Foreign Exchange, Fractional Cointegration and the Implied-Realized Volatility Relation

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January 2007[*]

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[*]Acknowledgement: Our thanks to Jerry Coakley, Yuval Millo, Lucio Sarno and Mark Wohar for their helpful comments and encouragement. The opinions expressed in the paper and any remaining errors remain our responsibility.

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Abstract

Almost all relevant literature has characterized implied volatility as a biased predictor of realized volatility. This paper provides new time series techniques to assess the validity of this finding within a foreign exchange market context. We begin with the empirical observation that the fractional order of volatility is often found to have confidence intervals that span the stationary/non-stationary boundary. However, no existing fractional cointegration test has been shown to be robust to both regions. Therefore, a new test for fractional cointegration is developed and shown to be robust to the relevant orders of integration. Secondly, employing a dataset that includes the relatively new Euro markets, it is shown that implied and realized volatility are fractionally cointegrated with a slope coefficient of unity. Moreover, the non-standard asymptotic distribution of estimators when using fractionally integrated data is overcome by employing a bootstrap procedure in the frequency domain. Strikingly, tests then show that implied volatility is an unbiased predictor of realized volatility!

EFMA classifications: 350, 410
JEL classification: C14, C22, C32, F31, G14.
Keywords: market efficiency, traded volatility, narrow band least squares, fractional cointegration, bootstrap

I. Introduction

Market efficiency in options markets is typically examined by estimating the following regression

$$\sigma_{t+\tau}^{RV} = \alpha + \beta\sigma_t^{IV} + u_{t+\tau} \tag{1}$$

where σ_t^{IV} is the implied volatility (IV) over a period of time τ and $\sigma_{t+\tau}^{RV}$ is the realized volatility (RV). Unbiasedness holds in (1) when $\alpha = 0$, $\beta = 1$ and $u_{t+\tau}$ is serially uncorrelated. Of course, unbiasedness is a sufficient condition for market efficiency but is not necessary in the presence of either a constant or a time-varying option market risk premium.

Conventional tests in the previous literature have generally led to the conclusion that IV is a biased forecast of RV in the sense that the slope parameter in (1) is not equal to unity (see, inter alia, Christensen and Prabhala, 1998, and Poteshman, 2000). This conclusion is found to be robust across a variety of asset markets (see Neeley, 2004) and has thus provided the motivation for several attempted explanations of this common finding. Popular suggestions include computing RV with low-frequency data (Poteshman, 2000); that the standard estimation with overlapping observations produces inconsistent parameter estimates (Dunis and Keller, 1995, Christensen et al., 2001); and that volatility risk is not priced (Poteshman, 2000, and Chernov, 2006). However, Neeley (2004), evaluates these possible solutions and finds that the bias in IV is not removed.

Of course, the optimality of the estimation procedure applied to (1) depends critically on the order of integration of the component variables. Given the acknowledged persistence in individual volatility series, the recent literature suggests they are well represented as fractionally integrated processes (see, inter alia, Anderson et al., 2001a and

2001b). Notably Bandi and Perron (2006), Christensen and Nielson (2006) and Nielsen (2006) have begun to examine the consequences of this approach for regression (1).

Employing stock market data, Bandi and Perron (2006), Christensen and Nielson (2006) and Nielsen (2006) suggest that IV and RV are fractionally cointegrated series[1]. Interestingly, Bandi and Perron (2006) stress the fractional order of volatility is found in the non-stationary region whereas Christensen and Nielson (2006) and Nielsen (2006) indicate the stationary region. However, each conclusion could be considered questionable given 95% confidence intervals would include both regions. In any case, Marinucci and Robinson (2001) stress that it is typically difficult to determine the integration order of fractional variables because a smooth transition exists between stationary and non-stationary regions. Christensen and Nielson (2006) and Nielsen (2006) note that when the fractional nature of the data is accounted for a slope parameter of unity in equation (1) cannot be rejected. Bandi and Perron (2006), noting the non-standard asymptotic distribution of conventional estimators in the non-stationary region, cannot test the relevant null hypothesis although they also claim their results give support to the unbiasedness hypothesis.

This paper extends the empirical work of Bandi and Perron (2006), Christensen and Nielson (2006) and Nielson (2006) in three steps. Firstly, we employ data for several foreign exchange markets including the relatively new Euro market. Importantly, the IV data collected is traded on the market (and hence is directly observable). Since these data are directly quoted from brokers, they avoid the potential measurement errors associated with the more common approach (see, inter alia, Christensen and Prabhala, 1998) of

[1]Although this recent work predominantly investigates stock markets, Bandi and Perron (2006) also analyse options on Deutsche Mark/US Dollar futures. Finding similar results to those for stock markets they suggest that fractional cointegration in the implied-realized relation is a stylised fact.

backing out implied volatilities from a specific option-pricing model.

Secondly, the possibility of fractional cointegration is examined formally using a new adaptation of the recently developed semi-parametric technique of Hassler et al. (2006) [hereafter HMV]. Under certain assumptions HMV prove that a residual-based log periodogram estimator, where the first few harmonic frequencies have been trimmed, has a limiting normality property. In particular, this methodology provides an asymptotically reliable testing procedure for fractional cointegration when the fractional order of regressors are strongly non-stationary. However, given the noted empirical uncertainty, (foreign exchange) volatility may present an integration order that violates the assumptions for the HMV test, as well as other fractional cointegration tests. To circumvent this uncertainty, we develop, examine and apply a fractional cointegration test robust to both stationary and (weak and strong) non-stationary regions.

