Capital Structure and Default Probability

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Abstract

This paper focuses on the historical default probability in a structural model with jumps, more precisely, when the firm assets dynamics are modeled by a double exponential jump diffusion process. It relies on the Leland and Toft approach (see Leland [1994a, 1994b] and Leland and Toft [1996]) as formalized by Hilberink and Rogers [2002], explains how to compute the default probability and examines its sensitivity with respect to fundamentals such as leverage and debt maturity. Because of a jump risk that can not be hedged, a risk neutral measure has to be chosen. The Esscher measure is chosen and with this choice it is proved that when changing universe (from the historical world to the risk-neutral one) the same kind of dynamics for the assets value prevails.

Introduction

The existence of an optimal capital structure is probably one of the main issue in finance. The riskiness of debt, its default probability and the market value of its credit spreads are some of the most important questions which both academics and practitioners have to face. From Modigliani and Miller [1958] to Morellec and Smith [2005], there has been a considerable amount of

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articles on this subject. Agency costs, signaling theory and bankruptcy costs have been advocated to explain the presence of an optimal debt. The introduction of bankruptcy costs in the analysis dates back to Baxter [1967]. The idea is to consider that tax advantages are traded off against the likelihood of incurring bankruptcy costs. The analysis of default and the dramatic development of credit derivatives have also given birth to a huge literature. The two traditional theoretical approaches to default rely on the so-called intensity and structural models. In the former class, default occurs as a pure random event, see Jarrow, Lando and Turnbull [1997] or Duffie and Singleton [1999], for representative works. In the latter class, the structural approach, default is associated with the first time the firm assets cross a threshold - Black and Scholes [1973] and Merton [1974] pioneered this framework. Longstaff and Schwartz [1995], Nielsen, Saà-Requejo and Santa-Clara [1993], and Collin-Dufresne and Goldstein [2001], amongst others, introduced stochastic interest rates in the framework. Madan and Unal [1998] introduced links between the two main approaches, and Jarrow and Protter [2004] showed that they can be embedded into a same model containing different informational assumptions. These formalizations have extensively been used to study credit spreads and to price many credit derivatives. In this paper we consider bankruptcy costs and the structural approach to default.

In this approach, two assumptions are fundamental: the dynamics of the firm assets value and the endogeneity or exogeneity of the default triggering threshold. The most convenient hypothesis is to suppose the assets value follows a geometric Brownian motion, see Merton [1974], Leland [1994a, 1994b], Leland and Toft [1996]. It gives closed form solution but generally fails to explain some empirical facts. For example Eom, Helwege and Huang [2003] show that the Leland and Toft model which relies on this hypothesis overpredicts credit-spreads for long-maturity bonds and underpredicts them for short maturity bonds. Besides, this assumption can not explain why credit spreads go to zero as maturity decreases to zero, an empirically observed fact. To take into account this feature, jumps can be introduced and the firm assets value can be modeled by the exponential of a Lévy process. This Lévy process can be a spectrally negative process and then would jump only downwards (see Hilberink and Rogers [2002]), or it can jump in both directions, up or down, with stable processes (see Le Courtois and Quittard-Pinon [2003]) or double exponential jump processes (see Dao [2005] and Chen and Kou [2005]). The latter situation is quite manageable and permits closed form or easy to compute solutions for debt, firm value equity and optimal barrier level. The barrier or threshold triggering default can be exogenous, for example it can be, as in Merton [1974], the debt face value, or a fraction of it, as in Longstaff and Schwartz [1995] or endogenous and a criterion must be chosen, as for example the maximization of equity value, as it is assumed in Black and Cox [1976] or in Leland and Toft [1996]. In this case the manager acts as a decision maker, as far as bankruptcy is concerned. The barrier depends on fundamentals such as: assets volatility, payout rate, interest rate, amount of debt, bankruptcy costs, recovery rates and corporate tax rates.

In this paper we focus on default probability. This point has been extensively analyzed with statistical methods. It begins with Altman [1968], but references to structural models are more recent. The well known KMV methodology with its concepts of distance to default, expected default frequencies (which we call in this paper default probability) is strongly inspired by Merton [1974] and Longstaff and Schwartz [1995]. Leland [2004] compares different default probabilities generated by three models: Leland and Toft, Longstaff and Schwartz and KMV. Obtaining these default probabilities is particularly important for ranking bonds according to their risk and it can be seen as an alternative to ratings; indeed, rating agencies do not reveal publicly their methodologies.

In this paper we keep the structure of these models, we use the Hilberink and Rogers [2002] formalization, and choose the Kou [2002] double exponential jump diffusion process. To a large extent these models essentially center their analysis on market value. To obtain arbitrage free values these models only use risk neutral dynamics. But when the question arises to compute real-world default probabilities, reference to a risk neutral measure is of little help. In the case of assets modeled by a geometric Brownian motion, this is not a difficult problem, but when the dynamics incorporate jumps, it becomes more difficult for at least two reasons: first, there are many candidates to be risk neutral measures, second, in the Leland and Toft framework the endogenous barrier level is given with parameters in a risk neutral universe. Two questions must therefore be answered: what risk neutral should be chosen, and how can the historical parameters be determined. One of the main contributions of this paper is to suggest a correspondence between the actual parameters and the risk neutral ones. We choose the risk-neutral Esscher measure. This measure has nice properties and has successfully been applied in insurance and finance, see Bühlmann [1980], Gerber and Shiu [1994] or Le Courtois and Quittard-Pinon [2004]. We show that if the dynamics of the assets value is a double exponential jump diffusion process in the real world, it remains a process of the same type in the Esscher risk neutral universe, and we find simple relations between the coefficients. Then it is possible to study default probabilities.

