DOES THE VIX JUMP?

IMPLICATIONS FOR PRICING AND HEDGING VOLATILITY RISK

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Abstract

Implied volatility indices are becoming increasingly popular as a measure of market uncertainty and as a vehicle for developing derivative instruments to hedge against unexpected changes in volatility. Although jumps are widely considered as a salient feature of volatility, their implications for volatility options and futures are not yet fully understood. This paper provides evidence indicating that the empirical behavior of the VIX equity implied volatility index over a period of 10 years is well approximated by a square root mean reverting process with jumps. By augmenting the popular Longstaff and Grunbichler (1996) option pricing model, we show that incorrectly omitting jumps may cause considerable problems to pricing and hedging.

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1. Introduction

Volatility is undoubtedly the most important variable in finance. It appears consistently across a wide spectrum of theories and applications in asset pricing, portfolio theory, risk management, derivatives, corporate finance, investment evaluation and econometrics. Most of our obsession with the analysis of volatility has to do with the simple fact that it is not directly observable. A myriad of alternative measures and approaches have been developed in academia and industry in order to empirically measure volatility.

A fascinating recent development has been the treatment of volatility as a distinct asset which can be packaged in an index and traded using volatility swaps, futures and options. Volatility derivatives are considered by some to "*have the potential to be one of the most important new financial innovations*" (Grunbichler and Longstaff, 1996). Traditionally, derivatives have allowed investors and firms to hedge against factors such as market volatility, interest rate volatility and foreign exchange volatility. Volatility derivatives provide protection against volatility risk, that is, unexpected changes in the volatility level itself. Such changes may arise as a response to changes in macroeconomic or microeconomic conditions (see, for example, Copeland et al., 2000). A widely cited example of the importance of volatility risk concerns the remarkable shift in volatility that followed the 1987 crash.

The first volatility index, named VIX (currently termed VXO), was introduced in 1993 by the the Chicago Board Options Exchange (CBOE). This was estimated from implied volatilities from at-the-money options on the SP100 index using a methodology proposed by Whaley (1993). The CBOE adopted a new methodology in 2003 to calculate VIX as an average of out-of-money option prices across all available strikes on the S&P 500 index. Several other implied volatility indices have been developed ever since, including: the VXN and VXD in the CBOE, the VDAX-NEW in Germany, the VX1 and VX6 in France, the VSTOXX in the Eurex, the VSMI in Switzerland, the MVX in Canada, etc. Volatility derivatives have been traded over the counter for several years, mainly as volatility swaps. However, only recently, in March 2004, the Chicago Board of Exchange (CBOE) introduced volatility futures on the implied volatility measured by the VIX index. The CBOE has announced the imminent introduction of volatility futures on the implied volatility index VXD along with volatility options. Eurex, has launched in September 2005 three new volatility futures on the VDAX-NEW, VSTOXX and VSMI volatility indices.

Options and futures written on a volatility index were first suggested by Brenner and Galai (1989, 1993) as a response to the growing need for instruments to hedge volatility risk. It has been argued that volatility derivatives make the markets more complete since they expand the available of investment opportunities and allow direct hedging of volatility risk, without necessarily resorting to dynamical adjustments. Traditionally, volatility could be traded via at-the-money straddles, whose value increases with volatility. But straddles have the disadvantage of creating both market and volatility exposure. The market effect can be removed by rolling forward, however this is done at uncertain future market levels and trading costs. In contrast, volatility derivatives allow pure volatility exposure by design. Volatility indices are particularly useful in monitoring market expectations. The popular financial press, eg. CNBC, Barrons, Wall Street Journal, regularly quotes the VIX volatility index as an "investor fear gauge". Regulatory bodies and central banks, such as the Bank of England, have used the VIX to depict equity uncertainty and relate it to subsequent movements in other variables, such as swap spreads.[1] Volatility derivatives have a wide range of important applications for all market participants. Investment funds employ volatility derivatives for vega hedging their portfolios against movements in volatility. Certain classes of investors, such as convertible bond arbitrage funds and structured product issuers, can use these derivatives to insure against their structural exposure to volatility. Investors can employ them to partially insure against shifts in transaction costs and tracking error penalties, both of which increase during periods of high uncertainty. Investment managers may use these derivatives to hedge against the risks of a so-called high-correlation environment. This is because, asset correlations have been found to increase significantly during periods of high volatility, making active asset picking and portfolio diversification very difficult. As volatility is a key input for risk management and capital adequacy models, such as the VaR, volatility derivatives could be used by banks as a shield against shifts in volatility and correlation during stress market conditions. Since shifts in equity risk have a significant impact on risk premia, firms could employ volatility derivatives to protect themselves from unexpected changes in cost of capital. Although not available yet, bond and foreign exchange volatility indices and derivatives, would allow firms that are exposed to volatility in these markets to hedge against changes in volatility. Finally, ample liquidity in this market is provided by traders and hedge funds since volatility derivatives can provide the most efficient and low-cost way for speculating against changes in volatility.

A number of recent empirical studies have examined the properties of implied volatility indices (e.g., Fleming et al., 1995; Moraux et al., 1999; Whaley, 2000; Blair, et

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[1] Bank of England, *Quarterly Bulletin*, Winter 2003.

al. 2001; Corrado and Miller, 2003; Simon, 2003, and, Giot, 2005). This research has demonstrated the practical importance of at-the-money implied volatility as an efficient, yet biased, forecast of future realized volatility. There has been also been a growing interest in modeling the time series dynamics of the autonomous implied volatility process. Bakshi et al. (2005) estimated various general specifications of diffusion processes with a non-linear drift and diffusion component. The author considers the squared implied volatility index VIX as a proxy to the unobserved instantaneous variance. Wagner and Szimayer (2004) investigated the presence of jumps in implied volatility by estimating an autonomous mean reverting jump diffusion process using data on the implied volatility indices VIX and VDAX. They found evidence of significant positive jumps in implied volatilities. However, they adopted the rather restrictive assumption that the volatility jump size is constant rather than being random. Finally, Dotsis et al. (2005) examined the ability of alternative popular continuous-time diffusion and jump diffusion processes to capture the dynamics of eight major European and U.S. volatility indices. They found that the best model in terms of fitting was a mean reverting process with random upward and downward jumps.

In response to the developments in the industry and academia, Grunbichler and Longstaff (1996) developed the first models for the valuation of futures and European-style options written on instantaneous volatility. The authors assumed that the underlying volatility followed a mean reverting square root process, similar to that used earlier by Heston (1993). Detemple and Osakwe (2000) provided analytical formulas to price both American and European-style volatility options assuming a mean-reverting in log volatility model. The discrete time analogs in the limit of the volatility process used by these two studies are the GARCH and EGARCH process, respectively. Heston and

Nandi (2000a) derived analytical solutions in both discrete and continuous time for pricing European options written on variance. These were based on a discrete-time GARCH volatility process and its continuous time counterpart developed by Heston and Nandi (2000b). Recently, Daouk and Guo (2004), studied the valuation of volatility options based on a Switching Regime Asymmetric GARCH process for the underlying.

Motivated by the increasing importance of volatility derivatives, this paper examines three main issues. Firstly, it extends the empirical literature on volatility indices using daily data on the VIX index for a period of 10 years. It confirms previous findings of mean reversion and heteroskedasticity and provides new evidence concerning stationarity, long-memory, non-normality and jump behavior. In line with previous research (eg., see Wagner and Szimayer, 2004; Dotsis et al., 2005) estimation results using the VIX data shows that the empirical fit of the popular mean reverting square root process proposed by Grunbichler and Longstaff (1996) can be significantly improved by the addition of jumps. We provide new evidence showing that if the jump occurring is conditioned on the level of the index, model performance is further enhanced. Simulation analysis suggests that the addition of jumps enables the process to produce highly non-normal distributions with higher moments that closely resemble those of the actual data. The possibility and implications of jumps has been examined by the literature dealing with the joint dynamics of volatility and asset returns (eg., Duffie *et al*, 2000; Eraker *et al*, 2003; Eraker, 2004). They have been used in order to better capture salient features of asset returns such as skewness and leptokurtosis. The implications of jumps are interesting since they constitute a source of systematic, rather than unsystematic, risk, and, they challenge conventional hedging strategies assuming smooth diffusion variations. Moreover, jumps are considered particularly important for accurately pricing short-term options, since pure diffusions are not capable of producing realistic levels of higher moments at short horizons.

Secondly, the paper studies closed form expressions for pricing futures and European options on volatility assuming a mean reverting square root process with jumps. It is demonstrated that when jumps are allowed to depend on the level of volatility, pricing is also possible via numerical analysis. The option pricing model proposed nests as a special case the model by Longstaff and Grunbichler (1996). It is based on the same volatility dynamics as those implied by the so-called "double jump" processes, where both the underlying asset price and the instantaneous volatility follow jump-diffusion processes (see, for example, Duffie et al., 2000; Bakshi and Cao, 2004; Broadie et al., 2004; Eraker, 2004). These models have been shown to be superior in terms of fitting traded index and equity options price series. From a risk management perspective, it makes sense to use the same model for the autonomous volatility process and the joint dynamics of volatility and asset returns.

Thirdly, the paper assesses the potential implications for volatility derivative pricing and hedging of incorrectly omitting jumps from the diffusion process for volatility. It is demonstrated that prices and hedge ratios may differ substantially. The model without jumps in volatility (ie., the Longstaff and Grunbichler, 1996, model) significantly undervalues (overvalues) short (long) maturity options, on average, by 25% (14%), respectively. Moreover, it is far more sensitive to changes in the underlying with the delta hedging parameter being twice as large. As argued by Daouk and Guo (2004), even though empirical analysis of option mispricing is not yet possible due to the lack of data, it is imperative to thoroughly understand all the issues related to pricing and hedging derivatives on volatility, prior to their introduction to the market. This will ensure the smooth and successful operation of the market and the effective use of volatility derivatives by investors.

