_\varepsilon}{\sigma_\varepsilon - c}\right)^{-2} = \sigma_\varepsilon(s-t+b)^{-2} \tag{4.16}$$

The model is now extended by introducing exchange rate risk and the possibility of an arbitrary correlation between the spot price and the futures convenience yield along the lines of the Miltersen and Schwartz model. With this extension the driving process is now a three-dimensional $\mathbf{Q}$-Wiener process and we again allow for a possibly non-zero constant $\eta \geq 0$ in the formulation of σ_ε. Specifically we

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assume the following diffusion terms

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_x(t) = \sigma_x \begin{pmatrix} \rho_{Sx} \\ \sqrt{1-\rho_{Sx}^2} \\ 0 \end{pmatrix} \tag{4.17}$$

and

$$\sigma_\varepsilon(t,s) = \sigma_\varepsilon \left((s-t+b)^{-2} + \eta \right) \begin{pmatrix} \rho_{S\varepsilon} \\ \frac{\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon}}{\sqrt{1-\rho_{Sx}^2}} \\ \sqrt{1-\rho_{S\varepsilon}^2 - \frac{(\rho_{x\varepsilon}-\rho_{Sx}\rho_{S\varepsilon})^2}{1-\rho_{Sx}^2}} \end{pmatrix} \tag{4.18}$$

Written as quadratic variation and mixed variation terms this means that the diffusion terms are given by

$$d\langle S\rangle_t = \sigma_S^2 S_t^2 dt \tag{4.19}$$

$$d\langle x\rangle_t = \sigma_x^2 x_t^2 dt \tag{4.20}$$

$$d\langle \varepsilon(\cdot,s)\rangle_t = \sigma_\varepsilon^2 \left((s-t+b)^{-2} + \eta \right)^2 dt \tag{4.21}$$

$$d\langle S,x\rangle_t = \rho_{Sx}\sigma_S\sigma_x S_t x_t dt \tag{4.22}$$

$$d\langle S,\varepsilon(\cdot,s)\rangle_t = \rho_{S\varepsilon}\sigma_S\sigma_\varepsilon S_t \left((s-t+b)^{-2} + \eta \right) dt \tag{4.23}$$

$$d\langle x,\varepsilon(\cdot,s)\rangle_t = \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon x_t \left((s-t+b)^{-2} + \eta \right) dt \tag{4.24}$$

Thus, by using equation (3.3), one gets that

$$\begin{aligned}
\| s_F(u,T) \|^2 &= \| \sigma_S(u) + \sigma_x(u) - \int_u^T \sigma_\varepsilon(u,s)ds \|^2 \\
&= \sigma_S^2 + \sigma_x^2 + 2\rho_{Sx}\sigma_S\sigma_x + \sigma_\varepsilon^2 \left([(s-u+b)^{-1}]_u^T - \eta(T-u) \right)^2 \\
&\quad + 2\left([(s-u+b)^{-1}]_u^T - \eta(T-u) \right) \left(\rho_{S\varepsilon}\sigma_S\sigma_\varepsilon + \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon \right) \\
&= \sigma_S^2 + \sigma_x^2 + 2\rho_{Sx}\sigma_S\sigma_x + \sigma_\varepsilon^2 \left((T-u+b)^{-2} + b^{-2} - 2\left(b(T-u+b) \right)^{-1} \right) \\
&\quad + \sigma_\varepsilon^2 \left(\eta^2(T-u)^2 - 2\eta(T-u)\left((T-u+b)^{-1} - b^{-1} \right) \right) \\
&\quad + 2\left((T-u+b)^{-1} - b^{-1} - \eta(T-u) \right) \left(\rho_{S\varepsilon}\sigma_S\sigma_\varepsilon + \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon \right) \tag{4.25}
\end{aligned}$$

