Deciding what and when to seed: Mean reverting process and Real Options

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Abstract

This paper analyzes the components of the market price for an agricultural land. After using several processes to match the data generating process for the different crop prices, we incorporate a real option on the land to model the irreversibility on the potential selling decision of the land owner. After this, we model the farmer's decision to rotate the crop. Our empirical results show that the value of the put option on the land is non trivial while the one on the crop rotation is negligible.

Keywords: valuation, agricultural land, put option, Monte Carlo simulation, irreversibility

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1 Introduction and literature review

This paper tries to answer the question: how much one should pay for a piece of agricultural land? As the land's value should be closely tied to the market price of the products that can be obtained from the land, our ultimate goal is trying to improve the benchmark to price agricultural land.

What is the problem in valuing agricultural land? Several factors should be taken into account to perform an accurate valuation:

1. The first problem is the characterization of the process followed by the asset. As we just said, the value of the asset, that is the piece of land, depends on the price of the crops that you seed. Those prices usually have a quite particular feature, since they are supposed to follow some kind of mean reversion process.

2. The second characteristic is that there is some benefit or premium from holding the underlying product, whether is it soya, corn, wheat, or other, rather than any contract for the product. This implies that we should include the convenience yield as another factor of the process for the price of the different crops.

3. The third interesting feature is that, to keep the quality of the land, it is of utmost importance to rotate the crops. In our analysis we will contemplate two different cases:

   (a) The first alternative will allocate a percentage to each of the crops and rotate it every year in an ad hoc way. For instance, assuming we have three possible crops (soya, corn and wheat), we would allocate 33 percent to each and rotate them on a triangular basis.

   (b) The alternative approach would be to look for the optimal way to decide those percentages, depending on the existing prices.

   So, if the price of one of the crops is too high (in comparison with its historical mean), you may wish to seed only this crop for as long as the difference persists. But, as seeding consecutively the same crop damages severely the quality of the soil, this cannot be done for a long period. Therefore, there could be some "optimal" value for waiting until the appropriate time comes to seed a particular crop.

4. There is some value from owning the land, other than the value of the land's products. The owner of the agricultural land has a certain "advantage" since it is usually very difficult to buy one at a "fair" price. While this shortage in agricultural land supply might be the result of some behavioral feature (like some emotional attachment to

the asset or some rational decision to leave a bequest to the future generation), the consequence on pricing agricultural lands could be important. This advantage can be modelled as a put option on the land, and the product generated by the land is the dividend or convenience yield that only the owner of the product has the possibility to enjoy.

Thus, we should include all these components in order to calculate accurately the value of the land.

There is a large literature examining the time path of commodities prices. It has been suggested that some sort of mean reverting process provides a better description of the price path for many commodities. As noted by Schwartz (1997), in an equilibrium setting we would expect that when prices are relatively high, supply will increase as the higher cost producers of the commodity will enter into the market putting downward pressure on prices. Conversely, when prices are relatively low, the higher cost producers will exit the market putting upward pressure on prices.

In that paper, Schwartz (1997) proposed a framework to test whether commodities prices were mean reverting and to obtain the value of the corresponding parameters. In particular, this author tested whether copper, oil and gold followed mean reverting processes using three different models. He found out that both cooper and oil were mean reverting but this was not the case with gold. He also used the discounted cash flow (DCF) criteria as well as the real options approach to capital budgeting for a project involving copper extraction from a mine. Whenever it was not take into account the mean reversion process, he found that the Discounted Cash Flow technique made investment decision too early (that is, when prices are too low) while the Real Options approach made that decision too late.

In other words, whenever we do not consider the mean reversion feature, we may conclude that an investment is worth enough just because prices are too high in comparison with the historical mean when using the DCF approach. On the other hand, on average, we will wait until prices are too high if we use the RO approach.

Thus, it is quite important to be sure that the crops indeed follow a mean reverting process and estimate the parameters of the process. The three processes suggested by Schwartz (1997) are the following:

1. A one-factor model where the commodity spot price follows an Ornstein-Uhlenbeck type of mean reversion.
2. A two-factor model that extends the previous one considering as a second factor the

convenience yield, modelled as a mean-reverting process that can be correlated with the spot price.

3. The third model is an extension of the second one by allowing for stochastic interest rates.

The three models are analytically tractable and imply a linear relationship between the logarithm of futures prices and the underlying factor.

Unfortunately there are some problems involving the O-U kind of processes. One of the problems is that, in the long run, the expected price does not converge to the long-run price. Also, negative values are not necessarily ruled out. Finally, the process does not satisfy a very reasonable assumption, namely, the homogeneity condition. This condition states that, if the price of one unit of the good reverts to some mean value, the price of two units of the same good should revert to twice that same value.

Robel (2001) proposed the Inhomogeneous Geometric Brownian Motion (GBM) as a way to satisfy the homogeneity condition and to preclude the possibility of negative values almost sure.

Following this suggestion, this paper extends the two factor model of Schwartz (1997) incorporating the inhomogeneous geometric brownian motion of Robel (2001) for the convenience yield.

Once this has been done, the next task is to value the option to sell the land. Since once you sell the land is very difficult to acquire it back, i.e. it has a large cost, the decision is in some sense irreversible. Then, we should incorporate this feature into the price of the land, in order to value it. Finally, we should also price the options for seeding the different crops optimally. Since the underlying asset is a real asset, these are not financial options but real options.