The remainder of the paper is structured as following. The next section analyses the empirical behavior of the VIX for a period of 10 years. Section 3, describes the mean-reverting square root volatility process along with two jump diffusion extensions. It also discusses estimation issues and presents an empirical application using the VIX data. Section 4, develops valuation formulae for volatility futures and European options when the underlying volatility follows a mean-reverting jump-diffusion process. It also discusses the properties of these models and explores the potential importance of jumps from the perspective of pricing and risk management, respectively. The final section concludes the paper.

[INSERT FIGURE 1 HERE]

2. Empirical Properties of the VIX

We use data over the complete life of the VIX volatility index, from 1/2/1990 to 9/13/2005, a total of 3,957 closing daily prices[2]. The VIX is traded in the CBOE and is constructed from out-of-the-money (OTM) puts and calls of 2 options nearest to 30 days expiries, covering a wide range of strikes. The construction of the VIX is independent of the model used to price the OTM options. It represents the implied volatility of a synthetic option that is at-the-money and has a "constant" maturity, 30 calendar days to expiry at any point in time. Figure 1 depicts the evolution of the VIX and its first

[2] Data are drawn from the website of the CBOE.

differences for the period under study. The time series plots suggest a volatile mean-reverting behavior for the levels with violent swings. First differences appear heteroskedastic with a number of spikes.

The summary statistics of the series, shown in Table 1, largely confirm this behavior. The VIX ranges between about 9% to 45%, with an average of 19.6%. The higher moments suggest a leptokuyrtotic distribution skewed to the right for both levels and differences. The Jarque-Bera test rejects the normality assumption with a high level of confidence. Autocorrelations die out slowly in levels, something consistent with a smooth, possibly mean reverting process. Differences appear anti-persistent with small negative short-term autocorrelations. The highly significant squared autocorrelations strongly suggest heteroskedasticity.

[INSERT TABLE 1 HERE]

[INSERT TABLE 2 HERE]

Given that simple brownian motion processes have also been employed in the literature to model volatility indices, we examine the stationarity of the VIX levels. The Augmented Dickey-Fuller (Dickey and Fuller, 1979) and Phillips-Perron (1988) tests reject the null hypothesis of a unit root with a high level of confidence. However, the null hypothesis of stationarity cannot be accepted on the basis of the Kwiatkowski-Phillips-Schmidt-Shin (KPSS, 1992). Since the KPSS test is sensitive to long-memory (see, for example, Lee and Schmidt, 1996)] and motivated by relevant empirical findings in the literature with respect to long-memory in historical volatility (eg., Ding, Granger and Engle, 1993; Baillie et al. 1996, by Breidt, et al., 1998), we examine further this possibility. Lo's (1991) modified R/S test statistic for long range dependence is significant at the 5% level with a value of 9.4875. The Geweke and Porter-Hudak (1983) log-periodogram method implemented with the trimming and smoothing options proposed by Robinson (1995), produced an estimate of fractional unit root *d* equal to 0.7236 (p = 0.0736). One must view this evidence with caution since long-memory tests are sensitive to a variety of factors such as structural breaks, outliers, regime switching and nonlinear transformations (see, for example, Diebold and Inoue, 2001; Engle and Smith, 1999; Dittmann and Granger, 2002). Moreover, it is possible that long-memory behavior is the result of aggregation in constructing the VIX. Granger (1980) pointed out that the summation of low-order ARMA processes will yield ARMA processes of increasing, and eventually infinite order which can be well approximated using an ARFIMA model. Notwithstanding, on the basis of the results presented, the possibility of long-memory characteristics in the VIX cannot be excluded.

[INSERT FIGURE 2 HERE]

We proceed in examining the unconditional distribution of the VIX levels and differences. As shown by the results contained in Figure 2, the unconditional distribution of the VIX closely resembles the shape of a highly skewed distribution, such as the chi-squared. The distribution of differences is clearly leptokurtotic. Fitting a variety of distributions via maximum likelihood is consistent with these suggestions, the results given in Table 3. The distribution of VIX levels appears to be well approximated by a skewed *t*-student and a Gamma distribution, the latter nesting the chi-squared as a special case. The log-normal also appears to fit relatively well the VIX levels. The unconditional distribution of VIX differences is well approximated by a *t*-student.[3] The normal distribution offers a relatively poor fit for both levels and differences.

[INSERT TABLE 3 HERE]

A more detailed breakdown of the unconditional distributions is presented in Table 4. Given that the standard deviation of differences is around 0.0122 with a mean very close to zero, we can observe 20 distinct four-standard deviation events, 8 downward and 12 upward. Under a normal distribution, which is consistent with some diffusion models of volatility, the variance implies that these events should occur with probability under 0.005% or once in about every 80 years. Here, we observe a much higher probability of occurrence, 100 times higher, of over 0.5%, or, once in every 164 days. These findings are expected, given the fat-tails in the ΔVIX distribution and could be due also to jumps in the underlying process. One must be careful in interpreting large negative changes as downward jumps since they could also be the result of heteroskedasticity and mean reversion.[4] Finally, we can also see that the likelihood of large upward movements in volatility seems to increase with the volatility level. For example, large volatility changes over 5% appear with probability 0.29% (4/1,369),

[3] Although results are not shown here, differences remain highly non-normal even if estimated as logarithmic ratios.

[4] Statistically signiginant evidence of downward jumps has been reported in the literature for interest rates (Das, 2002) and volatility (Dotsis et al., 2002).

2.34% (6/256) and 6.9% (2/29) for volatility levels in the [0.2, 0.3), [0.3, 0.4) and [0.4, 0.5) range, respectively.

[INSERT TABLE 4 HERE]

3. Diffusion and Jump Diffusion Processes for the VIX

One of the simplest processes to model volatility is the Mean Reverting Gaussian Process (also called Ornstein – Uhlenbeck). It was initially proposed in order to capture the mean reverting empirical property of volatility (eg., Hull and White, 1987; Stein and Stein, 1991; Scott, 1987; Brenner, Ou and Zhang, 2001). Under this process, the implied volatility changes are normal, something that is clearly rejected from our empirical analysis of the VIX. Moreover, this process has the significant disadvantage of allowing negative values. Detemple and Osakwe (2000), among others, have employed the Mean-Reverting Logarithmic Process whereby the unconditional volatility distribution follows a log-normal distribution. However, our results indicate that although the VIX unconditional distribution resembles a lognormal, the differences are highly non-normal. One of the most popular alternative processes that have been developed in the literature is the Mean Reverting Square Root Process (SR)[5]:

$$dV_t = (a + bV_t)dt + \sigma\sqrt{V_t}dZ \tag{1}$$

[5] See Hull and White (1988), Heston (1993), Ball and Roma (1994), Heynen et al. (1994), Grunbichler and Longstaff (1996), Bates (2000), and, Jones (2003).

Heston and Nandi (2000b) have shown that a degenerate case of the SR can be obtained as a limit of a particular GARCH-type process. This process should be able to capture the two basic empirical characteristics of the VIX: mean reversion and heteroskedasticity. Furthermore, volatility under the SR follows a non-central Chi-squared distribution (see Cox *et al.*, 1985), which is consistent with our analysis of the VIX unconditional distribution. However, since the preliminary analysis suggests also the possibility of upward jumps in the VIX, we consider two types of mean reverting processes augmented with upward jumps. One with constant probability of jump (SRJ) and one with the probability of jump being proportional to the level of implied volatility (SRPJ). Although we cannot exclude the possibility of abrupt downward movements also, these cannot be readily included in our model as jumps since they would allow negative values for the underlying, something non-admissible for option pricing purposes. We do not attempt to account for long-memory or more complicated nonlinear dynamics in the data, since these have been examined in detail by other studies and are outside the focus of this paper (see, for example, Bakshi et al., 2004; Daouk and Guo, 2004; Bollerslev and Mikkelsen, 1996).

Under the actual probability measure *P*, the SRJ and SRPJ is given by:

$$dV_t = k\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}dW_t + ydq_t \tag{2}$$

where dW_t is a standard Brownian motion dq_t is a compound Poisson process and y is the jump amplitude. In the SRJ process dq_t has a constant arrival parameter λ, whereas in the SRPJ process the arrival parameter is proportional to V_t, that is, $\Pr\{dq_t=1\}= \lambda V_t dt$.

dW and dq are assumed to be independent processes. We further assume that the jump size is drawn from exponential distribution:

$$f(y) = p\eta e^{-\eta y} 1_{\{y \geq 0\}} \tag{3}$$

where $1/\eta$, is the mean of the upward jump. The exponential distribution allows us to capture upward jumps in implied volatility and derive the characteristic function in closed form. The one sided exponential distribution adopted is a version of the double exponential distribution used by Kou (2004) in modeling the dynamics of stock and index prices. Duffie et al. (2000) prove that, under technical regularity conditions, the characteristic function for affine diffusion/jump diffusion processes has the following exponential affine form:

$$F(V_t, \tau; s) = \exp\left(A(\tau; s) + B(\tau; s)V_t\right) \tag{4}$$

Thus, the characteristic function of the SRJ can be derived as[6]

$$A(\tau; s) = a(\tau, s) + z(\tau, s) \tag{5}$$

$$a(\tau; s) = -\frac{2k\theta}{\sigma^2} \times log\left(\frac{k - \frac{1}{2} i\sigma^2 s\left(1 - e^{-k\tau}\right)}{k}\right) \tag{6}$$

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[6] This characteristic function has also been used for estimating purposes by Bakshi and Cao (2005).

$$z(\tau;s)=\frac{2\lambda p}{2k-\eta_1\sigma^2}\times log\left(\frac{k-\frac{1}{2}i\sigma^2 s+is\left(\frac{\sigma^2}{2}-\frac{k}{\eta_1}\right)e^{-k\tau}}{k-\frac{isk}{\eta_1}}\right) \tag{7}$$

and,

$$B(\tau;s)=\frac{ksie^{-k\tau}}{k-\frac{1}{2}i\sigma^2 s\left(1-e^{-k\tau}\right)} \tag{8}$$

In the case of the SRPJ, the coefficients $A(\tau;s)$ and $B(\tau;s)$ cannot be solved in closed form and are found numerically (see the appendix). The SR, SRJ and SRPJ are estimated via Maximum Likelihood (ML).[7] In addition to consistency and increased efficiency, ML estimation has been found able to disentangle the diffusion from the jump component.

