Optimal Tax-Timing and Asset Allocation when Tax Rebates on Capital Losses are Limited*

Marcel Marekwica†

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Abstract

Since Constantinides (1983) it is well known that in a market where capital losses qualify for tax rebates indefinitely it is optimal to realize losses immediately. However, the US-tax code as well as many other tax-codes around the world restricts tax rebates on capital losses. This paper shows that Constantinides' result generalizes to markets in which capital loss deduction is limited. Nevertheless, in such markets an investor who is not "locked-in"will hold substantially less risky assets than in markets with unlimited capital loss deduction. The less capital losses qualify for tax rebates, the lower the equity exposure.

JEL Classification Codes: G11, H21, H24

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†Johann Wolfgang Goethe University, Department of Finance, Postfach 111932 (Uni-Pf. 58), D-60054 Frankfurt am Main, Germany, Phone: ++49-798-25130, Fax: ++49-798-25228, Email: marekwica@finance.uni-frankfurt.de

1 Introduction

The tax rules private investors are facing are a potentially important factor influencing the household portfolio structure. Even though pre-tax returns are usually reported in newspapers, on television or on the internet, only after-tax returns should have an impact on investment decisions of private investors as only these returns have an impact on consumption.

In many countries around the world capital gains are taxable at a constant capital gains tax-rate. On the other hand capital losses only qualify for tax rebates it they do not exceed a certain upper bound (limited capital loss deduction). Under current US-tax code this upper bound is \$ 3,000. Both these limitations and the opportunity to defer the taxation of capital gains have an impact on the risk-return-profile of an asset and thus have an impact on the investor's asset allocation decision.

According to the seminal work of Constantinides (1983), it is optimal to realize losses immediately if capital loss deduction is unlimited and if "wash-sale rules" do not apply.[1] This paper generalizes his result to tax-systems in which capital loss deduction is limited and the remaining losses can be carried over indefinitely as a loss carryforward. It shows that in such tax-systems Constantinides' result to realize a loss immediately (and potentially building up a loss carryforward) remains an optimal tax-timing strategy. It further shows that compared to tax-systems with unlimited capital loss deduction investors usually cannot attain the same level of wealth as in tax-systems with limited capital loss deduction. This is due to the fact that in systems with unlimited capital loss deduction the investor effectively receives some tax-refund in cash which he can invest to earn some returns on the tax rebate. In a tax-system with limited capital loss deduction, however, he might end up with some loss carryforward remaining that does not pay any interest. This in turn implies that in tax-systems with limited capital loss deduction investors will hold less risky assets than in tax-systems with unlimited capital loss deduction.

The remainder of this paper is organized as follows. Section 2 reviews the related literature, section 3 introduces the model and shows that selling all assets facing a loss is an optimal tax-timing strategy in tax-systems with limited capital loss deduction as well. Section 4 analyzes the impact of capital gains taxation in tax-systems with limited and unlimited capital loss deduction and concludes that in tax systems with limited capital loss deduction investors will hold significantly less risky assets. Section 5 concludes.

[1] A transaction is termed a "wash sale" if a stock is sold to realize a capital loss and repurchased immediately. Under current US tax-rules "wash sales" do not qualify for the capital loss deduction if the same stock is repurchased within thirty days before or after the sale.

2 Related Studies

The taxation of capital gains has several impacts on the tax-timing decisions of private investors. First, it reduces the expected after-tax return which might lead some investors to decide not to invest their funds but to consume them. Second, investors having high unrealized capital gains in some assets might not want to sell them to avoid paying the capital gains tax and thus get "locked-in". This tax-saving behavior can result in portfolios that are not well diversified. Especially for older investors the disadvantages of badly diversified portfolios might be outweighed by the forgiveness of capital gains under current US tax-law when bequested. Third, the deferral of capital gains taxation results in compound returns on the postponed taxes and thus decreases the effective capital gains tax-rate (Chay, Choi, and Pontiff (2006)). Fourth, if capital loss deductions are limited, the risk-return profile of the assets changes in a disattracting manner compared to the case with unlimited capital loss deduction which results in a lower exposure to the risky asset. According to Poterba (1987) such loss-offset constraints are binding to about twenty percent of US taxpayers.

Many assets face both profits from capital gains and dividends which are taxed at different rates. These two types of profits differ in two ways. On the one hand dividends are taxable the year they are obtained while capital gains are taxable the year when the asset is sold and the gains are realized. In case of a bequest capital gains are even entirely forgiven under current US tax-law. On the other hand dividends are usually subject to a higher tax-rate than capital gains.

On the assumption that the returns of each asset are subject to the same tax-rate Auerbach and King (1983) show that an optimal portfolio is a weighted average of a market portfolio and a portfolio that is chosen on the basis of tax considerations ignoring risk. Thus, expected changes in the capital gains tax-rate result in vast realizations of capital gains (Auerbach (1988)).

While according to Shefrin and Statman (1985), Odean (1998), Barber and Odean (2000), and Barber and Odean (2003) investors tend to hold assets incurring losses too long and tend to sell assets with unrealized capital gains too early which clearly violates efficient tax-planning, the study by Jin (2006) finds selling decisions by institutions serving tax-sensitive clients to be sensitive to cumulative capital gains. Barclay, Pearson, and Weisbach (1998) show that fund managers manage tax liabilities to attract new investors. Bergstresser and Poterba (2002) find fund flows to be sensitive to tax burdens. This finding suggests that private investors might be more subject to the disposition effect than institutional investors. Ivkovich, Poterba, and Weisbenner (2005) compare the realizations of capital gains in taxable and tax-deferred accounts and find a strong "lock-in" effect for capital gains in the former type of accounts. This finding in turn suggests that private investors might ride efficient tax-timing strategies. On the other hand the empirical evidence in Seyhun and Skinner (1994) suggests that the $ 3,000 limit on capital loss deductions represents an important constraint on tax-reduction strategies and investors tend to follow very simple tax-timing strategies like realizing losses early and postponing gains. However, in their study, about 90% of the investors seem to follow simple buy-and-hold strategies. Other studies on the relation between taxation and investor behavior include Blouin, Radey, and Shackelford (2003) and the survey of Poterba (2002).

Under current US tax laws it is not allowed to short a security in which one has a long position to avoid realizing capital gains. Investors realizing such a "shorting-the-box-strategy" are treated as if they had sold the long position and hence their capital gains are taxed. Gallmeyer, Kaniel, and Tompaidis (2006) address this issue and propose tax-management strategies to circumvent the capital gains tax. Their so-called trading flexibility strategy minimizes future tax-induced trading costs by shorting one of several stocks even if none of the stocks has an unrealized gain. This strategy is in particular useful if the benefits from holding a well-diversified portfolio are outweighed by the expected future rebalancing costs. This is in particular the case if two assets are highly correlated. To circumvent the "shorting-the-box-strategy" Stiglitz (1983) suggests selling (or shorting if necessary) highly correlated assets instead of realizing capital gains. Nevertheless this shorting-strategy can be subject to significant costs that have to be taken into account.

Dammon, Spatt, and Zhang (2001) on the other hand show that selling an asset with an unrealized capital gain can be an optimal tax-timing strategy. According to their study the diversification benefit of reducing a volatile position can significantly outweighed the tax cost of selling the asset with an unrealized capital gain. The results of Dammon, Dunn, and Spatt (1989) suggest that the value of the option to realize long-term gains to regain the opportunity of realizing short-term losses is negatively related to the stocks price volatility.

Constantinides (1984) shows that if short-term capital gains are taxed at a higher tax-rate than long-term capital gains it can be optimal to sell assets with an unrealized capital gain as soon as they qualify for long-term treatment in order to regain the opportunity of producing short-term losses. Dammon and Spatt (1996) extend his approach by allowing the number of trading periods before a short term position becomes a long term position to be greater than one. In particular they show that contrary to intuition, it can be optimal to defer small short-term losses even in the absence of transaction costs. This finding is due to the fact that realizing these losses and repurchasing the asset restarts the short-term holding period and thus the time the investor has to wait until potential future gains qualify for long-term treatment. Under plausible parameter values, they find that it can be optimal for investors to defer realizing short-term capital losses of about 10% in the absence of transaction costs. Constantinides and Scholes (1980) argue, that even when an investor sells an asset with an unrealized capital gain, he can defer this gain by hedging.

