Generalized Sharpe Ratios and Portfolio Performance Evaluation

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Abstract

In this paper using the expected utility theory and the approximation analysis we derive a formula for the most natural extension of the Sharpe ratio which takes into account the skewness of distribution. The ranking statistic based on the adjusted for skewness Sharpe ratio preserves the standard Sharpe ratio for normal distribution, decreases ranking of distributions with left-tail risk, and improves ranking of distributions with right-tail potential. We illustrate the use of the adjusted for skewness Sharpe ratio by comparing the performances of portfolios with manipulated Sharpe ratios and the performances of hedge funds.

Key words: Sharpe ratio, skewness, kurtosis, portfolio performance evaluation.

JEL classification: G11.

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1 Introduction

The Sharpe ratio is a commonly used measure of portfolio performance. However, because it based on the mean-variance theory, it is valid only for either normally distributed returns or quadratic preferences. In other words, the Sharpe ratio is a meaningful measure of portfolio performance when the risk can be adequately measured by standard deviation. The Sharpe ratio can lead to misleading conclusions when return distributions are skewed, see Bernardo and Ledoit (2000). For example, it is well known that the distribution of hedge fund returns deviates significantly from normality (see, for example, Brooks and Kat (2002), Agarwal and Naik (2004) and Malkiel and Saha (2005)). Evaluation of the performances of hedge funds using the Sharpe ratio seems to be dubious. Moreover, recently a number of papers have shown that the Sharpe ratio is prone to manipulation (see, for example, Spurgin (2001) and Goetzmann, Ingersoll, Spiegel, and Welch (2002)). The manipulation of the Sharpe ratio consists largely in selling the upside return potential, thus creating a distribution with high left-tail risk.

The literature on performance evaluation that tries to take into account higher moments of distribution is vast one. Most distribution-based alternative performance measures focus only on downside risk caused by negative skewness. Motivated by a common interpretation of the Sharpe ratio as a *reward-to-risk* ratio, many researches replace the standard deviation in the Sharpe ratio by an alternative risk measure. For example, Sortino and Price (1994) and Ziemba (2005) replace standard deviation by downside deviation. Dowd (2000) and Gregoriou and Gueyie (2003) use Value-at-Risk (VaR) measure instead of standard deviation. Another possibility is to use conditional VaR instead of VaR. Many researches have introduced different ad-hoc performance measures. Some examples are: Stutzer (2000) introduced the Stutzer index which is based on the behavioral hypothesis that investors aim to minimize the probability that the excess returns over a given threshold will be negative. The Omega ratio was introduced by Keating and Shadwick (2002). This measure is expressed as the ratio of the gains with respect to some threshold to the loss with respect to the same threshold. Kaplan and Knowles (2004) introduced the Kappa measure which generalizes the Sortino and Omega ratios. The drawbacks of these measures are the following: (a) the ranking of portfolios based on most of these measures depends heavily on the choice of a threshold; (b) almost all of these measures take into account only downside risk, the upside return potential (that is, positive skewness) is not appreciated; (c) whereas the Sharpe ratio is based on the expected utility theory which is the cornerstone of the modern finance, all these alternative performance measures lack any solid theoretical underpinning.

In this paper using the expected utility theory and the approximation analysis we derive a formula for the most natural extension of the Sharpe ratio which takes into account the skewness of distribution. We denote this ratio as the adjusted for skewness Sharpe ratio (ASR). The ASR is very easy to compute: besides the excess returns and standard deviation one needs to know the value of skewness. The ASR preserves the standard Sharpe ratio for zero skewness. Depending on the value and the sign of skewness the value of the ASR increases when skewness is positive and increases. On the contrary, the value of the ASR decreases when skewness is negative and increases (in absolute value). We indicate that the ASR justifies the notion of the Generalized Sharpe Ratio (GSR) introduced by Hodges (1998). The GSR seems to be the most general generalization of the Sharpe ratio that accounts for all the moments of distribution.

It is natural to use the ASR as a ranking statistic in the comparison of performances of different risky assets and portfolios. However, it turns out that the value of the ASR is not unique for all investors, but depends on the relationship between the investor's risk aversion to variance (widely known as the Arrow-Pratt measure of risk aversion) and the risk aversion to skewness. For example, the investors with quadratic utility exhibit risk aversion to variance but risk-neutrality to skewness. For the investors who exhibit a constant absolute risk aversion (to variance) the value of the absolute risk aversion to skewness equals the squared value of the absolute risk aversion to variance. For the investors with logarithmic utility the value of the absolute risk aversion to skewness is two times the value of the absolute risk aversion to skewness for the investors with constant absolute risk aversion. In short, two investors with different utility functions might have the same absolute risk aversion to variance but different risk aversions to skewness. To avoid the ambiguity, for practical purposes we suggest using the ASR computed for investors who exhibit either a constant absolute risk aversion (to variance) or a constant relative risk aversion (to variance) when the value of the coefficient of the relative risk aversion is sufficiently greater than 1. We show that in these two cases the value of the ASR is virtually the same.

Since we derive the formula for the ASR exploiting the approximation analysis, it is valid only for either small time intervals or small values of skewness. Since the Sharpe ratio is usually computed for annualized returns, to produce a correct ranking statistic which is of practical relevance we calibrate the formula for the ASR using the class of normal inverse Gaussian processes (see, for example, Barndorff-Nielsen (1998)) and a wide set of realistic parameters. Then using the formula for the calibrated ASR we illustrate how it can be applied in the comparison of the performances of portfolios with manipulated Sharpe ratios and the performances of hedge funds.

