Default and Non-Flat Reorganization Boundaries in Credit Risk Models

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Abstract

In the paper we study the debt valuation and non-flat reorganization boundaries when strategic default and strategic interaction between debt and equity holders are explicitly modeled. We obtain an approximate analytical solution for endogenously determined non-flat reorganization boundary. The increasing and convex reorganization boundary shows that the closer is the maturity, the more cautious are debt holders with respect to a drop of firm's asset value, as the recovery in the short period of time becomes less probable. We also derive a closed form solution of the debt value based on the model. The numerical results calculated from the solution show that the model is capable of producing term structures of risk premium that are consistent with some empirical findings. In addition, we show that institutional and legal changes that alter the bargaining power of debt and equity holders or the liquidation costs have twofold impact on the risk premium through recovery rate and default probability: the premium associated to the former is higher in magnitude than the one associated to the latter and they work in opposite directions.

JEL Classification: G12, G13, G33

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1 Introduction

An essential feature of debt contracts, in both corporate and bank loan markets, are covenants. A commonly observed covenant is the requirement that the borrower must pay contractual payments to the lender at a pre-specified time in the future, with or without intermediate periodic payments. In the case of intermediate periodic payments any missed or delayed disbursement is treated as default. When there are no intermediate periodic payments any type of financial distress, in the form of poor cash flows or devalued firm value (the latter is also a market signal of a firm's poor performance), is treated as default. Modeling of such covenants is critical in the debt valuation literature. [Merton, R. C. (1974)], [Black, F. and Cox, J. (1976)], [Leland, H. (1994)], and [Leland, H. and Toft, K. (1996)] are examples of articles that study the valuation consequences of such debt covenants.

A number of recent articles, such as [Anderson, R. and Sundaresan, S. (1996)], [Fan, H. and Sundaresan, S. (2000)], [Francois, P. and Morellec, E. (2004)] and [Mella-Barral, P. and Perraudin, W. (1997)], have taken into consideration strategic issues and renegotiation in the context of debt valuation. These papers incorporate, under the assumptions of costless renegotiation, debt reliefs granted to firms in financial distress. Since liquidation is costly and bondholders bear liquidation costs, there is a room for strategic default. Strategic default occurs each time the state variable, typically the value of the firm's assets, falls below some endogenous default threshold.

Unfortunately the literature on debt valuation has assumed that the endogenously determined default threshold is constant over time. When the debt has a finite maturity this assumption means that close to the maturity debt holders are as tolerant to the drop of firm's asset value as they would be at the time when the debt is issued. However, practice shows that close to the maturity debt holders are more cautious to the drop of firm's asset value, as the recovery in the short period is less probable.

In the paper we consider an economy in which the borrowers (equity holders) have exclusive access to a project that provides a continuous stream of cash flows. They finance the project with debt, which could result in potentially inefficient liquidation and financial distress. The driving force behind strategic behavior in our model is the presence of proportional costs of liquidation. We endogenously determine the optimal sharing rule between equity and debt holders upon default. With the optimal sharing rule we next characterize the non-flat reorganization boundary and provide an analytical solution for the two extreme situations: when sharing rule provides debt holders with either all assets or nothing upon default. The endogenously determined non-flat reorganization boundary in the former case characterizes the situation where everything is seized without costs when the firm is in the financial distress. Boundaries obtained in the two cases are the upper and the lower bounds, respectively, for any non-flat reorganization boundary when the optimal sharing rule is in between those extreme cases.

We also derive a closed form solution for the value of the debt. The risk premium analysis based on the valuation result show that the use of non-flat reorganization boundary is capable of producing terms structures that are consistent with some empirical findings.

Importantly, the debt valuation result allows us to conclude that the recovery risk premium plays dominant role in relation to default risk premium in the term structure of risk premium. When we decrease the optimal sharing rule it decreases the endogenously determined non-flat reorganization boundary, which in turn decreases the probability of default. Decreased probability of default implies lower risk premium. One the other hand, decreasing the sharing rule decreases the recovery rate of the debt and, hence, increases the recovery risk. The latter requires higher risk premium. The numerical results show that the net impact of the decreased sharing rule is the increase in risk premium. The conclusion based on the argument that runs in the direction of increased optimal sharing rule is similar.

The reminder of the paper is organized as follows: Section 2 presents valuation framework with non-flat reorganization boundary and debt-equity swap reorganization scheme in the form of bargaining game. In Section 3 we obtain expressions characterizing the non-flat default boundary and its particular solutions. The closed form solution for the value of the debt is obtained in Section 4. Numerical results of the term structure of risk premium based on the pricing formula are analyzed in Section 5. In Section 6 we conclude. All technical developments are left for the Appendix.

2 Model Setup

In this section we develop a model, which allows determining the value of the debt and non-flat reorganization boundary under a particular scheme known as debt-equity swap. The model is set in continuous-time framework. The following assumptions underline the model:

1. There is a firm which has equity and a single issue of finite maturity debt. The time of maturity is denoted by T and by the indenture of debt the firm promises to pay a total amount of K to debt holders on the specified time T. We assume also that there are no coupon payments.

2. The firm cannot issue any new senior (or of equivalent rank) claims nor can pay cash dividends or do share repurchase prior to the maturity of the debt.

3. We assume that the default-free term structure is flat and the instantaneous risk-less (default-free) rate is r per unit time.

4. The firm can be liquidated only at a cost. The proportional cost is α ($0 \leq \alpha \leq 1$). The debt holders have strict absolute priority upon bankruptcy. Liquidation happens when the value of assets of the firm reaches $g(t)$, a time varying trigger level. In that case a cost $\alpha g(t)$ is taken away by outsiders; debt holders receive the remaining $(1-\alpha)g(t)$; and equity holders receive nothing.

5. The dynamics for the value of the assets, V, through time can be described by a diffusion-type stochastic process with stochastic differential equation

$$\frac{dV}{V} = \mu dt + \sigma dW_t \tag{1}$$

   where μ is the instantaneous expected rate of returns per unit time, σ^2 is the instantaneous variance per unit time and W_t is a standard Brownian Motion.