The paper is organized as follows. Section 2 describes the basic setting, recalls the Leland and Toft framework as formalized by Hilberink and Rogers, and gives general formulae for debt, equity firm values and the optimal default barrier. Section 3 adapts this approach to the cases of geometric Brownian motion and double exponential jump diffusion processes, and gives the expression for default probability, in terms of a Laplace transform. This section contains the main technical result of the paper.

1 Basic Setting

In this section, we briefly present the Leland and Toft [1996] environment in which the firm evolves, as formalized by Hilberink and Rogers [2002]. So we introduce the firm asset value V, the debt value $\mathcal{D}$, the firm value $\mathcal{V}$ and the equity value $\mathcal{E}$. All these quantities refer to market values. We consider bankruptcy costs, taxes and we assume the existence of a riskless asset with a constant rate of return r, the interest rate in this economy.

We assume that the process V is given by the exponential of a Lévy process X: $V_t = V_0 e^{X_t}$. Thus X can be interpreted as the assets return process. In particular we assume that V follows the stochastic differential equation

$$dV = V_-((\mu(V,t) - \delta)dt + dh)$$

in the real world, or

$$dV = V_-((r - \delta)dt + d\hat{h})$$

in a risk neutral universe. The parameters $\mu(V,t)$ and δ are respectively the total expected rate of return on asset value and the constant fraction of value paid out to stockholders and bondholders. The processes h and $\hat{h}$ are martingales. We assume that V_0 is known.

The debt has two features: its time structure and its riskiness. Firstly, it is constantly retired and reissued, such that it has a constant facial value P and a total coupon rate C. Its time profile is of an exponential type. By definition the maturity profile is:

$$\varphi(t) = me^{-mt}$$

If bankruptcy never occurs, the proxy average maturity T of debt is:

$$T = \int_0^\infty t\varphi(t)\,dt = \int_0^\infty t\left(me^{-mt}\right)dt = \frac{1}{m}$$

The higher the debt retirement rate (or "roll-over rate") m, the shorter the average maturity $T = 1/m$ of the debt. If $m = 0$, principal is never retired, and debt has infinite maturity as in Leland (1994). As $m \to \infty$, the average maturity of debt T approaches zero. For more details see Leland and Toft [1996] or Hilberink and Rogers [2002]. This time structure gives a great flexibility to the model. Secondly, because debt is risky, a default mechanism and a recovery scheme must be specified. As far as the first point is concerned the modeling pertains to the class of structural models. Bankruptcy or default occurs at the first time τ the value of the firm assets V fall below a default barrier L. The value of this default barrier L will be determined endogenously and optimally later, by maximizing the equity value. By definition, the stopping time

$$\tau = \inf\{t \geq 0 : V_t \leq L\}$$

can also be expressed as:

$$\tau = \tau_l = \inf\{t \geq 0 : X_t \leq l\}$$

Indeed the crossing of the barrier V_B by the assets value process V is equivalent to the crossing of the return barrier $l = \ln(V_B/V_0)$ by the assets return X. As for the second point: recovery. We consider that all debts are of equal seniority and we assume a constant fraction of asset value α $(0 \leq \alpha \leq 1)$ lost in default. We suppose that, when the absolute priority rule (APR) is not respected, shareholders receive γ $(0 \leq \gamma \leq 1)$, a constant fraction of the residual asset value at default. Let us denote $\hat{\alpha}$ such that

$(1-\alpha)(1-\gamma) = 1-\hat{\alpha}$, so $\hat{\alpha}$ becomes the fraction of asset value lost at default, in this case.

The total market value of the firm, according to Modigliani-Miller, is expressed as the sum of the asset value plus tax benefits and minus bankruptcy costs. We denote by θ the corporate tax rate. The firm receives an income stream of θCdt, i.e., an expected present value of tax benefits $TB = \frac{\theta C}{r}\left(1 - e^{-r\tau}1_{\tau<\infty}\right)$. This assumption is rather an idealized treatment of tax. Leland and Toft (1996) introduced a tax cutoff level V_T, which reduces the tax rebate to 0 when $V < V_T$ and keep it at θCdt when $V \geq V_T$. This assumption intends to reflect the idea that when the coupons exceed the profits, it is not possible for the firm to claim the tax benefit on the coupon payments. Later, we take into account this assumption.

Finally, $\mathcal{E}$, the equity value, is by definition equal to the firm value minus the debt value:

$$\mathcal{E} = \mathcal{V} - \mathcal{D}$$

Using the theory of arbitrage in continuous time, and denoting by Q a risk neutral measure, Hilberink and Rogers (2002) prove that the debt value can be expressed as

$$\mathcal{D} = \frac{C+mP}{r+m} E_Q\left[1 - e^{-(r+m)\tau}\right] + (1-\hat{\alpha}) V_0 E_Q\left[e^{X_\tau - (r+m)\tau} 1_{\tau<\infty}\right] \tag{1}$$

assuming no cutoff for tax benefits, the value of firm can be expressed as

$$\mathcal{V} = V_0 + \frac{\theta C}{r} E_Q\left[1 - e^{-r\tau}\right] - \alpha V_0 E_Q\left[e^{X_\tau - r\tau} 1_{\tau<\infty}\right] \tag{2}$$

and the equity as

$$\begin{aligned} \mathcal{E} = {} & V_0 + \frac{\theta C}{r} E_Q\left[1 - e^{-r\tau}\right] - \alpha V_0 E_Q\left[e^{X_\tau - r\tau} 1_{\tau<\infty}\right] \\ & - \frac{C+mP}{r+m} E_Q\left[1 - e^{-(r+m)\tau}\right] + (1-\hat{\alpha}) V_0 E_Q\left[e^{X_\tau - (r+m)\tau} 1_{\tau<\infty}\right] \end{aligned} \tag{3}$$

Looking at these general expressions, it is clear that we can obtain solutions if we know how to compute the Laplace transform of the first passage time and of the one of the pair: first passage time and X. Note that these expressions are independent of the Lévy process chosen for the assets return.

