Real Option Price Bounds in Incomplete Markets

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Abstract

This paper applies the minimax deviations approach to real options in incomplete markets by constructing narrow bounds around the value of real options embedded in capital budgeting decisions. While it is straightforward to obtain the unique value of a real option with HARA utility functions, the parameters of risk-aversion are often subject to misspecification and raise concerns for practical uses. Noticing that investors allow deviation from parameters related to a benchmark pricing kernel, we derive narrow bounds. Comparison with the approaches in the literature clarifies advantages of the minimax bounds.

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1 Introduction

Asset pricing in complete markets is well established. The value of an asset (a focus asset) is obtained from the prices of other traded assets (basis assets). Continuous trading in the underlying assets makes it possible to replicate the asset's payoffs, and the present value of the focus asset must equal the price of the replicating portfolio in the absence of arbitrage opportunities. The value of the focus asset does not require specifying investors' risk preference and is obtained by discounting at the risk free rates the asset's expected payoff computed with risk-neutral probabilities, which are the probabilities that correctly price assets in a risk-neutral economy.

However, exact replication of the focus assets' payoffs is not readily available in many cases. Trading cannot be done continuously and is costly. The risk factors that affect an asset's payoffs are not represented by traded assets. This is often the case of real options, or options embedded in capital investment decisions. The bounds provided by the no-arbitrage approach are too wide to be useful on the value of the option because there are infinite elements in the set of admissible martingale measures that accurately price the option, the focus asset. Holding the option becomes inherently risky because the portfolio of basis assets does not perfectly hedge the option.

Cochrane and Saa-Requejo (2000) derive tighter bounds on option prices by imposing restrictions on Sharpe ratios or, equivalently, on the volatilities of the pricing kernel. Cases with high Sharpe ratios provide near-arbitrage opportunities and cannot last because investors bid to buy assets with high Sharpe ratios. One of the problems with this method is that a Sharpe ratio can be very low for arbitrage opportunities and cannot be reduced further with a typical threshold. In addition, it can easily construct bounds which conflict with risk-averse preference. Bernardo and Ledoit (2000) rule out investment opportunities whose attractiveness to a representative investor exceeds a specified threshold. Their measure of attractiveness is the ratio of the expectation of the positive parts of the payoff (gain) on an investment to the expectation of the negative parts of the payoff (loss). By varying the gain-loss ratio, they are able to accommodate any bounds, from unique values to no-arbitrage bounds. In addition to setting an arbitrary gain-loss ratio, this approach easily tolerates the narrow bounds which conflict with risk preference implied by the benchmark pricing kernels.

Mello and Pyo (2006) suggests a way of narrowing the bounds, that does not impose further exogenous restrictions as in the literature. Instead, their approach extracts additional information on the investor's confidence on the initial estimation of the preference parameters. They observe maximum deviations around a benchmark price and take a minimum of the two in a sense that a minimum deviation implicitly exhibit more confidence on a pricing implication on that side over the other side. Minimax bounds presented by Mello and Pyo (2006) are shown to be simple in derivation, consistent with risk-aversion, and efficient in tightness of the bounds.

While they illustrate advantages of minimax bounds on call option values without dynamic rebalancing, they do not have to calibrate utility functions because they use the connection between the Black-Scholes formula and exponential utility function as shown by Rubinstein (1976). Since real options embedded in capital budgeting decisions display characteristics different from call options on stock, this article presents a framework of constructing minimax bounds by specifically calibrating utility functions in equilibrium settings. To make the results robust, we utilize a versatile utility function: HARA class utility function. Although expressions for the minimax bounds are not neat, the framework applies the minimax bounds to real options and presents practical ways of deriving minimax bounds in most cases of utility functions.

We use a numerical example to illustrate how the minimax deviation suggested by Mello and Pyo (2006) is compared to those two approaches in the literature in the case of a real option embedded in a capital budgeting problem. It is not necessary to set up an arbitrary threshold for the bounds and the obtained bounds are consistent with risk preference reflected in a benchmark price. More importantly, the bounds are really tight around a benchmark price as opposed to those constructed with a typical threshold in the literature.

The article is organized as follows. First, we briefly review the minimax deviations approach to narrow bounds on asset prices in an incomplete market. Next we derive the value of an option to defer a capital investment decision in a dynamically incomplete market, making explicit assumptions about a representative investor's attitudes toward risk. We show that when pricing is embedded in a utility maximization framework, a unique martingale measure emerges that reflects the investor's willingness to pay for cash across states and time. Next, we derive the corresponding risk-adjusted probabilities, and note that the expressions for the risk-adjusted probabilities in incomplete markets closely resemble the expressions for the risk-neutral probabilities that would be obtained in complete markets. We show that there is a linear relationship between changes in the investor's coefficient of absolute risk aversion and changes in the values of the risk-adjusted probabilities. More important, changing the absolute risk-aversion leads to very small changes in risk-adjusted probabilities. This is useful, because it implies that less precise information about the investor's risk preferences, measured by its coefficient of absolute risk aversion, does not translate proportionally into less precise valuations. In Section 4 we show that the precise location of the option value depends on the risk-adjusted probabilities for the risks not priced in the market and the no-arbitrage bounds. Section 5 develops the methodology to tighten the bounds on the values of the option. Comparison with other methods is presented in Section 6. A numerical example is presented in Section 7. Section 8 concludes.

2 Review of the Minimax Bounds

Mello and Pyo (2006) presents the minimax deviation approach to narrow the bounds on asset prices in an incomplete market, assuming that the pricing kernel declines monotonically with the state variable, that is, a state claim that pays off when consumption is low (high) has a relatively high (low) value, since such claim allows the benchmark investor to smooth consumption across future states of nature.[1] A benchmark investor that makes a capital budgeting decision exhibits risk-aversion with utility functions u, where u is a continuously differentiable and concave von Newmann-Morgenstern utility function with marginal utility of consumption $u'\left(c_j^*\right) > 0$ with risk-aversion $u''\left(c_j^*\right) < 0$, and $\tilde{c}^* = (c_1^*, .., c_S^*) \in \mathcal{R}^S$ is an equilibrium state contingent consumption package. A pricing kernel m_j^* is given by

$$m_j^* = \frac{u'\left(c_j^*\right)}{E\left[u'\left(\tilde{c}^*\right)\right]} \cdot \frac{1}{1+r_f} \tag{1}$$

The pricing kernel, m_j^*, is subject to misspecification of parameter values and is approximated by the minimax bounds as

$$m_j^* \in \left(\frac{\hat{p}_j + e_j^*}{q_j\left(1+r_f\right)}, \frac{\hat{p}_j - e_j^{**}}{q_j\left(1+r_f\right)}\right) = \left(\frac{1}{1+r_F}, \frac{\hat{p}_j - e_j^{**}}{q_j\left(1+r_f\right)}\right) \tag{2}$$

where q_j is the objective probabilities, $\hat{p}_j$ the estimated risk-adjusted probabilities,and r_f the risk-free rate of return. The total deviation $e_j = e_j^+ - e_j^-$ can be decomposed into a positive part $e_j^+ = \max\left(e_j, 0\right)$ and a negative part $e_j^- = \max\left(-e_j, 0\right)$. The minimax deviation is computed as

$$e_j^* = \min\left(e_j^+,\ e_j^-\right) = \begin{cases} \min\left(q_j - \hat{p}_j,\ \hat{p}_j\right) & if \quad 0 \le \hat{p}_j \le q_j \\ \min\left(1 - \hat{p}_j, \hat{p}_j -\ q_j\right) & if \quad q_j < \hat{p}_j \le 1 \end{cases} \tag{3}$$

[1] See Mello and Pyo (2006) for details.

and the other deviation on the pricing kernels, e_j^{**} is computed by solving the following equations:

$$u\left(\hat{p}_j + e_j^*\right) - u\left(\hat{p}_j\right) = u\left(\hat{p}_j\right) - u\left(\hat{p}_j - e_j^{**}\right) \quad \text{if } q_j - \hat{p}_j \leq \hat{p}_j \tag{4}$$

$$u\left(\hat{p}_j\right) - u\left(\hat{p}_j - e_j^*\right) = u\left(\hat{p}_j - e_j^{**}\right) - u\left(\hat{p}_j\right) \quad \text{if } q_j - \hat{p}_j > \hat{p}_j \tag{5}$$

3 The Value of a Real Option in Incomplete Markets

In the next three sections we apply the minimax approach described above to narrow the bounds of a real option value in incomplete markets. We illustrate the approach with an option to defer a real investment decision.

3.1 A Risky Project

Consider a risky project requiring the following alternative capital outlays, depending whether the investment occurs now, I_N, or is deferred one period, I_D:

$$\begin{aligned} I_N &= \left(I_0,\ I_1 \times I\left\{V_M \geq W_M\right\}\right) \text{ if invested immediately, or} \\ I_D &= \left(0,\ \left(I_0(1+r_f) + I_1\right) \times I\left\{V_M \geq W_M - I_0(1+r_f) - I_1\right\}\right) \text{ if deferred.} \end{aligned}$$

where the rate r_f denotes the constant risk free rate of interest, and $I\left\{V_M \geq W_M\right\}$ is an indicator function with V_M, the value of the project, and W_M, the selling price of the project at time 1. $M \in \{u, d\}$, where u indicates the up state in the binomial tree for variable M, and d the down state.