The implementation of ML estimation requires the knowledge of the transition density function. In general, the addition of a jump component does not permit the derivation of the density function in closed form. However, for the case of the SR, the addition of exponential jumps permits the derivation of the characteristic function in closed form. Duffie et al. (2000) prove that under technical regularity conditions, the characteristic function for affine diffusion/jump diffusion processes, such as the SRJ and SRPJ, has an exponential affine form, which can be derived in closed form or numerically. The required conditional density function can be obtained by means of Fourier inversion of the characteristic function. Maximizing the likelihood function via Fourier inversion, though computationally intensive, provides asymptotically efficient

[7] For a description of ML estimation for such processes see also, for example, Das (2002), Ait-Sahalia (2004) and Dotsis et al. (2005).

estimates of the unknown parameters (see Singleton, 2001 for a discussion and applications).

Suppose that $\{V_t\}_{t=1}^{T}$ is a discretely sampled time series of implied volatilities. Assume we stand at time *t*, and τ denotes the sampling frequency of observations. Then, the Fourier inversion of the characteristic function $F(V(t),\tau;s)$ provides the required conditional density function $f[V(t+\tau)|V(t)]$:

$$f[V(t+\tau)|V(t)] = \frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{-isV(t+\tau)}F(V(t),\tau;s)]ds \tag{9}$$

where Re denotes the real part of complex numbers. For a sample $\{V(t)\}_{t=1}^{T}$, the conditional log-likelihood function to be maximized is given by:

$$\Im = \max_{\{\Theta\}} \sum_{t=1}^{T} \log\left(\frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{-isV(t+\tau)}F(V(t),\tau;s)]ds\right) \tag{10}$$

where $\Theta=\{\kappa, \theta, \sigma, \lambda, \eta\}$ is the set of parameters to be estimated. The standard errors of the ML estimators are retrieved from the inverse Hessian evaluated at the obtained estimates.

As discussed previously, the implied volatility is distributed according to a non-central chi-squared distribution under the SR. The transition density is given by:

$$f(V(t+\tau)|V(t),\Theta) = ce^{-u-v}(u/v)^{q/2} I_q\left(2(uv)^{1/2}\right) \tag{11}$$

where $c \equiv 2k/(\sigma^2(1-e^{k\tau}))$, $u \equiv cV(t)1-e^{k\tau}$, $v \equiv cV(t+\tau)$, $q = 2k\theta/\sigma^2 - 1$ and $I_q(\bullet)$ is the modified Bessel function of the first kind of order *q*. The set of parameters *Θ* to be estimated is *Θ*={*κ, θ, σ*}.

4. Estimation Results

Table 5 shows the ML estimation results using the VIX sample. For each process we report: the estimated parameters (annualized), the asymptotic *t*-statistics (within brackets), the Akaike Information Criterion (AIC), the Bayes Information Criterion (BIC), and, the log-likelihood values. The SRPJ appears to have the highest log-likelihood value amongst the competing models. Since the models are hierarchically nested, the likelihood ratio (LR) test can be employed to compare relative goodness-of-fit. We find that the likelihood of the SRJ is significantly higher than that of the SR, the LR test statistic being 318.5 (the critical value at the 1% level from a Chi-squared with two degrees of freedom is 9.21). Allowing the probability of jumps to be proportional to volatility, causes a further statistically significant improvement in likelihood (LR= 73.74). The information criteria also suggest that the addition of jumps in proportion to the volatility level improves fitting ability.

[INSERT TABLE 5 HERE]

The SRPJ suggests an average jump frequency ($\lambda\bar{V}$) over 50 per year, assuming an average daily-implied volatility $\bar{V} = 19.57\%$, with an average jump magnitude (*1/η*) equal to 1.25%. The SRJ implies a smaller frequency of jumps, 19 per year, of larger magnitude, 1.77%. The parameters are comparable in magnitude to those estimated by Dotsis et al. (2005) for the VIX of a shorter sample. In line with previous studies (eg., Das, 2002), we find that the addition of jumps to the SR decreases the estimated volatility (σ) and long-run average volatility (θ) of the underlying process, implying that jumps account for a substantial component of variability. Moreover, incorporating jumps, especially if they are conditioned on the level, increases the estimated speed of mean reversion (κ). One explanation for this could be that a higher mean reversion is needed in order to force the process after a jump to revert back to a realistic level, especially since volatility is lower.

[INSERT TABLE 5 HERE]

In order to further asses the ability of the fitted models to represent the original series, we undertake a simulation experiment along the lines of Pan (2002), Jones (2003) and Eraker (2004). More specifically, we examine if the estimated models have the ability to generate unconditional distribution behavior in levels and differences that is consistent with that of the VIX index. More specifically, we estimate the sample skewness and kurtosis for the VIX data using three sampling intervals: daily, weekly and monthly. Then, we use Monte Carlo simulation to approximate the finite sample distribution of the skewness and kurtosis coefficients for the estimated three processes estimated. The sample size and sampling interval for each simulation are selected accordingly to the VIX sample. Finally, the quantiles of the empirical distribution can be used in an exact finite sample hypothesis test based on the null of a given diffusion process.

The results, summarized in Table 6, show that a reasonable amount of unconditional non-normality in levels is implied by all three processes under study. However, the higher moments of the SR differences are unrealistically smaller than those of the actual VIX levels. For example, at a daily level the kurtosis of the ΔVIX is over 9, while the 99% percentile for the SR is only 3.6. The simulated differences of the SPJ and SRPJ are capable of producing unconditional kurtosis coefficients that are consistent with those of the original data. However, the SPJ skewness term structure pattern follows closer that of the VIX differences, although it falls somewhat short for the daily and monthly data. However, we believe that the size of this inconsistency does not warrant serious concerns.

5. Pricing of Volatility Derivatives

In this section, we derive analytical formulae for pricing option and futures contracts on volatility when the underlying follows a mean-reverting square root process with jumps (SRJ). Pricing when the probability of jumps is proportional to the volatility level (SRPJ) is also possible numerically, but is not undertaken here in order to preserve simplicity. Rather, we examine some of the properties of the analytical models and investigate the potential implications for pricing and hedging of incorrectly omitting jumps from a SR process.

5.1. Volatility Futures

Before proceeding to futures valuation, we must rewrite equation (2) under the risk neutral probability measure *Q*. Following Heston (1993), Grunbichler and Longstaff (1996) and Pan (2002), we assume that the volatility risk is proportional to the current level of volatility, i.e., $\zeta_V V_t$. A similar risk premium ζ_j is assumed to be associated with jumps.[8] So, the volatility process under the risk neutral measure is given by:

$$dV_t = \left(k(\theta - V_t) - \zeta_V V_t - \zeta_j\right)dt + \sigma\sqrt{V_t}dz + Jdq \quad (12)$$

or, equivalently,

$$dV_t = k^*(\theta^* - V_t) + \sigma\sqrt{V_t}dz + Jdq \quad (13)$$

where $k^* = k + \zeta_V$ and $\theta^* = \dfrac{k\theta - \zeta_j}{k + \zeta_V}$.

Now denote $F_t(V,T)$ the price of a futures contract on V_t at time t with maturity T. Under the risk-adjusted equivalent martingale measure *Q*, $F_t(V,T)$ is determined by the conditional expectation of V_T at time T. This expectation is conditional on the information up to time t:

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[8] Our approach is similar to those of Bakshi *et al* (2004), Pan (2002), Eraker(2004), where they use similar assumptions as far as the jump risk premium is concerned in the case of stock returns.

$$F_t(V,T) = E_t^Q(V_T), \qquad t < T \tag{14}$$

As the conditional density function is not known in closed form, the characteristic function can be used to derive the expectation of $E_t^Q(V_T)$. This is done by differentiating the characteristic function once with respect to *s* and then evaluating the derivative at *s*=0.

$$E_t\left(V_T\right) = V_t e^{-k^*(T-t)} + \theta^*(1-e^{-k^*(T-t)}) + \frac{\lambda}{k^*}(1-e^{-k^*(T-t)})\frac{1}{\eta} \tag{15}$$

Equation (15) consists of three terms: the first and the second correspond to the diffusion part of the SRP, while the third term corresponds to the jump part. The following equation, corresponding to the Grünbichler and Longstaff (1996) volatility futures model, describes the expected value of volatility under the SR:

$$E_t\left(V_T\right) = V_t e^{-k^*(T-t)} + \theta^*(1-e^{-k^*(T-t)}) \tag{16}$$

We can see that the only difference between equations (15) and (16) is the term $\frac{\lambda}{k^*}(1-e^{-k^*(T-t)})\frac{1}{\eta}$. This allows a direct comparison between the diffusion and the jump diffusion formulae. Recall that $E\left(y\right) = \frac{1}{\eta} > 0$, then the volatility futures price under the SRP will be greater than the price delivered by its diffusion counterpart, the Grünbichler and Longstaff (1996) volatility futures model. The magnitude of the difference depends on the average size of the jumps $1/\eta$ as well as on the number of the jumps λ.