Closed forms solution are given in the case of geometric Brownian motion (see Leland and Toft [1996]), in the case of a double exponential jump diffusion (see Kou and Chen [2005]), and numerical solutions with spectrally negative processes (see Hilberink and Rogers [2002]) and in the case of stable processes (see Le Courtois and François Quittard-Pinon [2003]) can be obtained.

To end this section let us emphasize what we believe is in the heart of this approach when thinking in terms of market value. What has firstly to be done is to determine endogenously P, C and V_B. So, we need three equations. The first one is obtained when we demand that the initial market value of debt is equal to par. The first equation is therefore:

$$P = \mathcal{D}\left(P,\, C,\, V_B,\, V_0\right), \tag{4}$$

the second equation is given by the definition of the leverage $\mathcal{L}$:

$$\mathcal{L} = \frac{\mathcal{D}\left(P,\, C,\, V_B,\, V_0\right)}{\mathcal{V}\left(P,\, C,\, V_B,\, V_0\right)}, \tag{5}$$

and the third equation is given by the optimality criterion which is to maximize equity value. This first order condition writes:

$$\frac{\partial \mathcal{E}}{\partial V_0}\,|_{V_0=V_B} = 0. \tag{6}$$

As noted earlier V_0 is assumed to be known. To compute the triplet (P, C, V_B) we have to solve the system of equations (4), (5) and (6). The analyses on the capital structure, credit spreads or default probability come from this resolution. In this paper we focus on the latter, important, question.

2 The Default Probability

As far as one is interested in valuing the main determinants of firm value, the right referential is the risk neutral universe. But now, as we want to investigate the bankruptcy probability, we have to turn back to the historical or actual universe. We shall consider two cases. Firstly, we assume that the assets price is modeled by a geometric Brownian motion, secondly by a double exponential jump diffusion process, which we will sometimes call for brevity a Kou process. In order to compute the default probability, we need to calculate the first passage time probability density function for the firm assets value to reach a default barrier V_B at time t from a starting value $V_0 > V_B$. This first passage time law is explicit in the first case (GBM), whilst numerical computations have to be done in the second one (Kou processes).

Recall the notations $l = \ln\left(\frac{V_B}{V_0}\right)$, and $\tau_l = \{\inf \quad t/V(t) \leq V_B\}$. Here we are facing a new problem: the optimal endogenous barrier is given with parameters in a risk neutral universe. In the case when the firm assets value follows a geometric Brownian motion, this is innocuous: we are working in a Black and Scholes economy where there is a unique martingale measure equivalent to the real universe probability, and the parameters are well identified. Incorporating jumps into the model is more realistic but introduces two difficulties. Firstly, the market is no more complete, so an e.m.m. has to be chosen. Secondly, if a choice is made, what is the process followed by the assets value in this risk neutral universe? We consider this question in this section and suggest a complete solution.

Technically, a key tool for our analysis is the Laplace exponent of our return process X. By definition, it is defined by the function G such that $E[e^{\beta X(t)}] = e^{tG(\beta)}$. The Lévy-Khintchine representation theorem gives a canonical characterization of a Lévy process, identifying a drift term, a Brownian component, and a jump component. This formula writes, for a finite-variation process, as:

$$G(\beta) = a\beta + \frac{1}{2}\sigma^2\beta^2 + \int_R (e^{\beta y} - 1)\nu(dy)$$

where a is the drift, σ refers to the Brownian component, and ν is the Lévy measure identifying the jumps. For more details see for instance Satō [1987]. Given ρ, the roots in β of the equation $G(\beta) = \rho$ will be of a central importance in our analysis.

2.1 Geometric Brownian Motion

In this setting, the assets value in the historical world are given by:

$$dV = V((\mu - \delta)dt + \sigma dz)$$

and in the risk neutral world by:

$$dV = V((r - \delta)dt + \sigma d\hat{z})$$

where z and $\hat{z}$ are standard Brownian motions, and V_0 is known.

In this Black and Scholes economy, equations (1), (2), and (3) give the following market values for the debt:

$$\mathcal{D} = \frac{C + mP}{r + m}\left(1 - \left(\frac{V_B}{V_0}\right)^{\beta_{1,r+m}}\right) + (1 - \hat{\alpha})\, V_B \left(\frac{V_B}{V_0}\right)^{\beta_{1,r+m}},$$

the firm:

$$\mathcal{V} = V_0 + \frac{\theta C}{r}\left(1 - \left(\frac{V_B}{V_0}\right)^{\beta_{1,r}}\right) - \alpha V_B \left(\frac{V_B}{V_0}\right)^{\beta_{1,r}},$$

and the equity:

$$\begin{aligned}\mathcal{E} = V_0 &+ \frac{\theta C}{r}\left(1 - \left(\frac{V_B}{V_0}\right)^{\beta_{1,r}}\right) - \alpha V_B \left(\frac{V_B}{V_0}\right)^{\beta_{1,r}} \\ &- \frac{C + mP}{r + m}\left(1 - \left(\frac{V_B}{V_0}\right)^{\beta_{1,r+m}}\right) + (1 - \hat{\alpha})\, V_B \left(\frac{V_B}{V_0}\right)^{\beta_{1,r+m}}\end{aligned}$$

where $-\beta_{1,\rho}$ is the negative root of the equation:

$$\frac{1}{2}\sigma^2\beta^2 + \left(r - \delta - \frac{1}{2}\sigma^2\right)\beta = \rho \tag{7}$$