The real option framework was introduced in the finance literature by Brennan and Schwartz (1985) to evaluate the decision of extracting minerals. In this case, the uncertainty of the mineral prices was the source of value from waiting, whereas the excavation was time consuming. The authors showed that the Net Present Value method would lead to a non-optimal extraction of minerals because there is a value from waiting that is lost once the excavation begins. As you can always stop the excavation, this example is very simple but presented a new way to evaluate projects and to deal with capital budgeting: the issue is that there is some value associated to the fact that once you own an asset you have the opportunity to delay any relaated project until solving the uncertainty.

The real option value is basically a premium over the expected net value of the project, reflecting the opportunity cost of investing now and foregoing the option to delay investment until more information about the future becomes available. Real Options usually involve capital investment and natural resources, but have also been applied to real estate development decisions. The rationale behind using real options in real estate is similar to what happens in financial options: there is uncertainty about future prices and there is some value in having the possibility of delay a decision until some of the uncertainty is resolved.

There are three characteristics that give value to a real option: irreversibility, uncertainty and timing. In the case we are interested, the owner of an agricultural land has uncertainty about the future price of the crop. Since once the land has been sold is quite complicated to buy it again at a negligible cost, there is some potential value in waiting for the optimal moment to sell. So just as in corporate investment, holding a farmland involves a real put option because the farmer has the right (but not the obligation) to keep the land for farming or sell it to another for the same or other use.[1]Exercising this options means that the owner is willing to sell his land and hence close the door to all future opportunities that might be brought by the land being maintained.

There is a second type of option given by the decision on which crop to seed. As mentioned before, the quality of the soil decreases quite fast if you seed the same crop consecutively. Then, seeding a certain kind of seed provides the option to seed a different one in a later stage.

So far, the literature of Agricultural Real Estate has priced the selling option by subtracting the DCF from the Market Value assuming that the difference reflects the value of the Real Option (see Isgin and Forster (2004). However, this procedure can lead to large errors. Since the Data Generating Process of the underlying asset is (by definition) unknown, without a more systematic way to compute the real option value of owning a land, the result will be of no interest. The explanation of this fact is that there are many factors (like, for instance, taxes or weather) that can not be included in the analysis without risking the numerical solution to the problem.

Additionally, to the best of our knowledge, the rotation of crops has not been studied deeply in the Agricultural Real Estate literature. So far, the literature has been interested only in timber's harvesting problem for rotation of the harvesting (see, for instance, Insley and Rollins (2005). While the problems share similar features, the presence of different prices for the different crops jointly with the possibility of introducing shocks in the production function might make the problem interesting enough to be studied separately.

[1]Among the different uses we could mention urban development, agropecuary uses and organic products

The remainder of the paper is organized as follows. Section 2 presents the different processes for the crop prices. Section 3 introduces the Crop Rotation Model. Section 4 includes the empirical methodology and the numerical results. Finally, Section 5 summarizes the main conclusions.

2 Stochastic models for the crop price

This section describes three different processes for the crop prices and compares them to the usual benchmark case, which is a geometric brownian motion, which is not mean reverting. Then we derive the stochastic partial differential equation of a portfolio holding an asset function of these processes. Later we will incorporate the possibility of more than one crop per land, with the possibility of correlation among the different processes.

We will assume that the land has a 30 years horizon of use and that, after this period, the land is totally depreciated.[2] In this part we will also assume that the land is a) seeded either fully with one crop forever or b) seeded in equal proportions of the different crops rotating them every period in an "ad hoc" way.

2.1 Benchmark process

We will start the analysis with the benchmark process, that is, a process without mean reversion and a constant convenience yield, which will be a Geometric Brownian Motion:

$$dX_t = (r - c)X_t dt + \sigma X_t dW_t \tag{1}$$

where X_t is the value of one unit of the crop, r is the instantaneous risk-free interest rate, c is the convenience yield, σ is the volatility of the crop return and dW_t is a standard Brownian process.

Then, the discounted cash flow of owning a land that has been seeded with this kind of crop is given by

$$NPV_0 = DCF(X_t) - K = X_t \sum_{j=1}^{30} e^{-cj} - C \sum_{j=1}^{30} e^{-rj} - K \tag{2}$$

where C indicates the production costs (assumed to be constant) and K is the strike price of the option to sell the land. This strike price can be interpreted as the opportunity cost of

[2]This assumption might sound quite strong but is done only to assign some final day to the project. Additionally, if we would add more years, the relative weight of this additional period would be very low. But, without loss of generality, we could certainly drop this assumption and value a perpetuity instead.

owing the land. Additionally, as we will assume that, ex ante, there should be no economic profits, this strike price will be set as the one that makes the NPV equal to zero

Equation (2) can be rewritten as

$$DCF(X_t) = X_t\beta_1 - \beta_2$$

where

$$\beta_1 = \sum_{j=1}^{30} e^{-cj}, \quad \beta_2 = C\sum_{j=1}^{30} e^{-rj} + K$$

Now, at the beginning of each period, the land owner has to make a decision regarding whether to sell or not the land. If the strike price is higher than the DCF, then the farmer will sell the land. On the other hand, if the strike price is lower than the DCF, then the farmer retains the land one more period. Since the realization of the new prices could lead to a higher DCF, the option leads the farmer to a better decision making.

How to value then the option to sell the land? We will start assuming that the put option lasts for the entire life of the project. Additionally, we will assume that the option to sell the land can be exercised once every year. The motivation for this Bermuda feature is because the farmer only collects the crop once per year, so the uncertainty is solved at that moment.

Thus, the value of the land to the owner is $DCF(X_t) + V(X_t)$, where the term $V(X_t)$ denotes the value of the put option on the land.