Finally, the dynamics of the futures price are given by applying Ito's Lemma to (13):

$$\begin{aligned} dF_t &= -\lambda e^{-k^*(T-t)} E\left[y\right] dt \\ &+ e^{-k^*(T-t)} \sigma \sqrt{\left(F_t e^{k^*(T-t)} - \theta^* (e^{k^*(T-t)} - 1) - \frac{\lambda}{k^*} (e^{k^*(T-t)} - 1) E\left[y\right] \right)} dW_t \\ &+ y e^{-k^*(T-t)} dq_t \end{aligned} \tag{17}$$

We can see that the drift, volatility and jump structures have been modified. The mean reversion structure has vanished while the jump structure is scaled downwards by an exponential term.

The futures pricing formula (15) has the following limiting properties:

i. $$\lim_{\tau \to 0} E_t\left(V_T\right) = V_t, \tag{18}$$

ii. $$\lim_{\tau \to +\infty} E_t\left(V_T\right) = \theta^* + \frac{\lambda}{k^* \eta}, \tag{19}$$

iii. $$\lim_{V \to 0} E_t\left(V_T\right) = \theta^* (1 - e^{-k^*(T-t)}) + \frac{\lambda}{k^*} (1 - e^{-k^*(T-t)}) \frac{1}{\eta}. \tag{20}$$

Equation (18) shows the standard convergence property of the futures price to the spot price at maturity. Equation (19) shows that as the time-to-maturity increases, the futures price tends to the constant long-run volatility mean $\left(\theta^* + \frac{\lambda}{k^* \eta} \right)$. The latter means that as time-to-maturity increases, futures prices are becoming less sensitive to current volatility changes. This feature of volatility futures prices is in contrast to those of futures prices on stocks or stock indices, where future prices move in an almost one-to-one analogy to spot prices. Finally, equation (20) shows that as volatility tends to zero, futures price does not converge to zero, as in the case of futures on stocks or stock indices. The intuition of the above properties of futures prices are related to the mean reverting nature of volatility. Irrespective of the changes in the current value of *V*, there is a growing probability with time that *V* is going to revert to its long run mean prior to expiration of the contract.

5.2. Volatility Options

In order to obtain the valuation formula for a European volatility call, we follow the approach of Bakshi and Madan (2000).The price $C(V_t,\tau;K)$ of the call option with strike price *K* and τ time to maturity is given by:

$$C(V_t,\tau;K) = e^{-r\tau} e^{-k^*\tau} V_t \Pi_1(t,\tau) + e^{-r\tau}(1-e^{-k^*\tau})\left(\theta^* + \frac{\lambda}{k^*\eta}\right)\Pi_1(t,\tau) - e^{-r\tau} K \Pi_2(t,\tau) \qquad (21)$$

The Π_1 and Π_2 probabilities are determined by

$$\Pi_j(t,\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi K} \times g_j(V_t,\tau;\phi)}{i\phi}\right] d\phi \qquad (22)$$

where

$g_1(V_t,\tau;\phi)=\dfrac{F(V_t,\tau;\phi)}{F_\phi(V_t,\tau;0)}$ and $g_2(V_t,\tau;\phi)=e^{r\tau}F(V_t,\tau;\phi)$. $F_\phi(V_t,\tau;0)$ is the first derivative of $F(V_t,\tau;\phi)$ with respect to φ, evaluated at φ=0.

The call pricing formula (21) has the following limiting properties:

i. $$\lim_{\tau\to 0} C(V_t,\tau;K)=\max\left(V_t-K,0\right), \tag{23}$$

ii. $$\lim_{\tau\to +\infty} C(V_t,\tau;K)=0, \tag{24}$$

iii. $$\lim_{V\to +\infty} C(V_t,\tau;K)=+\infty. \tag{25}$$

iv. $$\lim_{V\to 0} C(V_t,\tau;K)\neq 0. \tag{26}$$

Equation (23) shows the standard convergence property of the option price to the option's payoff at maturity. Equation (24) shows that for very long maturities the volatility call option is going to be worthless, just as it was the case in the models of Detemple and Osakwe (2000), and Grunbichler and Longstaff (1996). This is due to the mean reverting nature of volatility. In the long-run, volatility will revert to it's long run mean $\left(\theta^*+\dfrac{\lambda}{k^*\eta}\right)$:

$$E\left[\max\left(\lim_{\tau\to\infty}\left(V_{t+\tau}\right)-K,0\right)\right]=\max\left(\left(\theta^*+\frac{\lambda}{k^*\eta}\right)-K,0\right) \text{ since } \lim_{\tau\to\infty}\left(V_{t+\tau}\right)=\theta^*+\frac{\lambda}{k^*\eta}$$

In addition,

$$\lim_{\tau\to\infty} e^{-r\tau}=0$$

Hence, using equation (21) we can see that the value of the volatility call tends to zero as $\tau\to\infty$. The latter can be visualized in Figure 3, which shows the value of the volatility call as a function of the time-to-maturity τ. We can see that volatility call options, in contrast to standard options, are concave functions of volatility; i.e. as T increases the value of the volatility call initially increases and then flattens out.

[INSERT FIGURE 3 HERE]

Equation (25) shows that as volatility grows to infinity, the price of the option tends to infinity too, just as in the case of the plain vanilla options. Finally, equation (26) shows that as V_t tends to zero, the volatility option price does not converge to zero. This is in contrast to the case of a standard European call, where its value tends to zero as the price of the underlying tends to zero (Merton, 1973). Once again, this can be attributed to the mean reverting nature of volatility. As soon as volatility becomes zero, it will become non-zero so as to return to its long-run mean $\left(\theta^* + \frac{\lambda}{k^*\eta}\right)$. Therefore, as $V_t \to 0$, volatility options retain their time value (the intrinsic value is zero). The latter, is in line with the results of Grunbichler and Longstaff (1996), but it contrasts the equivalent result of Detemple and Osakwe (2000), where a similar option does not have a value. This is because in the Detemple and Osakwe (2000) model, V_t has an absorbing barrier at zero, due to the multiplicative structure of the logarithmic process. The evolution of the value of the volatility call, as a function of the underlying volatility V_t, is shown in Figure 4. We can see that for $V_t = 0$ long-term volatility options still have value. In addition, volatility call options are increasing functions of volatility, but in contrast to standard options, the rate of growth decreases as time to maturity T increases.

[INSERT FIGURE 4 HERE]

We now turn our attention to *delta,* the sensitivity of the call price with respect to V_t. The magnitude of delta is related to volatility call option hedging effectiveness. The highest the *delta*, the more sensitive to volatility changes is the volatility call. Figures 5 and 6 show the *delta* of the volatility call as a function of the volatility V_t and time-to-maturity τ, respectively. We can easily observe that that delta is always positive. The magnitude of *delta* depends on the level of volatility and the time-to-maturity. For deep ITM (OTM) calls, the delta of longest (shortest) expiry volatility call is highest. It can also be observed that as τ increases the value of the volatility call delta decreases and flattens out. This implies that the sensitivity of the volatility call option price to volatility decreases as time-to-maturity increases. In other words, as time to maturity increases, the volatility call option loses its hedging effectiveness. The important implication of the latter result is that long maturity volatility calls are not effective for hedging or trading volatility purposes[9].

[INSERT FIGURE 5 HERE]

[INSERT FIGURE 6 HERE]

Figure 7 show the value of a volatility call option as a function of volatility for three different levels of moneyness. Both diffusion (model of Grunbichler and Longstaff, 1996)

[9] Grunbichler and Longstaff (1996), and Detemple and Osakwe (2000) came up with similar results for volatility options.

and jump-diffusion models are examined. We can see that for short maturities, the diffusion model underprices the volatility call. In contrast, for longer maturities, the diffusion model overprices the volatility call. The latter occurs because in the jump-diffusion model, the volatility of the process consists of two parts: the diffusion and the jump part. The jump part affects the value of the volatility call mainly in the short-run, whilst the diffusion part affects the value of the volatility call mainly in the long-run.[10] On the other hand, the volatility of the diffusion model is driven only by the diffusion part. Note that although the total volatility is almost the same for both diffusion and jump diffusion model, σ is significantly larger in the case of the diffusion model. In this manner, the diffusion model underprices the volatility call for short maturities where jumps in volatility still affect the call value.

[INSERT FIGURE 7 HERE]

Figures 9 and 10 depict the *delta* of both diffusion and jump-diffusion models as a function of τ and V_t, respectively. Interestingly, the *delta* of the diffusion model is significantly higher in all cases, except when we consider deep OTM options. The latter indicates that the diffusion model is more sensitive in volatility changes than the jump-diffusion model. The explanation follows from the assumption that volatility jumps do not depend on the current level of volatility. On the other hand, diffusion volatility

[10] Das and Sundaram (1999) and Pan (2002) provide similar results in the case of index options, where jumps improve the pricing mainly of the short terms options. The pricing of intermediate and long maturity options is mainly improved by the assumption that volatility of returns is stochastic.

depends on the current level of volatility through the term $\sqrt{V_t}$. Differentiation shows that the delta of the volatility calls depends mainly on σ rather than λ or η. The above finding has important implications in terms of hedging. Suppose that you have a long position in a call option and you use volatility options in order to hedge the vega risk of your position. Recall that diffusion model overestimates the *delta* of the volatility option. So, if you incorrectly use the diffusion model to calculate the *delta*, then you will use less volatility options for hedging than those that actually are required.