Nevertheless, all of these studies assume that capital loss deduction is unlimited. This paper takes the limited capital loss deduction explicitly into account, generalizes the finding of Constantinides (1983) that losses should be realized immediately to a tax-system with limited capital loss deduction in section 3 and shows the impact of such limitations on optimal asset allocation in section 4.

3 Analytical Results

Following Constantinides (1983), a market is considered, in which investors are price takers and only trade at equilibrium prices, there are no transaction costs and the tax-system allows for "wash-sales". Unrealized gains remain untaxed, while realized capital gains reduced by realized capital losses are taxable.

The difference between a realized capital gain and a realized capital loss in period k is called a net capital gain G_k if $G_k \geq 0$ or a net capital loss if $G_k < 0$. A net capital gain is taxable at tax-rate τ. As in Constantinides (1983) it is assumed that there is only one capital gains tax-rate and it is not distinguished between long-term and short-term capital gains. In contrast to Constantinides it is assumed that capital losses only quality for a tax-rebate up to a certain amount. For a net capital loss an investor only receives a tax rebate in period k for that part of the net capital loss G_k not exceeding a maximum amount of $M_k \geq 0$ dollars in period k in absolute value. This rebate is paid at the end of the period, when the investor's net capital gain is known. Even though in tax-law the maximum loss deduction M_k is usually a constant and thus independent from the time index k, the theoretical analysis with time dependent M_k allows for a deeper understanding of the problem. The net capital gain (or loss) T_k in period k that is subject to the capital gains tax is given by

$$T_k := \max\left(G_k + L_{k-1}; -M_k\right) \tag{1}$$

Realized net losses that extend the amount of M_k can be indefinitely carried forward to the following periods. Thus, the loss L_k that can be carried forward from period k to period $k+1$

is given by

$$L_k := \min\left(G_k - T_k + L_{k-1}; 0\right) \tag{2}$$

In a one-period model the evolution of an investor's wealth in a tax-system with limited capital loss deduction is thus equal to that of an investor in a tax-system with unlimited capital loss deduction who has sold a put-option at time zero with strike $P_1 - P_0 - M_1$ and participation rate of τ without receiving a premium for that put-option. However, in a tax-system with limited capital loss deduction the investor receives the loss carryforward according to Equation (2) as a kind of compensation when the put-option is in the money at time of maturity.

The Case with One Risky Asset

In the following the investment decision of an investor at time t is considered who has a loss carryforward of L_{t-1}. He has the opportunity to invest into a risk-free asset that pays an after-tax return of $r > 0$ per period. Let $R := 1 + r$ denote the gross after-tax return of the risk-free asset. Furthermore, the investor has the opportunity to invest into a risky asset. He wants to hold one unit of the risky asset until time $T > t$ that he attains alive with probability one. Let P_t denote the price of the risky asset at time t and assume the return of this asset only consists of capital gains, i.e. it does not pay any dividend or interest. Let P_t^* denote the purchase price of that asset at the end of period t. Thus, P_{t-1}^* is the purchase price before trading. Let $P_t' \in [\inf_{i\in[t,t+1)} P_i, \sup_{i\in[t,t+1)} P_i]$ be some price in time $[t, t+1)$. For simplicity it is assumed that the investor holds one single unit of the risky asset which is infinitely divisible for the entire investment horizon. In case $P_t' < P_{t-1}^*$ the investor could realize a net capital loss in period t. If that loss does not exceed M_t, that is $-\left(P_t' - P_{t-1}^*\right) \leq M_t$ the classical result of Constantinides (1983) applies and the investor should sell the asset to realize that loss and earn the tax-rebate on it.

If, however, the net capital loss exceeds M_t, i.e. $P_t' - P_{t-1}^* < -M_t$, the preconditions under which his result holds, are no longer full-filled. In the following it is shown that it remains optimal to sell the asset even though the investor does not necessarily attain the same wealth-level as in a tax-system with unlimited capital loss deduction.

The Relation between Wealth, Unrealized Gains and the Loss Carryforward

In tax-systems with limited capital loss deduction, an optimal asset allocation decision depends on the investor's total wealth before trading, his unrealized capital gains, his loss carryforward and the length of the remaining investment horizon. The key in understanding optimal tax-timing in such a tax-system is to understand the relation between the first three factors.

A loss carryforward L_{t-1} of one dollar in period t can be used in two ways. First, it can be subtracted from a realized capital gain to reduce capital gains taxes. Second, in the absence of a realized capital gain, the loss carryforward can be claimed as a net capital loss that is subject to the capital gains tax if $M_t > 0$ according to Equation (1). Thus, one dollar of loss carryforward can be shifted to τ dollars of wealth if $M_t \geq \tau$. Shifting the loss carryforward to wealth is a dominating strategy, as each dollar of loss carryforward can reduce future tax burden by not more than τ dollars. Furthermore, in contrast with the loss carryforward, the τ dollars of tax rebate can be reinvested. By investing the tax rebate in the risk-free asset, its value is always at least as high as the future tax burden of the unrealized capital gain.

Thus, if two investment strategies result in the same unrealized capital gains at some point in time t before trading, but one of them results in a higher pre tax wealth W_t before trading and the other in a higher loss carryforward L_{t-1}, the strategy with the higher pre tax wealth is at least as good as the strategy with the higher loss carryforward, if for every τ extra dollars of wealth W_t, the second strategy does not have more than one dollar of extra loss carryforward. Let $A \succeq B$ denote the fact that "A is at least as good as B", than this finding can also be expressed as

$$\begin{pmatrix} W_t = \tau \\ \\ L_{t-1} = 0 \end{pmatrix} \succeq \begin{pmatrix} W_t = 0 \\ \\ L_{t-1} = 1 \end{pmatrix} \tag{3}$$

An investor endowed with one dollar of unrealized capital gains U_t at the beginning of period t before trading and one dollar of loss carryforward can use his loss carryforward in two ways. It can either be used to realize the capital gain or it can be used to generate a net capital loss at time t and thus to earn a tax rebate of τ dollars if $M_t \geq \tau$. As shown above, the value of the tax rebate is at least as high as the future tax burden of the unrealized capital gain when invested in the risk-free asset. Realizing the net capital loss thus is a dominating strategy.

An investor who is neither endowed with that dollar of unrealized capital gain nor that dollar of loss carryforward can be considered an investor who has realized that capital gain and used his loss carryforward to avoid the capital gains tax. He then lacks the dominating opportunity of realizing the net capital loss and keeping the unrealized capital gain. Hence:

$$\begin{pmatrix} U_t = 1 \\ \\ L_{t-1} = 1 \end{pmatrix} \succeq \begin{pmatrix} U_t = 0 \\ \\ L_{t-1} = 0 \end{pmatrix} \tag{4}$$

The relation between wealth and unrealized capital gains follows from the relation between wealth and losses and the relation between unrealized gains and losses. If two investment strategies result in a loss carryforward of zero at some point in time t before trading, but the first of them results in a higher pre tax wealth and in higher capital gains than the other, the first strategy is at least as good as the second strategy, if for ever τ extra dollars of pre tax wealth, the unrealized capital gains of the first strategy does not extend one dollar. Then this is due to the fact that according to Equations (3) and (4) it holds that

$$\begin{aligned} & \begin{pmatrix} W_t = 0 \\ U_t = 1 \\ L_{t-1} = 1 \end{pmatrix} + \begin{pmatrix} W_t = \tau \\ U_t = 0 \\ L_{t-1} = 0 \end{pmatrix} \succeq \begin{pmatrix} W_t = 0 \\ U_t = 0 \\ L_{t-1} = 0 \end{pmatrix} + \begin{pmatrix} W_t = 0 \\ U_t = 0 \\ L_{t-1} = 1 \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} W_t = \tau \\ U_t = 1 \\ L_{t-1} = 0 \end{pmatrix} \succeq \begin{pmatrix} W_t = 0 \\ U_t = 0 \\ L_{t-1} = 0 \end{pmatrix} \end{aligned} \tag{5}$$

The economic intuition for this result is that each dollar of unrealized capital gains results in a tax burden of τ dollars. Furthermore, the τ dollars of wealth allow for earning the risk-free interest rate in forthcoming periods.