The project faces two different types of risks. The first risk is realized at time 1. This risk can be traded in the securities markets by means of a twin security whose payoffs are perfectly correlated with that source of risk. Accordingly, we call this risk *market risk*. The second risk starts to affect payoffs at time 1 and is realized at time 2, and is not traded in the securities market. Accordingly, we call this risk *private risk*. Combining both risks, we have four states and just two securities. With fewer securities than the number of states in the economy, the market is incomplete.

Upon the realization of market risk at time 1, the project has values $\tilde{V}_M = (V_u, V_d)$ with objective probabilities $q_1 = (q, 1-q)$, respectively. Similarly, the twin security, which has market value at time 0 of S_0, will generate payoffs at time 1 of $\tilde{S}_1 = \{S_u, S_d\}$. This security and the risk-free bond paying r_f are the basis assets used to obtain a market value for the project. As market risk is fully realized at time 1, after that date, and with no loss of generality, there is no more uncertainty as to the value of the twin security. At time 2, after the realization of private risk, the project ends, and there are four possible payoffs: $\tilde{V}_{MP} = (V_{uu}, V_{ud}, V_{du}, V_{dd})$ with objective probabilities $q_2 = \left(q^u, 1 - q^u, q^d, q^{1-d}\right)$. Note that $\tilde{V}_{MP}$ is the value after market risk M, and private risk P have been realized. Also, q^u is the probability of a good outcome for private risk conditional on a good outcome for market risk, q; similarly, q^d is the probability of a good outcome for private risk conditional on a bad outcome for market risk $1 - q$.

This example separates the risks affecting the project's payoffs by their time of realization. This differs from the usual characterization of market incompleteness, which considers all risks occurring simultaneously. Also, the different sources of risk are not necessarily independent of each other, and some degree of correlation is possible. We separate the timing of realization of the risks and allow for some degree of dependence among different risks to give the problem the flavor of a typical real option, as well as to highlight the investor's ability to change the distribution of the project's returns. This is what in the literature on real options is referred to as managerial flexibility.

At first, it might appear that the problem described can be reduced to two independent static problems in states u and d, beyond which the market becomes truly incomplete. Also, it might appear that there are no truly dynamic issues in the model. This is not so. The confusion arises because the traded risky security becomes risk-free after period one, to simplify the exposition. However, this simplification does not make the market complete before time 1, because private risk cannot be hedged at that date or at any other date. Finding the solution to two independent static problems in states u and d separate from each other is equivalent to a local maximization problem, and doing that means that the initial value is the solution to a local maximum, instead of a global maximum. For this reason, the initial value must be determined by maximization over the entire set of states and dates.

The problem cannot be solved by the no-arbitrage approach, since in the absence of traded assets that replicate the project's payoffs at time 2, it is not possible to obtain the risk-neutral probabilities (RNP) that would determine the value at time 1. On the other hand, it is not possible to figure out the values at time 1 in states u and d using only the value at time 0 and the risk-neutral probability derived from market risk. Therefore, one must first solve the optimization problem to obtain the project value at time 0 and then compute the the risk-adjusted probabilities using the well-known equivalence between the two approaches.

3.2 Equilibrium Values

Market incompleteness forces the investor in the project to bear risks that cannot be fully hedged in the capital markets. Therefore, the value of the project is not independent of its payoff distribution as well as of the investor's marginal rate of substitution for consumption across states and time. Because attitudes toward risk matter, the choice of utility function also matters.[2] We assume that preferences are described by an extended power utility function, also denoted by the class of HARA utility functions.[3] We do not impose additional assumptions on pricing kernels such as those used in Levy (1985)[4], because risk-aversion implicit in the utility function is equivalent to monotonicity of pricing kernels with the state variable (Perrakis and Ryan (1984)). Equivalently, risk-averse investors compute pricing kernels that are monotone in the state variable governing private risks[5].

The HARA class of utility functions of wealth exhibits increasing (IRRA) or decreasing (DRRA) relative risk aversion depending on whether the sign of the parameter a in the utility function is positive or negative:

$$U(X) = \sum_{t=0}^{2} \frac{1}{b-1}\left(a + b\frac{X_t}{(1+\rho)^t}\right)^{1-\frac{1}{b}} \tag{6}$$

Note that when $a = 0$, an extended power reduces to a narrow power utility function. An extended power utility function seems to accommodate the problem of Black-Scholes model mispricing reported by Jackwerth (2000) and Ait-Sahalia and Lo (2000)[6]. The utility function in Equation (6) is subject to the usual positivity conditions: $b > 0 : \frac{X_t}{(1+\rho)^t} > \max\left(-\frac{a}{b}, 0\right)$. a and b are the parameters characterizing the absolute risk aversion coefficient, $R_A(x) = -\frac{U''(x)}{U'(x)}$, and the relative

[2] Researchers valuing options have frequently related the narrow power utility function to the Black-Scholes valuation model. Rubinstein (1976), Brennan (1979), and more recently He and Leland (1993) and Ait-Sahalia and Lo (2000) note that the Black-Scholes model can be obtained in an equilibrium economy, when agents have a narrow power utility characterized by constant relative risk aversion (CRRA) and aggregate wealth and the underlying asset price are joint lognormally distributed.

[3] The HARA class of utility functions is very general and well established, but a different class could have been used. It is important to note that we do not need to specify the sign of the coefficient of relative risk aversion.

[4] Levy (1985) assumes that the pricing kernel declines monotonically with the state variable.

[5] The monotonicity relationship can decline or increase with the state variable depending upon the benchmark pricing kernel. More specifically, when the benchmark pricing kernel declines monotonically with the state variable, the pricing kernel is greater (lower) than one in lower (higher) states of private risks, and vice versa.

The case where the pricing kernel declines (increases) monotonically with the state variable occurs when the payoffs are positively (negatively) correated with the market risk. With positive (negative) correlation, the investor favors a higher payoff in "down" ("up") state for private risk, when the payoff of market risk is high (low). If the pricing kernel is not monotone with the state variable, the investor exhibits risk-loving preferences.

[6] According to these authors, differences in model and market prices cannot be captured by CRRA preferences.

risk aversion, $R_R(x) = -\frac{xU''(x)}{U'(x)}$. The cash payoffs are X_t at times $t = 0, 1, 2$, which are discounted at the rate ρ per period.

Since one of the risks affecting the project's payout is traded in the market, it is possible to form a portfolio of the riskless bond and the traded twin security to partially hedge the project's risks and improve the investor's welfare. This requires trading a portfolio of x_k shares of the risk-free asset and y_k shares of the twin security, a portfolio that changes with each different alternative action $k \in \{R, N, D\}$, where R means reject, N means invest immediately, and D means defer investment. The notation for the utility is U_k^{MP}, where M and P stand for market and private risk, respectively, with $M, P \in \{u, d\}$. Thus, U_D^{ud} represents the utility attained when the investor has deferred investing in the project until time 1, and the outcome for market risk was high and for private risk low.

To value the opportunity created by investing in the project with the option to defer, it is necessary to determine first the maximum expected utility that can be attained if the project is not taken, and use this as the benchmark against which one measures the improvement in the investor utility from taking the project. When the project is not taken, the investor only invests in the basis assets and returns depend solely on the realization of market risk. In Appendix A we present details of the derivations necessary to obtain the expressions for the values of each alternative, "invest immediately", V_N^*, "defer investment", V_D^*, and the option to defer, V_O^*.

3.3 Risk-Adjusted Probabilities

The value of this option to defer the investment can also be computed from the risk-adjusted probabilities. The risk-neutral probabilities associated with market risk can be determined as in Cox, Ross and Rubinstein (1979), and are p and $1 - p$, for an up and down shift, respectively. The one-to-one correspondence between the values and the risk-adjusted probabilities are derived in Appendix B. The risk-adjusted probability of a good outcome for private risk conditional on a good outcome from market risk p^u, and the risk-adjusted probability of a good outcome from private risk conditional on a bad outcome from market risk p^d are given by:

$$p^u = \frac{(1 + r_f)(Z_u + I_1) - V_{ud}}{V_{uu} - V_{ud}} \tag{7}$$

$$p^d = \frac{(1 + r_f)(Z_d + I_1) - V_{dd}}{V_{du} - V_{dd}} \tag{8}$$

where Z_u and Z_d are derived in Appendix B as:

$$Z_u = \frac{1+r_f}{p}[\phi - (1-p)\, I_0] \tag{9}$$

$$Z_d = \frac{1+r_f}{1-p}[\psi - (1-p)\, I_0] \tag{10}$$

The expressions for the risk-adjusted probabilities for private risk, p^u and p^d, closely resemble the expressions for the risk-neutral probabilities in complete markets. With complete markets, p^u would be replaced by $p^{u(C)} = \frac{(1+r_f)(V_u^C+I_1)-V_{ud}}{V_{uu}-V_{ud}}$ and p^d by $p^{d(C)} = \frac{(1+r_f)(V_d^C+I_1)-V_{dd}}{V_{du}-V_{dd}}$, where V_u^C and V_d^C are the project values at time 1 in complete markets. The only difference is the factors Z_u instead of V_u^C and Z_d instead of V_d^C. The values of Z_M are preference-dependent through the parameters ϕ and ψ.