6. There are no taxes, transaction costs and trading of assets takes place continuously in time[1].

The fact that the liquidation is costly and current legal arrangements provide opportunities for the borrower and the lender to avoid costly liquidation, bankruptcies are often resolved using exchange offers of different types. They include delayed or missed interest or principal payments, extension of maturity, debt-equity swap, debt holidays, etc. Essentially all these distressed exchanges and delayed payments can be considered as a value distribution between equity and debt holders. For concreteness and better illustration of the main findings we will concentrate on debt-equity swap. However, it is to be noted that in the absence of taxes there is no difference between most of the above mentioned reorganization schemes.

When reorganization results in a debt-equity swap, the firm becomes an all-euqity firm. Since we have assumed away taxes or the possibility of bankruptcy in the future, the total value of the firm is exactly the asset value V. At an endogenously determined reorganization boundary (which we have denoted by

[1]Note that in the absence of taxes there is no distinction between the value of the assets of the firm and the value of the firm.

$g(t)$), debt holders are offered a proportion of the firm's equity to replace their original debt contract. As a consequence the sharing will be given by

$$E(g(t),t) = \theta g(t), \qquad D(g(t),t) = (1-\theta)g(t) \tag{2}$$

where $E(\cdot)$ and $D(\cdot)$ are the value of equity and debt, respectively, θ is a parameter that reflects the sharing rule for the value of residual assets. In our setting parameter θ should satisfy $0 \leq \theta \leq \alpha$. If not then for debt holders it will be optimal to liquidate the firm and pay outsiders the proportional cost α.

In the existing literature on debt valuation with renegotiation the sharing rule is determined by a bargaining game. For expositional simplicity we consider the Nash bargaining solution.

Let us denote η as the equity holders' bargaining power, and $1-\eta$ is the debt holders' bargaining power. The Nash solution θ_N can be derived as follows. The incremental value for equity holders by continuing as opposed to liquidating is $\theta g(t) - 0$. The Incremental value to debt holders by accepting the debt-equity swap instead of forcing liquidation is $(\alpha - \theta)g(t)$. The Nash bargaining solution is characterized as:

$$\theta_N = \mathrm{argmax}_{0\leq\theta\leq\alpha} \left[\theta g(t)\right]^{\eta} \left[(\alpha-\theta)g(t)\right]^{1-\eta} = \eta\alpha. \tag{3}$$

The solution to the Nash bargaining game is characterized by the sharing rule, which is a function of the bargaining power η. Note, in addition, that once $\theta > \alpha$ then no bargaining can take place as the incremental value to the debt holder is negative. In the context of debt pricing this means that for debt holders the liquidation is less costly then reorganization, and, hence they will liquidate the firm whenever it defaults or is in a financial distress.

3 Reorganization Boundary

In this section we derive reorganization boundary $g(t)$, which is a function of t and changes over time. Time varying default boundary implies that the "loss given default" (LGD), which is equal to one minus the recovery rate in the event of default, is not constant but rather depends on time. This is by the fact that the recovery rate, defined by the ratio $(1-\theta)g(t)/K$, also changes over time.

It is not difficult to show (see for example [Merton, R. C. (1974)]) that the debt value satisfies the following partial differential equation:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 D}{\partial V^2} + rV\frac{\partial D}{\partial V} - rD + \frac{\partial D}{\partial t} = 0 \tag{4}$$

As the value of the asset V approaches infinity, debt becomes riskless and hence, the debt value is set to be the discounted face value of the debt, i.e.

$$\lim_{V \to \infty} D(V,t) = Ke^{-r(T-t)} \tag{5}$$

In the expression (5) $Ke^{-r(T-t)}$ is the price of riskless discount bond at time t, which promises a payment of K at time T in the future. The conditions for the non-flat reorganization boundary are given by

$$D(V,t)\Big|_{V=g(t)} = (1-\theta)g(t) \tag{6}$$

$$\frac{\partial D(V,t)}{\partial V}\Big|_{V=g(t)} = (1-\theta). \tag{7}$$

The equation (6) is the "value matching" condition, which says that on the endogenously determined non-flat boundary debt holders get $(1-\theta)$ proportion of the firms equity as a result of debt-equity swap reorganization. The equation (7) is the "smooth pasting" condition, which ensures the continuity of $D(V,t)$ on the boundary.

The complete description of the valuation equation requires an initial condition, which is given by

$$D(V,T) = \begin{cases} K & V \geq K \\ (1-\theta)V & V < K \end{cases}. \tag{8}$$

On the maturity of the debt the debt holders are either payed the promised payment of K (it happens only when the value of the firm is not smaller then the promised payment K) or the firm is reorganized (it happens when the value of the firm is not enough to pay the debt), in which case the debt holders receive $(1-\theta)$ proportion of the firms equity.

The partial differential equation (4) and the conditions (5)-(8) together are known as a free boundary problem, and a solution to such a problem gives the value of the debt with an indenture, which assumes that default and reorganization may occur any time between the issuance and maturity of the debt. It is convenient to reformulate the free boundary problem making the following notations $f(V,t) = Ke^{-r(T-t)} - D(V,t)$, $V = Ke^x$ and $h = \frac{2r}{\sigma^2}$. Then the partial differential equation (4) and the conditions (5)-(8) become

$$\frac{\partial^2 f}{\partial x^2} + (h-1)\frac{\partial f}{\partial x} - hf + \frac{2}{\sigma^2}\frac{\partial f}{\partial t} = 0 \tag{9}$$

$$\lim_{x\to\infty} f(Ke^x,t) = 0 \tag{10}$$

$$f(Ke^x,t)\Big|_{x=\ln\frac{g(t)}{K}} = Ke^{-r(T-t)} - (1-\theta)g(t) \tag{11}$$

$$\frac{\partial f(Ke^x,t)}{\partial x}\Big|_{x=\ln\frac{g(t)}{K}} = -(1-\theta)g(t) \tag{12}$$

$$f(Ke^x,T) = \left\{ \begin{array}{cc} 0 & x \geq 0 \\ K(1-(1-\theta)e^x) & x < 0 \end{array} \right. . \tag{13}$$

In line with the condition (10) it is natural to assume also that

$$\lim_{x\to\infty} \frac{\partial f(Ke^x,t)}{\partial x} = 0. \tag{14}$$

The next theorem characterizes the function f (for the notation and terminology one is referred to the Appendix).