This implies that

$$
\begin{aligned}
\Sigma_F^2 &= \int_0^t \parallel s_F(u,T) \parallel^2 du \\
&= \left(\sigma_S^2 + \sigma_x^2 + 2\rho_{Sx}\sigma_S\sigma_x\right)t + \sigma_\varepsilon^2\left([(T-u+b)^{-1}]_0^t + b^{-2}t + \frac{2}{b}[\log(T-u+b)]_0^t\right) \\
&\quad - \sigma_\varepsilon^2\left(\frac{\eta^2}{3}\left((T-t)^3 - T^3\right) + 2\eta\left[-(T-u) + b\log(T-u+b)\right]_0^t + \frac{\eta}{b}\left((T-t)^2 - T^2\right)\right) \\
&\quad - 2(\rho_{S\varepsilon}\sigma_S\sigma_\varepsilon + \rho_{x\varepsilon}\sigma_x\sigma_\varepsilon)\left(\frac{1}{b}t + [\log(T-u+b)])_0^t - \frac{\eta}{2}\left((T-t)^2 - T^2\right)\right)
\end{aligned}
\tag{4.26}
$$

Given the eight parameters of the model, listed in Table 2 below, it is now possible to price single-delivery forwards (or futures) European options by equations (3.16) and European options on flow forwards by equations (3.33) and (3.35). Assume that a single-delivery forward is defined as a forward for delivery on one single future day. This implies that all forward contracts with a delivery period extending one day is defined as a flow forward. As most electricity forwards have delivery periods equalling week(s), month(s) or quarter(s) of a year it is obvious that most forwards are, by this assumption, flow forwards. This naturally implies that most exchange traded electricity options are options with a flow forward as underlying contract. Thus, in the electricity market one would use equations (3.33) and (3.35) the opposite way compared to the oil market: In the oil market one would price options on futures portfolios based on the prices of single-delivery futures, while in the electricity market one would have market prices on flow forward options as the basis. And from this basis it is possible to price OTC single-delivery forwards and thus also OTC flow forward options.

[INSERT Table 2]

5 Monte Carlo simulation

5.1 Monte Carlo simulation and price comparison

In order to investigate the precision of the approximation approach for valuing options on futures portfolios we have performed Monte Carlo simulation[5] to find the prices of 11 call and 11 put options with the same underlying portfolio as in section 4.1, i.e. the three factor model with a futures portfolio consisting of 6 Brent crude futures with monthly delivery through the period October 2003$-$March 2004. The prices are calculated as of 30th May, 2003 (0), with the options expiring at 10th September, 2003 (t). The options only differ by the strike level which, as a percentage of the ATM strike level (157.46 DKK), belongs to the set $\{70\%, 80\%, 85\%, 90\%, 95\%, 100\%, 105\%, 110\%, 115\%, 120\%, 130\%\}$.

We have considered approximate option prices with three different durations for the representative single-delivery futures, namely δ^H, δ_A^H and the simple average of the two $\frac{\delta^H + \delta_A^H}{2}$. These prices are denoted by $C[\delta^H]$, $C[\delta_A^H]$ and $C[\frac{\delta^H + \delta_A^H}{2}]$, respectively, in the call case and analogously by $P[\delta^H]$, $P[\delta_A^H]$ and $P[\frac{\delta^H + \delta_A^H}{2}]$ in the put case. Moreover, we have considered the arithmetic average between the prices obtained by δ^H and δ_A^F. This price average exploits the observed fact that the δ^H-method has a negative price bias whereas the δ_A^H-method has a positive price bias.

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[5]50,000 pairs of antithetic sample paths have been simulated by an Euler approximation of the underlying futures' SDEs to price each of the options. The length of the time step was one calendar day. The calculations are performed in MAPLE.

At last, after having simulated and calculated all prices, we ran a simple OLS regression by regressing the whole sample of 22 MC prices on the corresponding prices $C[\delta^F]$ and $C[\delta_A^F]$ for the call prices and $P[\delta^H]$ and $P[\delta_A^H]$ for the put prices. Denoting the OLS parameter belonging to the $[\delta^H]$-variable ($[\delta_A^H]$-variable) by v_1 (v_2) we found the OLS estimates to be $\hat{v}_1 = 0.436$ and $\hat{v}_2 = 0.573$. Thus, more weight is attached to the $[\delta_A^H]$-variable. However, the hypothesis[6] $H_0 : v_1 = v_2 = 0.5$ of equal weights could not be rejected. It was accepted with a test probability of $1 - F_{F(2,20)}\left(\frac{(0.1110-0.1045)/2}{0.1045/20}\right) = 0.55$. In practice the interest is mainly in options that are close to being at-the-money. All methods, except the δ^H-method, appear to perform with a desirable accuracy in this region.