There are several important issues related to this benchmark model:

- The above stochastic process for the spot price implies that the convenience yield is constant. Thus, the model is not able to capture changes in the term structure of future prices. In fact, the convenience yield changes through time.
- This model also implies that the volatility of all future returns is equal to the volatility of spot returns.
- Finally, this process assumes that the variance of the spot price grows linearly with time, whereas a general equilibrium model would imply some mean reversion in spot commodity prices.

2.2 One-factor process: The Crop Price follows an Inhomogeneous Geometric Brownian Motion

We first describe the one-factor model where the crop price follows an Inhomogeneous Brownian Motion. This process is given by the expression

$$dX_t = \lambda(\overline{X} - X_t)dt + \sigma X_t dW_t \tag{3}$$

where λ is the speed of mean reversion, $\overline{X}$ is the long-run crop price, σ is the instantaneous volatility and dW_t is a Wiener process. A full description of this process can be found in the Appendix A. It is worth noting that now the convenience yield is no longer constant as it depends on the (stochastic) crop price.

As before, the land value for the owner is the sum of the DCF plus the option to sell the land. The DCF is the same as in equation (2), except that the DCF is now a function of the price given in equation (3), adapted to the risk neutral version.

The price of the option for selling the land is given by the following stochastic differential equation

$$\frac{1}{2}\sigma^2 X_t^2 V_{XX} + [\lambda(\overline{X} - X_t) - \rho\sigma\phi X_t]V_X + V_t - rV = 0$$

subject to the boundary condition for the perpetual option[3] at the last period to exercise the option, that is[4]

$$\max\left[0, \frac{K}{T} - DCF_T\right]$$

2.3 Two-factor model in which the stochastic convenience yield follows an Ornstein-Uhlenbeck mean-reverting process

The convenience yield, δ, is included now as an additional second factor. In this case, the convenience yield is assumed to follow an Ornstein-Uhlenbeck process. So, similar to Schwartz (1997), for the joint stochastic process, we have the following:

$$\begin{aligned} dX_t &= (\overline{X} - \delta_t)X_t dt + \sigma_1 X_t dW_1 \\ d\delta_t &= \kappa(\alpha - \delta_t)dt + \sigma_2 dW_2 \end{aligned}$$

[3]Here perpetual means that the option can be exercised until the end of the asset's life.

[4]As said previously, the strike price is computed such that the opportunity cost for owning the land compensates the present value of the cash flows. The strike is computed at the starting day although, as time goes by, it is decreased in a constant amount (K/T) per period. In other words, at any time $t(< T)$, we consider a value of $K\left(1 - \frac{t}{T}\right)$.

where $dW_1 dW_2 = \rho dt$.

In this case, we allow for a stochastic convenience yield, instead of making it a function of the stochastic price. In short, we assume the convenience yield follows a Ornstein-Uhlenbeck stochastic process. To obtain the risk neutral process, we should remove the market price of risk, where here is called μ. After this, we get the following:

$$\begin{aligned} dX_t &= (r - \delta_t) X_t dt + \sigma_1 X_t dW_1^* \\ d\delta_t &= [\kappa(\alpha - \delta_t) - \mu] dt + \sigma_2 dW_2^* \end{aligned}$$

where $dW_1^* dW_2^* = \rho dt$.

2.4 Two-factor model with stochastic convenience yield following an IHGBM process

The last model to be used is a two factor model, where the convenience yield follows an Inhomogeneous Geometric Brownian Motion. Its expression is as follows:

$$\begin{aligned} dX_t &= (\overline{X} - \delta_t) X_t dt + \sigma_1 X_t dW_1 \\ d\delta_t &= \lambda(\overline{\lambda} - \delta_t) dt + \sigma_2 \delta_t dW_2 \end{aligned}$$

where $dW_1 dW_2 = \rho dt$.

Converting this process to a risk-neutral one, the parameter for the mean reversion changes too, so now we have that:

$$\begin{aligned} dX_t &= (r - \delta_t) X_t dt + \sigma_1 X_t dW_1^* \\ d\delta_t &= [\lambda^*(\overline{\delta} - \delta_t) - \mu] dt + \sigma_2 \delta_t dW_2^* \end{aligned}$$

where $dW_1^* dW_2^* = \rho dt$.

3 Crop Rotation Model

Now we will introduce the possibility of rotating crops. Here the farmer will decide optimally each year which crop should be seed. We will assume that the farmer seeds the whole land with only one crop per period, introducing the possibility of crop rotation each period.[5]

[5]This assumption can always be relaxed since we can think of a percentage of the land being seeded.

Unlike the previous case where we allocated a certain amount of land to each crop and rotate them ad hoc, now we have a path dependent option.[6] Here the land value depends on the today's crop value which depends on the volume, which depends on what was seed previously, because of the quality of the soil problem. Path dependency complicates the (numerical) solution to the problem.

In areas where rotation is fairly short, the seeding decision proscribed by this problem will be expected to be quite different from the previous dummy decision model.

The price of the commodities are assumed to follow a known stochastic process. The land value is estimated assuming that the seeding will be determined optimally in the future no matter what the price path turns out to be.

The crop age, that is, the number of periods that the same crop has been seeded, α, will be given as

$$\alpha = t - t_h \tag{4}$$

where t_h is the starting period from when the seed has been seeded consecutively. We will assume that the volume produced is a (continuous) decreasing function of the number of periods that the same seed is used consecutively, being zero a lower bound.[7] So the state variables for the dynamic problem are the crop age and the prices. Another possibility would be to make the quantity of crop obtained from the agricultural land, Q, dependent on the α-value of other crops such that the productivity of the soil is path dependent. Anyway, differentiating equation (4), we have that $d\alpha = dt$.