Theorem 1 *The image of the function $f(Ke^x,t)$, which is denoted by $F(\omega,t) = \mathcal{F}(f(Ke^x,t))$, is given by*

$$F(\omega,t) = F(\omega,T)e^{-\frac{\sigma^2}{2}A(\omega)(T-t)} - \int_t^T \frac{\sigma^2}{2}B(\omega,\tau)e^{-\frac{\sigma^2}{2}A(\omega)(\tau-t)}d\tau \tag{15}$$

where $F(\omega,T) = \mathcal{F}(f(Ke^x,T))$ and

$$A(\omega) = \omega^2 + (h-1)\imath\omega + h \tag{16}$$

$$B(\omega,t) = \left[\left(h-1-\frac{2}{\sigma^2}\frac{g'(t)}{g(t)} - \imath\omega\right)\frac{Ke^{-r(T-t)}-(1-\theta)g(t)}{\sqrt{2\pi}}\right]e^{\imath\omega\ln\frac{(1-\theta)g(t)}{K}} - \tag{17}$$

$$-\frac{(1-\theta)g(t)}{\sqrt{2\pi}}e^{\imath\omega\ln\frac{(1-\theta)g(t)}{K}}.$$

Proof of Theorem 1 *See Appendix.*

The result of Theorem 1 shows that the value of the debt is decomposed into two components. The first term in the equation (15) characterizes the value of a debt with an indenture, which allows default only at the maturity, in which case the firm is reorganized through debt-equity swap reorganization scheme. The valuation result obtained in Theorem 2, below, is similar to the classical debt valuation result obtained in [Merton, R. C. (1974)], with only one difference. In Merton's result the reorganization scheme works in a way as if there are no liquidation costs and/or the debt holders have the whole bargaining power, while in the current context reorganization leads to sharing of defaulted firm's value between equity and debt holders at the maturity.

Theorem 2 *The original of the function*

$$F_1(\omega, t) = F(\omega, T)e^{-\frac{\sigma^2}{2}A(\omega)(T-t)}. \tag{18}$$

is given by

$$f_1(V, t) = Ke^{-r(T-t)}\Phi(d_-) - (1-\theta)V\Phi(d_+), \tag{19}$$

where

$$d_\pm = \frac{\ln\frac{K}{V} - (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{20}$$

and $\Phi(\cdot)$ *is the CDF of a standard Normal distribution. Moreover,*

$$D_1(V, t) = Ke^{-r(T-t)} - f_1(V, t) = Ke^{-r(T-t)}\Phi(-d_-) + (1-\theta)V\Phi(d_+). \tag{21}$$

Proof of Theorem 2 *See Appendix.*

Intuitively, the equation (18) can be interpreted as follows. The value of a function (in a complex plain) at any given time t is equal to the discounted value of the terminal value of the function, i.e. $F(\omega, T)$, where the discount factor is $\frac{\sigma^2}{2}A(\omega)$. The existence of the second term in the equation (15) shows that there is a premium associated with the possibility of early reorganization of the firm when it is in a financial distress. The second term also has an intuitive explanation. It is the sum (or the integral in the continuous time setting) of present values of net gains (losses) connected with reaching the reorganization boundary, where $\frac{\sigma^2}{2}B(\omega, \tau)$ has the interpretation of net gains (losses). Note that the discount factor is the same as in the first term. Once inverted, the second term in the equation (15), together with the equation (21), can be used to find the value of the debt $D(V, t)$. Moreover, one needs to use the inverted equations to derive the non-flat reorganization boundary. The next theorem provides a condition, which should be solved to find the unknown reorganization boundary.

Theorem 3 *The function $g(t)$ satisfies the following integral equation*

$$(1-\theta)\frac{g(t)}{K}(1+Erf(\eta_1)) - e^{-r(T-t)}\left(1+Erf(\eta_0)+\theta Erf\left(\eta_0-\frac{\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right)\right)+$$

$$+e^{-r(T-t)}sign(-\ln(1-\theta)) - (1-\theta)e^{-r(T-t)}Erf\left(\sigma_1\sqrt{T-t}-\frac{\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right)+ \quad (22)$$

$$+\int_t^T \frac{g(\tau)}{K}\frac{1}{\sqrt{\pi}}\left(\frac{\sigma_2}{\sqrt{\tau-t}}+\frac{\ln(1-\theta)}{\sigma\sqrt{2(\tau-t)^3}}\right)e^{-\left(\sigma_2\sqrt{\tau-t}-\frac{\ln(1-\theta)}{\sigma\sqrt{2(\tau-t)}}\right)^2}d\tau = 0$$

where

$$\eta_0 = \frac{\ln\frac{g(t)}{K}+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{2(T-t)}}; \qquad \eta_1 = \frac{\ln\frac{g(t)}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{2(T-t)}} \quad (23)$$

$$\sigma_1 = \frac{r-\frac{\sigma^2}{2}}{\sigma\sqrt{2}}; \qquad \sigma_2 = \frac{r+\frac{\sigma^2}{2}}{\sigma\sqrt{2}}. \quad (24)$$

Proof of Theorem 3 *See Appendix.*

The integral equation (22) with unknown function $g(t)$ is highly nonlinear and no methods are available to obtain an exact analytical solution for $g(t)$. In one special case, when $\theta = 1$ the exact analytical solution for $g(t)$ is readily available. This case corresponds to a situation where the debt holders get nothing as a result of debt-equity reorganization. With Nash bargaining solution this may happen when equity holders have the whole bargaining power (i.e. $\eta = 1$) and the proportional liquidation costs are equal to one. In practice, non of the situations happens. In general, the liquidation costs and what debt holders obtain upon firm's default are always positive. However, considering the case when $\theta = 1$ is important as it provides the lower bound for reorganization boundaries. With $\theta = 1$ the equation (22) reduces to

$$\mathrm{Erf}(\eta_0) = -1, \quad (25)$$

which is possible when $g(t) \equiv 0$. The intuition behind this result is as follows. Because debt holders get noting as a result of announcing the firm bankrupt, they are better off by letting the firm to continue its operation as for any given value of the firm's assets, whatsoever low level it is, there is a positive probability that it will outperform and repay the outstanding debt at the maturity.