Even though not everyone might agree with our application of the ASR (anyway, if one believes that the investors exhibit quadratic preferences, the value of skewness plays no role), still our study of performances is interesting in that it helps to find out how important is the impact of kurtosis in performance measurement. In particular, it is well known that the distributions of hedge fund returns exhibit often both high negative skewness and high positive kurtosis. Therefore, it seems plausible in the solution for the optimal portfolio choice problem to account for both skewness and kurtosis. Recent examples are Davies, Kat, and Lu (2005) and Jondeau and Rockinger (2006). To account for higher moments of distribution in our study of performances we also compute the GSR using empirical distributions. Our analysis suggests that the value of kurtosis has a negligible effect on the adjustment of the Sharpe ratio as compared to that of the value of skewness. The plausible explanation for this phenomenon is the fact that a high absolute value of skewness automatically induces high positive kurtosis. For example, in all distributions where one can define both skewness and kurtosis, the minimum possible kurtosis is computed in accordance to the formula $K_{\min} = 3 + cS^2$, where S is the value of skewness and c is some constant. What is not accounted for in the formula for the ASR is, actually, only the excess kurtosis which can be logically defined as the kurtosis minus the minimum possible kurtosis given the value of skewness.

The rest of the paper is organized as follows. In Section 2 using the approximation analysis we derive the formula for the adjusted for skewness Sharpe ratio. In the same section we show the obvious connection between the adjusted for skewness Sharpe ratio and the generalized Sharpe ratio. In Section 3 we test our approximated formula and find out that it is not quite good for all practical purposes. In the same section we calibrate the formula for a range of realistic parameters of distributions. In Section 4 using the adjusted for skewness Sharpe ratio and the generalized Sharpe ratio we compare the performances of the portfolios for which the shape of return distribution is far from normal. Section 5 concludes the paper.

2 Adjusted for Skewness Sharpe Ratio and Generalized Sharpe Ratio

In this section we apply the approximation analysis to derive the expression for the adjusted for skewness Sharpe ratio. At the end we indicate that our ASR justifies the notion of the generalized Sharpe Ratio introduced by Hodges (1998) which presumable accounts for all the moments of distribution of risky asset returns.

2.1 Optimal Portfolio Choice with Skewness

We consider an investor who wants to choose an optimal portfolio consisting of the risk-free and the risky asset. The returns of the risky asset over small time interval Δt are given by

$$x = \mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon,$$

where μ and σ are, respectively, the mean and volatility of the risky asset returns per unit of time, and ε is some (normalized) stochastic variable such that $E[\varepsilon] = 0$ and $\mathrm{Var}[\varepsilon] = 1$. The returns on the risk-free asset over the same time interval equal

$$r_f = r\Delta t,$$

where r is the risk-free interest rate per unit of time.

We further suppose that the investor has a wealth of w and invests a into the risky asset and $w - a$ into the risk-free asset. Consequently, the investor's wealth after Δt is given by

$$\widetilde{w} = a(x - r_f) + w(1 + r_f).$$

The investor's expected utility

$$E[U(\widetilde{w})] = E[U(a(x - r_f) + w(1 + r_f))],$$

where $U(\cdot)$ is some function. To shorten the subsequent notation, we denote $w_r = w(1 + r_f)$.

Now we apply the Taylor series expansion of $U(\widetilde{w})$ around w_r. Our intension is to keep all the terms up to $\Delta t^{\frac{3}{2}}$ in order to be able to account for skewness and disregard all the terms with higher powers of Δt (that is, we disregard all the higher moments of the distribution). This gives us

$$\begin{aligned} U(\widetilde{w}) \approx{}& U(w_r) + U^{(1)}(w_r)a(x - r_f) \\ &+ \frac{1}{2}U^{(2)}(w_r)a^2(x - r_f)^2 + \frac{1}{6}U^{(3)}(w_r)a^3(x - r_f)^3, \end{aligned}$$

where $U^{(i)}$ denotes the ith derivative. The expected utility is given now by

$$\begin{aligned} E[U(\widetilde{w})] \approx{}& U(w_r) + U^{(1)}(w_r)aE[(x - r_f)] \\ &+ \frac{1}{2}U^{(2)}(w_r)a^2E[(x - r_f)^2] + \frac{1}{6}U^{(3)}(w_r)a^3E[(x - r_f)^3]. \end{aligned} \tag{1}$$

Now observe that up to leading terms with $\Delta t^{\frac{3}{2}}$

$$E[(x - r_f)] = (\mu - r)\Delta t,$$

$$E[(x - r_f)^2] = E[(x - \mu\Delta t + \mu\Delta t - r\Delta t)^2] \approx E[(x - E(x))^2] = \left(\sigma\sqrt{\Delta t}\right)^2 E[\varepsilon^2],$$

$$E[(x - r_f)^3] = E[(x - \mu\Delta t + \mu\Delta t - r\Delta t)^3] \approx E[(x - E(x))^3] = \left(\sigma\sqrt{\Delta t}\right)^3 E[\varepsilon^3].$$

Now to shorten the subsequent notation we denote

$$\begin{aligned} e &= E[(x - r_f)], \\ v &= E[(x - E(x))^2], \\ s &= E[(x - E(x))^3]. \end{aligned}$$

Observe that e is related to the risk premium of the risky asset, v is related to the variance of the returns of the risky asset, and s is related to the third moment of distribution (that is, skewness) of the returns of the risky asset.