So the decision of the kind of crop to seed can be formulated as a dynamic problem, where the owner of the farm must decide in each period whether it would be better to stop with the kind of crop that is being used or continue with it. Calling $Q(\alpha)$ the volume obtained after α periods seeding the same crop consecutively, the Bellman equation for this situation would be:

$$\begin{aligned} & V(X_2, X_1, \alpha_1, \alpha_2) \\ = \;& \max[(X_1 - C)Q(\alpha_1) + (1 + r\Delta t)^{-1} E[V(X_1 + \Delta t, X_2 + \Delta t, \alpha_1 + \Delta\alpha, \alpha_2)] \\ & ; \quad (X_2 - C)Q(\alpha_2) + (1 + r\Delta t)^{-1} E[V(X_1 + \Delta t, X_2 + \Delta t, \alpha_2 + \Delta\alpha, \alpha_1)]] \end{aligned}$$

[6] A path dependent option is one whose value depends on the history of an underlying state variable and not just on its final value. Financial instruments with path-dependency features include, for instance, Asian options or barrier options.

[7] This is a shortcut, since actually it is not recommended to seed more than three periods the same crop, which means that the actual function should be discontinuous.

The first term in this equation represents the return, conditional on seeding the crop number 1, if continue seeding the current crop. It includes the net revenue from the current crop and the value of the land if tomorrow we would seed the same crop. The second expression is the value of the land if we seed a different crop.

From Itô's lemma, we have that the PDE describing V in the continuation region depends on the process followed by the crop price. For instance, when both crop prices follows an IHGBM process, we have the following equation

$$V_t + \frac{1}{2}\sigma^2 X^2 V_{XX} + \lambda(\overline{X} - X)V_X - rV + V_\alpha = 0$$

Unlike the usual PDE, we have here the term V_α as a state variable, which reflects the path dependency.

4 Empirical Estimation

4.1 Parameter estimation of the Commodity Prices

This section describes the procedure to estimate the parameters of the different processes. This estimation could be done by means of a simple OLS Regression or a SURE regression. The problem with the empirical implementation is basically that all the factors in the models we presented are not directly observable. For instance spot prices are basically non existing and, thus, we need to use futures contracts closest to maturity as a proxy for the spot prices.

The same problem appears with the convenience yield. In this case we will follow Gibson and Schwartz (1990) and will use the difference between two futures prices with different maturities to compute the convenience yield.

We will also need to adapt the models such that we can implement the corresponding numerical simulations. For such task we will rely on the Euler discretization, described as follows:

Assume that we face an stochastic differential equation of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

and that we wish to simulate values of X_T but do not know its distribution.[8] In particular, we simulate a discretized process, $[\widehat{X}_h, \widehat{X}_{2h}, \widehat{X}_{3h}, ..., \widehat{X}_{mh}]$, where m is the number of time steps, h is a constant and $mh = T$. The smaller the value of h, the closer our discretized

[8]Simulating an SDE means we simulate a discretized version of the SDE.

path will be to the continuous-time path we wish to simulate. The Euler scheme is intuitive and satisfies the following condition

$$\widehat{X}_{kh} = \widehat{X}_{(k-1)h} + \mu((k-1)h, \widehat{X}_{(k-1)h})h + \sigma((k-1)h, \widehat{X}_{(k-1)h})\sqrt{h}Z_k$$

where Z_k's are shocks distributed as i.i.d. standard normal random variables. With this type of discretization, we will estimate the parameters of each process from the data.

For each process, we will obtain different discretizations. Then, we have the following:

- For the process (1), the discretization would then be the following.[9] Working in logarithms, the exact equation is $X_t = X_{t-1}e^{((r-c)\Delta t + \sigma\sqrt{\Delta t}\epsilon)}$: where ϵ is a standard normal random variable. Then using logarithms and applying Itô's lemma we have that:

  $$\ln\left(\frac{X_t}{X_{t-1}}\right) = \left(r - c - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t$$

  Calling $\left(r - c - \frac{1}{2}\sigma^2\right)\Delta t \equiv a$ and $\sigma\sqrt{\Delta t}\epsilon_t \equiv e_t$, we now have the following expression

  $$\frac{X_t}{X_{t-1}} - 1 = a + e_t$$

  So, given a discretization of the time such that Δt is - for instance - one week, we will have the opportunity to exercisee the option once every 52 weeks because, as said before, the option has a Bermuda feature.

  Taking into account that Δt is one week and that $Var(e_t) = \sigma^2\Delta t$, we can recover all the parameters.

- A discretization for the Ornstein-Uhlenbeck process would look like the following:

  $$X_t - X_{t-1} = \lambda\mu\Delta t - \lambda X_{t-1}\Delta t + \sigma\sqrt{\Delta t}\epsilon_t$$

  Calling $a \equiv \lambda\mu\Delta t$, $b \equiv 1 - \lambda\Delta t$ and $e_t \equiv \sigma\sqrt{\Delta t}\epsilon_t$, we have the following regression

  $$X_t = a + bX_{t-1} + e_t$$

- Finally, a discretization of (3) will be

  $$X_t - X_{t-1} = \lambda\overline{X}\Delta t - \lambda X_{t-1}\Delta t + \sigma X_{t-1}\sqrt{\Delta t}\epsilon_t$$

  where, as usual, ϵ_t is distributed as a standard normal variable.