On the other extreme, where $\theta = 0$, debt holders get the whole equity as a result of debt-equity swap reorganization. In the case of Nash bargaining solution $\theta = 0$ when either there are no liquidation costs (i.e. $\alpha = 0$) or the debt holders have the whole bargaining power (i.e. $\eta = 0$). This situation is also not realistic. However, by considering it we obtain the upper bound for non-flat reorganization boundaries. In the case when $\theta = 0$ the equation (22) reduces to

$$\frac{g(t)}{K}(1+\mathrm{Erf}(\eta_1)) - e^{-r(T-t)}\left(1+\mathrm{Erf}(\eta_0)+\mathrm{Erf}(\sigma_1\sqrt{T-t})\right)+ \tag{26}$$

$$+\int_t^T \frac{g(\tau)}{K}\frac{\sigma_2}{\sqrt{\pi(\tau-t)}}e^{-\sigma_2^2(\tau-t)}d\tau = 0$$

We have derived an approximate analytical solution to the integral equation (26) and the next theorem provides the details of the derivation.

Theorem 4 *Assume $\theta = 0$. Then the approximate solution to the reorganization boundary $g(t)$ is given by*

$$g(t) = Ke^{A(t_1)+B(t_1)-C(t_1)}. \tag{27}$$

where

$$A(t_1) = \frac{e^{\sigma_0^2 t_1}}{r}\left(\sigma_1^2 Erf\left[\sigma_0\sqrt{t_1}\right] + \sigma_0\left(\sigma_1\left(1+Erf\left[\sigma_1\sqrt{t_1}\right]\right) + e^{rt_1}\sigma_2 Erfc\left[\sigma_2\sqrt{t_1}\right]\right)\right) \tag{28}$$

$$B(t_1) = \sigma_2 e^{\sigma_0^2 t_1} Erf\left[\sigma_0\sqrt{t_1}\right] * \left(\frac{e^{(\sigma_0^2-\sigma_1^2)t_1}}{\sqrt{\pi t_1}} + \sigma_2 e^{2\sigma_0\sigma_2 t_1} Erf\left[\sigma_2\sqrt{t_1}\right]\right) \tag{29}$$

$$C(t_1) = \frac{\sigma_2\sqrt{\sigma_0\sigma_2}}{\sqrt{2}r}e^{2\sigma_0\sigma_2 t_1} Erf\left[\sqrt{\sigma_0\sigma_2 t_1}\right] + 2\sigma_0\sigma_2 t_1 + 1 \tag{30}$$

and $\sigma_0 = \frac{\sigma}{\sqrt{2}}$ and $t_1 = T - t$.

Proof of Theorem 4 *See Appendix.*

The plot of the function $g(t)$ in Figure 1 shows that the reorganization boundary is increasing and convex function in t.

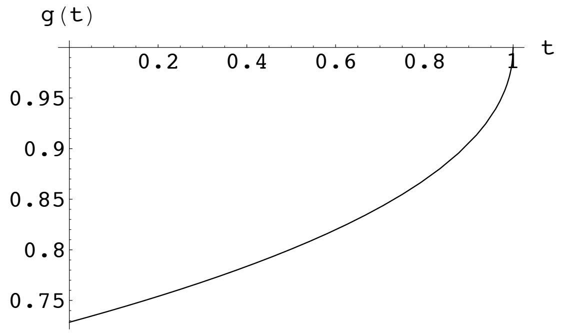

Figure 1: The plot of the reorganization boundary given in Theorem 4. The parameters are: $r = 0.06$, $\sigma = 0.2$, $K = 1$ and $T = 1$.

Compared with a flat boundary, the result of Theorem 4 shows that the closer is the maturity, more cautious are debt holders with respect to the drop of firm's value, as the recovery in the short period of time becomes less probable. On the other hand, closer to the debt issuance time debt holders are more reckless towards the drop of firm's value. The results of comparative statics of the reorganization boundary with respect to the model parameters, r, σ and T, are presented in Figure 6.

4 Debt Value

In this section we derive the closed form solution for the value of the debt with non-flat reorganization boundary. In Section 3 we have shown that the value of the debt has two components and provide the solution for the first term. In the next Theorem we derive the premium term and together with the result of Theorem 2 obtain the value of the debt.

Theorem 5 *The original of the integral expression in the equation (15) is given by*

$$f_2(V,t) = V\Phi\left(sign\left(\ln\frac{V}{(1-\theta)g(t)}\right)q_+\right) - \tag{31}$$

$$-Ke^{-r(T-t)}\left(\frac{sign\left(\ln\frac{V}{(1-\theta)g(t)}\right)-1}{2} + \Phi(q_-)\right)$$

where

$$q_{\pm} = d_{\pm} + \frac{\ln(1-\theta)}{\sigma\sqrt{T-t}}. \tag{32}$$

Moreover,

$$D(V,t) = \left[Ke^{-r(T-t)}\Phi(-d_-) + (1-\theta)V\Phi(d_+)\right] - \tag{33}$$

$$- \left[Ke^{-r(T-t)}\left(\frac{sign\left(\ln\frac{V}{(1-\theta)g(t)}\right)-1}{2} + \Phi(q_-)\right) - V\Phi\left(sign\left(\ln\frac{V}{(1-\theta)g(t)}\right)q_+\right)\right].$$

Proof of Theorem 5 *See Appendix.*

In the special case, when $\theta = 1$, the value of the debt becomes

$$D(V,t;\theta=1) = Ke^{-r(T-t)}\Phi(-d_-). \tag{34}$$

The intuition behind this result is as follows. When the debt holders get nothing as a result of reorganization, they discount only the expected payoff at the maturity, which is $K\Phi(-d_-)$. At the maturity debt holders get K, only when the firms value is not smaller then K, which happens with a probability $\Phi(-d_-)$.