The investor's objective is to choose a to maximize the expected utility

$$\max_a E[U(\widetilde{w})].$$

The first-order condition of optimality of a

$$\frac{dE[U(\widetilde{w})]}{da} = 0.$$

This gives us

$$U^{(1)}(w_r)e + U^{(2)}(w_r)av + \frac{1}{2}U^{(3)}(w_r)a^2 s = 0,$$

or

$$\frac{1}{2}\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)}a^2 s + \frac{U^{(2)}(w_r)}{U^{(1)}(w_r)}av + e = 0. \tag{2}$$

Note that

$$-\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)} = \text{ARA},$$

where ARA is the Arrow-Pratt measure of absolute risk aversion, or aversion to variance. Similarly,

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)}$$

can be interpreted as the investor's preference to the third moment of distribution, or (absolute) risk aversion to skewness, see, for example, Kane (1982).

Now we are going to consider two popular types of utility functions and the relationship between aversion to skewness and aversion to variance. First we consider the utility function which implies a constant absolute risk aversion to variance. This function (also known as the negative exponential utility) is defined by

$$U(w) = -\exp(-\gamma w),$$

where γ is the absolute risk aversion coefficient. That is,

$$-\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)} = \text{ARA} = \gamma.$$

Note that for this utility function

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} = \gamma^2 = \mathrm{ARA}^2.$$

Second we consider two utility functions which imply a constant relative risk aversion to variance. In case of the power utility the function is defined by

$$U(w) = \frac{w^{1-\rho}}{1-\rho},$$

where ρ is the relative risk aversion coefficient. With this utility

$$-\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)} = \mathrm{ARA} = \frac{\rho}{w_r},$$

and

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} = \frac{\rho(\rho-1)}{w_r^2} = \frac{\rho-1}{\rho}\mathrm{ARA}^2.$$

Now observe that when $\rho \gg 1$ then similarly as for the negative exponential utility we have

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} \approx \mathrm{ARA}^2.$$

In case of the logarithmic utility the function is defined by

$$U(w) = \log(w).$$

It is easy to check that for the logarithmic utility

$$-\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)} = \mathrm{ARA} = \frac{1}{w_r},$$

and

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} = \frac{2}{w_r^2} = 2\,\mathrm{ARA}^2.$$

Note that the investor with the logarithmic utility exhibits a higher aversion to skewness than the investor with the power utility. However, the considered utility functions do not exhaust all possible types and forms of the investor's utility function. In order to generalize our result we deduce that for every utility function we can write

$$\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} = b\,\mathrm{ARA}^2,$$

where b is some constant. For example, for the negative exponential utility and the power utility with $\rho \gg 1$ we have $b = 1$. For the logarithmic utility we have $b = 2$.

Now for every utility function we can rewrite the first-order condition of optimality (2) as

$$\frac{1}{2}a^2 b\gamma^2 s - a\gamma v + e = 0,$$

where γ denotes the investor's absolute risk aversion (ARA) at point with wealth w_r. The solutions for the quadratic equation above are given by

$$a_{1,2} = \frac{\gamma v \pm \sqrt{\gamma^2 v^2 - 2\gamma^2 bse}}{\gamma^2 bs} = \frac{v \pm \sqrt{v^2 - 2bse}}{\gamma bs}.$$

Now consider

$$\sqrt{v^2 - 2bse}. \tag{3}$$

Observe that v^2 is of order Δt^2, but $2bse$ is of order $\Delta t^{\frac{5}{2}}$. That is

$$2bse \ll v^2,$$

and, hence, we can apply the Taylor series expansion (we will use three terms around v^2) to the function (3). This gives us

$$\sqrt{v^2 - 2bse} \approx \sqrt{v^2} - \frac{1}{2}\frac{2bse}{\sqrt{v^2}} - \frac{1}{2}\frac{1}{4}\frac{(2bse)^2}{(v^2)^{\frac{3}{2}}} = \pm\left(v - \frac{bse}{v} - \frac{(bse)^2}{2v^3}\right).$$

Then the solutions for optimal a

$$a_{1,2} = \frac{v \pm \left(v - \frac{bse}{v} - \frac{(bse)^2}{2v^3}\right)}{\gamma bs}.$$

The correct solution (from the two ones) is that one which is closest to

$$a = \frac{e}{\gamma v} = \frac{\mu - r}{\gamma\sigma^2},$$

which is the solution for the optimal a when we ignore the terms with $U^{(3)}$ and higher derivatives of U. Consequently, we arrive to the solution

$$a = \frac{e}{\gamma v} + \frac{sbe^2}{2\gamma v^3} = \frac{e}{\gamma v}\left(1 + \frac{bse}{2v^2}\right). \tag{4}$$

This solution is the solution for optimal a when skewness is zero plus a correction term which accounts for the skewness.