[INSERT FIGURE 8 HERE]

[INSERT FIGURE 9 HERE]

6. Conclusions

Motivated by the growing literature on volatility derivatives and their imminent introduction in major exchanges, this paper examined the empirical relevance and potential impact of volatility jumps in autonomous volatility option pricing and risk management.

In line with previous research, empirical analysis of the VIX over a period of 10 years provided a wealth of evidence supporting the existence of some stationary, mean-reverting process with jumps. Motivated by the preliminary analysis, we concentrated on the popular mean-reverting square root process, originally proposed by Grunbichler and Longstaff (1996), and its augmentation by an upward jump. An ML estimation scheme was described and applied to the VIX data. The results suggested that the addition of jumps, especially if they are conditioned on the volatility level, improves significantly fitting ability. Moreover, simulation results suggested that the augmented model has the ability to produce non-normal distributions that closely resemble those of the original data. Closed form models for pricing futures and options were then developed assuming a square root mean reverting diffusion stochastic process that allows for positive jumps in volatility. The proposed volatility option pricing models appears to have comparable properties with existing models in the literature (Grunbichler and Longstaff 1996; Detemple and Osakwe, 2000). However, it was demonstrated that incorrectly omitting jumps in volatility may result in severe mispricing. In particular, in the case where there are upwards jumps in volatility, short (long) term volatility options are more expensive (cheaper) by about 25% (14%). In addition, volatility calls are far less sensitive to the changes of the underlying volatility by a factor of about two.

The findings in this paper do not necessarily support criticism against the specific structural form assumed by existing volatility future and option pricing models. Rather, they attempt to demonstrate that pricing derivatives on a volatility index should carefully account for salient features of the data since the results obtained are particularly sensitive to the model used to approximate the underlying dynamics. Testing against actual market prices will provide more definitive evidence on the merit of alternative pricing models. In the case of futures this is possible since some data do exist for futures on volatility indices (for a relevant application, see, for example, Dotsis et al., 2005). However, since no volatility options market data are yet available, we cannot fully test the empirical relevance of alternative option pricing models. However, it is crucial to fully understand the dynamics of the underlying and the implications of competing option pricing models in order to understand the peculiarities of this asset class and facilitate a smooth operation of the market when it operates.

We believe that much more research is needed on the practical usefulness of volatility derivatives, especially for corporate finance. Although some ideas have been proposed in the literature and discussed in this paper, it is not yet clear how financial managers can use these instruments and what the actual benefits they may expect are. This is not a trivial problem since the implications of volatility for a firm are so widespread, complicated and complex. For example, a short futures position on the VIX index buys insurance against changes in the volatility of the US equity market. A US firm assuming this position, would be affected directly and indirectly in a number of ways with respect to factors including: firm value, cost of equity, cost of debt, optimal finance mix, employee stock option value, value and effectiveness of existing hedges, value of investments, and investment hurdle rates. This complicates also the accounting treatment of the hedge relationship and effectiveness offered by volatility derivatives. For example, according to FAS 133, the statement issued by F.A.S.B. (Financial Accounting Standards Board) regarding accounting for derivative instruments and hedging accounting, three hedge relationships are recognized: fair value hedge, cash flow hedge and foreign currency hedge. The accounting treatment of derivatives depends on the hedge relationship they participate and the effectiveness of the hedge offered. In the case of volatility derivatives, the determination of the hedge relationship and the effectiveness is a very difficult task.

In closing, we would like to emphasize the growing need for introducing volatility indices and derivatives in more markets. Brenner and Galai (1989) first argued that volatility indices should be developed for equity, bond and foreign exchange markets.

However, the recent history has shown that significant volatility risk exists also in other important markets, such as, for example, the market for petrol and for electricity.

Appendix: Derivation of the characteristic function for SRPJ process

The conditional characteristic function $F(V_t, \tau; s) = E(e^{isV_{t+\tau}} \left| V_t ; \Theta \right.)$ of the SRPJ must satisfy the following Kolmogorov backward differential equation

$$\frac{\partial F}{\partial V_t} + k(\theta - V_t) + \frac{1}{2}\frac{\partial^2 F}{\partial V_t^2} V_t \sigma^2 - \frac{\partial F}{\partial \tau} + \lambda V_t \mathrm{E}\left[F(V_t + y) - F(V_t)\right] = 0 \tag{27}$$

subject to the boundary condition

$$F(V_t, \tau = 0; s) = e^{isV_t} \tag{28}$$

where $i = \sqrt{-1}$. Differentiating the characteristic function given by equation (4) yields

$$\begin{aligned} F_V &= BF \\ F_{VV} &= B^2 F \\ F_\tau &= F\left(A_\tau + VB_\tau\right) \end{aligned} \tag{29}$$

where the subscripts denote the corresponding partial derivatives.

Replacing equations (29) in equation (27) and rearranging yields

$$V_t \left(-kB - B_\tau + \frac{1}{2}\sigma^2 B^2 + \lambda \mathrm{E}\left[e^{yB} - 1\right] \right) + \left(k\theta B - A_\tau\right) = 0 \tag{30}$$

Also,

$$\mathrm{E}\left[e^{yB} - 1\right] = \int_0^{+\infty} \eta_1 e^{-\eta_1 y} e^{yB} dy - 1 = \frac{\eta_1}{\eta_1 - \mathrm{B}} - 1$$

Since $V_t \neq 0$, the expressions in the parentheses in equation (30) must equal zero. Therefore we obtain the following ordinary differential equations (ODEs)

$$-kB - B_\tau + \frac{1}{2}\sigma^2 B^2 + \lambda\left(\frac{\eta_1}{\eta_1 - \mathrm{B}} - 1\right) = \tag{31}$$

$$k\theta B - A_\tau = 0 \tag{32}$$

The ODEs cannot be solved in closed form. They are solved numerically subject to the boundary conditions $A(\tau = 0; s) = 0$, and $B(\tau = 0; s) = is$.

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Table 1. Descriptive Statistics

	VIX	ΔVIX
Mean	0.1957	-1.23E-05
Median	0.1856	-0.0004
Maximum	0.4574	0.0992
Minimum	0.0931	-0.0780
Std. Dev.	0.0639	0.0122
Skewness	0.9382	0.5647
Kurtosis	3.7411	9.1172
Jarque-Bera	671.14**	6,378.52**
$\rho(1)$	0.981	-0.041
$\rho(2)$	0.964	-0.088
$\rho(3)$	0.950	-0.057
$\rho^2(1)$	0.975	0.201
$\rho^2(1)$	0.950	0.189
$\rho^2(1)$	0.932	0.204

$\rho(q)$ and $\rho^2(q)$ are autocorrelation and squared autocorrelation coefficients at lag q, respectively. Two (one) stars denote significance at the 1% (5%) level.

Table 2. Unit Root Test results of VIX

Test	Null Hypothesis	Test Statistic
Augmented Dickey-Fuller	Unit Root	-3.7892**
Phillips-Perron	Unit Root	-4.9068**
Kwiatkowski-Phillips-Schmidt-Shin	Stationarity	1.4366**

The Augmented Dickey-Fuller (Dickey and Fuller, 1979) and the Phillips-Perron (1988) test the null hypothesis of a unit root. The Kwiatkowski-Phillips-Schmidt-Shin (1992) tests the null hypothesis of stationarity. An intercept is included in all test regressions. Two (one) stars denote significance at the 1% (5%) level.

Table 3. Log Likelihood of Alternative Distribution Models

	Parameters	VIX	ΔVIX
Normal	2	5,266.8	11,811.8
Log-normal	2	5,734.4	
t-student	1	5,295.3	12,221.3
Skewed *t*-student	2	5,841.5	12,225.6
Logistic	2	5,285.3	12,123.1
Exponential	2	5,051.4	
Gamma	3	5,763.7	
Extreme (max)	2	5,647.4	
Pareto	2	4,142.9	
Weibull	2	5,267.1	

Table 4. Conditional tabulation of VIX vs. ΔVIX

		VIX					
		[0, 0.1)	**[0.1, 0.2)**	**[0.2, 0.3)**	**[0.3, 0.4)**	**[0.4, 0.5)**	**Total**
ΔVIX	**[-0.1, -0.05)**	0	1	1	6	0	8
	[-0.05, 0)	2	1,244	661	114	8	2,029
	[0, 0.05)	3	1,053	703	130	19	1,908
	[0.05, 0.1)	0	0	4	6	2	12
	Total	5	2,298	1,369	256	29	3,957

Table 5. Model Estimation Results

Parameter	**SR**	**SRJ**	**SRPJ**
k	4.5496 (5.9778)	7.3800 (9.5121)	10.5004 (11.1326)
θ	0.1945 (19.9557)	0.1505 (21.7557)	0.1379 (24.0473)
σ	0.4048 (88.0705)	0.3502 (61.3238)	0.3294 (51.3363)
λ	-	19,4080 (4.5046)	263.8877 (9.1391)
1/η	-	0.0170 (8.2228)	0.0125 (4.5626)
Log-Likelihood	12,263.12	12,422.37	12,459.24
AIC	-24,520	-24,834	-24,908
BIC	-24,501	-24,803	-24,977