Here the Lévy-Khintchine formula writes

$$G(\beta) = \frac{1}{2}\sigma^2\beta^2 + \left(r - \delta - \frac{1}{2}\sigma^2\right)\beta \tag{8}$$

The formulae for debt, firm and equity come from well known results on geometric Brownian motion (see for example Elliott and Kopp [1999], page 196), and were first obtained by Leland and Toft [1996] *via* a partial differential equation.

Bankruptcy occurs the first time when $V_t = V_B$. Using standard results on Brownian motion, the first passage time law is given by:

$$P(\tau_l < t) = N\left(h_1(t)\right) + \exp\left\{-\frac{2\nu l}{\sigma^2}\right\} N(h_2(t)) \tag{9}$$

where $h_1(t) = \frac{l-\nu t}{\sigma\sqrt{t}}$ and $h_2(t) = \frac{l+\nu t}{\sigma\sqrt{t}}$ with $\nu = \mu - \delta - \frac{1}{2}\sigma^2$.

From Hilberink and Rogers (2002) the barrier level is:

$$V_B = \frac{\frac{C+mP}{r+m}\beta_{1,r+m} - \frac{\theta C}{r}\beta_{1,r}}{1 + \alpha\beta_{1,r} + (1-\hat{\alpha})\,\beta_{1,r+m}} \tag{10}$$

When the payoff ratio $\delta = 0$, $\hat{\alpha} = 0$, $\beta(r) = 1$ we obtain the closed form expression for L in the presence of tax cutoff level:

$$V_B = \frac{(C+mP)\,\beta_{1,r+m}}{(r+m)\left[rV_T\left(1 + \alpha\beta_{1,r} + (1-\hat{\alpha})\,\beta_{1,r+m}\right) + \theta C\beta_{1,r}\right]}$$

This formula, giving the optimal default barrier in presence of a tax cutoff, is equation (20) in Leland [1994b] as well as equation C11 in Hilberink and Rogers [2002]. However, when the payoff ratio $\delta \neq 0$, only the expression in equation C11 in Hilberink and Rogers [2002] applies.

2.2 Double Exponential Jump Diffusion Process

This case is a little bit more complicated. The barrier level is expressed with parameters in the risk neutral world. So we have to look for a correspondence between the actual and risk neutral universes. In order to do that, we assume that in the actual world, the firm assets value V follows the stochastic differential equation

$$\frac{dV}{V_-} = (\mu - \delta)dt + \sigma dz + d\left(\sum_{k=1}^{N_t} U_k\right)$$

where

- z is a standard Brownian motion
- N is a Poisson process with constant intensity rate λ
- $(1 + U_k)$ are strictly positive i.i.d. random variables
- $Y_k := \ln(1 + U_k)$ are i.i.d. random variables of double exponential distribution with density:

  $$f_Y(y) = p\,\eta_1\,e^{-\eta_1 y}\,1_{\{y \geq 0\}} \;+\; q\,\eta_2\,e^{\eta_2 y}\,1_{\{y<0\}}$$

  where η_1 and η_2 are positive numbers, and p and q positive numbers such that: $p + q = 1$
- All sources of randomness: N, z and the Y_k's are assumed to be independent.

Using Itō's lemma for semimartingales, we obtain:

$$V_t \;=\; V_0\,\exp\{X_t\} \;=\; V_0\,\exp\left\{at + \sigma z_t + \sum_{k=1}^{N_t} Y_k\right\}$$

with $a = \mu - \delta - \frac{1}{2}\sigma^2$, or:

$$V_t \;=\; V_0\,\exp\{at + \sigma z_t\}\prod_{k=1}^{N_t} Z_k$$

where $Z_k \;=\; e^{Y_k}$.

Recall also that the Lévy-Khintchine formula writes

$$G(\beta) = a\beta + \frac{1}{2}\sigma^2\beta^2 + \int_R (e^{\beta y} - 1)\nu(dy), \qquad \nu(dy) = \lambda f_Y(y)dy \tag{11}$$

with $E[e^{\beta X(t)}] = e^{tG(\beta)}$.

2.2.1 Changing Universe

In the current setting, the market is incomplete. The risk due to jumps cannot be hedged and there is no more a unique risk neutral measure. Among the measures equivalent to the historical probability measure, we choose the Essccher measure for which the discounted gain process at the interest rate r are martingales. The Esscher risk-neutral measure Q^h associated with the parameter h is defined by the Radon Nikodym density:

$$\frac{dQ^h}{dP} := \eta(t) = \frac{e^{hX(t)}}{E[e^{hX(t)}]}$$

The Laplace exponent $\hat{G}(\beta)$ is thus given in the risk-neutral world by:

$$E_h[e^{\beta X(t)}] = E[\eta(t)e^{\beta X(t)}] = e^{t\hat{G}(\beta)}$$

After some computations (see the Appendix), one obtains:

$$\hat{G}(\beta) = A\beta + \frac{1}{2}\Gamma^2\beta^2 + \int_R (e^{\beta y} - 1)e^{hy} f_Y(y)dy$$

and the characteristic triplet of the Kou process under the h Esscher measure is $(A, \Gamma, \hat{\nu})$, where:

$$A = a + \sigma^2 h \qquad \Gamma = \sigma \qquad \hat{\nu}(dy) = e^{hy}\nu(dy).$$

Consequently the price process which is a Kou process in the actual world remains a Kou process in the risk neutral universe, but with different parameters (whose expression will be given below in (14)).