These three findings on the relation between wealth, unrealized capital gains and the loss carryforward have a nice intuitive interpretation. Wealth allows for earning the risk-free rate in forthcoming periods. This is why wealth is preferred to a loss carryforward that does not earn any interest. Furthermore, a loss carryforward can be converted into wealth easier than a lower unrealized capital gain. Thus, a loss carryforward is closer to being converted into wealth than a lower unrealized capital gain. Hence, one can expect a loss carryforward to convert to wealth earlier than the lower unrealized capital gain and thus, the loss carryforward allows earning the risk-free interest rate earlier than the lower unrealized capital gain. This explains why wealth is preferred to a loss carryforward, which in turn is preferred to lower unrealized capital gains.

The Optimal Tax-Timing Strategy

To derive the optimal tax-timing strategy of an investor who holds one unit of the risky asset plus some risk-free bonds at time t, it suffices to analyze the case of an investor who holds only one unit of the risky asset. Even more, it suffices to analyze three tax-timing strategies in period t as all other strategies are linear combinations of these three strategies. First, the investor can sell the risky asset to realize the unrealized net capital loss, and immediately repurchase one unit of the risky asset (strategy one). Second, the investor can hold the asset and do no transactions in period t (strategy two). Third, the investor can sell that much of the risky asset that he realizes the maximum loss that can be offset in period t, and repurchase that much of the risky asset as he has sold (strategy three). In case that his loss carryforward L_{t-1} exceeds the limit on the loss deduction, he does not even have to sell any assets to realize the desired capital loss and strategies two and three coincide.

All other tax-timing strategies are linear combinations of these three strategies. Any strategy selling some fraction of the risky asset which is more than that of strategy three, but less that that of strategy one results in a portfolio and a loss carryforward that is a linear combination of those of strategy one and three. Accordingly any strategy selling some fraction of the risky asset which is less that that of strategy three, but more that that of strategy two results in a portfolio and a loss carryforward that is a linear combination of those of strategy two and three. To prove that strategy one is an optimal tax-timing strategy, it thus suffices to show that strategy one does at least as good as strategies two and three.

The three strategies only differ in their purchase prices of the risky asset, the loss carryforward and the investor's wealth after trading in period t. When the investor follows strategy one and sells the risky asset he realizes a net capital loss of $P'_t - P^*_{t-1}$ and increases his purchase price to $P^*_t = P'_t$. As $P'_t - P^*_{t-1} < -M_t \Rightarrow P'_t - P^*_{t-1} + L_{t-1} < -M_t$ his taxable net capital loss is

$$T^{(1)}_t = \max\left(P'_t - P^*_{t-1} + L_{t-1}; -M_t\right) = -M_t \tag{6}$$

Thus, his tax refund is $M_t\tau$ dollars. His remaining loss carryforward is given by

$$L^{(1)}_t = \min\left(P'_t - P^*_{t-1} + M_t + L_{t-1}; 0\right) = P'_t - P^*_{t-1} + M_t + L_{t-1} \tag{7}$$

If the investor follows strategy two and does not do any transactions in period t, his purchase

Table 1: **Comparison of tax-timing strategies**

	strategy 1	strategies 2,3a	strategy 3b
P_t^*	P_t	P_{t-1}^*	mixed
L_t	$P_t' - P_{t-1}^* + M_t + L_{t-1}$	$\min\left(L_{t-1} - \max\left(L_{t-1}; -M_t\right); 0\right)$	0
W_{t+1}	$P_{t+1} + M_t\tau$	$P_{t+1} - \max\left(L_{t-1}; -M_t\right)\tau$	$P_{t+1} + M_t\tau$
U_{t+1}	$P_{t+1} - P_t'$	$P_{t+1} - P_{t-1}^*$	$P_{t+1} - P_{t-1}^* + M_t + L_{t-1}$

price remains at $P_t^* = P_{t-1}^*$, his net capital gain is

$$T_t^{(2)} = \max\left(0 + L_{t-1}; -M_t\right) = \max\left(L_{t-1}; -M_t\right) \tag{8}$$

Thus, his tax refund is $\max\left(L_{t-1}; -M_t\right)\tau$. His remaining loss carryforward is

$$L_t^{(2)} = \min\left(0 - \max\left(L_{t-1}; -M_t\right) + L_{t-1}; 0\right) = \min\left(L_{t-1} - \max\left(L_{t-1}; -M_t\right); 0\right) \tag{9}$$

If the investor follows strategy three, he chooses his investment strategy such that his net capital loss is given by

$$T_t^{(3)} = -M_t \tag{10}$$

and thus his tax refund under strategy three is $M_t\tau$. His remaining loss carryforward is

$$L_t^{(3)} = 0 \tag{11}$$

Let $W_t^{(i)}$ denote the pre tax wealth in period t of strategy i ($i \in \{1, 2, 3\}$) before trading. Then

$$W_{t+1}^{(1)} = P_{t+1} + M_t\tau \tag{12}$$

$$W_{t+1}^{(2)} = P_{t+1} - \max\left(L_{t-1}; -M_t\right)\tau \tag{13}$$

$$W_{t+1}^{(3)} = P_{t+1} + M_t\tau \tag{14}$$

If the investor follows tax-timing strategy three two cases have to be distinguished. First, if $\max\left(L_{t-1}; -M_t\right) = -M_t$, then the loss carryforward L_{t-1} from period $t-1$ suffices to realize the desired net capital loss in period t. This case will be referred to as strategy three, case a), or just three a). In this case the investor does not have to do any transactions. Thus, if $\max\left(L_{t-1}; -M_t\right) = -M_t$ strategies two and three coincide.

Second, if $L_{t-1} > -M_t$, the investor still has to sell some of his risky assets. The amount of the risky assets he has to sell is equivalent to a fraction f of the risky asset, such that $-M_t = f\left(P_t' - P_{t-1}^*\right) + L_{t-1} \Leftrightarrow f = \frac{-M_t - L_{t-1}}{P_t' - P_{t-1}^*}$. This case will be referred to as strategy three, case b), or just three b).

Let $U_t^{(i)}$ denote the unrealized capital gains (or losses) in period t of strategy i ($i \in \{1, 2, 3\}$) before trading. Then

$$U_{t+1}^{(1)} = P_{t+1} - P_t \tag{15}$$

$$U_{t+1}^{(2)} = U_{t+1}^{(3a)} = P_{t+1} - P_{t-1}^* \tag{16}$$

$$\begin{aligned} U_{t+1}^{(3b)} &= P_{t+1} - \left(f P_t' + (1-f)\, P_{t-1}^*\right) \\ &= P_{t+1} - \left(\frac{-L_{t-1} - M_t}{P_t' - P_{t-1}^*} P_t' - \left(1 + \frac{-L_{t-1} - M_t}{P_t' - P_{t-1}^*}\right) P_{t-1}^*\right) \\ &= P_{t+1} - \left(\frac{-L_{t-1} - M_t}{P_t' - P_{t-1}^*} \left(P_t' - P_{t-1}^*\right) - P_{t-1}^*\right) \\ &= P_{t+1} - P_{t-1}^* + L_{t-1} + M_t \end{aligned} \tag{17}$$

Table 1 summarizes the properties of the three strategies.