Thirdly, given the non-standard asymptotic distribution of conventional estimators when using fractionally integrated data, we employ a bootstrap procedure to compute appropriate confidence intervals in (1). Again, this specifically overcomes the difficulties encountered by Bandi and Perron (2006) when estimators are applied in the non-stationary region.

Results employing the new fractional cointegration test confirm that foreign exchange RV and IV are fractionally cointegrated with a slope coefficient of unity. Strikingly, this result holds across a range of currencies. Moreover, tests using bootstrapped estimates then allow us to show that a slope parameter of unity in equation (1) cannot be rejected. In summary, and contrary to almost all previous research, foreign exchange implied volatility is shown to be unbiased. The paper is divided into five sections: Section 2 presents the empirical methodology; section 3 describes the data;

section 4, the results and finally, section 5 concludes.

II. Empirical methodology

A. Fractional integration

Many in the literature (see, inter alia, Bandi and Perron, 2006, Vilasuso, 2002, and Baillie et al. 1996) have suggested that asset price volatility is neither an I(1) nor an I(0) process but rather a fractionally integrated or I(*d*) process. The introduction of the autoregressive fractionally integrated moving average (ARFIMA) model by Granger and Joyeux (1980) and Hosking (1981) allows the modeling of persistence or long memory where $0 < d < 1$. A time series y_t follows an ARFIMA (*p, d, q*) process if

$$\Phi(L)(1-L)^d y_t = \mu + \Theta(L)\varepsilon_t, \quad \varepsilon_t \sim iid(0,\sigma^2) \tag{2}$$

where $\Phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ and $\Theta(L) = 1 - \theta_1 L - ... - \theta_q L^q$. Such models may be better able to describe the long-run behaviour of certain variables. For example, when $0 < d < 1/2$, y_t is stationary but contains long memory, possessing shocks that disappear hyperbolically not geometrically. Contrastingly, for $1/2 < d < 1$, the relevant series is non-stationary, the unconditional variance growing at a more gradual rate than when $d = 1$, but mean reverting.

The memory parameter d can be estimated by a number of different techniques. The most popular, due to its semi-parametric nature, is the log periodogram estimator (Geweke and Porter-Hudak, 1983; Robinson, 1995a) henceforth known as the GPH statistic. This involves the least squares regression

$$\log I(\lambda_j) = \beta_0 - d\log\{4\sin^2(\lambda_j/2)\} + u_j, \quad j = l+1, l+2, ..., m \tag{3}$$

where $I(\lambda_j)$ is the sample spectral density of y_t evaluated at the $\lambda_j = 2\pi j/T$

frequencies, T is the number of observations and m is small compared to T. Inter alia, Pynnönen and Knif (1998) and HMV, note that the least-squares estimate of d can be used in conjunction with standard t-statistics. For the stationary range, $-1/2 < d < 1/2$, Robinson (1995a, 1995b) demonstrated that the GPH estimate is consistent and asymptotically normally distributed. Additionally, Velasco (1999a, 1999b) shows that when the data are differenced, the estimator is consistent for $1/2 < d < 2$ and asymptotically normally distributed for $1/2 < d < 7/4$.

B. Fractional cointegration

As discussed in the introduction, some recent literature has presented the possibility that RV and IV are fractionally cointegrated. Fractional cointegration can be defined by supposing y_t and x_t are both I(d), where d is not necessarily an integer and the residuals, $u_t = y_t - \beta x_t$, are I($\delta = d - b$). When $b < d$, where b is also not necessarily an integer, series are fractionally cointegrated. Testing for fractional cointegration can be accomplished using a multi-step methodology (see HMV) where (i) the order of integration of the constituent series are estimated and tested for equality and (ii) the long-run equilibrium relationship is estimated[2] and the residuals examined for long-memory. Alternative methodologies include the joint estimation of memory parameters of the constituent series, the cointegrating residuals and the equilibrium relationship (see Velasco, 2003) or the use of bootstrap methods (see Davidson, 2005).

A frequently used approach is to adopt a multi-step methodology where the

[2]The long-run equilibrium relationship itself could be approximated by OLS, a fractional version of the Fully Modified method suggested by Kim and Phillips (2001), Gaussian semi-parametric estimation developed by Velasco (2003) or narrow band spectral estimates (see Robinson and Marinucci, 1998).

concluding step estimates the GPH statistic, δ, for the least squares residual of the equilibrium relationship (see Dittman, 2001). Inter alia, Tse et al. (1999) experimentally noted that t-statistics associated with $\hat{\delta}$ might not be normally distributed.

C. Nonstationary fractional cointegration

If d is in the *strongly* non-stationary region, HMV demonstrate that as long as $\hat{\beta}$ can converge fast enough, $\hat{\delta}$ possesses a limiting normal distribution[3] provided the very first harmonic frequencies are trimmed. Specifically, this entails setting $l > 0$ in (3). Moreover, Monte Carlo experiments show that trimming only one frequency, $l = 1$, provides a satisfactory normal approximation for the distribution of GPH statistic in finite samples. Of course, given (asymptotically) normal estimators, standard inference procedures can be legitimately applied.

A priori, it is useful to note the HMV test theoretically requires certain assumptions to hold to generate limiting normality for the distribution of $\hat{\delta}$. The most relevant to our discussion are listed below

$$0 \le \delta < 0.5 \tag{4}$$

$$\delta < d - 0.5 < 1 \tag{5}$$

$$0.72 < d < 1.5 \tag{6}$$

In particular, it should be stressed that condition (6) implies that for what might be termed the *weakly* non-stationary region (i.e. $0.5 < d < 0.72$), there is no limiting normal distribution theory. This is due to the slower convergence rate of $\hat{\beta}$.