As shown in the Appendix, the following martingale condition gives the risk neutral parameter h^* for the Esscher measure:

$$r = h^*\sigma^2 + \mu - \lambda\left[\frac{p\eta_1}{\eta_1 - h^*} + \frac{q\eta_2}{\eta_2 + h^*} - \frac{p\eta_1}{\eta_1 - (1 + h^*)} - \frac{q\eta_2}{\eta_2 + 1 + h^*}\right] \quad (12)$$

By identification of the characteristic exponent and after denoting by

$$\zeta = \frac{p\eta_1}{\eta_1 - h^*} + \frac{q\eta_2}{\eta_2 + h^*}, \quad (13)$$

we obtain (see again the Appendix) the correspondence between the historical and risk-neutral (hatted) parameters:

$$\left\{\begin{array}{r} \hat{p} = \frac{p\eta_1}{\zeta(\eta_1 - h^*)} \\ \hat{q} = 1 - \hat{p} \\ \hat{\eta_1} = \eta - h^* \\ \hat{\eta_2} = \eta_2 + h^* \\ \hat{\lambda} = \lambda\zeta \end{array}\right. \tag{14}$$

These relations are important and, as far as we know, new. The main result is the fact that if the firm assets value follows a Kou Process in the real world it follows again a Kou process in the risk neutral universe associated with the choice of the right Esscher measure, and the change of parameter can be done as exposed above. Indeed, the equations (14) give the way the coefficients of the historical universe and the coefficients in the risk neutral universe are precisely linked. It is not obvious that changing universe, a Kou process in a universe would remain of the same type in the other universe. Kou and Kou and Wang have noticed that a risk-neutral measure can be obtained using a rational expectation argument with a HARA utility function for the representative agent in a Lucas economy. Here our approach is quite different and conducted with the Esscher measure. To sum up we can say that, in the chosen risk neutral universe, we have:

$$X_t = \left(r - \delta - \frac{1}{2}\sigma^2 - \hat{\lambda}\hat{\xi}\right) t + \sigma \hat{z}_t + \sum_{k=1}^{\hat{N}_t} \hat{Y}_k$$

- where $\hat{z}$ is a standard Brownian motion,
- $\hat{N}$ is a Poisson process with constant intensity rate $\hat{\lambda}$,
- $\hat{Y}_k$ are strictly positive i.i.d random variables of double exponential distribution with density:

$$\widehat{f_Y}(y) \;=\; \hat{p}\,\hat{\eta}_1\, e^{-\hat{\eta}_1 y}\, 1_{\{y \geq 0\}} \;+\; \hat{q}\,\hat{\eta}_2\, e^{\hat{\eta}_2 y}\, 1_{\{y<0\}}$$

$$\hat{\eta}_1 > 1, \qquad \hat{\eta}_2 > 0, \qquad \hat{p}, \hat{q} > 0, \qquad \hat{p} + \hat{q} = 1,$$

- $\hat{\xi} = E_h[e^{\hat{Y}_1}] - 1 = \hat{p}\frac{\hat{\eta}_1}{\hat{\eta}_1 - 1} + \hat{q}\frac{\hat{\eta}_2}{\hat{\eta}_2 + 1} - 1,$

- and all the sources of randomness $\hat{N}$, $\hat{z}$ and $\hat{Y}_k$'s are assumed to be independent.

Thus in the risk neutral universe, the firm assets value follows

$$\frac{dV}{V_-} = (r - \delta)dt + \sigma d\hat{z} + d\hat{M}$$

where $\hat{M}$ is the compensated martingale associated with the compound Poisson process, that is where $\hat{M}$ is given by:

$$\hat{M}_t = \sum_{k=1}^{\hat{N}_t} (\hat{Z}_k - 1) - \hat{\lambda}\hat{\xi}t$$

Hence, we can write:

$$V_t \ = \ V_0 \ \exp\left\{\hat{a}t + \sigma\hat{z}_t\right\} \prod_{k=1}^{N_t} \hat{Z}_k$$

with $\hat{Z}_k = e^{\hat{Y}_k}$, $\hat{a} = r - \delta - \frac{\sigma^2}{2} - \hat{\lambda}\hat{\Phi}(1)$, and where the function $\hat{\Phi}$ is defined by:

$$\hat{\Phi}(x) := \hat{p}\frac{\hat{\eta}_1}{\hat{\eta}_1 - x} + \hat{q}\frac{\hat{\eta}_2}{\hat{\eta}_2 + x} - 1$$

The Laplace exponent of the firm assets process thus writes in the risk-neutral world as:

$$\hat{G}(\hat{\beta}) = \frac{1}{2}\sigma^2\hat{\beta}^2 + \hat{a}\hat{\beta} + \hat{\lambda}\hat{\Phi}(\hat{\beta}) \tag{15}$$

Now, for any $\rho \in (0, \infty)$, let $-\hat{\beta}_{3,\rho}$ and $-\hat{\beta}_{4,\rho}$ be the only two negative roots of the Laplace exponent equation

$$\hat{G}\left(\hat{\beta}\right) = \rho \tag{16}$$

where $0 < \hat{\beta}_{3,\rho} < \eta_2 < \hat{\beta}_{4,\rho} < \infty$. This equation is a quartic equation, i.e. a four degree polynomial equation. $\hat{\beta}_{3,\rho}$ and $\hat{\beta}_{4,\rho}$ will prove useful in the coming developments.