As for $\max\left(L_{t-1}; -M_t\right) = -M_t$ strategies two and three result in the same investment behavior in period t, strategies two and three a) do not differ. To show that strategy one is the dominating tax-timing strategy it thus suffices to show that strategy one dominates strategy two and strategy three b). With Equation (4) it holds for the relation between strategies one and three b) that

$$\begin{pmatrix} W_{t+1}^{(1)} \\ U_{t+1}^{(1)} \\ L_t^{(1)} \end{pmatrix} = \begin{pmatrix} P_{t+1} + M_t \tau \\ P_{t+1} - P_t' \\ P_t' - P_{t-1}^* + M_t + L_{t-1} \end{pmatrix} \succeq \begin{pmatrix} P_{t+1} + M_t \tau \\ P_{t+1} - P_{t-1}^* + M_t + L_{t-1} \\ 0 \end{pmatrix} = \begin{pmatrix} W_{t+1}^{(3b)} \\ U_{t+1}^{(3b)} \\ L_t^{(3b)} \end{pmatrix} \tag{18}$$

Thus, strategy one is at least as good as strategy three b). This is due to the higher flexibility of the loss carryforward that can be easier transferred to wealth and then earn the risk-free interest rate. With Equation (3) it holds for the relation between strategies two and one that

$$\begin{aligned}\begin{pmatrix} W_{t+1}^{(2)} \\ U_{t+1}^{(2)} \\ L_t^{(2)} \end{pmatrix} &= \begin{pmatrix} P_{t+1} - \max\left(L_{t-1}; -M_t\right)\tau \\ P_{t+1} - P_{t-1}^{*} \\ \min\left(L_{t-1} - \max\left(L_{t-1}; -M_t\right); 0\right) \end{pmatrix} \\ &\preceq \begin{pmatrix} P_{t+1} + M_t\tau \\ P_{t+1} - P_{t-1}^{*} \\ \min\left(L_{t-1} - \max\left(L_{t-1}; -M_t\right); 0\right) + M_t + \max\left(L_{t-1}; -M_t\right) \end{pmatrix}\end{aligned} \tag{19}$$

For both $L_{t-1} \geq -M_t$ and $L_{t-1} < -M_t$ it holds with $P_t' - P_{t-1}^{*} < 0$ and Equation (4) that

$$\begin{pmatrix} W_{t+1}^{(2)} \\ U_{t+1}^{(2)} \\ L_t^{(2)} \end{pmatrix} \preceq \begin{pmatrix} P_{t+1} + M_t\tau \\ P_{t+1} - P_{t-1}^{*} \\ L_{t-1} + M_t \end{pmatrix} \preceq \begin{pmatrix} P_{t+1} +' M_t\tau \\ P_{t+1} - P_t' \\ P_t' - P_{t-1}^{*} + M_t + L_{t-1} \end{pmatrix} = \begin{pmatrix} W_{t+1}^{(1)} \\ U_{t+1}^{(1)} \\ L_t^{(1)} \end{pmatrix} \tag{20}$$

Thus, strategy one is at least as good as strategy two, which shows that independent from the realization of future prices of equity and the relation of the maximum loss deduction M_t and the initial loss carryforward L_{t-1}, strategy one always does at least as good as strategies two and three. Furthermore, strategy one sometimes results in higher wealth than strategy two, by allowing to earn the risk-free interest rate in future periods on the loss carryforward converted to wealth. Hence, strategy one is an optimal tax-timing strategy and unrealized losses should be realized immediately.

Even more, if $P_t' \neq \inf_{i\in[t,t+1)} P_i$, the investor can still increase his realized loss in time $[t, t+1)$ by trading whenever the price of the asset is below his purchase price. In this case the above results with $P_t' = \inf_{i\in[t,t+1)} P_i$ apply.

In the proof it has thus far been assumed, that the risky asset does not pay any dividend or interest. If, however, the risky asset does pay some dividend or interest, both strategies are affected from these payments in the same way as in both tax-systems the investor holds one unit of the risky asset and hence receives the same amount of dividend or interest. Thus, an asset paying dividends affects investors in tax-systems with limited and unlimited capital loss deduction the same way. Hence, the results derived above also hold for risky assets whose returns contain of both capital gains and dividend or interest payments.

The Case with Multiple Risky Assets

In case the investor only holds one risky asset, it is optimal for the investor to realize losses at time t immediately in order to earn the interest on the tax rebate if $M_t > 0$. In case that $M_t = 0$, there is no tax rebate and the investor can only use a loss carryforward to reduce future realized capital gains. Thus, for the one-asset case when $M_t = 0$, both strategies one and three and any combination of them is optimal as there are no tax rebates that allow for earning a return.

In the one-asset case the investor never faces the situation in which one of his assets faces a capital gain and another faces a capital loss. Thus, in this case, a net capital loss that exceeds the amount of M_t can only be carried forward. If however the investor holds more than one risky asset he can use a loss carryforward L_{t-1} realized in some asset S_1 in period t to reduce his net capital gains from some other asset S_2 realized in period $k > t$ if he wants to reallocate his portfolio. As in contrast to strategy one, strategy three does not allow for this transfer of realized losses of some asset S_1 to some other asset S_2, strategy one dominates strategy three in the multiple-asset case. To make strategy one preferable to strategy three it is not even necessary to assume that the investor holds more than one risky asset in period t. It suffices if he holds more than one risky asset in some period k ($t < k \leq T$) with positive probability.

Attainable Wealth

As shown in Constantinides (1983) and above in both a tax system with limited and unlimited capital loss deduction it is optimal to realize losses immediately. Let $W_t^{(l)}$ denote the beginning of period t wealth before trading an investor can attain in a tax-system with limited capital loss deduction following the optimal tax-timing strategy, i.e. $W_t^{(l)} := W_t^{(1)}$. Let furthermore $W_t^{(u)}$ denote the corresponding wealth he can attain in a tax-system with unlimited capital loss deduction following the optimal tax-timing strategy to realize losses in the period they occur. As there are not loss carryovers in tax-systems with unlimited capital loss deduction, it is assumed, that $L_{t-1} = 0$ in the tax-system with limited capital loss deduction to make the two tax-systems comparable. In case an investor realizes a capital loss in period t that does not exceed M_t the evolution of his wealth from t to $t + 1$ is the same in both tax systems. If however the capital loss exceeds M_t, i.e. $P'_t - P^*_{t-1} < -M_t$, then

$$W^{(l)}_{t+1} = P_{t+1} + M_t \tau \tag{21}$$

$$U^{(l)}_{t+1} = P_{t+1} - P_t \tag{22}$$

$$L^{(l)}_t = P'_t - P^*_{t-1} + M_t \tag{23}$$

and

$$W^{(u)}_{t+1} = P_{t+1} - \left(P'_t - P^*_{t-1}\right) \tau \tag{24}$$

$$U^{(u)}_{t+1} = P_{t+1} - P'_t \tag{25}$$

$$L^{(u)}_t = 0 \tag{26}$$

With Equation (3) it holds that

$$\begin{pmatrix} W^{(u)}_{t+1} \\ U^{(u)}_{t+1} \\ L^{(u)}_t \end{pmatrix} = \begin{pmatrix} P_{t+1} + M_t\tau - \left(P'_t - P^*_{t-1} + M_t\right)\tau \\ P_{t+1} - P'_t \\ 0 \end{pmatrix} \succeq \begin{pmatrix} P_{t+1} + M_t\tau \\ P_{t+1} - P'_t \\ P'_t - P^*_{t-1} + M_t \end{pmatrix} = \begin{pmatrix} W^{(l)}_{t+1} \\ U^{(l)}_{t+1} \\ L^{(l)}_t \end{pmatrix} \tag{27}$$

Thus, not very surprisingly, an investment opportunity set with unlimited capital loss deduction is preferable to an investment opportunity set with limited capital loss deduction. The advantage of the investment opportunity set with unlimited capital loss deduction is the opportunity to get an unlimited tax rebate on capital losses and earn the interest on these losses, while in a tax-system with limited capital loss deduction no interest is paid on the loss carryforward. Furthermore, one dollar of cash at hand can be used a lot more flexible, than one dollar of loss carryforward, especially when the limits on the maximum loss deductions $M_1, \ldots, M_T$ are small.

4 Capital Gains Taxation and Asset Allocation

This section analyzes the impact of capital gains taxation on asset allocation decisions. As above investments for a given investment horizon in the absence of exogenous increases or decreases of wealth like income or consumption are considered. If there was no tax on capital gains, the classical result of Merton (1969) and Samuelson (1969) would apply and the investor's asset allocation would be the same in each period. Thus, each deviation of his asset allocation from this benchmark must be due to the taxation of capital gains. This section shows, how taxation of capital gains affects the investor's asset allocation and in particular how limitation of capital loss deduction does.