Figure (1) relates the changes in the values of p^u to the changes in risk preference represented by the absolute risk aversion (ARA), using the values shown in Table (1). The figure shows the linear relationship that exists between changes in the investor's ARA coefficient and changes in the values of the risk-adjusted probabilities. Furthermore, varying ARA leads to very small changes in the risk-adjusted probabilities. Note, also, that the difference between the objective probabilities and the risk neutral probabilities is largest at $q = 0.5$, rapidly declining as q values move away from 0.5. These findings are useful, because they imply that less precise information about the investor's risk preferences, measured by its ARA coefficient, does not translate proportionally into less precise valuations.

Table (1) shows that the differences between the values of the objective probabilities and those of the risk-adjusted probabilities are maximized when the objective probabilities are 0.5, declining to zero when the objective probabilities are either 0 or 1. This is because objective probabilities correspond to certain wealth, while risk-adjusted probabilities are related to certainty-equivalent wealth. It is well known that the difference between certainty-equivalent wealth and certain wealth is maximized at the midpoint of a range of possible payoffs, and is minimized at the extreme points of either lowest or highest payoffs.

4 Locating Option Values within the No-Arbitrage Bounds

The option values obtained from the risk-adjusted probabilities lie within the no-arbitrage bounds set by the risk-neutral probabilities for market risk only. In this section we show that the option value is expressed as a simple function of the no-arbitrage bounds and the risk-adjusted probabilities for private risk. To do this, we introduce two no-arbitrage bounds (NA) for the "defer investment" alternative: D_{NAU}, the upper bound, and D_{NAL}, the lower bound; and two no-arbitrage bounds for the "invest immediately" alternative: N_{NAU}, the upper bound, and N_{NAL}, the lower bound. With these bounds we determine the no-arbitrage bounds for the option to defer and then locate the option value within these no-arbitrage bounds.

Consider first the "defer investment" alternative. Under this alternative, the project is not taken if the outcome for market risk at time 1 is low, and the location within the bounds is given by the risk-adjusted probability for private risk, conditional on a good outcome for market risk, p^u.

To derive D_{NAU}, we use the discounted value of the highest terminal payoff, given a good outcome for market risk, with risk-neutral probability p. Similarly, the lower bound, D_{NAL}, is obtained by discounting the value of the lowest terminal payoff, given a good outcome for market risk:

$$D_{NAU} = p\left(-I_0 - \frac{I_1}{1+r_f} + \frac{V_{uu}}{(1+r_f)^2}\right) \tag{11}$$

$$D_{NAL} = p\left(-I_0 - \frac{I_1}{1+r_f} + \frac{V_{ud}}{(1+r_f)^2}\right) \tag{12}$$

Using $p^u = \frac{(1+r_f)(Z_u+I_1)-V_{ud}}{(V_{uu}-V_{ud})p}$, the location within the no-arbitrage bounds of the value of the "defer investment" alternative is given by the expression:

$$\begin{aligned} V_D^* &= p^u D_{NAU} + (1-p^u) D_{NAL} \\ &= p\left(-I_0 - \frac{I_1}{1+r_f} + \frac{V_{ud}}{(1+r_f)^2}\right) + \frac{p\,(V_{uu}-V_{ud})}{(1+r_f)^2} p^u \end{aligned} \tag{13}$$

where this last expression is exactly the same from either specifying preferences or using the risk-adjusted probabilities. This way of determining the value of the alternative "defer investment" is consistent with the other approaches because the triple (D_{NAU}, D_{NAL}, p^u) includes all the information necessary to obtain the exact value of the "defer investment" alternative.

For the "invest immediately" alternative, the no-arbitrage bounds are the discounted expected values using the highest possible payoffs, N_{NAU}, and using the lowest possible payoffs, N_{NAD}, for both outcomes of market risk, as follows:

$$N_{NAU} = -I_0 - \frac{I_1}{1+r_f} + \frac{pV_{uu}}{(1+r_f)^2} + \frac{(1-p)\,V_{du}}{(1+r_f)^2} \tag{14}$$

$$N_{NAL} = -I_0 - \frac{I_1}{1+r_f} + \frac{pV_{ud}}{(1+r_f)^2} + \frac{(1-p)\,V_{dd}}{(1+r_f)^2} \tag{15}$$

Recall that the risk-adjusted probability for private risk, conditional on a bad outcome for market risk, is $p^d = \frac{(1+r_f)(Z_d+I_1)-V_{dd}}{(V_{du}-V_{dd})p}$, which can also be expressed as

$$p^d = \frac{(1+r_f)^2}{V_{du}-V_{dd}}\left(\frac{V_N^*-V_D^*}{1-p}+I_0\right)+\frac{(1+r_f)\,I_1-V_{dd}}{V_{du}-V_{dd}} \tag{16}$$

Solving for the value of the "invest immediately" alternative, V_N^*, gives:

$$\begin{aligned}
V_N^* &= V_D^*+(1-p)\left[\frac{V_{du}-V_{dd}}{(1+r_f)^2}p^u-\frac{I_1}{1+r_f}+\frac{V_{dd}}{(1+r_f)^2}-I_0\right] \\
&= V_D^*+\left(N_{NAU}-N_{NAL}-\frac{p\,(V_{uu}-V_{ud})}{(1+r_f)^2}\right)p^u-(1-p)\left(\frac{I_1}{1+r_f}+I_0\right)+\frac{(1-p)\,V_{dd}}{(1+r_f)^2} \\
&= V_D^* \; + \; [N_{NAU}-N_{NAL}-(D_{NAU}-D_{NAL})]\,p^d-(1-p)\left(\frac{I_1}{1+r_f}+I_0\right)+\frac{(1-p)\,V_{dd}}{(1+r_f)^2} \\
&= V_D^* \; + \; [(N_{NAU}-D_{NAU})-(N_{NAL}-D_{NAL})]\,p^d+(N_{NAL}-D_{NAL}) \\
&= p^u D_{NAU}+(1-p^u)D_{NAL}-p^d\,(D_{NAU}-N_{NAU})-\left(1-p^d\right)(D_{NAL}-N_{NAL})
\end{aligned} \tag{17}$$

The value of the option to defer investment within the no-arbitrage bounds is then given by:

$$V_O = p^d\,(D_{NAU}-N_{NAU})+\left(1-p^d\right)(D_{NAL}-N_{NAL}) \tag{18}$$

In this expression, the value of the option to defer investment is obtained from two sets of no-arbitrage bounds and one risk-adjusted probability for private risk, p^d. Note that if $(D_{NAU}-N_{NAU})$ and $(D_{NAL}-N_{NAL})$ are treated as payoffs discounted at the risk free rate, the expression is exactly equal to that obtained using the binomial method to value options in complete markets, with p^d replacing the risk-neutral probability of an upward movement. The expression for the value of the option to defer does not require the risk-adjusted probability for private risk when market risk is high, p^u, because payoffs resulting from a high realization of market risk affect equally the "invest immediately" and the "defer investment" alternatives, cancelling each other out. The value of the option to defer investment comes from avoiding the losses in the downward market movement that are unavoidable under the "invest immediately" alternative. From the pairs of no-arbitrage bounds for the "defer investment" and the "invest immediately" alternatives, (D_{NAU}, D_{NAL}) and (N_{NAU}, N_{NAL}), we derive the no-arbitrage bounds for the option to defer as follows:

$$O_{NAU} = D_{NAL}-N_{NAL} = (1-p)\left(I_0+\frac{I_1}{1+r_f}-\frac{V_{dd}}{(1+r_f)^2}\right) \tag{19}$$

$$O_{NAL} = D_{NAU}-N_{NAU} = (1-p)\left(I_0+\frac{I_1}{1+r_f}-\frac{V_{du}}{(1+r_f)^2}\right) \tag{20}$$

The upper (lower) bound is derived from both the lower (upper) bounds for the “invest immediately” and the “defer investment”. To get the upper bound on the option to defer we need to determine the maximum loss from investing immediately. This maximum loss is given by the difference between the lower bounds of the two investing alternatives. Consistency requires that if a bad outcome for private risk is realized, it must affect both N and D. Conversely, to get the lower bound on the option value, we need to determine the minimum loss from the “invest immediately” alternative, which is given from the upper bounds of the two alternatives.

5 Tightening the Bounds with the Minimax Deviations Approach

5.1 No-Arbitrage Bounds

The no-arbitrage bounds are obtained with information about the project payoffs and the basis assets. These bounds, however, do not provide information about pricing private risks, because such risks are not traded in the securities markets. In the presence of private risk, the no-arbitrage bounds are often too wide to be of economic consequence. To obtain precise values for assets subject to private risk in incomplete markets it is necessary to specify investor preferences toward risk. However, in many cases it is difficult to know these preferences accurately.

The trade-off between more precise valuations from specifying preferences and the possibility of misspecification has prompted attempts to narrow the no-arbitrage bounds. Here we develop a method to narrow the bounds on option values in incomplete markets based on the idea of minimax deviations from benchmark preferences.