2.2.2 Explicit debt, firm market values and barrier with a Kou process

Using the approach in Hilberink and Rogers [2002] and results from Kou and Wang [2004], explicit formulae can be obtained for the market values of debt, firm, equity and optimal barrier level (see Chen and Kou [2005] or Dao [2005]). Here are the expressions for the debt:

$$\mathcal{D} = \frac{C+mP}{r+m} - \left(\frac{C+mP}{r+m}\frac{\hat{\beta}_{4,r+m}}{\hat{\eta}_2} - (1-\hat{\alpha})\,V_B\frac{\hat{\beta}_{4,r+m}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\eta}_2-\hat{\beta}_{3,r+m}}{\hat{\beta}_{4,r+m}-\hat{\beta}_{3,r+m}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{3,r+m}}$$
$$- \left(\frac{C+mP}{r+m}\frac{\hat{\beta}_{3,r+m}}{\hat{\eta}_2} - (1-\hat{\alpha})\,V_B\frac{\hat{\beta}_{3,r+m}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\beta}_{4,r+m}-\hat{\eta}_2}{\hat{\beta}_{4,r+m}-\hat{\beta}_{3,r+m}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{4,r+m}}, \tag{17}$$

the one for the firm value:

$$\begin{aligned}\mathcal{V} &= V_0 + \frac{\theta C}{r} - \left(\frac{\theta C}{r}\frac{\hat{\beta}_{4,r}}{\hat{\eta}_2} + \alpha V_B\frac{\hat{\beta}_{4,r}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\eta}_2-\hat{\beta}_{3,r}}{\hat{\beta}_{4,r}-\hat{\beta}_{3,r}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{3,r}} \\ &- \left(\frac{\theta C}{r}\frac{\hat{\beta}_{3,r}}{\hat{\eta}_2} + \alpha V_B\frac{\hat{\beta}_{3,r}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\beta}_{4,r}-\hat{\eta}_2}{\hat{\beta}_{4,r}-\hat{\beta}_{3,r}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{4,r}},\end{aligned} \tag{18}$$

and the one for the equity:

$$\begin{aligned}\mathcal{E} &= V_0 + \frac{\theta C}{r} - \left(\frac{\theta C}{r}\frac{\hat{\beta}_{4,r}}{\hat{\eta}_2} + \alpha V_B\frac{\hat{\beta}_{4,r}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\eta}_2-\hat{\beta}_{3,r}}{\hat{\beta}_{4,r}-\hat{\beta}_{3,r}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{3,r}} \\ &- \left(\frac{\theta C}{r}\frac{\hat{\beta}_{3,r}}{\hat{\eta}_2} + \alpha V_B\frac{\hat{\beta}_{3,r}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\beta}_{4,r}-\hat{\eta}_2}{\hat{\beta}_{4,r}-\hat{\beta}_{3,r}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{4,r}} \\ &- \frac{C+mP}{r+m} + \left(\frac{C+mP}{r+m}\frac{\hat{\beta}_{4,r+m}}{\hat{\eta}_2} - (1-\hat{\alpha})\,V_B\frac{\hat{\beta}_{4,r+m}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\eta}_2-\hat{\beta}_{3,r+m}}{\hat{\beta}_{4,r+m}-\hat{\beta}_{3,r+m}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{3,r+m}} \\ &+ \left(\frac{C+mP}{r+m}\frac{\hat{\beta}_{3,r+m}}{\hat{\eta}_2} - (1-\hat{\alpha})\,V_B\frac{\hat{\beta}_{3,r+m}+1}{\hat{\eta}_2+1}\right)\frac{\hat{\beta}_{4,r+m}-\hat{\eta}_2}{\hat{\beta}_{4,r+m}-\hat{\beta}_{3,r+m}}\left(\frac{V_B}{V_0}\right)^{\hat{\beta}_{4,r+m}}\end{aligned} \tag{19}$$

The optimal barrier level is:

$$V_B^1 = \frac{\frac{C+mP}{r+m}\hat{\beta}_{3,r+m}\hat{\beta}_{4,r+m} - \frac{\theta C}{r}\hat{\beta}_{3,r}\hat{\beta}_{4,r}}{1+\alpha\left[-1+\left(\hat{\beta}_{3,r}+1\right)\left(\hat{\beta}_{4,r}+1\right)\right]+(1-\hat{\alpha})\left[-1+\left(\hat{\beta}_{3,r+m}+1\right)\left(\hat{\beta}_{4,r+m}+1\right)\right]}\frac{\hat{\eta}_2+1}{\hat{\eta}_2} \tag{20}$$

Now, if the absolute priority rule (APR) is respected, i.e. $\gamma = 0$, the optimal default barrier becomes:

$$V_B^2 = \frac{\frac{C+mP}{r+m}\hat{\beta}_{3,r+m}\hat{\beta}_{4,r+m} - \frac{\theta C}{r}\hat{\beta}_{3,r}\hat{\beta}_{4,r}}{\alpha\left(\hat{\beta}_{3,r}+1\right)\left(\hat{\beta}_{4,r}+1\right)+(1-\alpha)\left(\hat{\beta}_{3,r+m}+1\right)\left(\hat{\beta}_{4,r+m}+1\right)}\frac{\hat{\eta}_2+1}{\hat{\eta}_2}$$