It is obvious that the prices given by our approximate option pricing formula is a non-linear function of the portfolio's stochastic duration δ^H or δ_A^H. Hence, one could also suspect that the linear relationship among the variables in the above OLS regression is not valid. However, we found neither signs of misspecification (applying Ramsey's RESET test), residual autocorrelation (applying Harvey's F-test for first-order autocorrelation) nor residual conditional heteroscedasticity (applying Engel's F-test for first-order ARCH). Consequently, the linear relationship seems applicable for practical purposes. Taking the call case as an example, we conclude that it appears reasonable to maximize the precision of the approximate option pricing formula by a direct convex combination of $C[\delta^H]$ and $C[\delta_A^H]$ instead of searching for an optimal convex combination of δ^H and δ_A^H that could be used in the $C[\cdot]$ function as the "optimal" approximate option price. According to our results, cf. Tables 3 and 4 and the test of H_0, choosing the convex combination $v_1 = v_2 = 0.5$ as default is not a bad choice.

[INSERT Table 3]

[INSERT Table 4]

Additionally, it is a useful guide that the δ^H-method seems to produce prices that are too low while the δ_A^H-method seems to produce prices that are too high.

5.2   Details about the example

The assumed portfolio consists of the 6 futures contracts for delivery of $\frac{1}{6}$ barrel in either of the months October, November, December 2003 and January, February and March 2004. Thus, a total of 1 barrel is delivered during the period October 2003−March 2004. The exercise price, expressed in DKK, is set equal to the futures price of the contract with time to maturity equal to the stochastic duration of the portfolio. By doing so the option is by construction at the money when the investor initially enters into it. The basic IPE data for the example are presented in Table 5. The contracts numbered $4, 5, \ldots, 9$ constitute the portfolio.

Additional information needed is taken from the Danish Central Bank (*Nationalbanken*) and the US Central Bank (*Federal Reserve*) according to whom the official exchange rate as of 30th May 2003 was 628.02 DKK/100 USD, the 6-month CIBOR interest rate was 2.30% p.a. and the 6-month Eurodollar interbank interest rate was 1.15% p.a. These 6-months rates have been used as flat rates throughout the whole period of interest, thus assuming constant interest rates in this period.

The parameters $\sigma_S, \sigma_\varepsilon, \rho_{S\varepsilon}, \kappa_\varepsilon$ and η have been estimated implicitly by finding the combination of parameter values that minimizes the sum of squared deviations between squared annual model volatility

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[6]Clearly, we are aware that the use of standard statistical tools in this setting is to be interpreted with considerable care.

(i.e. $\frac{\Sigma_{F_i}^2}{t_i}$ for $i = 1, 2, \ldots, 11$) and squared observed implied annual option volatility for the 11 observed implied option volatilities reported in Table 5. The resulting parameter estimates are given in Table 6.

[INSERT Table 5]

[INSERT Table 6]

6 Conclusion

In the first part of the paper we showed cross currency relations for forward and futures prices as well as convenience yields. We showed that whenever the forward price and the futures price deviate in one of the countries there is no simple currency translation between the futures prices. This holds a fortiori when the forward prices and the futures prices deviate in both countries. Under the same circumstances, futures convenience yields deviate, whereas forward convenience yields are always identical across currency denominations.

In the second part of the paper we showed how to find the stochastic duration of energy futures portfolios and flow forwards and how to approximately price European options with them as underlying instruments, also when the options are denominated in another currency than the trading currency for the underlying instrument. This has been done in a HJM framework which is well suited for modeling the term structure in energy futures markets. Making assumptions about log-Gaussian spot prices and exchange rates and Gaussian convenience yields, we found explicit pricing formulas for the futures portfolio and flow forward options. Suitable models for the crude oil and electricity markets were used as examples for implementation of the suggested numerical approach.