Consider the case that pricing kernels monotonically decline in the state variable. When investor’s preferences admit infinite risk-loving with respect to private risks, the value of the option coincides with the upper bound of the no-arbitrage value (NAU). This upper bound assumes values of $\left(p^u, p^d\right) = (1, 1)$ for the probabilities associated with private risk. Conversely, if investor’s preferences exhibit infinite risk aversion with respect to private risks, the value of the option equals the lower bound of the no-arbitrage bound (NAL), which assumes values of $\left(p^u, p^d\right) = (0, 0)$ for the probabilities of private risk.[7] Absent any arbitrage opportunities, less extreme risk preferences

[7] A risk-neutral investor is not willing to pay anything to avoid a given risk. If an investor is more risk-averse (seeking) than another investor, it would pay more to avoid (take) a given risk than other investor would. That is, a more risk-averse (seeking) investor assigns a lower (higher) value to an uncertain project than a less risk-averse (seeking) investor. At the extreme case, if an investor exhibits infinite risk aversion (seeking), it is willing pay any amount over (below) the guaranteed minimum (maximum) to avoid (take) risk. Hence, an investor with infinite risk

generate option values within these bounds.

5.2 Risk-Averse Bounds

If investors are risk averse, it is possible to narrow the bounds immediately to an interval that excludes any attitude deviating from risk-aversion. To see this, consider the case of no risk-aversion with respect to private risks. The value of the option to defer is easily computed by using the objective probabilities for private risk (q^u, q^d) and the risk-neutral probabilities for market risk, p. An investor who displays risk aversion with respect to private risks and uses the benchmark pricing kernel puts a value on the option that lies below the value implied by the risk-neutral case just defined, $(q^u,\ q^d,\ p)$. If the option values lie below, the investor puts a value on the option that lies between the upper bound defined by the risk-neutral case just defined, $(q^u,\ q^d,\ p)$, and the lower bound defined by the infinite risk aversion case, $(p^u = 0,\ p^d = 0,\ p)$. Hence, with a monotonic decreasing pricing kernel and risk aversion, the no-arbitrage upper bound (NAU) can be replaced by the risk-averse upper bound (RAU) defined by (q^u, q^d, p), which coincides with the highest bound for an investor who displays risk-aversion. This bound is easily computed. First, the values of the "defer investment" and "invest immediately" alternatives are given using $(q^u,\ q^d,\ p)$:

$$V_{D(q^u)} = \frac{p}{1+r_f}\left(\frac{q^u V_{uu} + (1-q^u)\,V_{ud}}{1+r_f} - I_1\right) - pI_0 \tag{21}$$

$$\begin{aligned} V_{N\left(q^u,q^d\right)} &= \frac{pq^u\,(V_{uu}-V_{ud})}{(1+r_f)^2} + \frac{pV_{ud}}{(1+r_f)^2} - \frac{I_1}{1+r_f} - I_0 \\ &\quad + \frac{(1-p)\,q^d\,(V_{du}-V_{dd})}{(1+r_f)^2} + \frac{(1-p)\,V_{dd}}{(1+r_f)^2}. \end{aligned} \tag{22}$$

$V_{D(q^u)}$ equals D_{RAU}, and $V_{N\left(q^u,q^d\right)}$ equals N_{RAU}. Thus, the lowest risk-averse bound for the value of the option to defer becomes:

$$O_{RAL} = D_{RAU} - N_{RAU} \tag{23}$$

It is possible to narrow the bounds even further by raising the lower bound from the current no-arbitrage bound, O_{NAL}, to a new risk-averse bound, O_{RAL}, by considering that some levels of risk-aversion are implausibly high. Option values computed from risk-adjusted probabilities for private risk with values close to zero imply that the investor is approximately infinite risk-averse. It is therefore reasonable to assume that for such investors their risk preferences are far from risk-neutral, resulting in the exclusion of values derived under risk neutrality. On the other hand, when

aversion (seeking) assigns the value at the lower (higher) bound.

the correct risk-adjusted probabilities are close to the objective probabilities, investors are unlikely to exhibit infinite risk-aversion, and option values associated with very high levels of risk-aversion can reasonably be rule out from the risk-averse bounds.

5.3 Minimax Bounds

The difficulty is where to draw the line limiting plausible levels of risk-aversion. To do this, we measure the minimax deviation from the risk preference of the investor. First, computing the deviations from p^u to either 0 (infinite risk-aversion) or q^u (risk-neutrality), we choose the minimum of the two deviations. Assuming that the inequality $0 \leq p^u < q^u$ holds, even if the underlying preferences that lead to p^u are initially misspecified, we set one bound on p^u with the minimum of the two maximum deviations, $[(0, p^u), (p^u, q^u)]$, based on the intuition that the minimax deviation indicates a greater confidence on the part of the investor about the correct value of the project. The minimum of the two maximum deviations gives an indication as to whether the risk preferences underlying p^u are closer to risk neutrality or to infinite risk aversion. Thus, if $q^u - p^u < p^u$, the upper bound is q^u, otherwise the lower bound is 0. Note that the confidence range does not provide guidance as to whether the investor is more or less risk-averse than that implied by the initial estimated parameters. Thus, it is reasonable to take symmetric changes in utility around p^u to allow for the equal likelihood of either higher or lower utility around the benchmark level, otherwise the benchmark utility level should itself move towards the larger interval of the two until the two intervals become balanced.

To represent investor's confidence range, we use minimax deviation $e^u = \min(q^u - p^u,\ p^u)$ as defined in Equation (3) for $0 \leq p^u \leq q^u$ and $e^u = \min(1 - p^u,\ p^u - q^u\ )$ for $q^u < p^u \leq 1$. This is then applied to the other bound by maintaining the same level of utility changes. Consider the case of $0 \leq p^u \leq q^u$. If $q^u - p^u < p^u$, we have $e^u = q^u - p^u$ with the upper bound as q^u. The lower bound is then set at $p^u - e^{u*}$. The last term e^{u*} is obtained by solving Equation (4) with p^u, q^u, e^u, e^{u*} replacing $\hat{p}_j, q_j, e_j^*, e^{**}$. In this case the risk-averse bounds are $(p^u - e^{u*}, q^u)$. Similarly assuming $0 \leq p^d < q^d$, the risk-averse bounds for private risks with bad outcome of market risks are $\left(p^d - e^{d*}, q^d\right)$ with e^{d*} computed by solving Equation (4).[8]

Using the minimax deviation e^u to narrow the option bounds, we turn our attention to risk-adjusted probabilities. A correct risk-adjusted probability is more likely to lie around the estimated risk-adjusted probabilities with the parameter values estimated by the investor. The final bounds

[8] The case with the other inequality $q^u < p^u \leq 1$ and $q^d < p^d \leq 1$ can be considered in a similar way.

become narrow as the computed risk-adjusted probability moves closer to each extreme. Indeed, when the computed risk-adjusted probability falls on either extreme, the risk-averse bounds are an exact point. On the other hand, if the computed risk-adjusted probability equals the mid-point of $(0, q^u)$, our approach cannot narrow the bounds at all.

The idea of minimax deviation is easy to apply. Once the investor obtains the lower bounds for the "defer investment" alternatives, the lower bounds for the "invest immediately" are similarly computed as $\left(q^d - e^{d*}, q^d + e^d\right) = \left(q^d - e^{d*}, q^d\right)$.

$$D_{RAL} = V_{D(p^u - e^{u*})} \tag{24}$$

$$N_{RAL} = V_{N\left(q^u, q^d - e^{d*}\right)} \tag{25}$$

The risk-averse bounds for the option to defer are obtained as

$$O_{RAU} = D_{RAL} - N_{RAL} = V_{D(p^u - e^{u*})} - V_{N\left(q^u, q^d - e^{d*}\right)} \tag{26}$$

$$O_{RAL} = D_{RAU} - N_{RAU} = V_{D(q^u)} - V_{N\left(q^u, q^d\right)} \tag{27}$$

The bounds on the value of the option are narrowed by the use of minimax deviation, e^u. From:

$$O_{NAU} = D_{NAL} - N_{NAL} = V_{D(0)} - V_{N(q^u, 0)} \tag{28}$$

$$O_{NAL} = D_{NAU} - N_{NAU} = V_{D(1)} - V_{N(q^u, 1)} \tag{29}$$

and $(O_{RAU},\ O_{RAL})$, the upper bound falls from O_{NAU}, and the lower bound increases from O_{NAL}, the no-arbitrage bounds:

$$O_{RAU} \leq O_{NAU} \tag{30}$$

$$O_{RAL} \geq O_{NAL} \tag{31}$$

and the value of real option lies in the narrower bounds:

$$V_O \in (O_{RAL}, O_{RAU}) \subset (O_{NAL}, O_{NAU}) \tag{32}$$

6 Comparison with Alternative Methods

Since the Sharpe ratio approach presented by Cochrane and Saa-Requejo (2000) does not rule out arbitrage opportunities, we focus the comparison on the gain-loss ratio method suggested by Bernardo and Ledoit (2000). Although Bernardo and Ledoit mention using a risk-neutral benchmark, they recognize the necessity of a benchmark model for precision as in Rubinstein (1976). The benchmark model would require them to specify a utility function with corresponding parameter values and then to select an appropriate gain-loss ratio to narrow bounds on the pricing kernels. However, they select the gain-loss ratio independent of risk preference implied in the chosen benchmark pricing kernels. The gain-loss ratio is based on the payoffs alone because it is the expectation of the project's positive excess payoffs divided by the expectation of its negative excess payoffs.