Recall that the skewness is defined by

$$S = \frac{E[(x - E[x])^3]}{E[(x - E[x])^2]^{\frac{3}{2}}} = \frac{s}{v^{\frac{3}{2}}}.$$

Using all these we can rewrite (4) as

$$a = \frac{\mu - r}{\gamma\sigma^2}\left(1 + \frac{bS}{2}\frac{\mu - r}{\sigma}\sqrt{\Delta t}\right).$$

Note that the Sharpe ratio is given by

$$SR = \frac{\mu - r}{\sigma}\sqrt{\Delta t}. \tag{5}$$

Therefore, we can rewrite the optimal solution as

$$a = \frac{SR}{\gamma\sigma\sqrt{\Delta t}}\left(1 + \frac{bS}{2}SR\right). \tag{6}$$

2.2 Adjusted Sharpe Ratio

Recall that the investor's expected utility is given by (1), the optimal amount invested in the risky asset is given by (6), and in addition recall (5) and that $\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)} = -\gamma$, and $\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)} = b\gamma^2$. Using all this we can rewrite the expression for the maximum expected utility as

$$\begin{aligned} E[U^*(\widetilde{w})] \approx U(w_r) &+ \frac{U^{(1)}(w_r)}{\gamma}SR^2\left(1 + \frac{bS}{2}SR\right) \\ &- \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2\left(1 + \frac{bS}{2}SR\right)^2 + \frac{U^{(1)}(w_r)}{\gamma}\frac{bS}{6}SR^2\left(1 + \frac{bS}{2}SR\right)^3. \end{aligned}$$

Again in the equation above we want to keep only the leading terms with up to $\Delta t^{\frac{3}{2}}$. Since S does not depend on Δt and SR is of order $\Delta t^{\frac{1}{2}}$, in the resulting expression we can only have the terms with up to SR^3. If we do this, we get

$$E[U^*(\widetilde{w})] \approx U(w_r) + \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2\left(1 + \frac{bS}{3}SR\right). \tag{7}$$

Consequently, for any investor the higher the value of

$$SR^2\left(1 + \frac{bS}{3}SR\right),$$

the higher the expected utility. By analogy with the Sharpe ratio, we introduce the adjusted for skewness Sharpe ratio

$$ASR = SR\sqrt{1 + \frac{bS}{3}SR}. \tag{8}$$

Observe in addition that the expected utility (7) can be rewritten now as (using a sort of Taylor series "contraction")

$$E[U^*(\widetilde{w})] \approx U\left(\frac{1}{2\gamma}ASR^2 + w(1 + r_f)\right).$$

For the negative exponential utility this implies

$$E[U^*(\widetilde{w})] \approx -\exp\left(-\frac{1}{2}ASR^2 - \gamma w(1+r_f)\right). \tag{9}$$

For the power utility where, recall $\gamma = \frac{\rho}{w(1+r_f)}$, the maximum expected utility is given by

$$E[U^*(\widetilde{w})] \approx \frac{\left(w(1+r_f)\left(1+\frac{1}{2\rho}ASR^2\right)\right)^{1-\rho}}{1-\rho}.$$

2.3 Generalized Sharpe Ratio

The purpose of this subsection is to show that our ASR justifies the notion of the Generalized Sharpe Ratio introduced by Hodges (1998) and further developed and used by, for example, Madan and McPhail (2000) and Cherny (2003).

In particular, Hodges points out (previously it was also observed by Bucklew (1990)) that for *normally distributed* risky asset returns and the investor with zero wealth and the negative exponential utility function the (standard) Sharpe ratio can be computed (for a motivation see equation (9)) using

$$\frac{1}{2}SR^2 = -\log\left(-E[U^*(\widetilde{w})]\right),$$

where $E[U^*(\widetilde{w})]$ is the maximum attainable expected utility. Using this identity Hodges conjectures that for *any distribution* of the risky asset returns the GSR can be computed using

$$\frac{1}{2}GSR^2 = -\log\left(-E[U^*(\widetilde{w})]\right). \tag{10}$$

Note that the GSR preservers the standard Sharpe ratio for normal distribution of returns.

Observe that our ASR can be interpreted as a particular form of the GSR for the case where the investor is risk-neural for all the moments of distribution greater than the third one. The GSR, in its turn, can be interpreted as the generalization of the ASR which accounts for all the moments of distribution of the risky asset returns. It is important to remind that the GSR was introduced originally for investors with constant absolute risk aversion. Our ASR suggests that the same value of GSR can be obtained also for an investor with the power utility in case $\rho \gg 1$. An investor with the logarithmic utility, however, puts more weight on the higher moments of distribution. Using the ASR as an example, we can deduce that such an investor punishes more severely negative skewness and appreciates more positive skewness as compared with, for example, an investor with constant absolute risk aversion.

3 Test and Calibration of the Formula for Adjusted for Skewness Sharpe Ratio

Our ultimate goal is to use the ASR as a measure of comparison of performances of different risky assets and portfolios. However, the value of the ASR is not unique and depends on the relationship between the investor's risk aversions to variance and to skewness. To avoid the ambiguity we suggest computing the ASR for the investors who exhibit either a constant absolute risk aversion or a constant relative risk aversion when the value of the risk aversion coefficient is sufficiently greater than 1, that is, when $\rho \gg 1$. In these cases the value of coefficient b in the formula for the ASR equals to 1.

Now recall that the ASR formula (8) was derived using the approximation technique. In this section we want to find out how good is this formula for practical purposes. To test the formula we need first to choose a suitable probability distribution. Our tests show that the ASR formula (8) is not good when either skewness or the Sharpe ratio is large. Therefore we calibrate the formula for a range of realistic parameters of the risky asset probability distribution.

3.1 Choosing a Suitable Probability Distribution

To test the formula for the ASR we need to have a probability distribution where we can define the value of skewness. *Since the ASR does not depend on a particular type of probability distribution, we can choose any suitable probability distribution.* There are many possible probability distributions that can suit our purpose. Some example are: a class of *stable* distributions which generalize the normal distribution (see, for example, Samorodnitsky and Taqqu (1994) or Uchaikin and Zolotarev (1999)), the *variance gamma* distribution introduced by Madan and Seneta (1990), and the *normal inverse Gaussian* distribution (see, for example, Barndorff-Nielsen (1998)). Our choice here is the normal inverse Gaussian (NIG) distribution. The reasons for this choice are the following: (1) the NIG distribution has an explicit expression for the probability density function (for example, there are no explicit formulas for general stable densities); (2) for the NIG distribution we have explicit formulas for finding the parameters of the distribution via the values of the first four moments of the distribution, see explanation below.