When $\theta = 1$ the second term in the brackets is zero. As we change θ, it becomes positive and increases reaching its maximum when $\theta = 0$. In other words, the premium, deducted from the pricing of the debt without a possibility of early reorganization, is the highest when debt holders have the whole bargaining power or there are no liquidation costs.

The plot of the value of the debt for extreme cases, premium and Merton's value of the debt against value of the firm is presented in Figure 2. As value of the firm goes to infinity the premium term vanishes, and the values of the debt for any given θ converges to $Ke^{-r(T-t)}$, that is to the discounted face value of the debt. Merton's value of the debt is always above the value of the debt derived in Theorem 5. This means that the risk premium obtained in our paper and discusses in the next section are always higher than the risk premium obtained in [Merton, R. C. (1974)].

5 Risk Premium

The term structure of risk premium for a low leveraged firm, with $V = 1050$ and $K = 100$, is illustrated in Figure 3. The debt ratio K/V equal to 9.5% in this illustration is similar to the market-value debt ratio of firms whose senior debt ratings

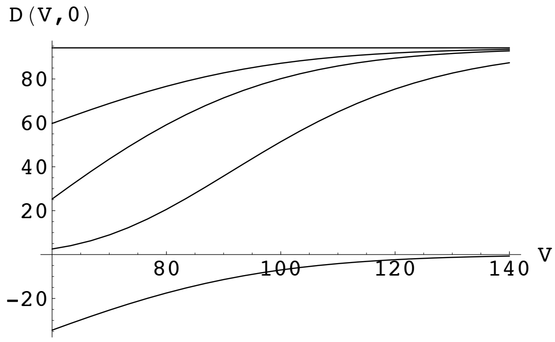

Figure 2: Debt value for extreme cases, premium and Merton's value of the debt plotted against value of the firm. The parameters are: $r = 0.06$, $\sigma = 0.25$, $T = 1$, $t = 0$ and $K = 100$.

from [Standard and Poor's (2001)] are AA. The expected recovery rate for senior unsecured debt is typically 48% according to [Carty, L. and Lieberman, D. (1998)]. Thus, we set $\theta = 0.52$. The other parameters used in the calculations are $\sigma = 0.25$ and $r = 0.06$. The asset volatility $\sigma = 0.25$ is close to the median for publicly listed companies. The term structure of risk premium based on our results is upward sloping in Figure 3. At short maturities its shape is flat. This finding is similar to the empirical studies in [Sarig, O. and Warga, A. (1989)]. They find that the term structure of the risk premium is upward sloping for high rating pure discount bonds.

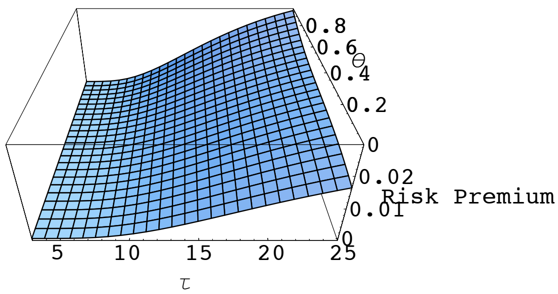

Figure 3: The terms structure of risk premium of a low leveraged firm (AA) for various values of sharing parameter. The parameters are: $r = 0.06$, $\sigma = 0.25$, $V = 1050$ and $K = 100$.

Figure 3 also shows that the risk premium based on non-flat reorganization boundary increases with θ. In Section 3 we have shown that with increasing θ the default boundary decreases and in the limit when $\theta = 1$ it becomes flat and equal to zero. Decreasing default boundary implies lower default risk. However, positive relation between risk premium and θ shows that not only the default risk but the recovery risk have an impact on risk premium. More importantly, the risk premium related to the recovery risk is higher in magnitude than the risk premium related to the default risk and the sign of the former determines the net impact. In other words, the recovery risk plays more important role in determining the risk premium than default risk.

The term structure of risk premium for a medium leveraged firm, with $V = 317$ and $K = 100$ is illustrated in Figure 4. The debt ratio $K/V = 31.5\%$ is similar to the market-value debt ratios of firms whose senior debt ratings from [Standard and Poor's (2001)] are BBB. Other parameters are the same as those used in Figure 3. The shapes are consistent with the empirical findings by [Sarig, O. and Warga, A. (1989)] that the term structure is humped for medium rating bonds.

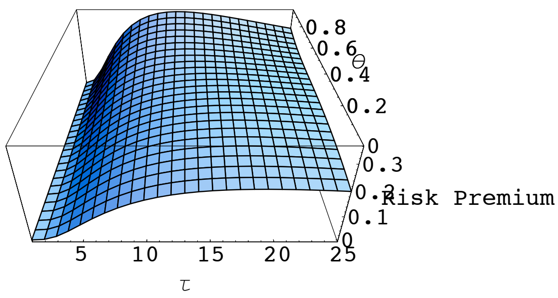

Figure 4: The term structure of risk premium of a medium leveraged firm (BBB) for various values of sharing parameter. The parameters are: $r = 0.06$, $\sigma = 0.25$, $V = 317$ and $K = 100$.

Similar to Figure 3, Figure 4 also shows that the risk premium increase with θ. This finding also shows that recovery rates have important impact on the risk premium for medium leveraged firms. In addition, the term structure of risk premium becomes more humped with high θ.