The framework is quite general and still analytically tractable. The generality is underpinned by the fact that prices on forwards and futures do not have to follow a log-Gaussian process, as they do in our examples. As long as they follow a process where it is possible to price European options on single-delivery forwards or futures analytically, it allows application of our option pricing approximation to options on a (flow) forward or a futures portfolio. Thus, prices can be allowed to follow e.g. a CIR-like process, giving rise to non-central χ^2 distributed prices, or exponential affine term structure models that are well-known from the fixed income literature.

Thus, it is possible to extend the framework to a multi-commodity situation with several energy commodities. In this situation it is of course still essential to be able to calculate the duration of any (multi-commodity) portfolio as a combined risk measure for the portfolio. It could be calculated in an extended model compared to our model in this paper, but still along exactly the same lines outlined here. Option sensitivity parameters, which are highly relevant for practical hedging purposes, could still be easily calculated due to the analytical tractability of the model. Moreover, it would be possible to value single-commodity and cross-commodity energy options in the same model, which is an attractive and consistency-procuring feature.

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Table 1: *The eight parameters for the oil market model classified after if they can be estimated by inference based on observed variables. In this case the parameter is marked by a † as Observable. Otherwise it is marked as Non-observable.*

Parameter	Observable	Non-observable
σ_S	†	
σ_x	†	
ρ_{Sx}	†	
σ_ε		†
κ_ε		†
$\rho_{S\varepsilon}$		†
$\rho_{x\varepsilon}$		†
η		†

Table 2: *The eight parameters for the electricity market model classified after if they can be estimated by inference based on observed variables. In this case the parameter is marked by a ◇ as Observable. Otherwise it is marked as Non-observable.*

Parameter	Observable	Non-observable
σ_S	◇	
σ_x	◇	
ρ_{Sx}	◇	
σ_ε		◇
b		◇
$\rho_{S\varepsilon}$		◇
$\rho_{x\varepsilon}$		◇
η		◇

Table 3: *Results of MC simulating the values of call options on the futures portfolio, consisting of six Brent oil futures of different maturities, from section 2.5. 11 Different strike prices have been used, ranging from 70% to 130% of the ATM strike level (157.46 DKK). $\hat{v}_1 = 0.426$ and $\hat{v}_2 = 0.573$.*

Strike	Call price	Price deviation				
	MC	$C[\delta^H]$	$C[\delta_A^H]$	$C[\frac{\delta^H+\delta_A^H}{2}]$	$\frac{C[\delta^H]+C[\delta_A^H]}{2}$	$\hat{v}_1 C[\delta^H] + \hat{v}_2 C[\delta_A^H]$
70%	46.66	0.14	0.03	0.08	0.09	0.03
80%	31.63	0.02	-0.01	0.00	0.02	-0.02
85%	24.74	-0.07	-0.03	-0.06	-0.05	-0.06
90%	18.50	-0.08	0.05	-0.03	-0.02	-0.02
95%	13.32	-0.19	0.01	-0.11	-0.09	-0.09
100%	9.12	-0.20	0.04	-0.10	-0.08	-0.07
105%	5.84	-0.06	0.18	0.04	0.07	0.08
110%	3.72	-0.15	0.08	-0.06	-0.04	-0.02
115%	2.06	0.05	0.23	0.13	0.15	0.16
120%	1.11	0.09	0.23	0.14	0.16	0.17
130%	0.35	0.00	0.06	0.03	0.03	0.03
SSD		0.146	0.154	0.077	0.077	0.080

Table 4: *Results of MC simulating the values of put options on the futures portfolio, consisting of six Brent oil futures of different maturities, from section 2.5. 11 Different strike prices have been used, ranging from 70% to 130% of the ATM strike level (157.46 DKK). $\hat{v}_1 = 0.426$ and $\hat{v}_2 = 0.573$.*