[9] We should be careful with the Δt chosen. If this is too large, then it is quite likely that the approximation would be a bad one.

Applying Itô's lemma to the usual log transformation gives us the following equation

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = -\left(\lambda - \frac{1}{2}\sigma^2\right)\Delta t + \frac{\lambda\overline{X}}{X_{t-1}}\Delta t + \sigma\sqrt{\Delta t}\epsilon_t$$

Calling $a \equiv -(\lambda - \frac{1}{2}\sigma^2)\Delta t$, $b \equiv \lambda\Delta t\overline{X}$, $e_t \equiv \sigma\sqrt{\Delta t}\epsilon_t$ and taking into account that Δt is one month, we can recover all the parameters from the regression:

$$\frac{X_t}{X_{t-1}} - 1 = a + b\frac{1}{X_{t-1}} + e_t$$

After having performed these regressions, we can use the estimated parameters to generate a large number of paths for the processes to get the expected values. For instance, to estimate $E[f(X_T)]$, we can use the following algorithm:

Set $t = 0;\widehat{X} = X_0$
for $j = 1$ to n
for $k = 1$ to $\frac{T}{h} = m$

generate $Z \sim N(0,1)$

set $\widehat{X} = \widehat{X} + \mu(t,\widehat{X})h + \sigma(t,\widehat{X})\sqrt{h}Z$
set $t = t + h$
end for
set $f_j = f(\widehat{X})$
end for

set $\widehat{\theta}_n = \frac{(f_1+...+f_n)}{n}$
set $\widehat{\sigma}_n^2 = \frac{\sum_{j=1}^{n}(f_j-\widehat{\theta}_n)^2}{n-1}$
set approximate $100(1-\alpha)$ $CI = \widehat{\theta}_n \pm z_{1-\frac{\alpha}{2}}\frac{\widehat{\theta}_n}{\sqrt{n}}$
The discretization error can be defined as $D = |E[f(X_T)] - E[f(\widehat{X}_T)]|$ and we have to ensure that is small enough, otherwise we will have very bad estimates of $E[f(\widehat{X}_T)]$.

For the multidimensional SDE, we will have vectors and matrixes. In this case, if there are n correlated Brownian motions driving the SDE's then at each time step, t_i, we must generate n i.i.d. standard normal random variables. We would use a Cholesky decomposition to generate $X_{t_{i+1}}$.

4.2 Numerical solution to the sell option and the optimal rotation scheme

The valuation of the put option on the land is harder than that of a simpler European option as the sell option is a Bermuda-type one. Thus, at each exercise time, the holder of this option must compare the profit for exercising the option against the profit if the option is exercised in a later stage (continuation value).

Longstaff and Schwartz (2001) proposed a method named Least-Squares Monte Carlo (LSM) to evaluate this continuation value using OLS regressions to provide a measure of the expected value one period later. The expected value is just the estimated value. As the process is iterated backwards in time starting at the expiration period, the right exercise moment can be obtained and, then, the option price can be computed. The regressions can be done through a linear combination of orthogonal basis such as, for instance, Chebichev polynomials.[10]

For the rotation problem, the question is a little bit more complicate. First of all, the Bellman equation involves too many states variables. Then, to find a numerical solution we should reduce the number of variables. For instance, as only relative prices matter, we could get rid of one state variables by expressing everything in terms of the price ratio or differences in prices. If we assume that the relative age of the crop is the relevant variable, we can get rid of another state variable. This assumption is not very strong since, if you have seed the same crop for too many periods, we will have to seed another one to let the soil recover from its exhaustive use. We could also set that, whenever we decide to change the crop, the variable *alpha* goes to zero.

With these changes, we will try to find a numerical solution to the dynamic programming problem. Since an explicit solution to the real option might be too complicated to be obtained, we will rely on numerical methods for its solution. In particular, we can use the Least Squares Monte Carlo technique proposed in Longstaff and Schwartz (2001) or the Crank Nicolson method for solving the second order PDE's that will be obtained.

We will assume that, whenever you decide to change the crop, you forgo the dividend or payoff in that period. Then, you will never exercise the option at time T and change to another crop. We will also assume that there is a constant rate of depreciation from seeding the same crop consecutively, as a way to match the decrease in the productivity of the soil. Therefore, the relevant backward decision process is the following:

[10]Moreno and Navas (2003) proved that the results do not depend on the chosen basis

- At time T, it is never optimal to exercise the option. Therefore, you get the market price of whatever was seeded in time $T-1$.
- At time $T-1$, we should compare $P_1Q(\alpha)+\beta E[(P_1+\Delta t)Q(\alpha+\Delta t)|I_t]$ against $\beta E[(P_2+\Delta t)Q(0)|I_t]$.

  If the first value is higher than the second one, we do not exercise the option at that period. Then, at time $T-2$, we have to compare $P_1Q(\alpha)+\beta E[(P_1+\Delta t)Q(\alpha+\Delta t)+\beta(P_1+2\Delta t)Q(\alpha+2\Delta t)|I_t]$ versus $\beta E[(P_2+\Delta t)Q(0)+\beta(P_2+2\Delta t)Q(\Delta t)|I_t]$ and so on.

Since it is pretty likely that, for a high α, there will not be a price path such that the option is not exercised, we will consider that the relevant period for the option is that when the option has some value.

Thus, the call option will be "*in the money*" whenever the termination value is higher than the continuation one.

4.3 Empirical Results

The crops to be considered in our empirical application are wheat, soya, and corn.