In a tax-system with unlimited capital loss deduction, capital gains and capital losses are treated symmetrically. That is, no matter if a capital gain or a capital loss is realized, the investor is confronted with the same taxable treatment. In tax-systems with limited capital loss deduction, however, there is an asymmetric taxation of capital gains and losses. While capital gains are taxed at the capital gains tax-rate without any limits, capital losses qualify for tax rebates only up to a certain amount. Thus, the investor receives the fraction $1 - \tau$ of potential capital gains, but bears the entire risk for losses exceeding the maximum loss deduction when he has no loss carryforward. The compensation for this risk comes as a loss carryforward. However, in contrast to tax rebates, the investor cannot earn any interest on the loss carryforward.

Furthermore, he also bears the risk that he has no use for the entire amount of loss carryforward in forthcoming periods. This risk is especially important, if the remaining investment horizon is short. Thus, compared to a tax-system with unlimited capital loss deduction, an investment into a risky asset offers the same opportunities to the investor when returns are positive, but bears higher risks when returns are negative. Thus, in a tax-system with limited capital loss deduction, investors will hold less risky assets than in a tax-system with unlimited capital loss deduction when they are not endowed with a loss carryforward.

The size of the advantage that results from the opportunity to invest in a tax-system with unlimited capital loss deduction instead of a tax-system with limited capital loss deduction depends on five factors. First, it depends on $M_1, \dots, M_T$, the amounts up to which realized losses qualify for tax rebates. The higher these values the lower the advantage of the tax-system with unlimited capital loss deduction. Second, it depends on the capital gains tax-rate τ. The higher τ, the higher the tax rebates and thus the more advantageous the tax-system with unlimited capital loss deduction. Third, it depends on the evolution of the price of the risky asset, $P_1, \dots, P_T$. The earlier and the higher capital losses that exceed $M_1, \dots, M_T$, the bigger the advantage of the opportunity to invest in a tax-system with unlimited capital loss deduction. Thus, the more volatile the risky asset, the bigger the disadvantage of being confronted with a tax-system with limited capital loss deduction. Hence, in tax-systems with limited capital loss, investors will decrease their holdings in risky assets the stronger, the more volatile they are. Fourth, the lower the risk-aversion of an investor the higher his exposure to risky assets. Thus the higher his disadvantage when being confronted with a tax-system with limited capital loss deduction. Fifth, the higher the risk-free rate the higher the disadvantage of the loss carryforward not paying any interest and thus, the higher the advantage of the tax-system with unlimited capital loss deduction.

If, however, the investor is endowed with a loss carryforward, this has a positive impact on his risk-return profile. In this case, an investment into a risky asset is more preferable than without a loss carryforward as it allows the investor to earn an amount of capital gains not exceeding the loss carryforward tax-free. Thus, if the investor is endowed with some loss carryforward, there is no longer a dominating relation between the two tax-systems from the investor's point of view. In the tax-system with limited capital loss deduction, on the one hand he bears the risk that the treasury does not participate in high losses via tax rebates. On the other hand, he has the opportunity of earning some capital gains tax-free.

Numerical Evidence

Following Dammon, Spatt, and Zhang (2001) it was assumed, that the investor can only trade at time $0, 1, \dots, T$ and that the purchase price used to compute capital gains is the average weighted historical purchase price. If P_t^* denotes the tax basis of a risky asset (also referred to as equity in the following) after trading at time t, then this tax basis is defined by

$$P_t^* = \begin{cases} \frac{q_{t-1}P_{t-1}^* + \max(q_t - q_{t-1}, 0)P_t}{q_{t-1} + \max(q_t - q_{t-1}, 0)} & \text{if } P_{t-1}^* < P_t \\ P_t & \text{if } P_{t-1}^* \geq P_t \end{cases}. \tag{28}$$

This specification takes the fact into account, that in it is optimal to realize capital losses (i.e. $P_{t-1}^* \geq P_t$) immediately, which decreases the average purchase price from P_{t-1}^* to P_t. If, however, the investor is endowed with an unrealized capital gain (i.e. $P_{t-1}^* < P_t$) the change in his tax-basis depends on his trading in period t. If the investor sells some assets (i.e. $q_t < q_{t-1}$), his tax basis remains unchanged. If instead, the investor buys some assets (i.e. $q_t > q_{t-1}$) his tax basis is a weighted average of the previous tax basis and the purchase price P_t of the new asset. As according to the optimal tax-timing strategy losses shall be realized immediately, the investor's realized capital gains in period t is given by

$$G_t = \left(1_{P_{t-1}^* > P_t} q_{t-1} + 1_{P_t \geq P_{t-1}^*} \max\left(q_{t-1} - q_t, 0\right)\right) \cdot \left(P_t - P_{t-1}^*\right) \tag{29}$$

In the tax-system with unlimited capital loss deduction, the optimization problem is the same as in Dammon, Spatt, and Zhang (2001). In the tax-system with limited capital loss deduction, the investor's optimizing problem is

$$\max_{q_t} \mathbb{E}\left[U\left[W_T'\right]\right] \tag{30}$$

s.t.

$$W_t = q_{t-1} \cdot (1+d_t)\, P_t + b_{t-1} \cdot R, \qquad t = 0, \ldots, T \tag{31}$$

$$W_t = \tau T_t + q_t \cdot P_t + b_t \qquad t = 0, \ldots, T-1 \tag{32}$$

$$W_T' = W_T - \tau T_T \tag{33}$$

$$q_t \geq 0 \qquad t = 0, \ldots, T-1 \tag{34}$$

$$q_T = 0 \tag{35}$$

where d_t is the after-tax dividend of equity, given the initial holding of bonds b_{-1}, stocks q_{-1}, the initial tax-basis P^*_{-1}, the price of one unit of the stock P_0 and his initial loss carryforward L_{-1}. According to Equation (30) the investor maximized his utility he derives from terminal wealth after taxes. To have the classical result of Merton (1969) and Samuelson (1969) as a benchmark, the investor is assumed to have no bequest motive.

Equation (31) defines the investor's beginning of period t wealth as the sum of his wealth in stocks and his wealth in bonds before trading at time t, including the after-tax interest and dividend income, but before any capital gains taxes resulting from trading at time t.

Equation (32) is the investor's budget constraint at time t. If the investor trades equity he possibly has to pay some capital gains tax or gets some tax-refund on his net capital gain T_t as defined in Equation (1).

By letting X_t denote the vector of the investor's state variables at time t, the Bellmann equation for the maximization problem can be written as follows:

$$V_t(X_t) = \max_{q_t} \mathbb{E}_t\left[V_{t+1}\left(X_{t+1}\right)\right] \tag{36}$$

for $t = 0, \ldots, T-1$ subject to Equations (1), (2), (28), (29) and (31) to (35). The state variables needed to solve that problem at time t are the investors beginning-of-period-wealth W_t before trading, his initial loss carryforward L_{t-1}, his tax basis P^*_{t-1} and the number of stocks q_{t-1} he is holding at the beginning of period t before trading. Thus, his vector of state variables can be represented as

$$X_t = \left[P_t, W_t, L_{t-1}, P^*_{t-1}, q_{t-1}\right] \tag{37}$$

For the numerical analysis it is assumed, that M_t is a negative multiple of W_t, i.e. $m_t := \frac{M_t}{W_t}$ is some negative real value and constant in time, i.e. $m := m_t \, (t \leq T)$. This assumption, which is made to reduce the number of state-variables, is in contrast with the US-tax code, that allows a constant amount up to \$ 3,000 to qualify for a tax-rebate. In this case, the impact of the limitation of capital loss deduction depends on the absolute wealth level as well. However, this relation can also be captured by varying m. Nevertheless, except for $M = 0$, m cannot capture the impact of an increase in total wealth on the relation between M_t and W_t. In tax-systems where $M_t > 0$ is constant, investors should c.p. hold a lower proportion of equity with increasing wealth as otherwise they run a higher risk to end up with capital losses exceeding M_t.