The limitation of using a quantity based on payoffs alone to narrow bounds is its potential conflict on the selected benchmark pricing kernels, which reflect risk-aversion of a utility function. The bounds constructed by gain-loss ratios could include pricing kernels that imply both risk-aversion and risk-loving. The consequence is that a risk-averse investor allows pricing kernels reflecting both risk-aversion and risk-loving, which is an inherent conflict in the gain-loss ratio method. For example, the method would construct the bounds on the estimated risk-adjusted probability, p^u, as (a, b) using a gain-loss ratio such as 2. When we have an inequality of $p^u < q^u$, it is expected to have any bounds contained in $(0, q^u)$ because the other intervals $(q^u, 1)$ will represent risk-loving preference for this pricing kernel. However, the upper bound, b, can be easily greater than q^u and we have an inequality $a < p^u < q^u < b$. Then, one sub-interval (a, q^u) represents risk-aversion and the other sub-interval (q^u, b) risk-loving, while p^u exhibits risk-aversion from a utility function. Thus, the latter interval (q^u, b) violates the implicit assumption in the benchmark pricing kernels of risk aversion. The case of an inequality, $p^u > q^u$, can be similarly argued.

The minimax deviation approach constructs the narrow bounds around p^u such as $(p^u - e^{u*}, q^u)$: $p^u - e^{u*} < p^u < q^u$ if $p^u < q^u$, including the interval for risk-aversion alone. The minimax deviation approach is robust in the sense that it does not require any arbitrary quantity such as a Sharpe ratio or a gain-loss ratio. This is consistent with the no-arbitrage principle as opposed to the Sharpe ratio method; and is easily extended to a dynamic framework as in Cochrane and Saa-Requejo (2000). Furthermore, it is free from inherent conflicts as opposed to the gain-loss ratio method by Bernardo and Ledoit (2000).

7 Numerical Example

We use a numerical example to illustrate how the two approaches in the literature are compared with that by Mello and Pyo (2006) in the case of a real option embedded in a capital budgeting problem. The real (as opposed to financial) feature of the option arises from two assumptions: 1) the realizations of the different risks may not be necessarily independent, and one source of risk may have an on the range of possible realizations of the other source of risk; and, 2) the investor has the flexibility to make subsequent investment decisions only after having observed the realization of the first risk as presented below.

We assign the following values to the variables already defined in the previous part:

$r_f = 8\%; \quad \rho = 10\%; \quad \underline{I} = (I_0, I_1) = (90,\ 10)\,;$

$\underline{S} = (S_0,\ S_1) = (S_0,\ (S_u, S_d)) = (20,\ (36, 12))\,;$

$q_1 = (q, 1-q) = (0.5,\ 0.5)\,; \quad q_2 = \left(q^u, 1-q^u, q^d, 1-q^d\right) = (0.5,\ 0.5,\ 0.75,\ 0.25)\,;$

$V_{MP} = (V_{uu}, V_{ud}, V_{du}, V_{dd}) = (221.40,\ 199.80,\ 88.56,\ 45.36).$

We obtain the risk-neutral probability of a high outcome for market risks from the twin security,

$$p = \frac{1.08 \times 20 - 12}{36 - 12} = 0.4 \tag{33}$$

The option to defer has a zero payoff in a high realization of market risk, and some positive payoffs in the event of a bad outcome for market risk. This is because both alternatives, "Invest Immediately" and "defer investment", have the same payoffs under a good outcome of market risk, but the "invest immediately" alternative incurs losses in case of a bad outcome of market risk. The option to defer gains from realizing a low outcome of market risks by avoiding investment in this case. The pricing kernel, m_1, corresponds to a high outcome of market risks, while the pricing kernel, m_2, corresponds to a low outcome of market risks and a high outcome of private risks, and the pricing kernel, m_3, to a low outcome of both market risks and private risks. If the benchmark investor is risk-neutral, we have the following pricing kernel

$$m_1 = \frac{0.4}{0.5 \times 1.08^2} = 0.685871\,; \quad m_2 = \frac{0.6 \times 0.75}{0.375 \times 1.08^2} = 1.028807\,; \quad m_3 = \frac{0.6 \times 0.25}{0.125 \times 1.08^2} = 1.028807$$

where numerators reflect the risk-neutral probabilities from market risks, which are multiplied by the probability of private risks. The pricing kernel gives the risk-neutral benchmark investor the value of option to defer as 19.56,

$$\begin{aligned} V_O &= 0.5 \times 0.685871 \times 0 + 0.375 \times 1.028807 \times \left(90\,(1.08)^2 + 10 \times 1.08 - 88.56\right) \\ &\quad +0.125 \times 1.028807 \times \left(90\,(1.08)^2 + 10 \times 1.08 - 45.36\right) \\ &= 0 \times 0.375 \times 1.028807 \times 27.216 + 0.125 \times 1.028807 \times 70.416 = 19.56 \end{aligned} \tag{34}$$

7.1 Minimax Bounds

The minimax deviation developed by Mello and Pyo (2006) constructs bounds that are consistent with risk aversion, even if they do not impose additional restrictions on the pricing kernel. To illustrate this, consider the utility function given by (6) with $(a, b) = (210,\ 0.6)$. From (79), the risk-adjusted probabilities for private risk are computed as $\left(p^u, p^d\right) = (0.4908,\ 0.7314)$. The value of the option to defer is computed with (68) as $V_O^* = 19.97$, which is also obtained with $\left(1 - p, p^d\right) = (0.6, 0.7314)$ as

$$V_O^* = \frac{0.4 \times 0}{1.08^2} + \frac{0.6 \times 0.7314 \times 27.216}{1.08^2} + \frac{0.6 \times (1 - 0.7314) \times 70.416}{1.08^2} = 19.97 \tag{35}$$

Since $q^d - p^d = 0.75 - 0.7314 = 0.0186 < 0.7314 = p^d$, we let the minimax deviation be $e^{d*} = 0.0186$, which results in the utility loss of 0.000053406 from the benchmark price. By allowing the utility gain of 0.000053406 on the other side of the benchmark price, which is computed from the risk-adjusted probability, 0.7314, we obtain the risk-adjusted probability for the lower bound as $p^d - e^{d**} = 0.7314 - 0.0373 = 0.6941$.

$$O_{RAL} = \frac{0.4 \times 0}{1.08^2} + \frac{0.6 \times 0.75 \times 27.216}{1.08^2} + \frac{0.6 \times 0.25 \times 70.416}{1.08^2} = 19.56 \tag{36}$$

$$O_{RAU} = \frac{0.4 \times 0}{1.08^2} + \frac{0.6 \times 0.6941 \times 27.216}{1.08^2} + \frac{0.6 \times (1 - 0.6941) \times 70.416}{1.08^2} = 20.80 \tag{37}$$

The option risk-averse bounds (19.56, 20.80) are significantly narrower than the no-arbitrage bounds (14.00, 36.22).

7.2 Good-Deal Bounds with Sharpe Ratio

Since the good deal bounds by Cochrane and Saa-Requejo impose the volatility constraint on the pricing kernels, we compute pricing kernels corresponding to the upper bound and the lower bound for the value of the option to defer. The pricing kernel of the lower bound and its volatility, $\sigma(m)$, are:

$$m_1 = \frac{0.4}{0.5 \times 1.08^2} = 0.685871\,; \quad m_2 = \frac{0.6 \times 1}{0.375 \times 1.08^2} = 1.371742\,; \quad m_3 = \frac{0.6 \times 0}{0.125 \times 1.08^2} = 0$$

$$E(m) = 0.5 \times 0.685871 + 0.375 \times 1.371742 + 0.125 \times 0 = 0.857339; \quad \sigma(m) = 0.453661$$

The pricing kernel generates the lower bound of the value of the option to defer as

$$O_{NAL} = 0.5 \times 0.685871 \times 0 + 0.375 \times 1.371742 \times 27.216 + 0.125 \times 0 \times 70.416 = 14.00 \tag{38}$$

The pricing kernel of the upper bound and their volatility, $\sigma(m)$, are:

$$
\begin{aligned}
m_1 &= \frac{0.4}{0.5 \times 1.08^2} = 0.685871; \quad m_2 = \frac{0.6 \times 0}{0.375 \times 1.08^2} = 0; \quad m_3 = \frac{0.6 \times 1}{0.125 \times 1.08^2} = 4.115226 \\
E(m) &= 0.857339; \quad \sigma(m) = 1.271639
\end{aligned}
$$

The pricing kernel generates the upper bound of the value of the option to defer as

$$
O_{NAU} = 0.5 \times 0.685871 \ \times 0 + 0.375 \times 0 \ \times 27.216 + 0.125 \times 4.115226 \times 70.416 = 36.22 \tag{39}
$$

From the two pricing kernels with their volatilities (0.453661, 1.271639), we have the no-arbitrage bounds $(14, 00,$ 36.22) around the unique price of 19.56 for the risk-neutral benchmark investor.