If the debt is perpetual with coupon C and if the APR is respected, i.e. $m=0$, $P=0$ and $\gamma=0$, then the expression for V_B becomes:

$$V_B^3 = \frac{C(1-\theta)}{r}\frac{\hat{\eta}_2+1}{\hat{\eta}_2}\frac{\hat{\beta}_{3,r}}{\hat{\beta}_{3,r}+1}\frac{\hat{\beta}_{4,r}}{\hat{\beta}_{4,r}+1} \tag{21}$$

2.2.3 Determination of the first passage time law

We now are equipped to determine the default probability. The Laplace transform of the cumulative distribution of the first passage time through the barrier is given by:

$$\int_0^\infty e^{-\rho t} P(\tau_l \leq t)\, dt = \frac{1}{\rho}\int_0^\infty e^{-\rho t} dP(\tau_l \leq t) = \frac{1}{\rho} E_P\left[e^{-\rho \tau_l(X)}\right] \tag{22}$$

and the Laplace transform is known from Kou and Wang as follows:

$$E_P\left[e^{-\rho\tau_l(X)}\right] = \frac{\eta_2-\beta_{3,\rho}}{\eta_2}\frac{\beta_{4,\rho}}{\beta_{4,\rho}-\beta_{3,\rho}}e^{l\beta_{3,\rho}} + \frac{\beta_{4,\rho}-\eta_2}{\eta_2}\frac{\beta_{3,\rho}}{\beta_{4,\rho}-\beta_{3,\rho}}e^{l\beta_{4,\rho}},$$

where $-\beta_{3,\rho}$ and $-\beta_{4,\rho}$ are the two negative roots of the following Laplace exponent function $G(\cdot)$:

$$G^P(\beta) = \frac{1}{2}\sigma^2\beta^2 + a\beta + \lambda^P\left(\frac{p^P\eta_1^P}{\eta_1^P-\beta} + \frac{q^P\eta_2^P}{\eta_2^P+\beta} - 1\right) \tag{23}$$

with the convention:

$$a = \mu - \delta - \frac{\sigma^2}{2}$$

Kou, Petrella and Wang [2005] give an explicit solution, which can improve the computation in inverting the Laplace transform. We use the upperscript P to remind that the parameters here are in the historical world. The quantities $p^P$and $q^P \geq 0$, with $p^P + q^P = 1$ represent the probabilities of upward and downward jumps, and $1/\eta_1^P$ and $1/\eta_2^P$ are the average upward and downward jumps under the objective probability measure P. The cdf of the first passage time cannot be obtained in closed form, so a numerical inversion is needed (for this purpose, one may use for example the Gaver Stehfest, as suggested by Kou and Wang).

3 Empirical Relevance

3.1 The Gross Impact of Introducing Jumps

Let us first start with a study of the overall impact of the introduction of jumps on cumulative default probabilities. For comparison purposes, we take the same firm and market parameters as in Leland [2004], except $T = 10Y$ which is replaced by $m = 0.11$. The reason for this replacement can be found in the fact that in his empirical analysis, Leland implements the Leland and Toft model [1996]. By contrast, we will compare the CDPs obtained with our new method to the ones obtained with a Leland [1994b] diffusive Gaussian model.

μ	δ	r	V_0	α	τ	m
0.12	0.06	0.08	100	0.3	0.15	0.11

Table 1: Common Parameters

P	p	η_1	η_2	σ_G
43.3	0.5	5	5	0.23

Table 2: Specific Parameters

$$< \ln\left(\frac{V_1}{V_0}\right) > = \sigma^2 + 2\lambda\left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right)$$

Figure 1: Cumulative Default Probabilities

3.2 Fitting Real-World Curves

Conclusion

Appendix

In this setting the market is incomplete. The risk due to jumps cannot be hedged and there is no more a unique risk-neutral measure. This is always the case when the price process is of a geometric Lévy process type, excepted of course the Brownian motion case. Many candidates for suitable martingale measure equivalent to the real world probability measure have been suggested and from a theoretical point of view, different criteria can be chosen based on hedging arguments or distance minimization. Hellinger distance, L^2 distance, entropy or Kullback Leibler distance have frequently been put forward. In this paper we choose the Esscher measure associated with the coupound return of the asset price process. It has been introduced by Gerber and Shiu [bbbb]. We motivate our choice for theoretical, economic and convenient reasons. From a mathematical point of view, this measure is the nearest equivalent martingale measure to the historical probability measure in the sense of power metric. From an economic point of view, the risk-neutral universe being not unique, prices will rely on the attitude of economic agents toward risk. Using the expected utility or the neo-Bernoulli theory, a fair price can be obtained with the marginal utility principle and it can be shown that the fair price or indifference price given by a power utility or logarithmic utility function can be expressed via the coumpound return Esscher measure. See Gerber and Shiu [ii] and Davis [EEE]. But the main motivation is from an operational perspective: most of the usual Lévy processes used in financial modelling, generalized hyperbolic processes, coumpound Poisson processes, Normal Inverse Gaussian proceses, variance gamma and CGMY processes remain of the same kind in this particular risk-neutral universe. Furthermore the passage of one set of parameters of these processes in the historical universe to the set of parameters in the coumpound return Esscher risk-neutral universe is very simple. There is a general formula for the generic triplet of the risk-neutral Lévy process, we explicit it in the case of the Kou process. The asset price can be expressed either as an exponential of a Lévy process X or as the Doléans Dade exponential of another process $\tilde{X}$. Of course thes two proessess are linkes together and relations between their charactezristic triplet are easily obtained. This leads to another choice for the Esscher measure the so-called simple return Esscher measure, defined with $\tilde{X}$. This measure has an extremal entropy property: it is the closest measure to the historical probability in the sense of the Kullback-Leibler distance. Regarding the economic side, this measure is associated with exponential utility functions. With this measure Lévy processes in the actual universe remain Lévy processes in this new risk-neutral universe. A General formula is available for the generic triplet of the process X, under the simple return Esscher measure, however the particular type of the process is not necessarily kept. For example a Kou process does not remain such, so we can't use the endogenous default barrier obtained with the Kou process in our previous analysis. Furtheremore, the default probability which is our main concern does not seem easily available contrary to the case with the coumpound return Escher measure. Another advantage of our choice is its flexbility to adjust to actual data. The important question of the model ability to reproduced observed market expected default frequencies is adressed in the paragraph devoted to empirical relevance. Let us now return to the precise definition of the Esscher measure.