Table 6. Unconditional Higher moments of actual and simulated distributions

		Levels			Differences		
		Daily	**Weekly**	**Monthly**	**Daily**	**Weekly**	**Monthly**
Actual							
	Skewness	0.9382	0.9772	0.7865	0.5647	0.8939	-0.5693
	Kurtosis	3.7411	4.0192	3.0652	9.1172	8.8547	6.9027
SR							
Skewness	1%	0.0632	0.0410	0.0789	-0.1862	-0.2428	-0.2683
	5%	0.1933	0.1513	0.1823	-0.1600	-0.1888	-0.1999
	10%	0.2602	0.2457	0.2525	-0.1430	-0.1556	-0.1655
	50%	0.5063	0.5190	0.5274	-0.0884	-0.0321	-0.0378
	90%	0.8645	0.8591	0.8751	-0.0304	0.0781	0.0987
	95%	1.0098	0.9626	1.0060	-0.0151	0.1139	0.1318
	99%	1.1788	1.2113	1.2683	0.0158	0.1748	0.1948
Kurtosis	1%	2.4033	2.4330	2.3927	3.0300	2.8109	2.7819
	5%	2.5757	2.5765	2.5718	3.0846	2.9123	2.8985
	10%	2.6517	2.6851	2.6556	3.1212	2.9656	2.9599
	50%	3.1338	3.1369	3.1466	3.2702	3.2071	3.2210
	90%	4.1161	4.1893	4.1510	3.4310	3.5461	3.5721
	95%	4.6273	4.6437	4.7423	3.4910	3.6659	3.6722
	99%	5.7824	6.3340	6.1775	3.6029	3.9490	3.9055
SRJ							
Skewness	1%	0.1397	0.1084	0.0703	0.6726	0.1193	-0.3613
	5%	0.2286	0.2150	0.1931	0.8342	0.2329	-0.1801
	10%	0.2791	0.2706	0.2452	0.8909	0.2961	-0.1092
	50%	0.4923	0.5197	0.5082	1.1496	0.4682	0.1478
	90%	0.8005	0.7997	0.8437	1.5066	0.6995	0.4071
	95%	0.9095	0.8686	0.9429	1.6348	0.7905	0.4894
	99%	1.0633	1.0462	1.2365	1.8717	0.9159	0.6995
Kurtosis	1%	2.5292	2.5405	2.3649	6.0200	3.1745	2.4982
	5%	2.6978	2.6783	2.5360	6.7553	3.4387	2.6733
	10%	2.7857	2.7897	2.6549	7.2171	3.5903	2.7855
	50%	3.2483	3.2480	3.1624	9.2736	4.2507	3.2473
	90%	4.2213	4.1638	4.3336	13.0505	5.4779	4.1107
	95%	4.5481	4.5619	4.8878	14.3873	5.9301	4.4890
	99%	5.5437	5.3391	6.1345	18.9351	7.0055	5.1983
SRPJ							
Skewness	1%	0.1985	0.2048	0.1298	0.6414	0.0913	-0.3959
	5%	0.2900	0.2865	0.2459	0.7157	0.1805	-0.2150
	10%	0.3368	0.3356	0.3090	0.7573	0.2111	-0.1447
	50%	0.5797	0.5801	0.5820	0.9014	0.3549	0.1221
	90%	0.8608	0.8694	0.9180	1.1006	0.5316	0.3714
	95%	0.9686	0.9673	1.0361	1.1627	0.5869	0.4408
	99%	1.2635	1.2128	1.3778	1.2834	0.7360	0.6316
Kurtosis	1%	2.5730	2.5458	2.3895	4.9888	3.1293	2.4830
	5%	2.7311	2.7248	2.5870	5.3320	3.2467	2.6728
	10%	2.8403	2.8269	2.6902	5.5287	3.3413	2.7698
	50%	3.3298	3.3419	3.3094	6.4981	3.7832	3.2742
	90%	4.4017	4.4068	4.5604	8.1753	4.5429	4.0679
	95%	4.7510	4.8235	5.0704	8.7837	4.7770	4.5023
	99%	6.1520	6.0101	7.0421	9.8599	5.8960	5.3777

Figure 1. The VIX Index and first differences (ΔVIX)

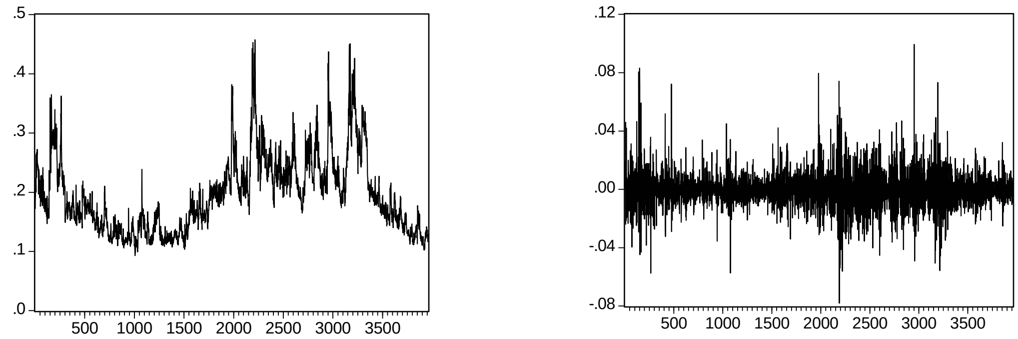
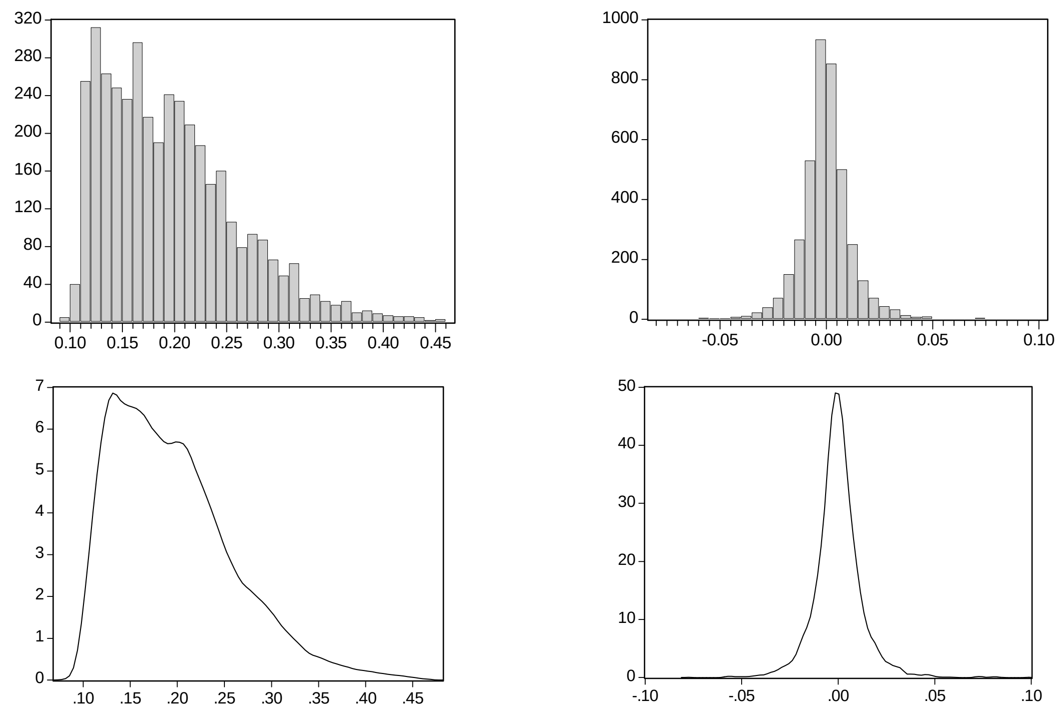


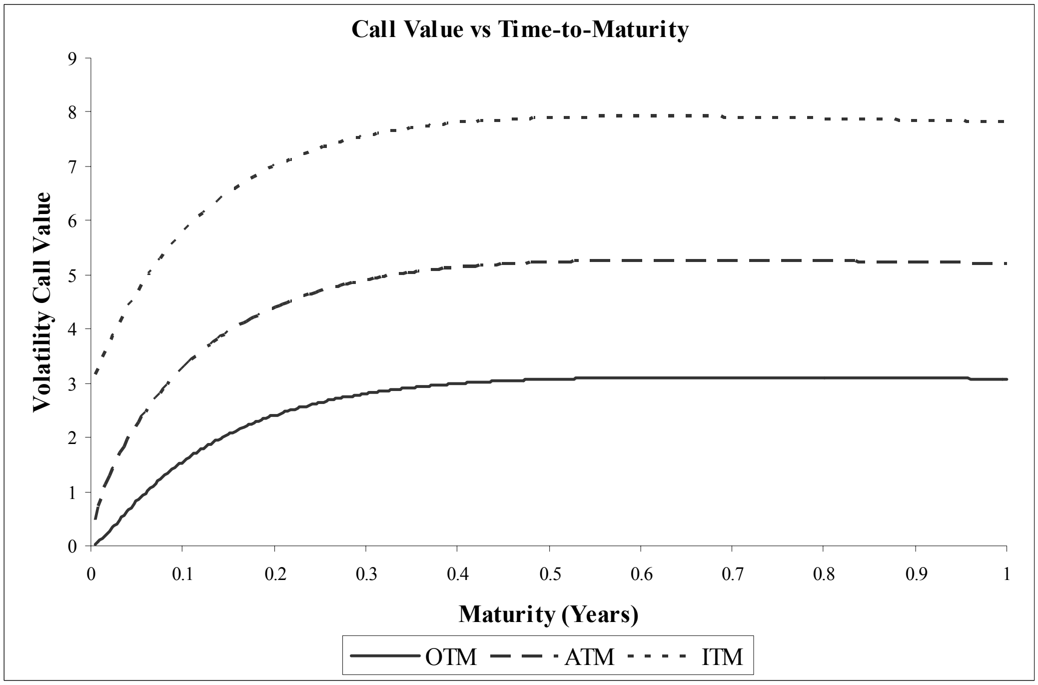
Figure 2. Histograms and Kernel Distributions