China is the most important producer of wheat. It produces 16 percent of the total amount. India and USA produce 12 percent each, and France, Australia, and Russia 6 percent each.

Soya is produced in many countries. USA produces almost 50 percent of the world soya output. Brazil has a market share of 18 percent and Argentina and China account for another 10 percent each. These are the countries where a bad or good weather season will have a significant impact on the soya price.

United States also produces a significant amount of the world output of corn sharing a 43 percent of the production. China, Brazil and Argentina produce 18 percent, 6 percent and 2 percent, respectively. Also, USA is the most important corn exporter, with 60 percent of the total exports.Additionally, Argentina, China and Brazil export 15 percent, 5 percent and 4 percent, respectively.

The following graphs show the soya, wheat and corn prices, convenience yield and volatility from 1996 to 2006. As we can see, all the prices seem to be strongly mean reverting. Also, the convenience yield for these commodities shows a mean-reverting pattern.

Since mean reversion implies stationarity, the OLS regression parameters should be consistent, provided that the model is the correct one.

[INSERT FIGURES 1 TO 3 ABOUT HERE]

The time series for the different prices is long enough to capture at least two changes in the economic cycle. One is in 1995/96 and the other one is after 2001. This guarantees enough variation in the price as we can see from the volatilities charts.

Additionally, the idea that economic slumps are followed by a drop in the price are confirmed by the charts, since the 1997 recession led to a significant drop in the price of the three different crops. Yet soya had a somehow different behavior. The super rates at which China grew sustained the demand for that crop, which is one of the main foods for the chinese.

The following tables include the main statistics for the processes

Moments for the prices

	Mean	St Dev.	Asymmetry	Kurtosis	Max	Min
Wheat	120.5474	17.1826	0.5426	-0.6430	166.43	90.84
Soya	178.4769	37.6238	1.6216	2.9923	336	120.5
Corn	89.1951	11.0365	1.105971	1.02849	130.11	68.8

Correlation matrix of the prices

	Wheat	Soya	Corn
Wheat			
Soya	0.7161		
Corn	0.713	0.7328	

The following tables present the results of the corresponding regressions.

Wheat Regressions Results			
	One Factor GBM Model		
constant	0.0000096221	alpha	0.036206733
Variance	0.000268111	sigma	0.259930656
	One Factor IHGBM Model		
constant	-0.007049087	lambda	1.742688948
beta	0.834525753	$\overline{X}$	120.6758613
variance	0.00026731	sigma	0.259542133
	Two Factor Model		
constant	4.05e-05	alpha	0.044033
beta	-7.31e-05	beta	1
variance	0.00026823	sigma	0.259988424
	OU Two Factor Model		
constant	0.03996306	kappa	4.957674307
beta	-0.095339891	alpha	0.419164102
variance	0.010134632	sigma	0.72594248
	Two Factor IHGBM Model		
constant	-0.035860816	lambda	1.8464762443
beta	-0.037236857	$\overline{\delta}$	-1.038371719
variance	1.571140913	sigma	9.038768029

Soya Regressions Results			
One Factor GBM Model			
constant	-0.00012676	alpha	0.002836971
Variance	0.00027603	sigma	0.23741353
One Factor IHGBM Model			
constant	-0.004133785	lambda	1.006955883
beta	0.473736433	$\overline{X}$	118.5569133
variance	0.000275856	sigma	0.263658411
Two Factor Model			
constant	6.98e-05	alpha	0.052194
beta	4.23e-03		
variance	0.000274632	sigma	0.263072565
OU Two Factor Model			
constant	0.133213014	kappa	6.927076706
beta	0.063412222	alpha	-0.476021228
variance	28.00300442	sigma	38.1596517

Corn Regressions Results			
One Factor GBM Model			
constant	-0.0000575879	alpha	0.013487287
Variance	0.000222218	sigma	0.236640844
One Factor IHGBM Model			
constant	-0.009167264	lambda	2.282271271
beta	0.801340193	$\overline{X}$	0.236132234
variance	0.000221264	sigma	0.236132234
Two Factor Model			
constant	-7.17e-04	alpha	-0.152770
beta	1.76e-03		
variance	0.000222087	sigma	0.236571381
OU Two Factor Model			
constant	0.37972871	kappa	4.92781662
beta	-0.094765704	alpha	0.400702676
variance	0.0011424663	sigma	0.70767469
Two Factor IHGBM Model			
constant	-0.18027616	lambda	0.374360337
beta	0.078834652	$\overline{\delta}$	0.437299373
variance	1.172393716	sigma	7.807974977

The regressions results are consistent with the idea of mean reversion. For instance, the estimated growth rate in the standard GBM is very low. Also, the O-U one factor model shows strong mean reversion for the three processes, as well as when we calibrated with two factors.

The IHGBM for two factors throws some wear results. This is mainly because the process discards negative values for the convenience yield, something that it is not reflected in the data.

Except for the soya process, all other coefficients are significant at 10 percent or less. Anyway, in order to have some idea of what the value would have been, we are going to use it as if it were statistically different from zero. Nevertheless we will have to discard the two-factor model with convenience yield following an IHGM because the process did not converge.

With the above parameters we simulated 10,000 price paths for each process. We then calculated the value of the discounted cash flow at an interest rate of 7 percent and we evaluated the Bermuda put option. The results are the following[11]:

	Wheat		*Soya*		*Corn*	
	K	Put	K	Put	K	Put
One-factor model						
GBM	3524	1537	2743	1153	1649	1482
IHGBM	1858	767	1854	750	1360	543
Two-factor model						
OU	391	92	16215	7541	220	49
IHGBM	19134	8942	NA	NA	197.36	40.46

As we can see, the sell option for the owner is by no means insignificant. The value of being the owner has some strategic content which is, at least, 20 percent of the total value of the land (see the corresponding strike price). Obviously, when the process is not necessarily mean reverting, as with the GBM, the put value will be higher. But the ratio of put value to total value is not that different from the other processes.