[3]See Appendix for an outline of relevant theorem in Hassler et al. (2006).

D. Stationary fractional cointegration

If $d < 0.5$, OLS estimates of β are inconsistent suggesting the above approach may be inappropriate. However, Robinson and Marinucci (1998) and Christensen and Nielson (2006) have shown that narrow band least squares (NBLS) estimation can result in an estimator $\hat{\beta}_z$ that is consistent and normally distributed. To explain NBLS consider first that a matrix form of (1) could be written

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \tag{7}$$

where $\boldsymbol{\beta}$ is a 2×1 vector of unknown coefficients and $\mathbf{u}$ is a $T \times 1$ vector of disturbances. Additionally define the complex $T \times T$ Fourier matrix, $\mathbf{V}$, which has as its *jth* element

$$v_{jt} = T^{-1/2}\left[\exp\{-i\lambda_{j-1}t\}\right], \quad j,t = 1,...,T \tag{8}$$

and presents the frequencies

$$\lambda_j = \frac{2\pi j}{T} \tag{9}$$

A transformation to the frequency domain (see Harvey, 1993) can be made by pre-multiplying the observation matrices in (7) by $\mathbf{V}$ and expressing the transformed model as

$$\dot{\mathbf{y}} = \dot{\mathbf{X}}\boldsymbol{\beta} + \dot{\mathbf{u}} \tag{10}$$

OLS estimation of (10) will produce identical estimates to that of (1). Note however different frequency components may be omitted by removing $T - z$ corresponding transformed observations. This is band spectrum regression and it can be shown that the β in (1) will be estimated by the statistic

$$\hat{\beta}_z = \left[\sum_{j=0}^{z} I_{\sigma_{IV}}(\lambda_j) \right]^{-1} \sum_{j=0}^{z} I_{\sigma_{IV}\sigma_{RV}}(\lambda_j), \quad 0 \le z \le T-1 \tag{11}$$

where $I_{\sigma_{IV}}(\lambda_j)$ is the sample spectral density of IV and $I_{\sigma_{IV}\sigma_{RV}}(\lambda_j)$ is the cross-spectrum between IV and RV[4]. Additionally, such band spectrum regression is called NBLS if

$$\frac{1}{z} + \frac{z}{T} \to 0 \ \ as\, T \to \infty \tag{12}$$

A number of assumptions are required for the generation of a limiting normal distribution for $\hat{\beta}_z$. These include

$$d > 0 \tag{13}$$

$$\delta \ge 0 \tag{14}$$

$$d + \delta < 0.5 \tag{15}$$

Christensen and Nielson (2006) recently employed a multi-step methodology, where the concluding step semi-parametrically estimates δ for the NBLS residual of the equilibrium relationship. Hypothesis testing is then conducted on $\hat{\delta}_z$ as if the residuals were observed. Although possibly an appropriate procedure (see Nielsen, 2006) this has not been ascertained in the literature[5].

E. Boundary fractional cointegration

The fractional order of volatility has typically been found to have confidence intervals that span the stationary/non-stationary boundary (i.e. $0.3 < d < 0.7$). In any case,

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[4]Therefore, $\hat{\beta}_{T-1}$ is a special case, equivalent to the OLS estimate of β in (1).

[5]However, Nielsen (2006) uses a local Whittle QMLE to jointly estimate the integration order of the regressors, the integration order of the residuals and the coefficients of the cointegrating vector. Under certain conditions, this estimator is shown to be asymptotically normal for the stationary region.

Marinucci and Robinson (2001) suggest determining the stationarity or otherwise of time series variables is often difficult. This difficulty is particularly pronounced in a fractional context, where a smooth transition exists between stationary and non-stationary regions. However, the use of the fractional cointegration methodology discussed above relies on the identification of the appropriate region. Furthermore, point estimates for d are often found in the weakly non-stationary region (i.e. $0.5 < d < 0.72$), a result for which there is no limiting normal distribution theory. Therefore, we require an estimator which is robust in finite samples to orders of integration for d that span the boundary. As a first step, it would seem more appropriate to use NBLS rather than OLS to estimate the equilibrium relationship. As discussed above NBLS, in contrast to OLS, provides a consistent estimator in the stationary region. However, analogously to OLS, NBLS is consistent in the non-stationary region (see Marinucci and Robinson, 2001). Furthermore, in the non-stationary region, NBLS generally converges faster than OLS (see Marinucci and Robinson, 2001) and thus may resolve the lack of a normal distribution for $\hat{\delta}$ in the weakly non-stationary region shown by HMV. In a second step, it would appear useful to examine the effect of trimming on the distribution of $\hat{\delta}_z$.

To assess the distribution of $\hat{\delta}_z$ across the boundary we performed a simple simulation. Let x_t be generated by an ARFIMA $(0, d, 0)$ series

$$(1-L)^d x_t = \varepsilon_{1t} \tag{16}$$

where the fractional difference operator is defined by the Maclaurin series

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(-d+j)L^j}{\Gamma(-d)\Gamma(j+1)} = \sum_{j=0}^{\infty} d_j L^j; \;\; d_j = \frac{(j-1-d)d_{j-1}}{j}; \;\; d_0 = 1 \tag{17}$$

and $\Gamma(.)$ is the gamma function. To avoid the initial conditions effect, sample sizes