In particular, a random variable X follows the NIG distribution with parameter vector $(\alpha, \beta, \eta, \delta)$ if its probability density function is

$$f(x; \alpha, \beta, \eta, \delta) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x-\eta))}{\pi\sqrt{\delta^2 + (x-\eta)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\eta)^2}\right),$$

where

$$\gamma = \sqrt{\alpha^2 - \beta^2}$$

and K_1 is the modified Bessel function of the third kind with index 1. η and δ are ordinary parameters of location and scale whereas α and β determines the shape of the density. In particular, β determines the degree of skewness. For symmetrical densities $\beta = 0$. The conditions for a viable NIG density are: $\delta > 0$, $\alpha > 0$, and $\frac{|\beta|}{\alpha} < 1$. The mean, variance, skewness and kurtosis of X are

$$\begin{aligned}
\mu &= E[X] = \eta + \delta\frac{\beta}{\delta}, \\
\sigma^2 &= \mathrm{Var}[X] = \delta\frac{\alpha^2}{\gamma^3}, \\
S &= \mathrm{Skew}[X] = 3\frac{\beta}{\alpha\sqrt{\delta\gamma}}, \\
K &= \mathrm{Kurt}[X] = 3 + \frac{3}{\delta\gamma}\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right).
\end{aligned} \tag{11}$$

The equations (11) can be solved explicitly for the parameters of the NIG distribution. After tedious but straightforward calculations we can obtain

$$\begin{aligned}
\alpha &= \frac{\sqrt{3K - 4S^2 - 9}}{\sigma^2(K - 5/3S^2 - 3)^2}, \\
\beta &= \frac{S}{\sigma(K - 5/3S^2 - 3)}, \\
\eta &= \mu - \frac{3S\sigma}{3K - 4S^2 - 9}, \\
\delta &= 3\sigma\frac{\sqrt{3K - 5S^2 - 9}}{3K - 4S^2 - 9}.
\end{aligned} \tag{12}$$

Note that to get meaningful parameters of the NIG distribution the following condition must be satisfied

$$K > K_{\min} = 3 + \frac{5}{3}S^2. \tag{13}$$

3.2 Testing the Formula for ASR

The ASR formula was derived using the approximation technique under assumption that Δt is rather small. In practice, the Sharpe ratio is usually computed for annualized returns, that is, when $\Delta t = 1$. In this case, how good is the formula for ASR for practical purposes? We can answer this question by comparing the (approximate) solution given by (8) with the exact numerical solution. The exact numerical solution can by computed by maximizing the investor's expected utility and, then, if we employ the negative exponential utility (with zero initial wealth of the investor, that is, when $w = 0$)

$$ASR = \sqrt{-2\log(-E[U^*(\widetilde{w})]}. \tag{14}$$

We assume that the risky assets returns follow the NIG distribution. To find the maximum expected utility, we solve numerically the following problem

$$E[U^*(\widetilde{w})] = \max_a \int_{x_{\min}}^{x_{\max}} -\exp(-\gamma a(x - r_f))f(x; \alpha, \beta, \eta, \delta)dx.$$

Observe that the NIG distribution is defined on the whole real line, but for numerical computations we need to limit the range for x to some $x \in [x_{\min}, x_{\max}]$.

We use the following model parameters: $\Delta t = 1$ year, $r_f = 0.05$, $\sigma = 0.20$, $\mu_1 = 0.15$ which gives $SR_1 = 0.5$ and $\mu_2 = 0.25$ which gives $SR_2 = 1.0$. We vary the skewness $S \in [-3, 3]$ and keep the kurtosis at minimum which is given by (13). Note that the value of the coefficient of absolute risk aversion, γ, does not influence the ASR, hence it might be chosen arbitrary. Figure 1 illustrates the results of comparison of the approximated solution for the ASR and the exact numerical solution. From the figure we see that even for not small Δt the approximated ASR is close to the exact ASR when either skewness is small or the Sharpe ratio is not high. For example, when $SR = 0.5$ and $S = \pm 1$ the discrepancy between the approximated ASR and the exact ASR amounts to 1.25%. When $S = \pm 2$ the discrepancy between the approximated ASR and the exact ASR increases to 6.1%. When $SR = 1.0$ and $S = \pm 1$ the discrepancy between the approximated ASR and the exact ASR amounts to 5.25%. When we increase the skewness to $S = \pm 2$ the discrepancy between the approximated ASR and the exact ASR increases to 27.8%. Note that for $SR = 1$ and $S = -2$ the error produced by the approximated ASR is roughly the same as the error between the exact ASR and the SR. For larger values of negative skewness the standard SR becomes closer to the exact ASR than the approximated ASR.

3.3 Calibrating the Formula for ASR

The tests of the formula for the ASR have shown that when skewness of distribution or the standard Sharpe ratio is large, then the ASR formula provides not quite accurate results. In particular, the greater the skewness the more the approximated ASR underestimates the real ASR. The degree of underestimation increases as the standard Sharpe ratio increases. To increase the precision of the ASR formula we suggest calibrating the formula of the ASR to the exact numerical solution.