Are our estimation close to the market price of agricultural lands? The following table reports the prices of the agricultural land in Argentina for the period 1977-2006. Our estimation was done for a land that begins to produce in the period 2005 - 2006.

Agricultural Lands Prices in Argentina					
Year	USD per Ha.	Year	USD per Ha.	Year	USD per Ha.
1977	1,985	1987	1,457	1997	4,042
1978	1,483	1988	1,550	1998	4,858
1979	2,427	1989	1,696	1999	4,000
1980	3,001	1990	2,058	2000	3,950
1981	1,868	1991	2,292	2001	3,592
1982	1,388	1992	2,592	2002	3,000
1983	2,070	1993	2,129	2003	3,900
1984	1,990	1994	2,254	2004	5,360
1985	1,655	1995	2,400	2005	6,100
1986	1,575	1996	3,142	2006	7,500

Source: Margenes Agropecuarios

Part of the variation observed in land prices can not be attributed just to an update in the future price paths given the new information available to traders. In fact, as mentioned

[11]Recall that these results are in US dollars per ton

briefly in the introduction, there are several variables that we are not incorporating into the analysis. For instance, we can mention weather conditions, soil productivity and governments policies regarding taxes.

Anyway, here we treat land productivity as a constant. In more detail, we will use an average of the productivity in the last years to compare our results with the actual data.

Average Productivity of the soil in Argentina

Crop	Tons per HA
Wheat	2
Soya	2.8
Corn	3

Source: Margenes Agropecuarios

Using this average productivity we can transform the obtained results to compute what the predicted value of the lands would be in USD per HA. Still, it should be clear that the two-factor model for the soya should not be used as the result is too far away from reality. We should also recall that so far our assumption is that we are seeding the land split equally among the three crops. Thus, the results are:[12]:

Predicted values for the land (USD per Ha)

GBM	IHGBM	2OU	2IHGBM
10677	7132	24508	30919

As we can see, both the one-factor IHGBM and the two-factor OU throw results that are close to the values that we observe in reality, with the two-factor model more accurate, while the GBM and the two-factor IHGBM are far away from what we observe as market prices. It should be noted that, for the case of Corn, the two-factor model with IHGBM in the convenience yield is accurate enough. So, maybe a combination of the process is more relevant to get the "correct price".

The reason for the bad performance of the two-factor IHGBM is probably due to an incompatibility of the two processes. IHGBM forces convenience yield to be positive, but we know that this is not so in reality. In particular, for both wheat and soya, the convenience yield is negative in a large part of the sample.

The results for the optimal way of rotating the crops are also interesting. As crops are rotated ad hoc, a relatively small value in the call option price would indicate that the farmer's decision is not far away from optimality.

[12]We will use the one-factor IHGBM as the relevant process for the Soya two-factor models

In fact, as we can see in the following table, our empirical findings support in some way the behavior of the farmers. For instance, the call option to seed wheat when you are currently seeding soya is worth only $2.6062 per Ha.

Call Options on the rotation of crops			
	Wheat	Soya	Corn
Wheat	-	2.2642	0.053
Soya	2.6062	-	0.1621
Corn	7.1612	110.15	-

The explanation for these finding is possibly linked to the high correlation among the different crops. Because of this, it is quite unlikely that waiting for the right moment will provide a substantially higher benefit.

5 Conclusions

The price of an asset should be tied to its dividends, payoff or cash-flows. Agricultural lands are in no sense different to this. We have introduced and tried different process for the cash-flows from an agricultural land to check which one is best suited for predicting land values. We found out that the models with higher forecasting power are the one-factor IHGBM and the two-factor Ornstein-Uhlenbeck.

We have also priced the put option for the land and we evaluated the decision of which crop to seed. The put option on the agricultural land is nontrivial, as it represents around 40 percent of the land value.

On the other hand, the decision to rotate the crops in an ad hoc way has proven to be a wise decision, since the increase in the value from rotating the crops in an optimal (much more sophisticated) way is almost negligible. Thus, the way in which things are usually carried on in any agricultural farm is not far away from optimality.

We think that the predictability of the model will increase considerably if we include the possibility of jumps in the productivity of the soil or to match the weather changes as well as improvements in the production function.

For further research, several variables can be incorporated into the analysis. For instance, we can mention three of them as the most relevant ones: weather conditions, soil productivity and governments policies regarding taxes. Regarding wether, the distributions of rains could be incorporated into the model as there exists a historical probability matrix that provides enough information to obtain future paths. On the other hand, the soil productivity changes appear to arrive in jumps. As government policy is always treated as a random shock in macroeconomic models, this might not affect results that much. Thus, any future improvements in the set up presented here could arise from considering these variables.