$t = 1,...,T + w$ are generated and the first $w = 1000$ observations removed. Additionally, $\sum_{j=0}^{\infty} d_j L^j$ is approximated by allowing $d_j = 0$ when $j > 1000$. The true regression model is

$$y_t = x_t + v_t; \quad (1-L)^{\delta} v_t = \varepsilon_{2t} \tag{18}$$

and is estimated by NBLS

$$y_t = \hat{\alpha}_z + \hat{\beta}_z x_t + \hat{v}_t \tag{19}$$

GPH statistics $\hat{\delta}_z$ are then computed for $\hat{v}_t$ and the statistics below calculated

$$\frac{\hat{\delta}_z - \delta}{se(\hat{\delta}_z)} \tag{20}$$

In the experiments two-sided tests at the 1%, 5% and 10% level are calculated. To allow comparison with HMV we set $m = T^{0.5}$, allowed the trimming parameter $l \in (0,1)$ and use 2000 replications. Indeed, simulations not reported here, and using OLS instead of NBLS, replicate the Monte Carlo results of HMV. In our reported simulations we also vary the NBLS estimation by applying both $z = T^{0.3}$ and $z = T^{0.75}$. Recent work has recommended the use of a low number of frequencies (see Christensen and Nielson, 2006, Marinucci and Robinson, 2001, and Robinson and Marinucci, 1998) and thus the two settings will allow some assessment of this approach. To begin with Tables 1 and 2 show the size of the GPH tests, without trimming, employing different values of d and δ.

[Insert Tables 1 and 2]

The results above clearly show that NBLS/GPH methodology, without trimming, does not typically produce a normally distributed test statistic. Tests are particularly oversized when a relatively small number of frequencies ($z = T^{0.3}$) are employed[6]. This casts some doubt on the recent approach used in the literature and the twin assertions that z should be relatively low and hypothesis testing can be conducted on $\hat{\delta}_z$ as if the residuals were observed. Tables 3 and 4 below repeat the previous simulations, although now with $l = 1$.

[Insert Tables 3 and 4]

Tables 3 and 4 clearly show that with trimming, the new methodology produces an approximately normally distributed test statistic. Strikingly, this result holds for when d is in the stationary region or the weakly non-stationary region. The reduction in size bias is particularly marked in the $z = T^{0.3}$ case. These finite sample results have clear implications for testing option market efficiency using regression (1). They suggest that as long as trimming is employed an NBLS/GPH methodology can be legitimately used to assess whether RV and IV are fractionally cointegrated. We shall use this result in empirical tests of data discussed below.

III. Data

Daily and monthly time series of RV and IV were constructed from daily data for the period January 1991[7] to September 2005. As in Dunis and Keller (1995), Dunis and Huang (2002), and Sarantis (2005), IV is measured by at-the-money, one-month forward,

[6]Marinucci and Robinson (2001) also examine the NBLS estimate of the cointegrating vector and associated semiparametric methods for testing for the existence of fractional cointegration. Using a Monte Carlo approach, as one would expect, their Hausman test is shown to be similarly oversized. It should be noticed that their Monte Carlo investigation only examined the 5% size over the strongly non-stationary region (i.e. $0.8 < d < 1.2$). See Marinucci and Robinson (2001) Table 11.

[7]The choice of start date was governed by the availability of IV data.

market quoted volatilities at close of business in London, obtained from brokers by Reuters. These `traded' implied volatilities[8] measure the market's expectation about the future volatility of the spot exchange rate for six currencies: Sterling/US dollar, US dollar/Swiss Franc, US dollar/Yen, Euro/Yen, Euro/Sterling and Euro/US dollar[9]. As currency volatility has now become a traded quantity in financial markets, it is therefore directly observable on the marketplace. The databank is maintained by CIBEF at Liverpool John Moores University. As noted in the introduction, since these data are directly quoted from brokers, they avoid the potential biases associated with backing out.

Given IV data for each specific day, as in Christensen and Hansen (2002), RV is calculated over the remaining one month of the option

$$\sigma_{t+\tau}^{RV} = \sqrt{\frac{252}{\tau-1}\sum_{i=1}^{\tau}(r_{t+i}-\bar{r}_t)^2} \tag{21}$$

where $r_t = \ln(S_t / S_{t-1})$, τ is the relevant number of trading days[10] and S_t is the closing (London time) average of bid and ask quotes for the spot exchange rates[11]. The raw daily

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[8]Implied volatilities are also annualised rates so that a quoted volatility of 5 per cent typically translates to a monthly variance rate of $(0.05^2)(21/252)$. The calculations assume that annualised rates refer to a 252 trading day year.

[9]The three Euro IV series begin in January 1995 and comprise Deutsche Mark volatility until December 1998; after the introduction of the Euro in January 1999, actual Euro volatility are used. The splicing together of Deutsche Mark and Euro series is to ensure we have enough observations (particularly when using monthly observations) to usefully employ semi-parametric estimation. For the exchange rate series, we compute 'synthetic' euro returns until December 1998 using the fixed Euro/Deutsche Mark rate of 1.95583 agreed at the EU Brussels summit in May 1998 for the change over to the Euro on 31/12/1998 (i.e. combined with the 'time-varying' US dollar/Deutsche Mark rate to produce the synthetic Euro/US dollar rate). We did not use data prior to 1995 as the period 1992-1994 was one of sharp appreciation and revaluations of the Deutsche Mark versus the other ERM currencies prior to its stabilisation over 1995-1998: using the fixed Euro/Deutsche Mark exchange rate agreed in Brussels in May 1998 to compute 'synthetic' euro returns prior to January 1995 would therefore have been problematic.

[10]Assumed to be 21 days.