The general description of the calibration technique (which is also a sort of an approximation) we propose to employ can be found in Judd (1998)

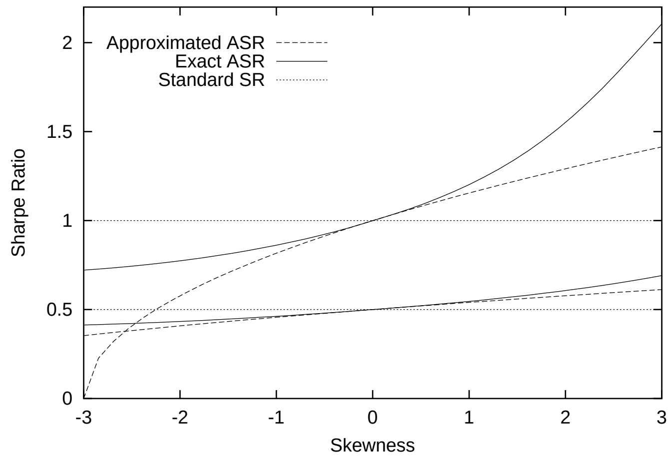

Figure 1: The exact and the approximated value of the ASR versus the skewness of distribution for two values of the standard Sharpe ratio.

Chapter 6. This technique was successfully applied by, for example, Zakamouline (2006) for finding an approximated closed-form solution for the utility-based option hedging strategy in the presence of transaction costs.

We assume the following functional form of the ASR when skewness is positive

$$ASR = SR\sqrt{1 + \lambda S^{\phi} SR^{\psi}}. \tag{15}$$

Similarly, when skewness is negative

$$ASR = SR\sqrt{1 - \lambda |S|^{\phi} SR^{\psi}}. \tag{16}$$

So the idea is to find the values of the parameters λ, ϕ, and ψ which produces the best fit to the exact numerical solution which is computed using

$$ASR^2 = -2\log\left(-E[U^*(\widetilde{w})]\right).$$

Then, if the skewness is positive, our functional assumption implies

$$ASR^2 = SR^2\left(1 + \lambda S^{\phi} SR^{\psi}\right).$$

From here

$$\left(\frac{ASR}{SR}\right)^2 - 1 = \lambda S^{\phi} SR^{\psi}.$$

After the linearizing log-log transformation of the equation above it takes the form

$$\log\left(\left(\frac{ASR}{SR}\right)^2 - 1\right) = \log(\lambda) + \phi\log(S) + \psi\log(SR).$$

We measure the goodness of fit using the L^2 norm. This largely amounts to using the techniques of ordinary linear regression after the log-log transformation. That is, we find the parameters λ, ϕ, and ψ by solving the problem

$$\min_{\lambda,\phi,\psi} \sum \left(\log\left(\left(\frac{ASR}{SR}\right)^2 - 1 \right) - \log(\lambda) - \phi \log(S) - \psi \log(SR) \right)^2 .$$

There is no good reason to believe that our simple functional specification for the ASR can produce a decent fit for all possible sets of distribution parameters. However, such a functional form yields quite a nice fit to the exact numerical solution over some narrow intervals of parameters. We restrict our attention to the following intervals of the parameters (keeping $\Delta t = 1$): $\mu - r \in [0, 0.30]$, $\sigma \in [0.05, 0.30]$ and $S \in [-3.5, 3.5]$. Since some combinations of $\mu - r$ and σ can produce unrealistically high Sharpe ratios, we limit our calibration to the cases when $SR \leq 2.5$. Also note that we limit our calibration to the cases with only positive risk premiums, that is when $\mu - r > 0$.

λ	0.5008 ± 0.0115
ϕ	1.4739 ± 0.0274
ψ	1.3110 ± 0.0336

(a) Positive skewness

λ	0.2359 ± 0.0029
ϕ	0.6703 ± 0.0146
ψ	0.6918 ± 0.0179

(b) Negative skewness

Table 1: 95% confidence interval for the best-fit parameters of the calibrated ASR.

The results of our estimation of the best-fit parameters are presented in Table 1. The goodness of fit of the calibrated ASR formula measured by R^2 statistics is 98%. After an insignificant rounding off the values of the best-fit parameters, the calibrated formulas for the ASR could be written as: when skewness is positive

$$ASR = SR\sqrt{1 + 0.50\, S^{1.47} SR^{1.31}}, \tag{17}$$

and when skewness is negative

$$ASR = SR\sqrt{1 - 0.24\, |S|^{0.67} SR^{0.69}}. \tag{18}$$

Figure 2 illustrates the results of comparison of the calibrated formula for the ASR and the exact numerical solution for the ASR. Now, for example, when $SR = 0.5$ and $S = \pm 2$, the maximum discrepancy between the calibrated ASR and the exact ASR amounts to 2.7% (without calibration it was 6.1%). When $SR = 1.0$ and $S = \pm 2$ the maximum discrepancy between the calibrated ASR and the exact ASR amounts to 1.6% (without calibration it was 27.8%). The biggest deviation of the value of the calibrated ASR from the exact value is observed for the cases with high positive value of skewness and rather high value of the standard Sharpe ratio.