Figure 5 shows the term structure of risk premium for a highly leveraged firm, with $V = 200$ and $K = 100$. The debt ratio $K/V = 50\%$ is similar to the market-value debt ratios of firms whose senior debt ratings from [Standard and Poor's (2001)] are BB. Other parameters are the same as those used in Figure 3. The depicted term structures exhibit steep upward slopes at short maturities and are downward

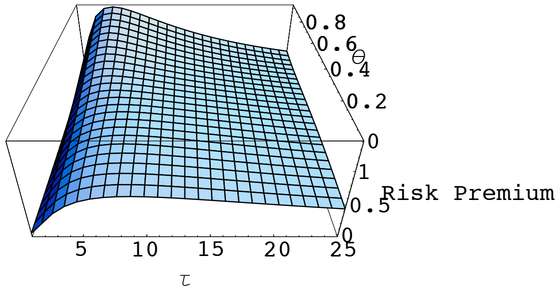

Figure 5: The term structure of risk premium of high leveraged firm (BB) for various values of sharing parameters. The parameters are: $r = 0.06$, $\sigma = 0.25$, $V = 200$ and $K = 100$.

sloping for maturities longer than a few years. The shapes are consistent with the empirical findings of the downward sloping term structure of low ratings bonds reported by [Sarig, O. and Warga, A. (1989)]. The phenomenon of high risk premium at short maturities is called "crisis-at-maturity" by [Johnson, R. (1967)]. This downward slope is generated by high initial default probabilities which are expected to decrease over time as the firm survives. In this case also, the risk premium increases with θ, leaving our conclusion about relative importance of recovery risk unchanged.

6 Conclusion

This paper provides a framework of early debt renegotiation with non-flat reorganization boundary. The bargaining powers of equity and debt holders can be varied to examine their effects on reorganization boundaries and risk premiums. The non-flat reorganization boundary is endogenously determined and approximate analytical expression for it is obtained. We also derived a closed form solution of the corporate bond price as a function of, among other model parameters, the sharing rule that is determined through bargaining process over the residual assets. The numerical results show that the model is capable of producing term structure of risk premium which are broadly consistent with some empirical findings. An innovative finding in our paper is the fact that the recovery rate and premium associated to it is much more important component of the risk premium than the default risk and default premium.

7 Appendix

7.1 Notation

In the text we have used the following notations. The imaginary number is denoted by $\imath = \sqrt{-1}$. Fourier transform (or image) of a function $f(t)$ is defined to be

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\imath\omega t} dt \tag{35}$$

and is denoted by $\mathcal{F}(f(t))$. The inverse Fourier transform (or the original) is defined to be

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-\imath\omega t} d\omega \tag{36}$$

and is denoted by $\mathcal{F}^{-1}(F(\omega))$. The definition and the notation of Laplace transform and inverse Laplace transform is given by, respectively

$$F(p) = \mathcal{L}(f(t)) = \int_{0}^{\infty} f(t) e^{-pt} dt \tag{37}$$

$$f(t) = \mathcal{L}^{-1}(F(p)) = \frac{1}{2\pi\imath} \int_{\gamma-\imath\infty}^{\gamma+\imath\infty} F(p) e^{pt} dp \tag{38}$$

where γ is an arbitrary positive constant chosen so that the contour of integration lies to the right of all singularities in $F(p)$. The sign $(*)$ stands for a convolution of two functions, which is defined to be

$$f * g = \int_{0}^{t} f(\tau) g(t-\tau) d\tau. \tag{39}$$

The error function $\mathrm{Erf}(x)$ is defined by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2} dz. \tag{40}$$

7.2 Proof of Theorem 1

Let us denote the Fourier transform of the function $f(Ke^x, t)$ by

$$F(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(Ke^x, t) e^{\imath\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{\ln\frac{(1-\theta)g(t)}{K}}^{\infty} f(Ke^x, t) e^{\imath\omega x} dx. \tag{41}$$

Then, we have

$$\mathcal{F}\left(\frac{\partial f(Ke^x, t)}{\partial x}\right) = -\imath\omega F(\omega, t) - \frac{Ke^{-r(T-t)} - (1-\theta)g(t)}{\sqrt{2\pi}} e^{\imath\omega \ln\frac{(1-\theta)g(t)}{K}}, \tag{42}$$

$$\mathcal{F}\left(\frac{\partial^2 f(Ke^x,t)}{\partial x^2}\right) = \omega^2 F(\omega,t) + \imath\omega \frac{Ke^{-r(T-t)} - (1-\theta)g(t)}{\sqrt{2\pi}} e^{\imath\omega \ln\frac{(1-\theta)g(t)}{K}} + \tag{43}$$

$$+\frac{(1-\theta)g(t)}{\sqrt{2\pi}} e^{\imath\omega \ln\frac{(1-\theta)g(t)}{K}},$$

$$\mathcal{F}\left(\frac{\partial f(Ke^x,t)}{\partial t}\right) = \frac{dF(\omega,t)}{dt} + \frac{Ke^{-r(T-t)} - (1-\theta)g(t)}{\sqrt{2\pi}} \frac{g'(t)}{g(t)} e^{\imath\omega \ln\frac{(1-\theta)g(t)}{K}}. \tag{44}$$

Plugging the expressions (41)-(44) in equation (9) and collecting the terms we obtain an ordinary differential equation in $F(\omega,t)$:

$$\frac{\sigma^2}{2}\frac{dF(\omega,t)}{dt} - A(\omega)F(\omega,t) = B(\omega,t) \tag{45}$$

where $A(\omega)$ and $B(\omega,t)$ are given as in Theorem 1. The solution to the ODE with the initial condition

$$F(\omega,T) = \mathcal{F}(f(Ke^x,T)) \tag{46}$$

yields the required result.

7.3 Proof of Theorem 2

To proof the theorem we use the convolution theorem, according to which the original of the product of two functions is equal to the convolution of the originals of the functions, i.e.

$$\mathcal{F}^{-1}(F(\omega)G(\omega))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}^{-1}(F(\omega))(z)\mathcal{F}^{-1}(G(\omega))(x-z)dz \tag{47}$$

We have

$$\mathcal{F}^{-1}(\mathcal{F}(f(Ke^x,T))) = f(Ke^x,T) \tag{48}$$

and

$$\mathcal{F}^{-1}\left(e^{-\frac{\sigma^2}{2}A(\omega)(T-t)}\right) = \frac{e^{-r(T-t)-\frac{\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+x\right)^2}{2\sigma^2(T-t)}}}{\sqrt{(T-t)\sigma^2}}. \tag{49}$$

By the convolution theorem and plugging the expression for $f(Ke^x,T)$ we obtain

$$f_1(V,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_0^K \frac{K-(1-\theta)z}{z} e^{-\frac{\left(\ln\frac{z}{V}-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dz. \tag{50}$$

Some algebra yields the required result.