[11]It should be noted that this is only a proxy for the true, but unknown RV. Alternatively, a methodology using intra-day foreign exchange data and following Anderson et al. (2001a) could be employed. However, given that IV is drawn from a daily sampling frequency it seems appropriate to calculate RV from an analogous frequency. Recent studies that have also used a daily frequency include Bandi and Perron (2006)

dataset thus consists of (2770) 3780 time series observations for each (euro) volatility series. Of course, as pointed out by Christensen and Prabhala (1998), overlapping data problems will beset estimation of equation (1) if daily datasets are employed. To circumvent this a monthly dataset is derived from the daily version. Specifically, IV data is taken only from the subsequent trading day after the final day used in the calculation of the previous RV figure. This allows the data to cycle through the calendar and the resulting dataset contains (126) 172 non-overlapping observations for each (euro) volatility series[12].

IV. Empirical results

GPH statistics[13] for the logarithm[14] of monthly[15] volatility series were estimated using

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and Christensen and Hansen (2002). Finally, Neeley (2004) has shown that using intra-day data does not explain the predictive bias in IV.

[12]Similarly in Bandi and Perron (2006) the monthly dataset is also derived from daily data and contains 152 observations. However, IV data is taken only from the closing value of each month. Although common practice, particularly in forward market analysis, this methodology does not ensure that periods of observation are strictly non-overlapping. For example, an IV figure drawn from the last trading day in January 1991 (Thursday 31st) would be matched with a RV figure calculated from 21 days of subsequent trading day returns (i.e. data up to and including Friday 1st March). Of course the next IV figure would be drawn from the last trading in February (Thursday 28th) causing subsequent periods of observation to overlap. In contrast, the cycling dataset suggested here ensures the non-overlapping nature of the data in construction. Additionally the cycling dataset does not draw data solely from one period of the month and therefore is not likely to be as susceptible to any intra-monthly seasonality. See Breuer and Wohar (1996) for an analogous application of cycling monthly datasets to the forward foreign exchange market.

[13]Note that the GPH statistic was estimated at $m = T^{0.75}$ following Maynard and Phillips (2001). As we have data at a monthly frequency, the use of much smaller bandwidths would produce standard errors too large to provide any meaningful information over the orders of integration we are interested in. Moreover, the estimated standard error of d is that derived by Geweke and Porter-Hudak (1983) and shown in equation (4) of HMV, who show it to be more appropriate than the conventional and Robinson (1995a, 1995b) alternatives.

[14]Natural logarithms of all volatility series were taken to minimise the possibility of non-normal variables as shown by, inter alia, Christensen and Hansen (2002).

[15]All empirical analysis is carried out on the monthly dataset to avoid the overlapping data problems discussed by Christensen and Prabhala (1998).

differenced data[16] and Ox version 3.3 (see Doornik, 1999) and are shown in Table 5. An alternative approach would be to specify fully parametric ARFIMA (p,d,q) models computed by exact maximum likelihood (EML). Of course this fully parametric approach is more efficient but will be inconsistent if the short-run dynamics are incorrectly specified[17]

[Insert Table 5]

Table 5 contains some interesting results. Firstly, the GPH point estimates of fractional differencing in foreign exchange volatility are spread over the range 0.71 to 0.29. Tests for $d=1$ and $d=0$ show that, in particular, the volatility series are fractionally integrated with $0<d<1$. These results confirm that foreign exchange market behaviour is analogous to stock market behaviour investigated by Bandi and Perron (2006), Christensen and Nielson (2006) and Nielsen (2006). Secondly, standard errors are such that it cannot be ascertained whether volatility series are typically stationary or non-stationary fractionally integrated processes. Thirdly, RV and IV series appear to have similar orders of integration. To examine this in more detail we test that the fractional orders of the constituent variables are equal by applying the homogenous restriction

$$H_0 \ : \ PD=0 \tag{22}$$

where $D=\begin{bmatrix} d_{RV} \\ d_{IV} \end{bmatrix}$ and $P=\begin{bmatrix} 1 & -1 \end{bmatrix}$. Robinson (1995a) noted the relevant Wald test statistic could be expressed as

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[16]The resulting estimate of d was then increased by 1. If $\hat{d}<0.5$ then d was re-estimated using data in levels. Also note that in (3) l is set equal to zero, indicating no trimming of the harmonic frequencies.

[17]Recent work by Nielsen (2006), Christensen and Nielson (2006) and Bandi and Perron (2006) all employ semi-parametric estimation of the long memory parameter.

$$\hat{D}'P'\left[(0,P)\left\{(Z'Z)^{-1}\otimes\Omega\right\}(0,P)'\right]^{-1}P\hat{D} \tag{23}$$

where Ω is residual variance-covariance matrix from (3), $Z=[Z_{l+1}\ldots Z_m]$ and $Z_j=[1,-\log\{4\sin^2(\lambda^2/2)\}]$. Table 6 contains the Wald test results

[Insert Table 6]

For all currencies the Wald test indicates equal fractional orders of d for RV and IV. Thus it is now useful to examine the fractional differencing parameter of the possible cointegrating relationship. Of course, given that volatility series clearly have confidence intervals for d that typically span the stationary/non-stationary boundary it would appear sensible to employ a fractional cointegration test robust to both these regions. Thus we next apply the new test proposed and examined in section II.E. Specifically, in a first step, regression (1) is estimated by NBLS and employing $z=T^{0.75}$. In a second step, the NBLS residuals from (1) are tested for their order of integration using GPH with $l=1$. The resulting estimates of δ, are shown below in Table 7

[Insert Table 7]

Interestingly, the point estimate of δ is always lower than the fractional parameter d of the constituent series, implying fractional cointegration. Furthermore, in all cases the null $\delta=0$ cannot be rejected. Therefore, using our robust test, we cannot reject the null of bi-variate fractional cointegration between RV and IV for any of the currencies.