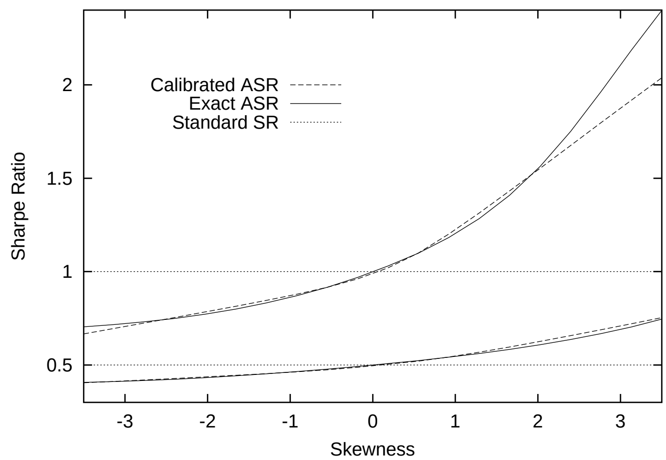

Figure 2: The exact and the calibrated value of the ASR versus the skewness of distribution for two values of the standard Sharpe ratio.

4 Portfolio Performance Evaluation Using ASR and GSR

In this section we use the ASR and the GSR to compare the performances of the portfolios for which the shape of return distribution is far from that of the normal probability distribution. In this case the use of the standard Sharpe ratio is misleading, see Bernardo and Ledoit (2000). We analyze the empirical probability distributions and report the return statistics where by kurtosis we always mean the *excess kurtosis* computed as $K - 5/3S^2 - 3$. That is, this is the part of the standard kurtosis, K, which is not accounted for in the formula for the ASR. We compute the ASR using the formulas (17) and (18) for the calibrated ASR. We compute the GSR using the *empirical* probability distribution in order to account for all the moments of a real probability distribution.

4.1 Performances of Portfolios with Manipulated Sharpe Ratios

Recently a number of papers have shown that the standard Sharpe ratio is prone to manipulation, see, for example, Spurgin (2001) and Goetzmann et al. (2002). In particular, Spurgin (2001) shows that managers can increase the Sharpe ratio by selling off the upper end of the return distribution. Goetzmann et al. (2002) identify a class of strategies that maximize the Sharpe ratio without requiring any manager skill. They show how to achieve the maximum Sharpe ratio by either selling out-of-the-money call options or selling both out-of-the-money call and put options.

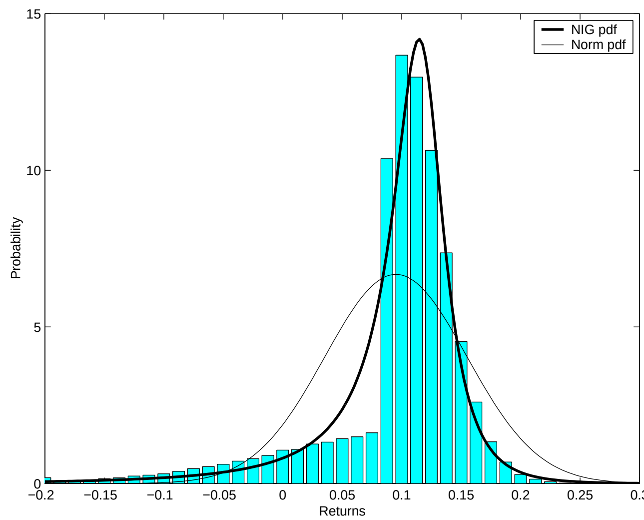

Figure 3: Empirical probability distribution of the portfolio consisting of the stock and short calls on the stock with fitted NIG and Normal distributions.

In this subsection we study the performances of the portfolios, described in Goetzmann et al. (2002), with manipulated Sharpe ratios using the (calibrated) ASR and the GSR. The benchmark portfolio is a stock which price process follows the Geometric Brownian Motion

$$P(t + \Delta t) = P(t) \exp\left((\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon\right),$$

where ε is a standard Normal variable. The model parameters are the following: $\mu = 0.15$, $r = 0.05$, $\sigma = 0.15$, and $\Delta t = 1$. The simplest manipulation strategy consists in selling 0.843 call options on one share of the stock with strike $1.0098P(t)$. A higher Sharpe ratio can be achieved by selling 2.58 put options with strike $0.88P(t)$ and selling 0.77 call options with strike $1.12P(t)$.

We simulate the benchmark portfolio and the portfolios with short options by generating 1,000,000 paths of the stock price and compute the return statistics as well as the different Sharpe ratios, see Table 2. Figure 3 illustrates the empirical probability distribution of the portfolio consisting of the stock and short calls. Easy to see that this strategy produces a return distribution which is far from the normal probability distribution. In particular, the return distribution has low standard deviation but significant negative skewness. The Sharpe ratio of this strategy is higher than the Sharpe ratio of the underlying stock. The strategy with both puts and calls produces an even higher Sharpe ratio.

However, if we look at the values of the ASR and GSR, it is easy to note that the strategies with manipulated Sharpe ratios show worse performances than the benchmark strategy. First observe that the values of ASR and GSR are close because the excess kurtosis of all these strategies is rather insignificant. Note that the ASR and the GSR of the benchmark strategy is greater than the SR since a lognormal probability distribution exhibits positive skewness. Then observe that the portfolio with the stock and short calls has lower ASR and GSR than the benchmark portfolio which indicates that short calls are, in fact, decrease the performance. The portfolio with short calls and puts has the highest Sharpe ratio but the lowest ASR and GSR.

Strategy	**Mean**	**Std**	**Skew**	**Kurt**	**SR**	**ASR**	**GSR**
Only stock	0.162	0.177	0.456	0.027	0.631	0.667	0.672
Stock and calls	0.095	0.061	-1.997	-1.276	0.731	0.624	0.627
Stock, puts and calls	0.139	0.120	-2.364	0.158	0.743	0.606	0.601

Table 2: Returns statistics and performances of different strategies.