To illustrate the application of Cochrane and Saa-Requejo to our example, we now compute the Sharpe ratio of the twin security with market risks alone because the twin security is considered as a basis asset:

$$
\begin{aligned}
E[X] &= 0.5\left(\frac{36-20}{20}\right) + 0.5\left(\frac{12-20}{20}\right) = 0.2 \\
Var[X] &= 0.5\,(0.8-0.2)^2 + 0.5\,(-0.4-0.2)^2 = 0.36; \quad \sigma[X] = (0.36)^{1/2} = 60\% \\
\text{Sharpe Ratio} &= \frac{0.2-0.08}{0.6} = 0.2
\end{aligned} \tag{40}
$$

The volatility constraint on the pricing kernels with twice the Sharpe ratio from the basis asset is

$$
\sigma(m) \leq \frac{h}{(1+r_f)^2} = \frac{c \times \text{Sharpe ratio}}{(1+r_f)^2} = \frac{2 \times 0.2}{1.08^2} = 0.342935 \tag{41}
$$

We need to reduce the volatility of the pricing kernel to satisfy the constraint. The first element of the pricing kernel is fixed because it only reflects market risks, which is determined by the basis asset. In case of the upper bound, we move m_2 upward from 0 and m_3 downward from 4.115226, while maintaining $E(m) = 0.857339$. The pricing kernel is given as

$$
m_1 = 0.685871; \quad m_2 = 0.786315; \quad m_3 = 1.756280; \quad E(m) = 0.857339; \quad \sigma(m) = 0.342935
$$

and generates the upper bound, 23.46. Similarly, the pricing kernel for the lower bound is obtained as

$$
m_1 = 0.685871; \quad m_2 = 1.271298; \quad m_3 = 0.301331; \quad E(m) = 0.857339; \quad \sigma(m) = 0.342935
$$

and produces the lower bound, 15.61. Thus, the volatility constraint with twice the Sharpe ratio results in the narrower bounds from $(14, 00,$ 36.22) to (15.61, 23.46) around the risk-neutral price of 19.56.

The good-deal bounds do not require a benchmark price and generates loose bounds with a typical threshold of twice the Sharpe ratio. The problem with these bounds is that the bounds conflict with a benchmark price, if any, of any risk-averse investor because the Sharpe ratio constraint constructs bounds around a risk-neutral benchmark price. From the obtained bounds of (15.61, 23.46) , one range of (15.61, 19.56) represents risk-loving preference and the other range of $(19.56, 23.46)$ exhibits risk-averse preference.

7.3 Pricing Bounds with Gain-Loss Ratio

We compute the benchmark pricing kernel from the risk-adjusted probability, 0.7314, of a high outcome of private risks with a low outcome of market risks computed in the previous part:

$$m_1^* = \frac{0.4}{0.5 \times 1.08^2} = 0.685871\,; \quad m_2^* = \frac{0.6 \times 0.7314}{0.375 \times 1.08^2} = 1.003292; \quad m_3^* = \frac{0.6 \times 0.2686}{0.125 \times 1.08^2} = 1.105350$$

The benchmark price of the option to defer is computed as

$$V_O^* = 0.5 \times 0.685871 \times 0 + 0.375 \times 1.003292 \times 27.216 + 0.125 \times 1.105350 \times 70.416 = 19.97$$

Since the gain-loss ratio is equivalent to the ratio of the maximum and minimum values of the pricing kernel, we focus on the pricing kernel. As in the previous part, the first element, m_1, is fixed at $m_1 = 0.685871$.

$$\begin{aligned} &0.5 \times 0.685871 + 0.375 m_2 + 0.125 m_3 = \frac{1}{1.08^2} = 0.857339 \\ &0.375 m_2 + 0.125 m_3 = 0.514403 \text{ and } m_2, m_3 > 0 \\ &m_3 = \frac{1}{0.125}\left(0.514403 - 0.375 m_2\right) = 4.115227 - 3m_2, \quad 0 < m_2 < 1.371742 \end{aligned} \tag{42}$$

$$\begin{aligned} \max \frac{E^*\left[\widetilde{x}^+\right]}{E^*\left[\widetilde{x}^-\right]} &= \min \frac{\sup\left(m_j/m_j^*\right)}{\inf\left(m_j/m_j^*\right)} = \min \frac{\sup\left(m_2/1.003292, (4.115227 - 3m_2)/1.105350\right)}{\inf\left(m_2/1.003292, (4.115227 - 3m_2)/1.105350\right)} \\ m_2 &> 0 \text{ subject to } \frac{\sup\left(m_2/1.003292, (4.115227 - 3m_2)/1.105350\right)}{\inf\left(m_2/1.003292, (4.115227 - 3m_2)/1.105350\right)} \leq \overline{L} \end{aligned} \tag{43}$$

The restriction results in

$$\frac{4.115227}{1.101710\overline{L} + 3} \leq m_2 \leq \frac{3.735307\overline{L}}{1 + 2.723038\overline{L}} \text{ and } m_3 = 4.115227 - 3m_2 \tag{44}$$

If we let $\overline{L} = 1$, we have the benchmark pricing kernel and the value of the option to defer,

$$\begin{aligned} m_2 &= \frac{4.115227}{1.\,106910 + 3} = 1.003295; \quad m_3 = 4.115227 - 3 \times 1.003295 = 1.105341 \\ V_O &= 0.5 \times 0.685871 \times 0 + 0.375 \times 1.003295 \times 27.216 + 0.125 \times 1.105341 \times 70.416 = 19.\,97 \end{aligned}$$

If we let $\overline{L} = 2$, we restrict the pricing kernel to be

$$\frac{4.115227}{1.101710\overline{L} + 3} = 0.790869 \quad \leq \quad m_2 \leq \frac{3.735307\overline{L}}{1 + 2.723038\overline{L}} = 1.158940 \tag{45}$$

$$0.638407 \quad \leq \quad m_3 \leq 1.742618 \tag{46}$$

The corresponding narrow bounds are (17.45, 23.41):

$$O_U = 0.5 \times 0.685871 \times 0 + 0.375 \times 0.790869 \times 27.216 + 0.125 \times 1.742618 \times 70.416 = 23.41$$

$$O_L = 0.5 \times 0.685871 \times 0 + 0.375 \times 1.158940 \times 27.216 + 0.125 \times 0.638407 \times 70.416 = 17.\,45$$

Figure (2) summarizes benchmark prices with alternative bounds obtained as above. The difficulty of relying on the gain-loss ratio constraint of Bernardo and Ledoit is that it may generate bounds that are inherently inconsistent with the risk preferences of the decision maker. This is because the gain-loss ratio constraint does not consider assumptions implied by a benchmark price, so does not rely on information regarding these preferences from a benchmark price, and bases the bounds on a value exclusively from the asset payoffs. Setting bounds based on payoffs alone is more likely to include prices that indicate inconsistent attitudes toward risk. Inside the bounds set by an $\overline{L}$ ratio, it is possible to find values that conform in a region with the investor displaying risk aversion, while in a neighboring region also inside the bounds set by the same $\overline{L}$ ratio, the values are consistent with the investor displaying a risk-loving preferences. Such anomalies are less desirable, and more so when the gain-loss ratio is effectively an individual-specific measure, impossible to separate from the decision-maker. The same investor cannot be simultaneously risk-averse and risk-loving with respect to the same project.

When we compare the result with that obtained from the gain-loss ratio of Bernardo and Ledoit, (17.45, 23.41) with a ratio of $\overline{L} = 2$, it becomes clear that their bounds include values that are not consistent with pricing implications on the benchmark pricing kernel when investors are risk-averse. Since the benchmark price for the option, 19.97, is obtained using a utility function that indicates risk-aversion, any consistent bounds must also exhibit risk aversion. The option to defer has the lowest risk-averse bound at 19.56. Any value below this number implies risk-loving on the part of the investor. The value of the option to defer comes from avoiding losses associated with premature implementation of the project. The higher the investor's risk aversion the greater the value assigned to these losses. Therefore, in the range of values (17.45, 19.56), which are possible with a Bernardo-Ledoit gain-loss ratio of $\overline{L} = 2$, the investor exhibits a risk-loving attitude, which is inconsistent with the economic assumptions embedded in the benchmark price.

8 Conclusion

Unlike asset pricing in a complete market, it is impossible to compute the unique values of certain assets using other liquid assets in an incomplete market. Relative pricing formulas typically provide pricing implications that are too loose to be economically meaningful. To improve precision, we necessarily specify investors' risk preferences and rely on arguments about the marginal rate of substitution for consumption across time and states. The weakness of a preference-based approach is the restriction imposed on the agents' attitudes toward risk, which makes it susceptible to misspecification. The challenge is to achieve a good balance between the level of precision in prices and the restrictions imposed on the economy.

We apply the minimax deviations approach presented by Mello and Pyo (2006) to preclude arbitrage opportunities and to improve pricing implications on real options in an incomplete market. Taking a versatile HARA class utility function, we calibrate a equilibrium framework to compute benchmark prices and derive minimax bounds on a real option value. We then compare the minimax deviation approach with the methods developed by Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000), in the case of a capital budgeting decision with an option to defer investment. We show that the method of Cochrane and Saa-Requejo compromises the range of the bounds and risk preference of a benchmark investor, if any. We also show that the method of Bernardo and Ledoit can give bounds on the value of the option that are inconsistent with the risk preferences of the decision maker. While current approaches in the literature exogenously impose threshold restrictions on the values to obtain narrower bounds, Mello and Pyo (2006) uses information inherent to the investor's risk preference on the initial parameter estimation to compute narrower bounds, which are tight and consistent with risk preference implied in a benchmark price.