VIX **ΔVIX**



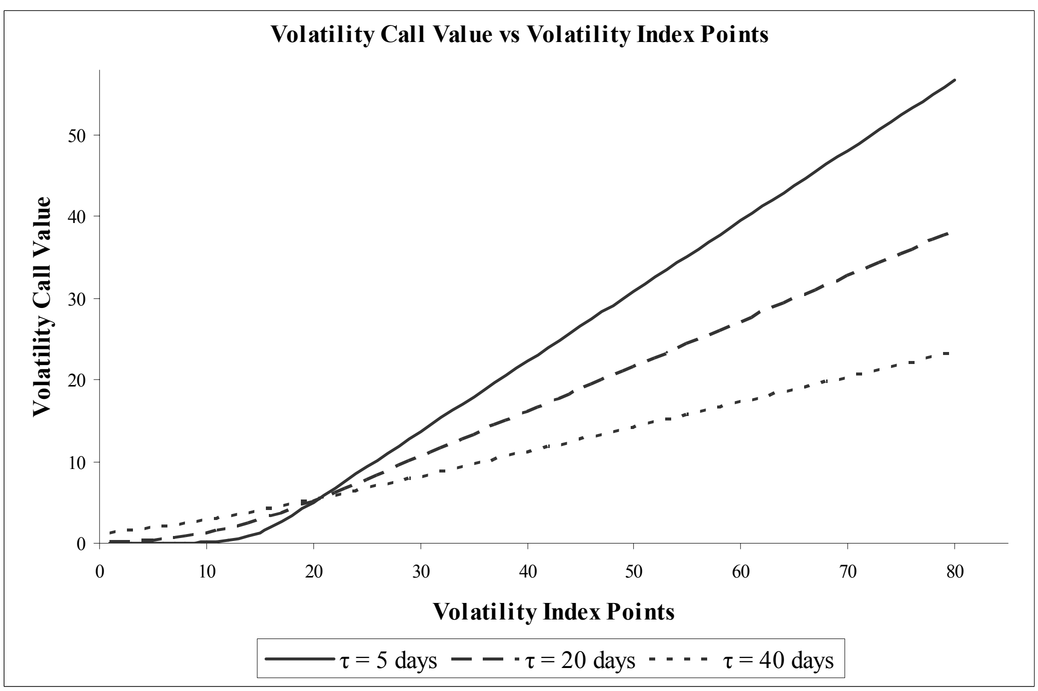
Densities were estimated with Epanechikov kernel functions over 100 points. The bandwidth was determined according to the method suggested by Silverman (1986).

Figure 3. The value of a volatility call as a function of time-to-maturity



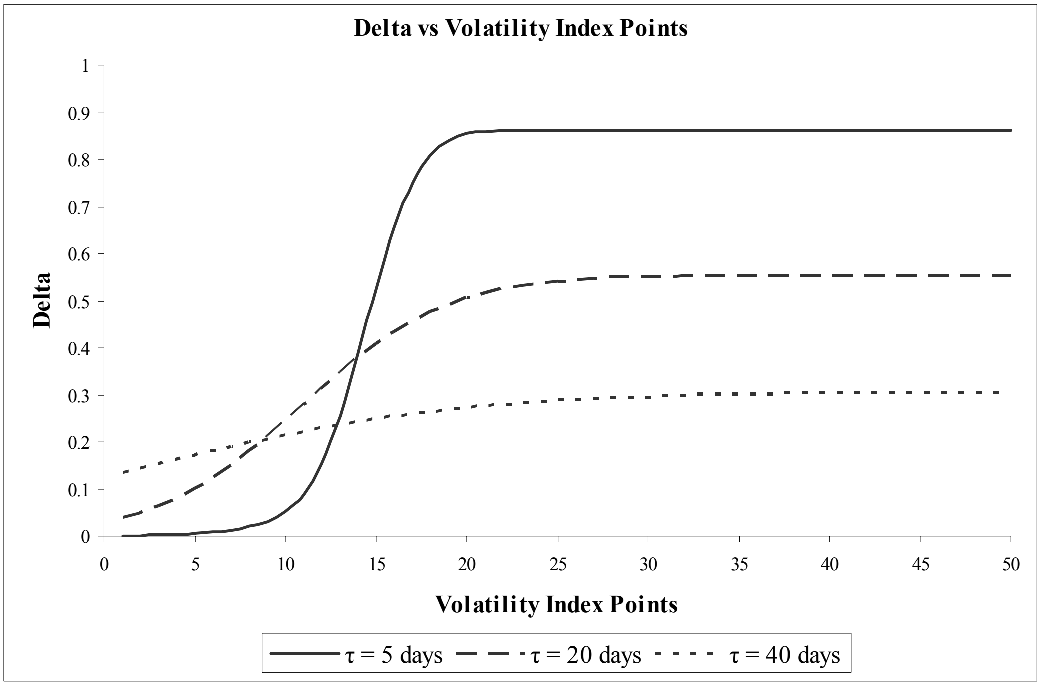
Estimated for three different moneyness levels K = 12(ITM), 15(ATM), 18(OTM). The figure is drawn for r = 5%, and V_t =15. k, θ, σ, η, λ are given from the third column of Table 5.

Figure 4. The value of a volatility call as a function of volatility



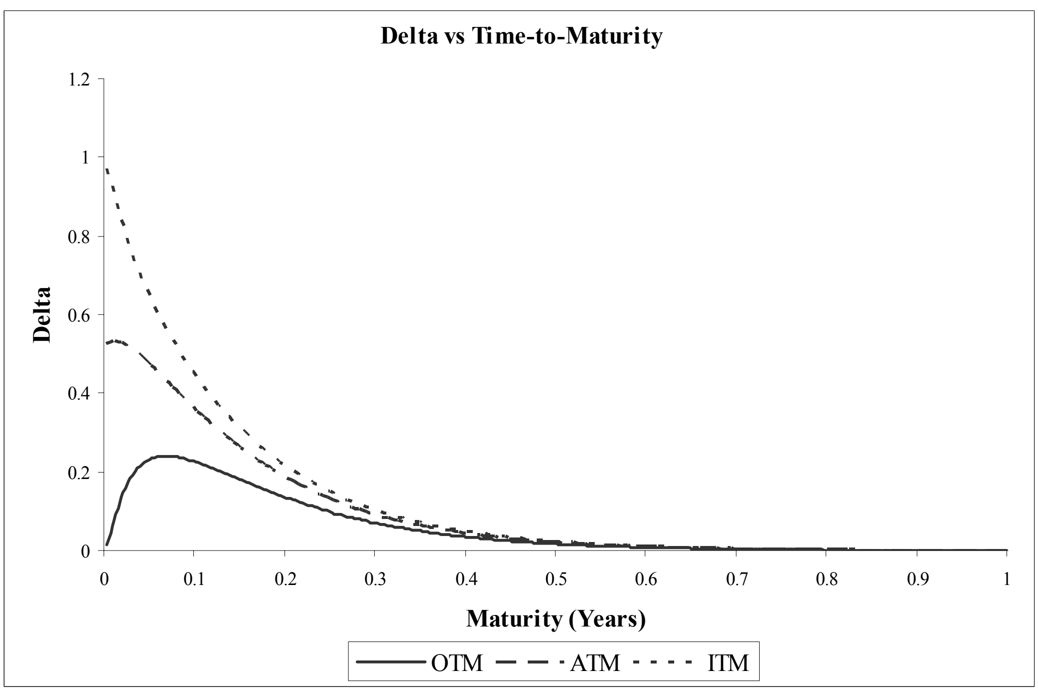
Estimates for three different maturities: τ = 5, 20, and 40 days. The figure is drawn for r = 5%, and K =15. k, θ, σ, η, λ are given from the third column of Table 6.

Figure 5. The delta of a volatility call as a function of volatility



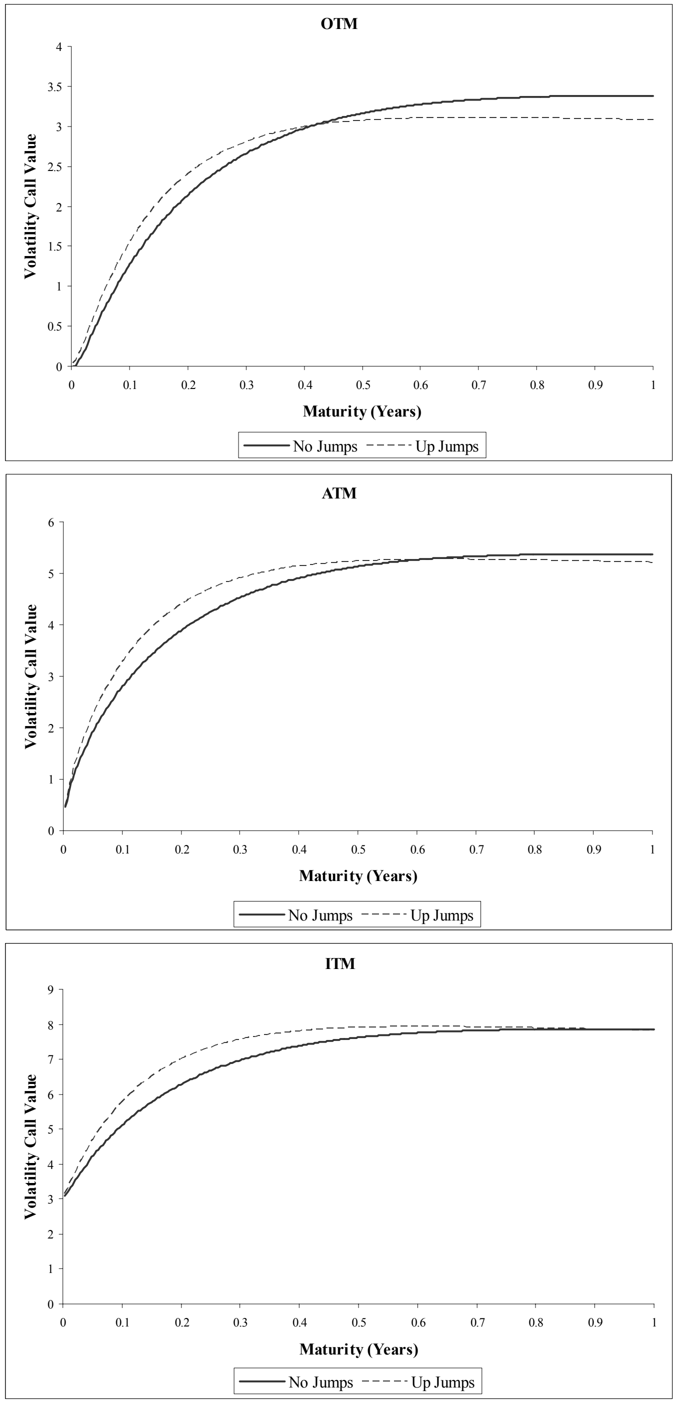
Estimates for three different maturities: τ = 5, 20, and 40 days. The figure is drawn for r = 5%, and K =15. k, θ, σ, η, λ are given from the third column of Table 5.