The fractional cointegration between RV and IV established above is a necessary but not sufficient condition for unbiasedness in the options foreign exchange market. Therefore we need to finally consider the intercept and slope parameters in (1). As a preliminary step and for comparative purposes we present conventional OLS estimates in Table 8

[Insert Table 8]

These suggest that, as has previous literature, IV is a biased predictor of RV in the foreign exchange market. However, the slope coefficients found are generally much closer to unity than those estimated in previous studies (see Neeley, 2004). As we are employing traded volatility for the first time, this suggests that perhaps the measurement error in 'backing out' implied volatility from option pricing models may have more effect on biasing parameters than previously acknowledged.

Of course, as already noted earlier, the fractional order of integration of volatility is likely to have an effect on the OLS estimation of (1). In particular, if $d < 0.5$ then OLS estimates will be inconsistent. However, even if $d > 0.5$, OLS will typically converge slower than NBLS. Therefore, Table 9 provides the NBLS[18] point estimates for (1)

[Insert Table 9]

NBLS parameter estimates are consistent but have non-standard limit distributions in the non-stationary region. To circumvent this a bootstrap procedure is employed to generate 90% and 95% confidence intervals for the slope coefficient in (1). Specifically, in the frequency domain, NBLS residuals $\dot{\mathbf{u}}$ are resampled with replacement and used to generate a bootstrapped dependent variable $\dot{\mathbf{y}}^*$. The new dependent variable is regressed on the original frequency domain regressors $\dot{\mathbf{X}}$ to get the bootstrapped coefficient vector $\boldsymbol{\beta}^*$. Using the bootstrap class in OX, 1000 bootstrapped slope coefficients were generated in this manner.

Strikingly, Table 9 shows that for all exchange rates the NBLS slope parameter is much closer to unity than the OLS version. In a similar vein, the NBLS constant

[18] Again employing $z = T^{0.75}$.

approaches zero. Furthermore, the 90% and 95% confidence intervals for the NBLS slope coefficient all include unity. It would appear that when we account for the fractional nature of the variables the bias in IV is completely removed!

V. Conclusions

Almost all relevant literature has characterized foreign exchange implied volatility (IV) as a biased predictor of realized volatility (RV). The cause of this bias has been the subject of much debate but in a recent working paper, Neeley (2004), the popular suggestions (i.e. overlapping data; use of low frequency data; and the non-pricing of volatility premia) are rejected.

A small strand of the literature (see Bandi and Perron, 2006, Christensen and Nielson, 2006, and Nielsen, 2006) has concentrated on the effect on the IV-RV relation of characterizing volatility series as fractionally integrated processes. This paper extends their work. We begin with the empirical observation that the fractional order of volatility is typically found to have confidence intervals that span the stationary/non-stationary boundary. However, no existing fractional cointegration test has been shown to be robust to both regions. Additionally, there is no limiting normal distribution theory currently developed for the weakly non-stationary region $(0.5 < d < 0.72)$.

As a first step, we develop, examine and apply a new test for fractional cointegration which is shown to be robust to the typically relevant orders of integration. Specifically, we adopt a simple multi-stage approach where pertinently (i) the cointegrating relationship is estimated by narrow band least squares (NBLS) and (ii) a trimmed residual log-periodogram estimator is then employed. It shown that in finite samples the new estimator is approximately normally distributed and can thus be used for inference in the volatility context. Strikingly, trimming is shown to dramatically reduce size when the literature recommended low number of bandwidths are used in the NBLS estimation. The normal approximation is shown to hold across the stationary/non-stationary boundary and as a consequence even holds in the weakly non-stationary region.

Secondly, whereas previous studies have concentrated primarily on equity markets, we employ data for foreign exchange including the relatively new Euro markets: Sterling/US dollar, US dollar/Swiss Franc, US dollar/Yen, Euro/Yen, Euro/Sterling and Euro/US dollar. Importantly, the IV data collected is traded on the market (and hence is directly observable). Since these data are directly quoted from brokers, they avoid the potential measurement errors associated with the more common approach (see, inter alia, Christensen and Prabhala, 1998) of backing out implied volatilities from a specific option-pricing model.

Finally, employing the developed estimator, it is shown that foreign exchange RV and IV are fractionally cointegrated with an approximate slope coefficient of unity. However, and as in previous research, the use of conventional estimators may provide non-standard limiting distributions for the slope coefficient and as a consequence no reliable testing procedure can be employed in this long-run context. These issues are resolved by constructing a bootstrap confidence interval in the frequency domain. The confidence intervals are subsequently not able to reject the hypothesis that, in fact, IV is a unbiased predictor of RV.

Neeley (2004) suggests that although the economic value of the information is very limited, foreign exchange IV is a biased predictor of RV. The results in this paper strongly suggest that, when we employ traded volatility data, account for its fractional nature and use appropriate confidence intervals, the bias disappears!

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Table 1:

1% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.022	0.027	0.029	0.021	0.023	0.021	0.021	0.021	0.021
0.7	-	0.024	0.025	0.021	0.024	0.019	0.021	0.018	0.016
0.6	-	-	0.019	0.017	0.020	0.017	0.019	0.019	0.015
0.5	-	-	-	0.015	0.015	0.017	0.018	0.014	0.014
0.4	-	-	-	-	0.012	0.015	0.017	0.016	0.014
0.3	-	-	-	-	-	0.015	0.016	0.016	0.016

5% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.073	0.074	0.074	0.073	0.076	0.074	0.069	0.066	0.064
0.7	-	0.066	0.070	0.067	0.066	0.065	0.066	0.060	0.060
0.6	-	-	0.064	0.067	0.063	0.062	0.059	0.059	0.056
0.5	-	-	-	0.055	0.057	0.061	0.056	0.059	0.052
0.4	-	-	-	-	0.054	0.056	0.051	0.057	0.055
0.3	-	-	-	-	-	0.052	0.050	0.051	0.053

10% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.127	0.124	0.123	0.126	0.126	0.128	0.127	0.122	0.115
0.7	-	0.118	0.122	0.126	0.120	0.117	0.121	0.116	0.112
0.6	-	-	0.115	0.120	0.118	0.114	0.108	0.113	0.102
0.5	-	-	-	0.113	0.119	0.113	0.108	0.106	0.102
0.4	-	-	-	-	0.107	0.109	0.102	0.103	0.096
0.3	-	-	-	-	-	0.102	0.099	0.098	0.090

Table 2:

1% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.049	0.042	0.043	0.040	0.042	0.040	0.038	0.035	0.037
0.7	-	0.049	0.050	0.037	0.043	0.037	0.037	0.040	0.034
0.6	-	-	0.047	0.042	0.042	0.046	0.043	0.039	0.035
0.5	-	-	-	0.044	0.041	0.040	0.042	0.036	0.037
0.4	-	-	-	-	0.042	0.044	0.036	0.038	0.036
0.3	-	-	-	-	-	0.046	0.042	0.037	0.038

5% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.114	0.108	0.100	0.096	0.101	0.105	0.100	0.092	0.090
0.7	-	0.114	0.112	0.096	0.099	0.094	0.101	0.096	0.095
0.6	-	-	0.122	0.102	0.099	0.101	0.095	0.092	0.085
0.5	-	-	-	0.120	0.104	0.094	0.096	0.089	0.079
0.4	-	-	-	-	0.110	0.104	0.097	0.090	0.080
0.3	-	-	-	-	-	0.107	0.103	0.094	0.088

10% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 0$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.181	0.170	0.158	0.158	0.153	0.167	0.157	0.159	0.149
0.7	-	0.179	0.171	0.164	0.160	0.156	0.165	0.159	0.158
0.6	-	-	0.186	0.177	0.166	0.167	0.155	0.157	0.151
0.5	-	-	-	0.187	0.174	0.163	0.158	0.154	0.150
0.4	-	-	-	-	0.187	0.171	0.158	0.145	0.139
0.3	-	-	-	-	-	0.173	0.161	0.149	0.138

Table 3:

1% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.014	0.018	0.018	0.016	0.017	0.017	0.015	0.014	0.013
0.7	-	0.019	0.016	0.016	0.018	0.016	0.015	0.013	0.013
0.6	-	-	0.017	0.018	0.018	0.018	0.012	0.013	0.011
0.5	-	-	-	0.019	0.017	0.015	0.011	0.015	0.011
0.4	-	-	-	-	0.015	0.016	0.013	0.012	0.012
0.3	-	-	-	-	-	0.017	0.015	0.013	0.013

5% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.067	0.063	0.059	0.061	0.055	0.056	0.055	0.050	0.047
0.7	-	0.064	0.061	0.061	0.059	0.054	0.058	0.054	0.050
0.6	-	-	0.061	0.064	0.059	0.057	0.057	0.053	0.053
0.5	-	-	-	0.062	0.062	0.059	0.056	0.055	0.054
0.4	-	-	-	-	0.056	0.058	0.061	0.057	0.050
0.3	-	-	-	-	-	0.055	0.059	0.055	0.051

10% Size of NBLS Tests ($T = 250$; $z = T^{0.75}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.114	0.120	0.109	0.110	0.105	0.106	0.108	0.102	0.102
0.7	-	0.118	0.116	0.108	0.106	0.110	0.112	0.106	0.103
0.6	-	-	0.113	0.111	0.102	0.103	0.111	0.113	0.104
0.5	-	-	-	0.111	0.102	0.110	0.115	0.112	0.101
0.4	-	-	-	-	0.102	0.107	0.113	0.109	0.105
0.3	-	-	-	-	-	0.101	0.110	0.108	0.107

Table 4:

1% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.015	0.015	0.017	0.017	0.018	0.017	0.016	0.017	0.015
0.7	-	0.021	0.018	0.015	0.014	0.014	0.015	0.016	0.014
0.6	-	-	0.020	0.016	0.018	0.018	0.016	0.016	0.013
0.5	-	-	-	0.016	0.018	0.016	0.016	0.015	0.015
0.4	-	-	-	-	0.016	0.015	0.012	0.014	0.013
0.3	-	-	-	-	-	0.017	0.016	0.012	0.012

5% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.062	0.060	0.059	0.060	0.057	0.054	0.056	0.053	0.055
0.7	-	0.062	0.065	0.059	0.056	0.056	0.055	0.059	0.058
0.6	-	-	0.069	0.061	0.062	0.054	0.055	0.060	0.058
0.5	-	-	-	0.066	0.061	0.061	0.061	0.058	0.057
0.4	-	-	-	-	0.068	0.064	0.068	0.062	0.056
0.3	-	-	-	-	-	0.067	0.071	0.070	0.063

10% Size of NBLS Tests ($T = 250$; $z = T^{0.3}$; $l = 1$)

$d \setminus \delta$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.8	0.108	0.106	0.107	0.107	0.103	0.101	0.107	0.104	0.105
0.7	-	0.113	0.115	0.110	0.111	0.099	0.101	0.105	0.111
0.6	-	-	0.115	0.109	0.107	0.107	0.105	0.106	0.108
0.5	-	-	-	0.118	0.119	0.113	0.111	0.107	0.103
0.4	-	-	-	-	0.123	0.126	0.120	0.114	0.109
0.3	-	-	-	-	-	0.173	0.161	0.149	0.138