Real options, perhaps more than other risky assets, have features that test the preference-free relative pricing approach. Often, the event on which the option payoff depends is not a traded asset. Thin trading of real assets may also make it impossible to hedge. The key question is then how to deal with private risk factors that can significantly impact the value of the option. While current approaches in the literature exogenously impose restrictions on the values to obtain narrower bounds, we illustrate that Mello and Pyo (2006) extracts additional information from the investor's confidence on the initial parameter estimation to compute narrower bounds.

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A Real Option Values in a Preference-Based Approach

The expected utility of not investing (or rejecting) the project, $E[U_R]$, is given by:

$$E[U_R] = qU_R^u + (1-q)\,U_R^d$$

where

$$U_R^M = \frac{1}{b-1}\left[(a-b[x_k+Sy_k])^{1-\frac{1}{b}} + a^{1-\frac{1}{b}} + \left(a + b\frac{(1+r_f)^2 x_k + S_M y_k (1+r_f)}{(1+\rho)^2}\right)^{1-\frac{1}{b}}\right]$$

$$E[U_R] = \frac{q}{b-1}\left(A_R^u\right)^{1-\frac{1}{b}} + \frac{(a-bx_R-bSy_R)^{1-\frac{1}{b}} + a^{1-\frac{1}{b}}}{b-1} + \frac{1-q}{b-1}\left(A_R^d\right)^{1-\frac{1}{b}} \tag{47}$$

where

$$A_R^M = a + b\frac{(1+r_f)^2 x_R + S_M (1+r_f)\,y_R}{(1+\rho)^2} \tag{48}$$

for $M \in \{u,d\}$. The maximum expected utility, U_R^*, results from choosing the optimal amounts x_R and y_R:

$$U_R^* = \max_{(x_R,y_R)} E[U_R] \tag{49}$$

subject to the positivity conditions. The constrained maximization problem is solved by applying the Kuhn-Tucker theorem, given that the expected utility function is concave, and the constraints are convex functions of x_R and y_R.

The derivatives of the expected utility are

$$\begin{aligned}
\frac{\partial E[U_R]}{\partial x_R} &= \frac{(1+r_f)^2}{(1+\rho)^2}\left(q\left(A_R^u\right)^{-\frac{1}{b}} + (1-q)\left(A_R^d\right)^{-\frac{1}{b}}\right) - (a-b[x_R+Sy_R])^{-\frac{1}{b}} = 0 \\
\frac{\partial E[U_R]}{\partial y_R} &= \frac{1+r_f}{(1+\rho)^2}\left(qS_u\left(A_R^u\right)^{-\frac{1}{b}} + (1-q)\,S_d\left(A_R^d\right)^{-\frac{1}{b}}\right) - S(a-b[x_R+Sy_R])^{-\frac{1}{b}} = 0
\end{aligned}$$

From these equations, and using the expressions above for A_R^u and A_R^d, we solve for the values of x_R and y_R that maximize the expected utility $E^*[U_R]$:

$$x_R^* = -\frac{\frac{A_1}{A_2} + A_3 \times (A_2 \times A_4 - A_1 \times A_5)}{A_2 \times (-A_3 \times A_5 + A_2 \times A_6)} \tag{50}$$

$$y_R^* = \frac{A_2 \times A_4 + A_1 \times A_5}{-A_3 \times A_5 + A_2 \times A_6} \tag{51}$$

where

$$\begin{aligned}
A_n &= \frac{(1+r_f)\,a^{I_{\{n=1,4\}}} b^{I_{\{n=2,3,5,6\}}} S_u^{I_{\{n=3\}}} S_d^{I_{\{n=6\}}}}{(1+\rho)^2} + (-a)^{I_{\{n=1\}}}\, b^{I_{\{n=2,3\}}} S^{I_{\{n=3,6\}}} \\
&\times \left[\frac{(1+r)^2}{(1+r_f)^2 (S_u - S_d)}\left(\frac{S(1+r_f)-S_d}{q}\right)^{I_{\{n=1,2,3\}}}\left(\frac{S_u - S(1+r_f)}{1-q}\right)^{I_{\{n=4,5,6\}}}\right]^{-b} \\
n &= 1,\ 2, ...,\ 6
\end{aligned}$$

In the expression above, $I_{\{n=1,4\}}$ is the indicator function of the events $\{n=1,\ 4\}$, i.e., $I_{\{n=1,\ 4\}}$ is 1 if the event $\{n=1 \text{ or } 4\}$ occurs; otherwise it is 0. Plugging the values x_R^* and y_R^* into (47) gives:

$$U_R^* = \frac{q}{b-1}\left(A_R^u\right)^{1-\frac{1}{b}} + \frac{\left(a - bx_R^* - bSy_R^*\right)^{1-\frac{1}{b}} + a^{1-\frac{1}{b}}}{b-1} + \frac{1-q}{b-1}\left(A_R^d\right)^{1-\frac{1}{b}} \tag{52}$$

Next, we evaluate the alternatives of investing I_0 in the project at time 0 and I_1 at time 1 – the "invest immediately" alternative and of investing all $I_0(1+r_f)+I_1$ at time 1 – the "defer investment" alternative.

The expected utility is:

$$E\left[U_k\right] = qq^u U_N^{uu} + q\left(1 - q^u\right)U_N^{ud} + (1-q)\,q^d U_N^{du} + (1-q)\left(1-q^d\right)U_N^{dd} \tag{53}$$

in the case of the "invest immediately", the state contingent utilities are:

$$U_N^{MP} = \frac{\left(a - b\left[I_0 + x_N + Sy_N + V_N\right]\right)^{1-\frac{1}{b}}}{b-1} + \frac{1}{b-1}\left(a - \frac{bI_1}{1+\rho}\right)^{1-\frac{1}{b}} + \frac{A_N^{MP}}{b-1} \tag{54}$$

where

$$\begin{aligned} A_N^{MP} &= \left(a + b\frac{V^{MP} + \left(1+r_f\right)^2 x_N + S_M\left(1+r_f\right)y_N}{(1+\rho)^2}\right)^{1-\frac{1}{b}} \\ M,P &\in \{u,d\} \end{aligned} \tag{55}$$

The expected utility of the "invest immediately" is then

$$E\left[U_N\right] = \frac{1}{b-1}\left(\left(a - b\left[I_0 + x_N + Sy_N + V_N\right]\right)^{1-\frac{1}{b}} + \left(a - \frac{bI_1}{1+r}\right)^{1-\frac{1}{b}} + J\right) \tag{56}$$

where

$$\begin{aligned} J &= qq^u\left(A_N^{uu}\right)^{1-\frac{1}{b}} + q\left(1 - q^u\right)\left(A_N^{ud}\right)^{1-\frac{1}{b}} \\ &\quad + (1-q)\,q^d\left(A_N^{du}\right)^{1-\frac{1}{b}} + (1-q)\left(1-q^d\right)\left(A_N^{dd}\right)^{1-\frac{1}{b}} \end{aligned} \tag{57}$$

The maximized expected utility is:

$$U_N^* = \max_{(x_N, y_N)} E\left[U_N\right] \tag{58}$$

subject to the positivity conditions. It is solved by choosing the appropriate quantities for x_N and y_N. The value of investing immediately in the project can be computed as the maximum amount of money, V_N^* , that the investor is willing to pay and still be left it with the same level of utility as when it decides to reject the project:

$$V_N^* = V_N \qquad \text{such that } U_N^* = U_R^* \tag{59}$$

From (56) and solving for V_N^* gives

$$V_N^* = \frac{a}{b} - (I_0 + x_N + Sy_N) - \frac{1}{b}\left((b-1)\,U_R^* - \left(a - \frac{bI_1}{1+\rho}\right)^{1-\frac{1}{b}} - J\right)^{\frac{b}{b-1}} \tag{60}$$

Next we consider deferring investing I_0 until time 1, when $I_0(1 + r_f) + I_1$ is either invested or not. The state-contingent utilities are

$$\begin{aligned} U_D^{MP} &= \frac{1}{b-1}\left(a - b\left[I_0 + x_D + Sy_D + V_D\right]\right)^{1-\frac{1}{b}} \\ &+ \frac{1}{b-1}\left[I_{\{M=u\}}\left(a - b\frac{I_1}{1+\rho}\right)^{1-\frac{1}{b}} + \left(A_D^{MP}\right)^{1-\frac{1}{b}}\right] \end{aligned} \tag{61}$$

where

$$A_D^{MP} = a + \frac{b\left(I_{\{M=u\}}V^{MP} + \left(I_{\{M=d\}}I_0 + x_D\right)(1+r_f)^2 + y_D\,(1+r_f)\,S_M\right)}{(1+\rho)^2} \tag{62}$$

And after simplification, the expected utility (53) under the "defer investment" alternative is:

$$\begin{aligned} E\left[U_D\right] &= \frac{1}{b-1}\left(a - b\left[I_0 + x_D + Sy_D + V_D\right]\right)^{1-\frac{1}{b}} + \frac{q}{b-1}\left(a - b\frac{I_1}{1+\rho}\right)^{1-\frac{1}{b}} \\ &+ \frac{qq^u}{b-1}\left(A_D^{uu}\right)^{1-\frac{1}{b}} + \frac{q\,(1-\ q^u)}{b-1}\left(A_D^{ud}\right)^{1-\frac{1}{b}} + \frac{(1-q)}{b-1}\left(A_D^{du}\right)^{1-\frac{1}{b}} \end{aligned} \tag{63}$$

The maximum expected utility, U_D^*, under the deferring investment alternative is given as

$$U_D^* = \max_{(x_D, y_D)} E\left[U_D\right] \tag{64}$$

subject to the positivity conditions. It is solved by choosing the appropriate amounts x_D and y_D. The value of the "defer investment" alternative is the maximum amount of money, V_D^*, that the investor is willing to pay and still be left with the same level of utility of not investing in the project.