With the assumption, that m is a constant, the above optimization problem can be simplified by normalizing with beginning-of-period-wealth W_t. Let $s_t := \frac{q_{t-1} P_t}{W_t}$ denote the fraction of the investor's beginning-of-period-wealth before trading invested into equity, $\alpha_t := \frac{q_t P_t}{W_t}$ the investor's fraction of beginning-of-period-wealth allocated to equity after trading, $b'_t := \frac{b_t}{W_t}$ the fraction of the beginning-of-period-wealth allocated to risk-free bonds after trading, $p^*_{t-1} := \frac{P^*_{t-1}}{P_t}$ the investors basis-price ratio, $t_t := \frac{T_t}{W_t}$ the fraction of the investor's beginning-of-period-wealth that is taxable at the capital gains tax-rate, $l_{t-1} := \frac{L_{t-1}}{W_t}$ the fraction of the investors loss carryforward to his beginning-of-period-wealth, $g_t = \frac{P_{t+1}}{P_t} - 1$ the capital gain on the stock in period t, and

$$R_t := \frac{\alpha_t \left(1 + d_t\right) \left(1 + g_t\right) + b'_t R}{\alpha_t + b'_t}, \tag{38}$$

the gross nominal return on the investor's portfolio after trading in period t after payment of taxes on dividends and interest, but before payment of capital gains taxes. Assuming the investor to have CRRA-utility with $\gamma \neq 1$

$$U\left[W_T\right] = \frac{W_t^{1-\gamma}}{1-\gamma} \tag{39}$$

with parameter of risk-aversion $\gamma \geq 0$, and defining $v_t(x_t) = \frac{V_t(X_t)}{W_t^{1-\gamma}}$ and $w_{t+1} = \frac{W_{t+1}}{W_t}$, the investor's optimization problem can be rewritten as

$$v_t\left(x_t\right)=\max_{\alpha_t}\mathbb{E}\left[v_{t+1}\left(x_{t+1}\right)w_{t+1}^{1-\gamma}\right] \tag{40}$$

s.t.

$$1=\tau t_t+\alpha_t+b_t' \qquad t=0,\ldots,T-1 \tag{41}$$

$$w_{t+1}=\left(1-\tau t_t\right)R_t \qquad t=0,\ldots,T-1 \tag{42}$$

$$\alpha_t\geq 0 \qquad t=0,\ldots,T-1 \tag{43}$$

in which the fraction of realized gains to beginning-of-period-wealth is given by

$$\delta_t:=\frac{G_t}{W_t}=\left(1_{p_{t-1}^*>1}s_t+1_{p_{t-1}^*\leq 1}\max\left(\alpha_t-s_t,0\right)\right)\cdot\left(1-p_{t-1}^*\right) \tag{44}$$

and p_t^* is given by

$$p_t^*=\begin{cases}\frac{s_t p_{t-1}^*+\max(\alpha_t-s_t;0)}{(s_t+\max(\alpha_t-s_t;0))(g_t+1)} & p_{t-1}^*<1\\ \frac{1}{g_t+1} & \text{otherwise}\end{cases} \tag{45}$$

At time T, the investor's value function takes the value

$$v_T=\left(1-\tau t_T\right)^{1-\gamma}. \tag{46}$$

This problem can be solved numerically using backward induction with state variables $x_t=[s_t,p_{t-1}^*,l_{t-1}]$. To do so, a $(101\times 100\times 31)$ grid is spanned. The optimization problem with unlimited capital loss deduction with state variables $x_t=[s_t,p_{t-1}^*]$ is solved with a 101×100 grid. For values between the grid, cubic spline interpolation is performed. To expedite computation, the spline function for each of the two problems is computed symbolically for each period t.
For the numerical analysis, it is assumed, that the risk-free rate is 6%, the return of equity is binomially distributed, comes with an expected capital gain of 7%, a standard deviation of 20% and a constant dividend rate of 2%. The tax-rate on interest and dividends is assumed to be 36%. The tax-rate on realized capital gains is assumed to be 20%. This choice of parameters follows Dammon, Spatt, and Zhang (2001) and Gallmeyer, Kaniel, and Tompaidis (2006). The length of the investment horizon is $T=10$ years. These parameter values are referred to as the base case scenario. For the base-case it is assumed, that the relative risk-aversion of the investor is $\gamma=3$. In the absence of the tax-timing option, i.e. for an asset whose returns are

reduced by 20% but no longer subject to the capital gains tax-rate, but whose volatility is kept constant, the optimal fraction of stocks is 20.1% in each period. All deviations from this value result from the tax-timing option as well as the limitation of capital loss deduction.

Limited versus Unlimited Capital Loss Deduction

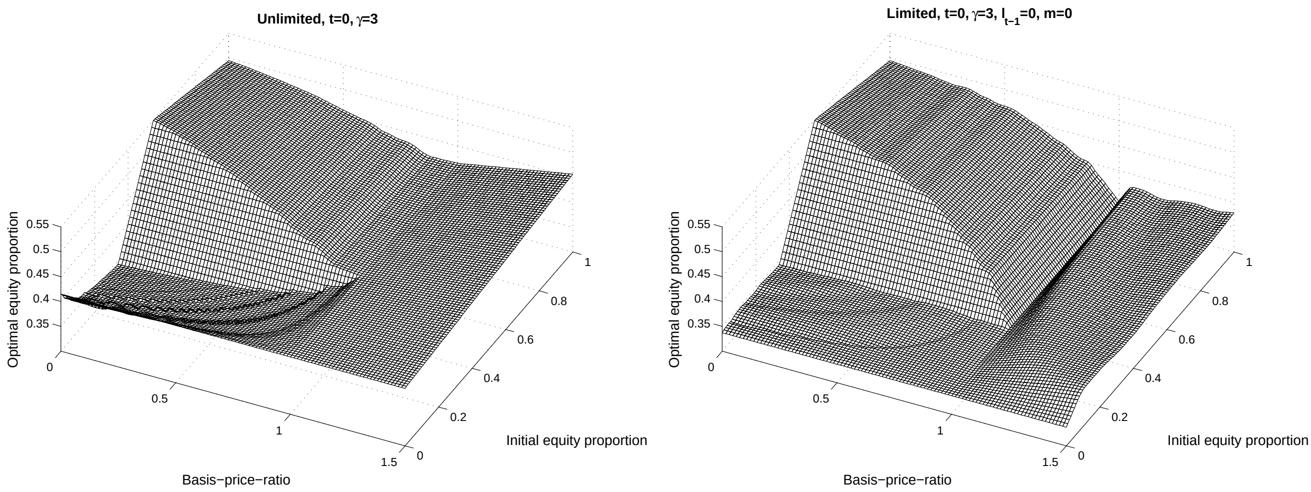

Figure 1: This figure shows the optimal equity proportion after trading α_t of an investor with risk-aversion of $\gamma = 3$ in a tax-system with unlimited capital loss deduction (left graph) and in a tax-system with limited capital loss deduction (right graph), when the investor is endowed with no initial loss carryforward ($l_{t-1} = 0$) and there is no tax rebate on capital losses ($m = 0$) as a function of the investor's initial equity exposure s_t before trading at time t and the basis-price-ratio p^*_{t-1} when the investment horizon is 10 years.

The left graph in Figure 1 shows the optimal asset allocation after trading α_t of an investor with risk-aversion $\gamma = 3$ in a tax-system with unlimited capital loss deduction. If the investor is neither endowed with an unrealized capital gain nor a loss (i.e. $p^*_{t-1} = 1$), the investor holds 41.5% of his beginning-of-period wealth in equity which is significantly more than in the benchmark case without capital gains taxes, which reflects the value of the tax-timing option. In particular, his equity proportion after trading does not depend on his fraction of stocks relative to beginning-of-period-wealth s_t, as for $p^*_{t-1} = 1$ the investor does not have to pay any capital gains taxes when rearranging his asset allocation. If the investor is endowed with a capital loss (i.e. $p^*_{t-1} > 1$), he receives some tax rebate on this capital loss, which increases his wealth after trading. With increasing beginning-of-period-wealth after trading and a constant fraction of stocks relative to that wealth, the fraction of stocks after trading relative to beginning-of-period wealth α_t increases. The higher the capital loss and the higher the fraction of stocks before trading s_t the higher the tax rebate and thus the higher the fraction of stocks relative to beginning-of-period-wealth before trading α_t.

If, however, the investor is facing an unrealized capital gain (i.e. $p^*_{t-1} < 1$), he has to decide, whether to realize that gain or to postpone it. Concerning this decision, there are two opposing effects. On the one hand, postponing the realization of the capital gain allows for earning compound returns on the taxes that have been postponed. On the other hand, postponing the realization of the capital gains can result in unbalanced portfolios. Especially if some asset has been performing well in the past, its fraction relative to the investor's total wealth can be substantial and so is its impact on the return of the portfolio. However, selling this asset to rebalance the portfolio would result in a substantial capital gains tax.