Table 5: GPH Tests for the d of Individual Volatility Series

		$\hat{d}$	$(\hat{d}-1)/\sigma_d$	$\hat{d}/\sigma_d$
UK£ /US$	RV	0.629 (0.111)	-3.34	5.67
	IV	0.617 (0.111)	-3.45	5.56
US$/SF	RV	0.290 (0.111)	-6.40	2.61
	IV	0.598 (0.111)	-3.62	5.39
US$/Yen	RV	0.482 (0.111)	-4.67	4.34
	IV	0.713 (0.111)	-2.59	6.42
Euro/Yen	RV	0.579 (0.129)	-3.26	4.49
	IV	0.712 (0.129)	-2.23	5.52
Euro/UK£	RV	0.413 (0.129)	-4.55	3.20
	IV	0.712 (0.129)	-2.23	5.52
Euro/US$	RV	0.421 (0.129)	-4.49	3.26
	IV	0.609 (0.129)	-3.03	4.72

Note: numbers in parentheses alongside the estimates for d are the standard errors σ_d.

Table 6: Wald Tests for the Equality of the GPH Estimates for Realized and Implied Volatility

UK£ /US$	US$/SF	US$/Yen	Euro/Yen	Euro/UK£	Euro/US$
0.004	2.627	1.602	0.524	1.930	1.225
[0.951]	[0.105]	[0.206]	[0.469]	[0.165]	[0.268]

Note: the Wald statistic has a $\chi^2(1)$ distribution. The figures in square brackets are p values.

Table 7: Robust Tests for the Integration Order of the (Level) Residuals in (1)

UK£ /US$	US$/SF	US$/Yen	Euro/Yen	Euro/UK£	Euro/US$
0.173	0.011	0.081	0.132	-0.050	0.179
(0.128)	(0.128)	(0.128)	(0.152)	(0.152)	(0.152)

Note: numbers in parentheses are standard errors.

Table 8: OLS estimates of (1)

	$\hat{\alpha}$	$\hat{\beta}$
UK£ /US$	-0.280	0.945
US$/SF	-0.537	0.791
US$/Yen	-0.571	0.790
Euro/Yen	-0.501	0.814
Euro/UK£	-0.600	0.797
Euro/US$	-0.631	0.780

Table 9: NBLS estimates of (1)

	$\hat{\alpha}$	$\hat{\beta}$	95% CI for β	90% CI for β
UK£ /US$	-0.100	1.019	[0.853 -1.199]	[0.876 -1.162]
US$/SF	-0.057	1.011	[0.800 -1.250]	[0.817 -1.223]
US$/Yen	-0.196	0.958	[0.800 -1.138]	[0.824 -1.108]
Euro/Yen	-0.162	0.967	[0.784 -1.133]	[0.817 -1.107]
Euro/UK£	-0.218	0.945	[0.803 -1.094]	[0.826 -1.071]
Euro/US$	-0.236	0.953	[0.704 -1.216]	[0.748 -1.167]

Appendix

Let y_t and x_t be integrated of order d and satisfy

$$y_t = \beta x_t + u_t, \quad \beta \neq 0, \quad t = 1,...,T$$

where $u_t \sim I(\delta)$.

Assumption 1 Allow for two cases

Case 1: If $\delta + d \geq 1$ then $\hat{\beta} - \beta = O_p(T^{\delta - d})$

Case 2: If $\delta + d < 1$ then $\hat{\beta} - \beta = O_p(T^{1-2d})$

These assumptions will hold if $\hat{\beta}$ is estimated by OLS and $d \in (0.5, 1.5)$, $\delta \in (0, d)$.

Assumption 2 Hassler et al. (2006) choose

$m \sim AT^a$, $l \sim BT^b$, $0 < b < a < 1$, $0 < A, B < \infty$

Assumption 3 The (pseudo) spectral density $f_{zz}(\lambda)$ of z_t, $z \in \{x, u\}$ $(d_x = d,\ d_u = \delta)$ satisfies, $0 < \gamma \leq 2$, $0 < G_z < \infty$

$$f_{zz}(\lambda) = G_z \lambda^{-2d_z}(1 + O(|\lambda|^{\gamma})) \quad \text{as } \lambda \to 0$$

and is differentiable within $(0, \varepsilon)$ of the origin with

$$\left| \frac{d}{d\lambda} f_{zz}(\lambda) \right| = O(|\lambda|^{-1-2d_z}) \quad \text{as } \lambda \to 0$$

This assumption will hold, under certain conditions, for fractional series with $f(\lambda) = (2\sin(\lambda/2))^{-2d_z} f^{*}(\lambda)$ and ARFIMA models with $\gamma = 2$.

Assumption 4 $\max\{0, (1 - d - \delta)/(d - \delta - 0.5)\} < b < a < 1, \quad d - \delta > 0.5$

Assumption 5 $b < a < 2\gamma/(1 + 2\gamma)$

Theorem Given assumptions 1,2,3 and 4 for Gaussian u_t and x_t, with $0 \leq \delta < 0.5$, $\delta < d - 0.5 < 1$, as $T \to \infty$ then

$$\log T \left(\hat{\delta}(\hat{u}) - \delta \right) \to_p 0$$

Moreover, given assumption 5 then

$$m^{1/2} \left(\hat{\delta}(\hat{u}) - \delta \right) \to_d N\left(0, \frac{\pi}{24}\right)$$

However, it should be noted that when $d + \delta < 1$, the slower convergence rate of $\hat{\beta}$ in assumption 1 provide regions (i.e. $0.5 < d < 0.72$) where no choices of m and l can be found to show $\sqrt{m}$ limiting normality of $\hat{\delta}$.