Figure 6. The delta of a volatility call as a function of time-to-maturity



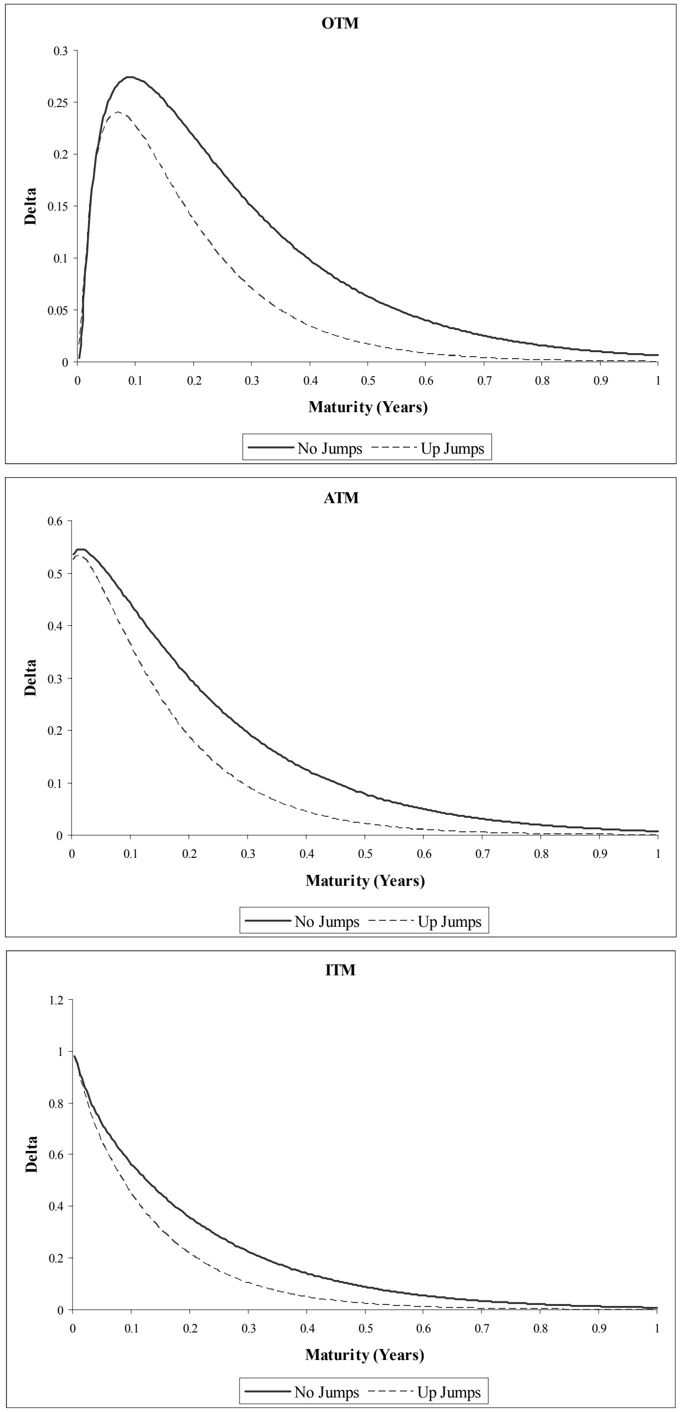
Estimated for three different moneyness levels K = 12(ITM), 15(ATM), 18(OTM). The figure is drawn for r = 5%, and V_t =15. k, θ, σ, η, λ are given from the third column of Table 5.

Figure 7. Value of the volatility call as a function of Time-to-Maturity



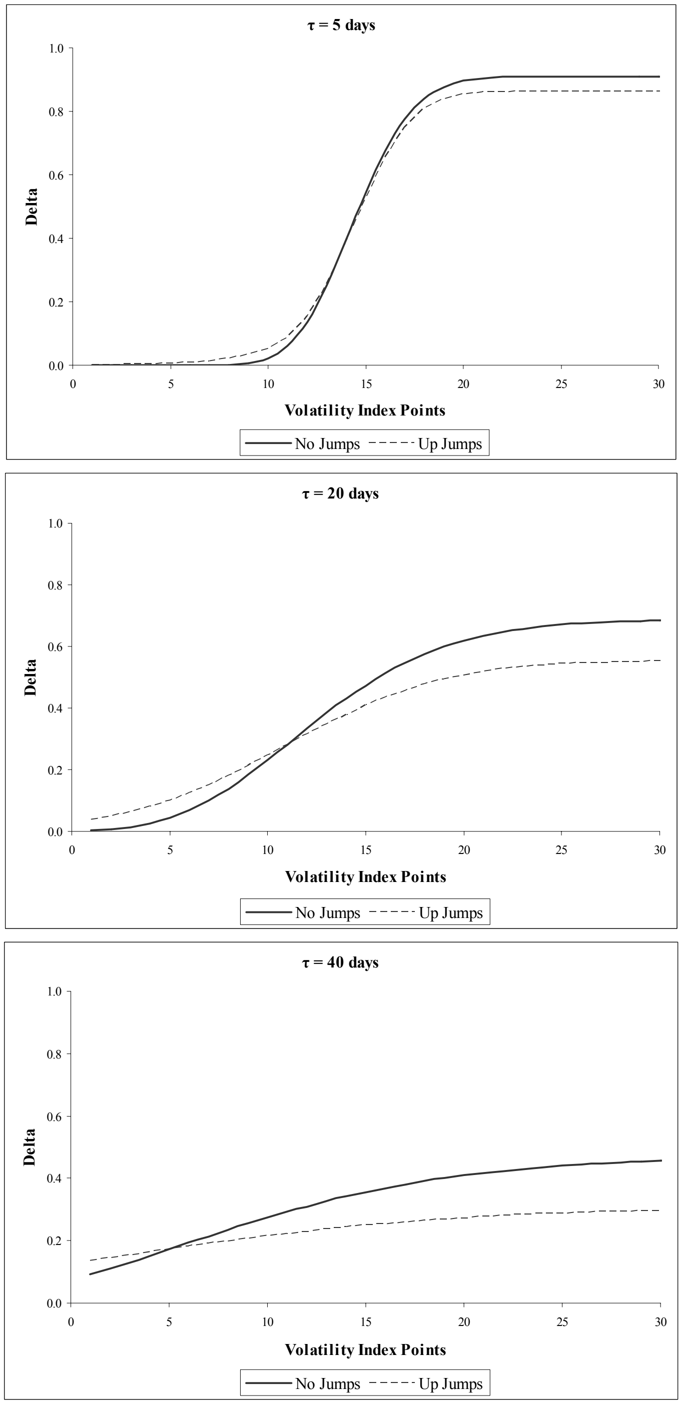
Estimated for three different moneyness levels K = 12(ITM), 15(ATM), 18(OTM). The solid line corresponds to the case where there are no jumps in the volatility process (model of Grunbichler and Longstaff, 1996) using the estimated k, θ, and σ (Table 5, second column). The dotted line corresponds to the case where there are upwards jumps in the volatility process using the estimated k, θ, σ, η, λ (Table 5, third column). Assume r = 5% and V_t =15.

Figure 8. Delta of the volatility call as a function of Time-to-Maturity τ



Estimated for three different moneyness levels K = 12(ITM), 15(ATM), 18(OTM). The solid line corresponds to the case where there are no jumps in the volatility process (model of Grunbichler and Longstaff, 1996) using the estimated k, θ, and σ (Table 5, second column). The dotted line corresponds to the case where there are upwards jumps in the volatility process using the estimated k , θ, σ , η , λ (Table 5, third column). Assume r = 5% and V_t =15.

Figure 9. Delta of the volatility call as a function of volatility V_t,



Estimated for three different maturities τ = 5, 20, and 40 days. The solid line corresponds to the case where there are no jumps in the volatility process (model of Grunbichler and Longstaff, 1996) using the estimated k, θ, and σ (Table 5, second column). The dotted line corresponds to the case where there are upwards jumps in the volatility process using the estimated k , θ, σ , η , λ (Table 5, third column). Assume r = 5% and K =15.