In case the investor is endowed with a low fraction of stocks before trading s_t, the investor's fraction of stocks after trading is decreasing in his fraction of stocks before trading s_t and increasing in his basis-price-ratio p^*_{t-1}. Even though his total fraction of stocks after trading is higher than his fraction of stocks before trading, it is not as high as if the investor was not facing a capital gain (i.e. $p^*_{t-1} = 1$). This is due to the reason, that the new basis-price-ratio p^*_t is defined as a weighted average of the beginning-of-period basis-price-ratio p^*_{t-1} and one. The lower p^*_{t-1} and the higher s_t, the lower p^*_t and thus the higher the risk of getting "locked-in" in with a significant amount in future periods.

With increasing fraction of stocks before trading, the investor sometime attains a level of stocks before trading s_t that exceeds his desired fraction of stocks after trading α_t. As long as his desired fraction of stocks after trading does not deviate too far from the fraction of stocks after trading α_t he has to hold, to avoid paying the capital gains tax, the rebalancing motive is outweighed by the opportunity to earn compound returns on the unrealized capital gains. In case the investor is endowed with an even higher fraction of stocks before trading s_t, the rebalancing motive outweighs the impact of the postponing of capital gains, and the investor's wealth after trading is reduced by the capital gains taxes, which is why his fraction of stocks after trading relative to beginning-of-period-wealth is the lower, the higher his fraction of stocks before trading.

The right graph in Figure 1 shows the optimal asset allocation of an investor with risk-aversion $\gamma = 3$ and no initial loss carryforward (i.e. $l_{t-1} = 0$) in a tax-system with limited capital loss deduction where realized capital losses can only be carried forward and deducted from future realized capital gains, but do not qualify for tax rebates (i.e. $m = 0$). Such a taxable treatment of realized capital losses is e.g. implemented in the German tax-code.

If the investor is neither endowed with an unrealized capital gain nor with an unrealized capital loss (i.e. $p^*_{t-1} = 1$), his equity exposure is about 33.7%, which is significantly above the benchmark case without the tax-timing option, but well below the 41.5% in the tax-system with unlimited capital loss deduction. This is due to the fact, that on the one hand in the tax-system with limited capital loss deduction, the investor bears the entire risk when having net capital losses, as he does not get any tax rebates on them. On the other hand, capital gains are treated the same way as in a tax-system with unlimited capital loss deduction. Hence, equity is less attractive in the tax-system with limited capital loss deduction which is why the investor holds a substantially lower fraction of his wealth in equity.

If the investor is facing an unrealized capital loss (i.e. $p^*_{t-1} > 1$), it is optimal to realize that loss immediately as shown above. The investor is then endowed with a loss carryforward which is the higher, the higher the net capital loss p^*_{t-1} and the higher the equity proportion before trading s_t. As this loss carryforward allows for earning some future capital gains tax-free, the investor chooses a higher equity proportion after trading compared to the case of $p^*_{t-1} = 1$. While in the tax-system with unlimited capital loss deduction the investor receives the loss carryforward in cash as a tax rebate, in the tax-system with limited capital loss deduction, it only has a value to the investor, if it can be subtracted from forthcoming capital gains. This is why in the tax-system with limited capital loss deduction the increase in the fraction of stocks is stronger than in the tax-system with unlimited capital loss deduction but remains at a certain level with increasing realized losses for not ending up with a portfolio that is too heavily invested into equity.

If, however, the investor is facing an unrealized capital gain (i.e. $p^*_{t-1} < 1$) and his fraction of stocks before trading is low, his fraction of stocks after trading will be about the same as without the capital gain and his basis-price-ratio after trading will decline. Compared to the tax-system with unlimited capital loss deduction, the danger of getting "locked-in" with a higher amount of wealth seems to be neglectable. The reason for this is that in the tax-system with limited capital loss deduction, each capital gain in equity increases his future capital gains tax, while each loss in equity decreases his future capital gains tax. This decrease in the future capital gains tax results in future capital gains and capital losses being treated equally and thus has a higher value for the investor than a loss carryforward, as the loss carryforward carries the risk of potentially remaining unused. The numerical result suggests that the opportunity of reducing the embedded capital gain outweighs the risk of getting locked in.

If the investor is facing an unrealized capital gain and his fraction of stocks before trading is high, the rebalancing motive outweighs the opportunity of earning compound returns on the unrealized capital gains and the results in the tax-system do not differ from those in the tax-system with unlimited capital loss deduction.

In general, the fraction of equity is at least as high in the tax-system with unlimited capital loss deduction as in the tax-system with limited capital loss deduction. The reason for this is that equity is more attractive in the latter tax-system due to the tax rebate on realized capital losses.

The Impact of a Loss Carryforward and Tax Rebates

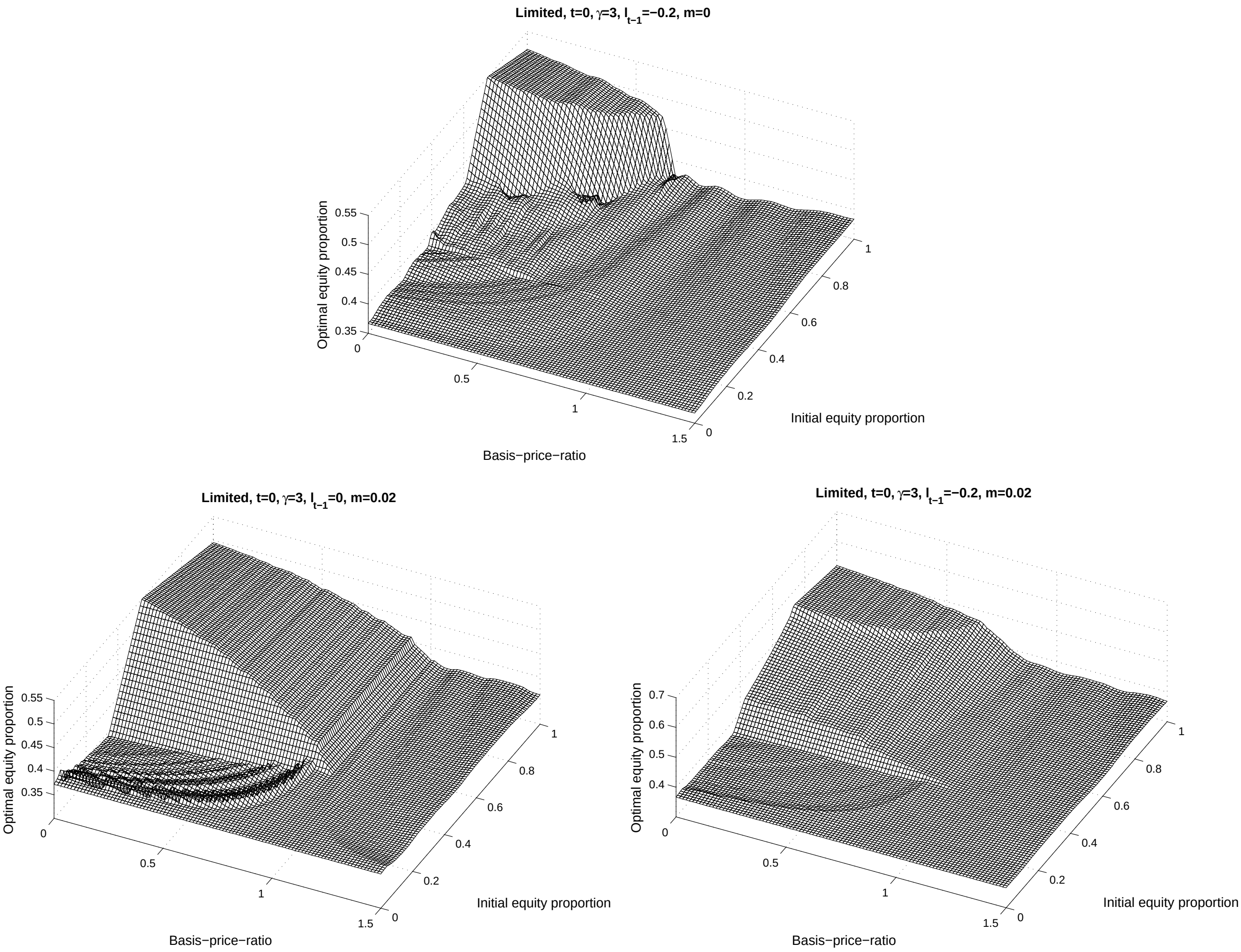

Figure 2: This figure shows the optimal equity proportion after trading α_t of an investor with risk-aversion of $\gamma = 3$ in a tax-system with limited capital loss deduction. The upper graph shows the asset allocation when the investor is endowed with an initial loss carryforward of $l_{t-1} = -0.2$ and there is no tax rebate on capital losses ($m = 0$) as a function of the investor's initial equity proportion s_t before trading at time t and the basis-price-ratio p^*_{t-1} when the investment horizon is 10 years. The lower graphs show his asset allocation if the tax system allows for a rebate on capital losses for up to 2% of present wealth ($m = 0.02$) for an investor with no initial loss carryforward ($l_{t-1} = 0$, left graph) and an initial forward of $l_{t-1} = -0.2$ (right graph).