4.2 Hedge Fund Performance Evaluation

It is well known that the distribution of hedge fund returns deviates significantly from normality. These deviations have been widely described in the literature, see, for example, Brooks and Kat (2002), Agarwal and Naik (2004) and Malkiel and Saha (2005). In particular, the hedge funds returns exhibit high negative skewness and positive excess kurtosis. This can be explained by various reasons, for example, an extensive use of options and option-like dynamic trading strategies. Consequently, the ranking of hedge funds using the standard Sharpe ratio is dubious since it only takes into account the first two moments of distribution. That is, the significant left-tail risk in the distribution is ignored.

In this subsection we evaluate the performances of the CS/Tremont Hedge Fund Indexes. The CS/Tremont indexes are based on the TASS database which tracks around 2,600 hedge funds. Using a subset of around 650 funds CS/Tremont calculates 13 indexes (in addition to the main index) which track every major style of hedge fund management. Our sample consists of monthly returns of the CS/Tremont indexes from January 1994 to November 2006. Table 3 reports the summary statistics and different Sharpe ratios. As we can see from the table, if we change the performance measure from the SR to either ASR or GSR, many funds exhibit shift in their ranking, but a few funds are stable in their ranking. For example, the Equity Market Neutral index is ranked first according to the SR, ASR and GSR due to the fact that the skewness of its distribution is positive. The worst funds that are ranked 11th, 12th, 13th, and 14th also retain their ranking even though some of their performances were adjusted downwards.

Hedge Fund Index	**Mean**	**Std**	**Skew**	**Kurt**	**SR**	**ASR**	**GSR**
CS/Tremont Hedge Fund Index	0.1055	0.0770	0.1144	2.2926	0.8556 (08)	0.8628 (05)	0.8227 (06)
Convertible Arbitrage	0.0871	0.0465	-1.3428	0.1629	1.0214 (06)	0.8566 (06)	0.8588 (04)
Dedicated Short Bias	-0.0065	0.1701	0.8302	0.8752	-0.2709 (14)	-	-
Emerging Markets	0.0984	0.1608	-0.6761	3.8714	0.3659 (12)	0.3486 (12)	0.3471 (12)
Equity Market Neutral	0.0960	0.0290	0.3339	0.1516	1.9452 (01)	2.1646 (01)	2.1883 (01)
Event Driven	0.1119	0.0557	-3.4013	4.7359	1.2999 (03)	0.7656 (09)	0.8460 (05)
Distressed	0.1294	0.0630	-2.8847	4.9123	1.4258 (02)	0.8750 (03)	0.9355 (03)
Multi-Strategy	0.1031	0.0597	-2.5307	5.8293	1.0629 (05)	0.7765 (08)	0.7623 (08)
Risk Arbitrage	0.0740	0.0410	-1.2296	4.0419	0.8394 (09)	0.7297 (10)	0.6982 (10)
Fixed Income Arbitrage	0.0628	0.0368	-3.0577	0.6955	0.6308 (11)	0.5009 (11)	0.4981 (11)
Global Macro	0.1329	0.1082	0.0379	2.9167	0.8620 (07)	0.8634 (04)	0.8125 (07)
Long/Short Equity	0.1175	0.1010	0.2227	3.7361	0.7721 (10)	0.7871 (07)	0.7372 (09)
Managed Futures	0.0661	0.1187	0.0514	0.3545	0.2231 (13)	0.2232 (13)	0.2241 (13)
Multi-Strategy	0.0925	0.0429	-1.1900	0.9031	1.2332 (04)	1.0231 (02)	1.0127 (02)

Table 3: Hedge fund returns statistics and performances. Data as of November 30, 2006. The means and standard deviations are annualized. Ratios are calculated using the rolling 90 day T-bill rate. Numbers in the brackets show the rank of a fund using a particular Sharpe ratio.

We believe that the ASR and GSR give not only a more correct ranking of hedge funds, but also (and this is more crucial) a more correct measure of performance. For example, the Event Driven Distressed index is ranked second by the SR and third by either ASR or GSR. The loss of one place seems to be insignificant. However, the performance of this index was downgraded from 1.4258 (SR) to 0.9355 (GSR), which means a loss of 35% of its original value due to high negative skewness and positive excess kurtosis. In addition, judging by the SR, the difference between the performance of the fund ranked 4th and that of the fund ranked 7th amounts to 0.3712 which is relatively large, while judging by the GSR the similar difference amounts to only 0.0463 which is insignificant. This means that these funds exhibit more or less similar performance.

It is also crucial to realize how important is the influence of skewness on a performance measure relative to the influence of excess kurtosis. As we can see from Table 3, as a rule the GSR of a fund is either on approximately the same level as the corresponding ASR or adjusted downwards relative to the ASR due to positive excess kurtosis. There are two exceptions from this rule: the GSR of Event Driven and Event Driven Distressed indexes are greater than their ASR. We believe that the main explanation of this phenomenon is the fact that our (calibrated) ASR tends to underestimate the real ASR when both the Sharpe ratio and skewness is large (see Figure 2 and subsection 3.3), which is true for these two indexes. This can be corrected by using several versions of the ASR which are calibrated for different intervals for the Sharpe ratio and skewness. Now observe that the maximum difference between an ASR and the corresponding GSR amounts to 9.5%, while the maximum difference between an SR and the corresponding GSR amounts to 53.6% (all relative to the value of a GSR). This implies that the correction for skewness in a performance measure is much more important than the correction for excess kurtosis and other higher moments of distribution.

5 Summary and Conclusion

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