7.4 Proof of Theorem 3

To proof the theorem we start from the equation (15). Integration by parts of the second term in the left hand side and some algebra yields

$$F_2(\omega,t) = -\frac{\theta K}{\imath\omega\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(T-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)-\ln(1-\theta)\right)} + \tag{51}$$

$$+\frac{Ke^{-r(T-t)}-(1-\theta)g(t)}{\imath\omega\sqrt{2\pi}} e^{\imath\omega \ln\frac{(1-\theta)g(t)}{K}} +$$

$$+\int_t^T \frac{(1-\theta)(rg(\tau)-g'(\tau))}{\imath\omega\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(\tau-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}\right)} d\tau -$$

$$-\frac{\sigma^2}{2}\int_t^T \frac{(1-\theta)g(\tau)}{\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(\tau-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}\right)} d\tau.$$

We have

$$\mathcal{F}^{-1}\left(-\frac{\theta K}{\imath\omega\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(T-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)-\ln(1-\theta)\right)}\right) = \tag{52}$$

$$= \frac{\theta K e^{-r(T-t)}}{2}\mathrm{Erf}\left(\frac{x+(r-\frac{\sigma^2}{2})(T-t)-\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right),$$

$$\mathcal{F}^{-1}\left(\frac{Ke^{-r(T-t)}-(1-\theta)g(t)}{\sqrt{2\pi}\imath\omega} e^{\imath\omega\ln\frac{(1-\theta)g(t)}{K}}\right) = \tag{53}$$

$$= -\frac{Ke^{-r(T-t)}-(1-\theta)g(t)}{2}\mathrm{sign}\left(x-\ln\frac{(1-\theta)g(t)}{K}\right),$$

$$\mathcal{F}^{-1}\left(\int_t^T \frac{(1-\theta)(rg(\tau)-g'(\tau))}{\imath\omega\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(\tau-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}\right)} d\tau\right) = \tag{54}$$

$$= -\int_t^T \frac{(1-\theta)(rg(\tau)-g'(\tau))e^{-r(\tau-t)}}{2}\mathrm{Erf}\left(\frac{x+(r-\frac{\sigma^2}{2})(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}}{\sigma\sqrt{2(\tau-t)}}\right) d\tau$$

$$\mathcal{F}^{-1}\left(\frac{\sigma^2}{2}\int_t^T \frac{(1-\theta)g(\tau)}{\sqrt{2\pi}} e^{-\left(r+\frac{\sigma^2\omega^2}{2}\right)(\tau-t)-\imath\omega\left(\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}\right)} d\tau\right) = \tag{55}$$

$$= \int_t^T \frac{(1-\theta)g(\tau)e^{-r(\tau-t)}}{2} \frac{\sigma_0}{\sqrt{\pi(\tau-t)}} e^{-\left(\frac{x+\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}}{\sigma\sqrt{2(\tau-t)}}\right)^2} d\tau.$$

Using expressions (19), (52)-(55) in the condition (11) we obtain

$$\frac{K}{2}e^{-r(T-t)}\left(\mathrm{Erfc}(\eta_0)+\mathrm{sign}\left(-\ln(1-\theta)\right)\right)-(1-\theta)\frac{g(t)}{2}\mathrm{Erfc}(\eta_1)+ \tag{56}$$

$$+(1-\theta)\frac{g(t)}{2}\mathrm{sign}\left(-\ln(1-\theta)\right)-\frac{\theta K}{2}e^{-r(T-t)}\mathrm{Erf}\left(\eta_0-\frac{\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right)+$$

$$+\int_t^T \frac{(1-\theta)(rg(\tau)-g'(\tau))e^{-r(\tau-t)}}{2}\mathrm{Erf}\left(\sigma_1\sqrt{\tau-t}-\frac{\ln(1-\theta)}{\sigma\sqrt{2(\tau-t)}}\right)d\tau+$$

$$+\int_t^T \frac{(1-\theta)g(\tau)e^{-r(\tau-t)}}{2}\frac{\sigma_0}{\sqrt{\pi(\tau-t)}}e^{-\left(\sigma_1\sqrt{\tau-t}-\frac{\ln(1-\theta)}{\sigma\sqrt{2(\tau-t)}}\right)^2}d\tau =$$

$$= Ke^{-r(T-t)}-(1-\theta)g(t).$$

Integrating by parts the first integral in the equation (56) and some algebra yields the required result.

7.5 Proof of Theorem 4

First of all denote $\tau = T-\tau_1$ and $t = T-t_1$ in the equation (26). Then the integral equation becomes

$$\frac{g(T-t_1)}{K}\left[1+\mathrm{Erf}(\eta_1(g(T-t_1)))\right]-e^{-rt_1}\left[1+\mathrm{Erf}(\eta_0(g(T-t_1))+\mathrm{Erf}(\sigma_1\sqrt{t_1})\right]+$$

$$+\int_0^{t_1}\frac{g(T-\tau_1)}{K}\frac{\sigma_2}{\sqrt{\pi(t_1-\tau_1)}}e^{-\sigma_2^2(t_1-\tau_1)}d\tau_1 = 0. \tag{57}$$