Strike	Put price	Price deviation				
	MC	$P[\delta^H]$	$P[\delta_A^H]$	$P[\frac{\delta^H+\delta_A^H}{2}]$	$\frac{P[\delta^H]+P[\delta_A^H]}{2}$	$\hat{v}_1 P[\delta^H] + \hat{v}_2 P[\delta_A^H]$
70%	0.04	0.01	0.02	0.01	0.01	0.01
80%	0.53	0.02	0.11	0.06	0.06	0.07
85%	1.49	-0.10	0.07	-0.03	-0.03	-0.01
90%	3.11	-0.16	0.09	-0.05	-0.04	-0.02
95%	5.73	-0.25	0.07	-0.11	-0.09	-0.07
100%	9.35	-0.25	0.11	-0.09	-0.07	-0.05
105%	14.05	-0.27	0.10	-0.11	-0.09	-0.07
110%	19.52	-0.13	0.22	0.03	0.05	0.05
115%	25.90	-0.14	0.16	-0.01	0.01	0.01
120%	32.72	-0.05	0.21	0.06	0.08	0.07
130%	47.53	-0.07	0.12	0.02	0.02	-0.01
SSD		0.280	0.185	0.045	0.034	0.025

Table 5: *Futures and option data from 30th May 2003 from IPE used for calculating the price of European options on a futures portfolio consisting of contracts No. $4, 5, \ldots, 9$. Prices are in USD.*

i	Contract	Price	Strike	Call price	Put price	Imp. vol.,% p.a.	Option expiry t_i	Futures expiry T_i
1	Jul 2003	26.32	26.50	0.69	0.87	40.66	10/06 2003	13/06 2003
2	Aug 2003	26.00	26.00	1.23	1.23	34.56	11/07 2003	16/07 2003
3	Sep 2003	25.74	25.50	1.61	1.37	32.33	11/08 2003	14/08 2003
4	Oct 2003	25.51	25.50	1.66	1.65	30.50	10/09 2003	15/09 2003
5	Nov 2003	25.28	25.50	1.72	1.94	29.49	13/10 2003	16/10 2003
6	Dec 2003	25.04	25.00	1.99	1.95	29.40	10/11 2003	13/11 2003
7	Jan 2004	24.77	25.00	1.97	2.20	28.68	11/12 2003	16/12 2003
8	Feb 2004	24.52	24.50	2.16	2.14	27.87	12/01 2004	15/01 2004
9	Mar 2004	24.29	24.50	2.02	2.23	26.10	09/02 2004	12/02 2004
10	Apr 2004	24.08	24.00	2.22	2.14	25.68	11/03 2004	16/03 2004
11	May 2004	23.88	24.00	2.18	2.30	25.29	08/04 2004	15/04 2004

Table 6: *Estimates of the parameters of the oil market model* $\sigma_S, \sigma_\varepsilon, \rho_{S\varepsilon}, \kappa_\varepsilon, \eta, \sigma_x$ *and* ρ_{Sx}. $\rho_{x\varepsilon}$ *has been set to zero. For comparison, estimated values for* $\sigma_S, \sigma_\varepsilon, \rho_{S\varepsilon}, \kappa_\varepsilon$ *for NYMEX crude oil futures from Schwartz (1997) are reported too. Also the SSD between observed squared annual implied IPE option volatility and squared annual model volatility is reported (it relates to the first five parameters only).*

Parameter estimate	Estimated value	Schwartz (1997)
$\hat{\sigma}_S$	0.4409	0.344
$\hat{\sigma}_\varepsilon$	1.7923	0.372
$\hat{\rho}_{S\varepsilon}$	0.9850	0.915
$\hat{\kappa}_\varepsilon$	8.5172	1.045
$\hat{\eta}$	0	-
$\hat{\sigma}_x$	0.1104	-
$\hat{\rho}_{Sx}$	-0.0015	-
$\hat{\rho}_{x\varepsilon}$	0	-
SSD	$6.88 \cdot 10^{-5}$	-