Appendix A

- The first moment for the Inhomogeneous Geometric Brownian Motion process (3 can be computed in the following way:

$$E(dX_t) = \lambda(\overline{X} - E(X_t))dt$$

Using the linearity of the expectation, we arrive at the following formula

$$\frac{dE(X_t)}{dt} = \lambda(\overline{X} - E(X_t))$$

)

Then,

$$e^{\lambda t}\left[\frac{dE(X_t)}{dt} + \lambda E(X_t)\right] = e^{\lambda t}\lambda\overline{X}$$

Integrating this equation, we can find the first moment.

$$\int_0^t e^{\lambda t}\left[\frac{dE(X_t)}{dt} + \lambda E(X_t)\right] dt = \int_0^t e^{\lambda t}\lambda\overline{X}dt$$

Solving that we get

$$E(X_t)e^{\lambda t} - E\left[X_0\right] = \overline{X}e^{\lambda t} - \overline{X}$$

Thus, we have that

$$E(X_t) = \overline{X} + (X_0 - \overline{X})e^{-\lambda t}$$

As expected, as time goes to infinity, $E(X_\infty) = \overline{X}$, corroborating that $\overline{X}$ is the long-term value at which the variable X converges to.

- For the second moment, defining $f(X_t) = X_t^2$ and applying Itô's lemma, we obtain

$$\frac{dE(X_t^2)}{dt} = 2\lambda\overline{X}E(X_t) + (\sigma^2 - 2\lambda)E(X_t^2)$$

Substituting for $E(X_t)$

$$\frac{dE(X_t^2)}{dt} + (2\lambda - \sigma^2)E(X_t^2) = 2\lambda\overline{X}\left(\overline{X} + (X_0 - \overline{X})e^{-\lambda t}\right)$$

Using $(2\lambda - \sigma^2)t$ as the integrating factor, we have

$$\int_0^t e^{(2\lambda-\sigma^2)t}\left[\frac{dE(X_t^2)}{dt} + (2\lambda - \sigma^2)E(X_t^2)\right] dt = \int_0^t e^{(2\lambda-\sigma^2)t}[2\lambda\overline{X}(\overline{X} + (X_0 - \overline{X})e^{-\lambda t})]dt$$

Solving the integral, we obtain

$$E(X_t^2)e^{(2\lambda-\sigma^2)t} - X_0^2 = \int_0^t 2\lambda\overline{X}^2 e^{(2\lambda-\sigma^2)t}dt + \int_0^t 2\lambda\overline{X}\left(X_0 - \overline{X}e^{(\lambda-\sigma^2)t}\right) dt$$

Then,

$$e^{(2\lambda-\sigma^2)t}E(X_t^2) = \frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2}(e^{(2\lambda-\sigma^2)t}-1) + \frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2}(e^{(\lambda-\sigma^2)t}-1) + X_0^2$$

Then,

$$E(X_t^2) = \frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2}\left(1-e^{(\sigma^2-2\lambda)t}\right) + \frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2}\left(e^{-\lambda t}-e^{(\sigma^2-2\lambda)t}\right) + X_0^2e^{(\sigma^2-2\lambda)t}$$

provided that $(2\lambda-\sigma^2)(\lambda-\sigma^2)\neq 0$.[13]

As $Var(X_t^2) = E[(X_t - E(X_t))^2]$, the second non-central moment can be computed as

$$\begin{aligned} Var(X_t) &= e^{(\sigma^2-2\lambda)t}\left(X_0^2 + \frac{2\lambda\overline{X}^2}{\sigma^2-2\lambda} + \frac{2\lambda\overline{X}(X_0-\overline{X})}{\sigma^2-\lambda}\right) \\ &+ e^{-\lambda t}\left(\frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2} - 2\overline{X}(X_0-\overline{X})\right) \\ &+ -e^{-2\lambda t}(X_0-\overline{X})^2 + \frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2} - \overline{X}^2 \end{aligned}$$

So $Var(X_\infty) = \frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2} - \overline{X}^2 = \frac{\sigma^2}{2\lambda-\sigma^2}\overline{X}^2$ This means that the speed of reversion should be high enough ($\lambda > \sigma^2/2$) for the variance to converge to a finite value.

The explicit solution to the Inhomogeneous Geometric Brownian Motion is:

$$X_t = e^{-(\lambda+\frac{\sigma^2}{2})t+\sigma W_t}\left[X_0 + \lambda\overline{X}\int_0^t e^{(\lambda+\frac{\sigma^2}{2})s-\sigma W_s}ds\right]$$

Now, since the above process is not risk-neutral, we should replace the drift with the growth rate in a risk-neutral world which is achieved by discounting the risk premium to obtain the correspondent risk-neutral process[14] The risk premium is $\rho\sigma\frac{r_m-r}{\sigma_m}X_t = \rho\sigma\phi X_t$

Then, the risk-neutral version is:

$$d\widehat{X_t} = [\lambda(\overline{X}-\widehat{X_t}) - \rho\sigma\phi X_t]dt + \sigma\widehat{X_t}dZ_t$$

Following the same method that in the previous part, the first two moments can be shown to be

$$E(\widehat{X_t}) = \frac{\lambda\overline{X}}{\lambda+\rho\sigma\phi}[1-e^{-(\lambda+\rho\sigma\phi)t}] + X_0e^{-(\lambda+\rho\sigma\phi)t}$$

[13] We actually have two more possibilities: If $\sigma^2-2\lambda=0$, then $E(X_t^2) = -(2k\overline{X}^2)t + 2(X_0-\overline{X})(e^{-\lambda t}-1) + X_0^2$. Alternatively, if $\lambda=\sigma^2$, then $E(X_t^2) = 2\overline{X}^2(1-e^{-\lambda t}) + 2\overline{X}(\overline{X}-X_0)te^{-\lambda t} + X_0^2e^{-\lambda t}$.

[14] The growth rate in a risk-neutral world is given by $r-\delta$, where δ is the convenience yield.

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