Denote $\eta_1(g(T-t_1))$ by $u(t_1)$ and express $\frac{g(T-t_1)}{K}$ and $\eta_0(g(T-t_1))$ in terms of $u(t_1)$. Then, the first two terms in the equation (57) and $\frac{g(T-\tau_1)}{K}$ in the integrand will be nonlinear functionals of unknown function u. Those functionals can be approximated by the linear functionals in u, in which case we obtain the following equation

$$-e^{\sigma_0^2 t_1}(1-\mathrm{Erf}[\sigma_0\sqrt{t_1}]+\mathrm{Erf}[\sigma_1\sqrt{t_1}])+\left(1+\sigma u(t_1)\sqrt{2t_1}\right)+ \tag{58}$$

$$+\sigma_2\int_0^{t_1}(1+\sigma u(\tau_1)\sqrt{2\tau_1})\frac{e^{(\sigma_0^2-\sigma_1^2)(t_1-\tau_1)}}{\sqrt{\pi(t_1-\tau_1)}}d\tau_1=0.$$

Luckily, we obtain an integral equation for which solution methods are available. Let $\mathcal{L}\left(\sigma u(t_1)\sqrt{2t_1}\right)=U(p)$ and making Laplace transform of other expressions in the integral equation (58) we obtain a linear equation in $U(p)$, which has the following solution

$$U(p)=\frac{-\sqrt{p}\sigma_0\sqrt{p-\sigma_0^2+\sigma_1^2}+p(\sigma_1-\sigma_2)+\sigma_0^2(\sqrt{p-\sigma_0^2+\sigma_1^2}+\sigma_2)}{p(p-\sigma_0^2)(\sqrt{p-\sigma_0^2+\sigma_1^2}+\sigma_2)}. \tag{59}$$

Then, $\mathcal{L}^{-1}(U(p))$ yields the required result.

7.6 Proof of Theorem 5

It is not difficult to show that

$$\int_t^T\frac{(1-\theta)(rg(\tau)-g'(\tau))e^{-r(\tau-t)}}{2}\mathrm{Erf}\left(\frac{x+(r-\frac{\sigma^2}{2})(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}}{\sigma\sqrt{2(\tau-t)}}\right)d\tau+$$

$$+\int_t^T\frac{(1-\theta)g(\tau)e^{-r(\tau-t)}}{2}\frac{\sigma_0}{\sqrt{\pi(\tau-t)}}e^{-\left(\frac{x+\left(r-\frac{\sigma^2}{2}\right)(\tau-t)-\ln\frac{(1-\theta)g(\tau)}{K}}{\sigma\sqrt{2(\tau-t)}}\right)^2}d\tau= \tag{60}$$

$$=\frac{(1-\theta)Ke^{-r(T-t)}}{2}\mathrm{Erf}\left(\sigma_1\sqrt{T-t}+\frac{x-\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right)-$$

$$-\frac{(1-\theta)g(t)}{2}\mathrm{sign}\left(x-\ln\frac{(1-\theta)g(t)}{K}\right)+$$

$$+\frac{Ke^x}{2}\mathrm{Erfc}\left(\mathrm{sign}\left(x-\ln\frac{(1-\theta)g(t)}{K}\right)\left(\sigma_2\sqrt{T-t}+\frac{x-\ln(1-\theta)}{\sigma\sqrt{2(T-t)}}\right)\right)$$

The result (60), together with the expressions (52) and (53) yields the result (31), while (31) together with (19) yields (33).

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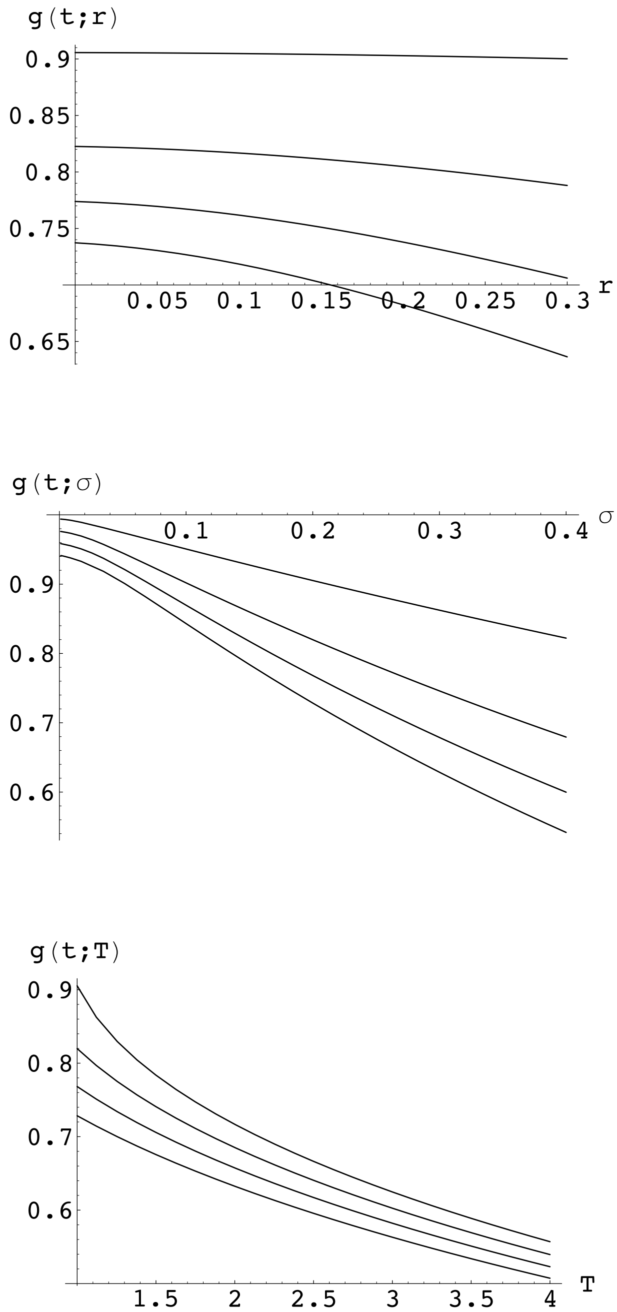

Figure 6: The results of comparative statics of the reorganization boundary $g(t)$ with respect to the model parameters. Each line represents a graph of $g(t)$, plotted against a parameter r (top panel), σ (middle panel) and T (bottom panel), when t is fixed to be 0.9, 0.6, 0.3 and 0, respectively.