Figure 2 shows the impact of variations in the initial loss carryforward l_{t-1} and the maximum capital loss deduction m in a tax-system with limited capital loss deduction for an investment horizon of ten years. In the upper graph the investor is assumed to have an initial loss carryforward of 20% of his current wealth (i.e. $l_{t-1} = -0.2$) and the tax-system is assumed not to provide any tax rebates on capital losses (i.e. $m = 0$). Compared to the case in which the investor has no loss carryforward (right graph in Figure 1) the investor increases his equity proportion to 36.7% when he is neither endowed with an unrealized capital gain nor an unrealized capital loss (i.e. $p^*_{t-1} = 1$). This is due to the fact that due to the loss carryforward the investor can realize some capital gains in equity without having to pay the capital gains tax. Thus, the risk-return profile in that asset becomes more attractive for him, which is why he increases his exposure to it. With decreasing basis-price-ratio p^*_{t-1} and increasing fraction of stocks before trading s_t, the investor realizes more of his unrealized capital gains than without a loss carryforward to get a portfolio that is better diversified. However, if his capital gains exceed his loss carryforward, similar to the case without an initial loss carryforward, the investor does not realize that much of his capital gains as to come up with the same fraction of stocks after trading as without such a high unrealized capital gain.

In the lower left graph of Figure 2, the optimal asset allocation of an investor with no initial loss carryforward (i.e. $l_{t-1} = 0$) is shown who is trading in a tax-system that allows for tax rebates on capital losses of up to 2% of the investor's beginning-of-period wealth (i.e. $m = 0.02$). If such an investor is neither endowed with an unrealized capital gain nor an unrealized capital loss (i.e. $p^*_{t-1} = 1$), his fraction of stocks after trading is about 36.6%, which is well above the level in a tax-system with no tax rebates (i.e. $m = 0$) and significantly below the level in a tax-system with unlimited capital loss deduction.

If the investor is endowed with a capital loss, his exposure to equity is slightly decreasing as soon as the capital loss exceeds the maximum amount qualifying for a tax rebate. This is due to the reason that the investor wants to avoid increasing his loss carryforward even more. If, however, his net capital loss is of significant height, the risk of not using this loss carryforward outweighs the risk of increasing it even further, which is why in this case; his optimal equity exposure is increasing again. If the investor is endowed with an unrealized capital gain (i.e. $p^*_{t-1} < 1$) and his fraction of stocks before trading is small, the investor increases his equity exposure. If his unrealized capital gain is small, he increases his equity exposure even above the level when not being endowed with a capital gain to decrease p^*_t that much, that he regains the opportunity of getting tax rebates on potential future capital losses. If, however, his unrealized capital gains are large, he decreases his equity exposure even below the fraction of stocks for $p^*_{t-1} = 1$ to reduce to risk of getting "locked-in" with a higher amount of wealth. If the investor faces a significant capital loss (i.e. $p^*_{t-1} > 1$) and ends up with a loss carryforward, he does not change his equity exposure after trading α_t significantly compared to the case of $p^*_{t-1} = 1$. Due to the tax rebate on the loss, his beginning-of-period-wealth after trading increases slightly. Thus, his fraction of stocks to beginning-of-period-wealth after trading decreases slightly. The reason for this decline is the risk of building up a loss carryforward which is that large that the investor can potentially not use it until the end of the investment horizon.

In the lower right graph of Figure 2, the optimal asset allocation of an investor with an initial loss carryforward of 20% of his initial wealth (i.e. $l_{t-1} = -0.2$) is shown who is trading in a tax-system that allows for tax rebates on capital losses up to 2% of the investor's beginning-of-period wealth (i.e. $m = 0.02$). If the investor is neither facing an unrealized capital gain nor an unrealized capital loss (i.e. $p^*_{t-1} = 1$), his fraction of stocks after trading is about 37.2%, which is slightly above the level in a tax-system with no tax rebates, and slightly above the level of an investor with no loss carryforward, and significantly below the level in a tax-system with unlimited capital loss deduction. Compared to the case with no loss carryforward (i.e. $l_{t-1} = 0$) the investor has the opportunity to sell some of his assets with an embedded capital gain without having to pay the capital gains tax. This is why the incline in the fraction of stocks after trading α_t of an investor with an embedded capital gain is a lot smoother than for an investor who does not have a loss carryforward. However, if the desired equity exposure only deviates slightly from the equity exposure the investor has accept a higher equity exposure to keep his loss carryforward for future periods. Compared to the case with no tax rebate on capital losses, the investor realizes a higher fraction of his embedded capital gains to regain the opportunity of getting tax rebates on potential future capital losses. If the investor is facing a capital loss (i.e. $p^*_{t-1} > 1$), he has about the same equity exposure after trading α_t as an investor without an unrealized capital gain or loss.

The Impact of Risk-Aversion and the Remaining Investment Horizon

The numerical results presented so far were computed for an investor with coefficient of risk-aversion $\gamma = 3$. Even though the absolute exposure of the investor to equity changes with varying coefficients of risk-aversion (not shown here), effects evoked by the capital gains tax are the same is the same in both tax-systems with limited an unlimited capital loss deduction. Furthermore, the effects of the tax-effects remain the same for varying investment horizons, but become more pronounced in their order of magnitude (not shown here). In particular, when the remaining investment horizon is very short and the tax-system does only allow for limited capital loss deduction, the investor seeks to avoid generating net capital losses in the forthcoming period when he does not have significant unrealized capital gains for not ending up with a loss carryforward he does not have any use for.

The paper has only considered total final after-tax wealth at the end of the given investment horizon in tax-systems with limited capital loss deduction, but not the remaining loss carryforward. If the end of the investment horizon is not the end of the investor's participation in capital markets, this remaining loss carryforward still has some value for the investor not exceeding $\tau \cdot L_T$. Assigning a value of $x \cdot L_T$ with $x \leq \tau$ to that value, increases the investor's equity exposure after trading.

5 Conclusion

In his seminal 1983 paper Constandinides shows that in a tax-system that does not permit "wash-sales" and allows unlimited capital loss deduction, and in a market without transaction costs it is always optimal for an investor to realize capital losses immediately. This paper generalizes his finding to the case of tax-systems with limited capital loss deduction. Due to the asymmetric taxable treatment of capital gains and losses investors tend to hold substantially less equity in tax-systems with limited than in tax-systems with unlimited capital loss deduction.

Inevitably, this paper neglects many important issues. Instead of considering that impact of an absolute bound M on capital loss deductions, it considered a bound $\frac{M}{W_t}$ relative to current beginning-of-period-wealth and thus did not take the impact of the absolute level of wealth on tax-timing into account. Furthermore, it did not take a bequest motive, different tax-rates on long-term and short-term capital gains and losses, transaction costs, more than one risky stock, inflation, consumption or labor income and potential shocks associated with it into account. Interesting avenues for further research are to include these factors into the model and computing tax-rates that allow investors to end up with the same level of utility in a tax-system with limited capital loss deduction for given wealth and tax-rates in a tax-system with unlimited capital loss deduction.

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