Let X be a Lévy process, the Q_h h-Esscher measure associated with X, is defined by

$$\frac{dQ_h}{dP} = \frac{e^{hX(t)}}{E\left(e^{hX(t)}\right)}$$

The Laplace exponent is $G(\beta)$ such that

$$E\left(e^{\beta X(t)}\right) = e^{G(\beta)t}$$

First consider the martingale condition: the discounted gain process must be a martingale under the risk neutral measure, therefore for $t > 0$, we have

$$V_0 = E_{Q_h}\left(V_0 e^{X(t)} e^{\delta t} e^{-rt}\right)$$

or

$$V_0 = V_0 e^{(\delta - r)t} \int_\Omega \frac{e^{(h+1)X(t)}}{E_P(e^{hX(t)})} dP$$

this conditiontion writes

$$V_0 = V_0 e^{(\delta - r)t} \frac{G_P(h+1)}{G_P(h)}$$

where we write G_P to refer to the real world probability measure. The martingale condition corresponds to the identification of the parameter h such that it is a solution of the following equation

$$\delta - r + G_P(h+1) - G_P(h) = 0 \tag{24}$$

We denote by $h*$ this solution. To find relations between the real world and the risk-neutral one we begin with the definition of the Laplace exponent under Q. For sake of simplicity we don't use the subscript h and write Q instead of Q_h.

$$E_Q\left(e^{\beta X(t)}\right) = \int_\Omega e^{\beta X(t)} \frac{e^{hX(t)}}{E_P(e^{hX(t)})}) dP$$

or:

$$e^{G_Q(\beta)} = E_P\left(\frac{e^{(\beta+h)X(t)}}{E_P(e^{hX(t)})}\right) = \frac{e^{G_P(\beta+h})}{e^{G_P(h)}}$$

so:

$$G_Q(\beta) = G_P(\beta+h) - G_P(h) \tag{25}$$

This equation gives the link between the actual measure P and the risk neutral measure Q. This risk neutral measure is the Esscher measure associated with the parameter h^* verifying the martingale condition (24).

Let us now come back to our jump diffusion process and apply these general results to our particular case

$$X(t) = (\mu - \delta - \frac{1}{2}\sigma^2)t + \sigma z(t) + \sum_{k=1}^{N(t)} Y_k$$

We know from Kou (55555) that with this process the Laplace exponent is

$$G(\beta) = \frac{1}{2}\beta^2\sigma^2 + \beta(\mu - \delta - \frac{1}{2}\sigma^2) + \lambda\left[\frac{p\eta_1}{\eta_1 - \beta} + \frac{q\eta_2}{\eta_2 + \beta} - 1\right] \tag{26}$$

The martingale condition (24) writes

$$r = h\sigma^2 + \mu + \lambda\left[\frac{p\eta_1}{(\eta_1 - h)(\eta_1 - (1+h))} - \frac{q\eta_2}{(\eta_2 + h)(\eta_2 + 1 + h)}\right] \tag{27}$$

which can be rewritten

$$r = h\sigma^2 + \mu - \lambda\left[\frac{p\eta_1}{\eta_1 - h} + \frac{q\eta_2}{\eta_2 + h} - \frac{p\eta_1}{\eta_1 - (1+h)} - \frac{q\eta_2}{\eta_2 + (1+h)}\right]$$

Now, using (25), (26), and after computations

$$G_Q(\beta) = \frac{1}{2}\beta^2\sigma^2 + \beta(\mu - \delta - \frac{1}{2}\sigma^2 + \sigma^2 h) - \lambda\zeta\left[\hat{p}\frac{\eta_1}{\hat{\eta_1} - \beta} + \hat{q}\frac{\eta_2 + h}{\hat{\eta_2} + \beta} - 1\right]$$

$$\zeta := \frac{p\eta_1}{\eta_1 - h} + \frac{q\eta_2}{\eta_2 + h},$$

$$\begin{cases} \hat{p} = \frac{p\eta_1}{\zeta(\eta_1 - h)} \\ \hat{q} = 1 - \hat{p} \\ \hat{\eta_1} = \eta_1 - h \\ \hat{\eta_2} = \eta_2 + h \\ \hat{\lambda} = \lambda\zeta \end{cases} \tag{28}$$

which, according to (26) shows that X is under the Esscher risk neutral measure a Kou process with parameters defined by (28) and with the choice $h = h^*$, the solution in h of equation (27).

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