Thus, the project value, V_D^*, for the "defer investment" alternative' is

$$V_D^* = V_D \qquad \text{such that } U_D^* = U_R^* \tag{65}$$

From (63) and solving for V_D^* yields,

$$V_D^* = \frac{a}{b} - (I_0 + x_D + Sy_D) - \frac{1}{b}D_D^{\frac{b}{b-1}} \tag{66}$$

where

$$\begin{aligned} D_D &= (b-1)\,U_R^* - q\left(a - b\frac{I_1}{1+\rho}\right)^{1-\frac{1}{b}} - qq^u\left(A_D^{uu}\right)^{1-\frac{1}{b}} \\ &\quad -q\left(1- \ q^u\right)\left(A_D^{ud}\right)^{1-\frac{1}{b}} - (1-q)\left(A_D^d\right)^{1-\frac{1}{b}} \end{aligned} \tag{67}$$

From the values of investing in the project immediately, V_N , and investing later, V_D, it is possible to determine the value of the option to defer, $V_O^* = V_D^* - V_N^*$:

$$\begin{aligned} V_O^* &= x_N - x_D + S\left(y_N - y_D\right) \\ &\quad -\frac{1}{b}\left[D_D^{\frac{b}{b-1}} + \left((b-1)\,U_R^* - \left(a - \frac{bI_1}{1+\rho}\right)^{1-\frac{1}{b}} - J\right)^{\frac{b}{b-1}}\right] \end{aligned} \tag{68}$$

B Risk-Adjusted Probabilities in Incomplete Markets

The risk-adjusted probabilities associated with private risks are determined by equating the value of the investment decision at time 1, after market risk has been realized, to the discounted expectation of the payoffs at time 2, after private risk has been realized:

$$V_u + I_1 = \frac{V_{uu}p^u + V_{ud}\left(1-p^u\right)}{1+r_f} \tag{69}$$

and solving for the risk-adjusted probability of a good outcome from private risk conditional on a good outcome from market risk, p^u, gives:

$$p^u = \frac{\left(V_u + I_1\right)\left(1+r_f\right) - V_{ud}}{V_{uu} \ - V_{ud}} \tag{70}$$

To compute p^u, V_u needs to be known. When the action is to defer investment, with value V_D^*, I_0 is set aside until time 1. If at time 1 the outcome for market risk is high, which happens with risk-neutral probability p, the value of the project is V_u given an implicit investment of $I_0\left(1+r_f\right)$, and before an investment of I_1 is made. If the outcome for market risk is low, the best course of action is not to implement the project. Consequently:

$$V_D^* = \frac{p[V_u - \left(1+r_f\right)I_0]}{1+r_f} \tag{71}$$

Solving for V_u gives:

$$V_u = \frac{\left(1+r_f\right)\left(V_D^* + pI_0\right)}{p} \tag{72}$$

Substituting (72) into (70) and using (71) above yields:

$$
\begin{aligned}
p^u &= \frac{1+r_f}{V_{uu}-V_{ud}}\left[\frac{(1+r_f)\left(V_D^*+pI_0\right)}{p}+I_1\right]-\frac{V_{ud}}{V_{uu}-V_{ud}} \\
&= \frac{(1+r_f)^2}{(V_{uu}-V_{ud})\,p}\left(\frac{a}{b}-x_D-Sy_D-\frac{1}{b}D_D^{\frac{b}{b-1}}-(1-p)\,I_0\right)+\frac{(1+r_f)\,I_1-V_{ud}}{V_{uu}-V_{ud}} \\
&= \frac{(1+r_f)}{(V_{uu}-V_{ud})}\left[\frac{1+r_f}{p}\phi-\frac{(1+r_f)(1-p)\,I_0}{p}+I_1\right]-\frac{V_{ud}}{V_{uu}-V_{ud}} \\
&= \frac{(1+r_f)(Z_u+I_1)-V_{ud}}{V_{uu}-V_{ud}}
\end{aligned}
\tag{73}
$$

where

$$
Z_u = \frac{1+r_f}{p}\phi-\frac{(1+r_f)(1-p)\,I_0}{p}=\frac{1+r_f}{p}\left[\phi-(1-p)\,I_0\right] \tag{74}
$$

$$
\phi = \frac{a}{b}-x_D-Sy_D-\frac{1}{b}D_D^{\frac{b}{b-1}} \tag{75}
$$

and D_D is given in (67).

The same procedure can be used to find the risk-adjusted probability of realizing a good outcome from private risk, conditional on a bad outcome from market risk, p^d. From

$$
V_d+I_1=\frac{V_{du}p^d+V_{dd}\left(1-p^d\right)}{1+r_f} \tag{76}
$$

we get

$$
p^d=\frac{(V_d+I_1)(1+r_f)-V_{dd}}{V_{du}-V_{dd}} \tag{77}
$$

where p^d depends on an intermediate value of the project, at time 1, when the outcome of the market risk is low, V_d. To get this value we use the expression that equates the discounted value of the project at time 1, to the value of the "invest immediately" alternative plus the investment necessary at time 0, I_0:

$$
V_N^*+I_0=\frac{pV_u+(1-p)\,V_d}{1+r_f} \tag{78}
$$

Using (72) and (60), we can solve for V_d . Substituting in (77) yields:

$$
\begin{aligned}
p^d &= \frac{(1+r_f)}{V_{du}-V_{dd}}\left[\frac{(1+r_f)\left(V_N^*-V_D^*+I_0\right)}{1-p}+I_1\right]-\frac{V_{dd}}{V_{du}-V_{dd}} \\
&= \frac{(1+r_f)^2}{(V_{du}-V_{dd})(1-p)}\left[\frac{a}{b}-(I_0+x_N+Sy_N)-\frac{1}{b}\left((b-1)\,U_R^*-\left(a-\frac{bI_1}{1+\rho}\right)^{1-\frac{1}{b}}-J\right)^{\frac{b}{b-1}}\right. \\
&\quad \left.-\frac{a}{b}+(I_0+x_D+Sy_D)+\frac{1}{b}D^{\frac{b}{b-1}}\right]+\frac{(1+r_f)^2 I_0+(1+r_f)\,I_1-V_{dd}}{V_{du}-V_{dd}} \\
&= \frac{(1+r_f)^2}{V_{du}-V_{dd}}\left[\frac{1+r_f}{1-p}\psi+(1+r_f)\,I_0+I_1\right]-\frac{V_{dd}}{V_{du}-V_{dd}} \\
&= \frac{(1+r_f)(Z_d+I_1)-V_{dd}}{V_{du}-V_{dd}}
\end{aligned}
\tag{79}
$$

where

$$\begin{aligned}
Z_d &= \frac{1+r_f}{1-p}\psi - (1+r_f)\, I_0 = \frac{1+r_f}{1-p}[\psi - (1-p)\, I_0] \\
\psi &= -\frac{1}{b}\left((b-1)\, U_R^* - \left(a - \frac{bI_1}{1+\rho}\right)^{1-\frac{1}{b}} - J\right)^{\frac{b}{b-1}} + x_D - x_N + S\,(y_D - y_N) + \frac{1}{b} D_D^{\frac{b}{b-1}}
\end{aligned} \tag{80}$$

and D_D is given in (67), and J in (57).

Table 1: Changes in risk-neutral probabilities over changes in Absolute Risk Aversion (ARA)

$q_u = 0.1$ $(ARA,\ p_u)$	$q_u = 0.3$ $(ARA,\ p_u)$	$q_u = 0.5$ $(ARA,\ p_u)$	$q_u = 0.7$ $(ARA,\ p_u)$	$q_u = 0.9$ $(ARA,\ p_u)$
(0.01928, 0.0956)	(0.01925, 0.2894)	(0.01922, 0.4871)	(0.01919, 0.6889)	(0.01916, 0.8951)
(0.01171, 0.0973)	(0.01169, 0.2935)	(0.01168, 0.4922)	(0.01167, 0.6934)	(0.01166, 0.8971)
(0.00841, 0.0980)	(0.00840, 0.2954)	(0.00840, 0.4944)	(0.00839, 0.6953)	(0.00838, 0.8979)
(0.00656, 0.0985)	(0.00656, 0.2964)	(0.00655, 0.4957)	(0.00655, 0.6963)	(0.00654, 0.8984)
(0.00538, 0.0987)	(0.00537, 0.2970)	(0.00537, 0.4964)	(0.00537, 0.6970)	(0.00537, 0.8987)
(0.00456, 0.0989)	(0.00455, 0.2975)	(0.00455, 0.4970)	(0.00455, 0.6975)	(0.00455, 0.8989)

For a given level of objective probabilities (q_u) of a good outcome with respect to private risks, the pairs, $(ARA,\ p_u)$ of absolute risk aversion and the corresponding risk-adjusted probability of a good outcome, are computed using the parameter values: $a = (150, 250, 350, 450, 550, 650)$ and $b = 0.6$.

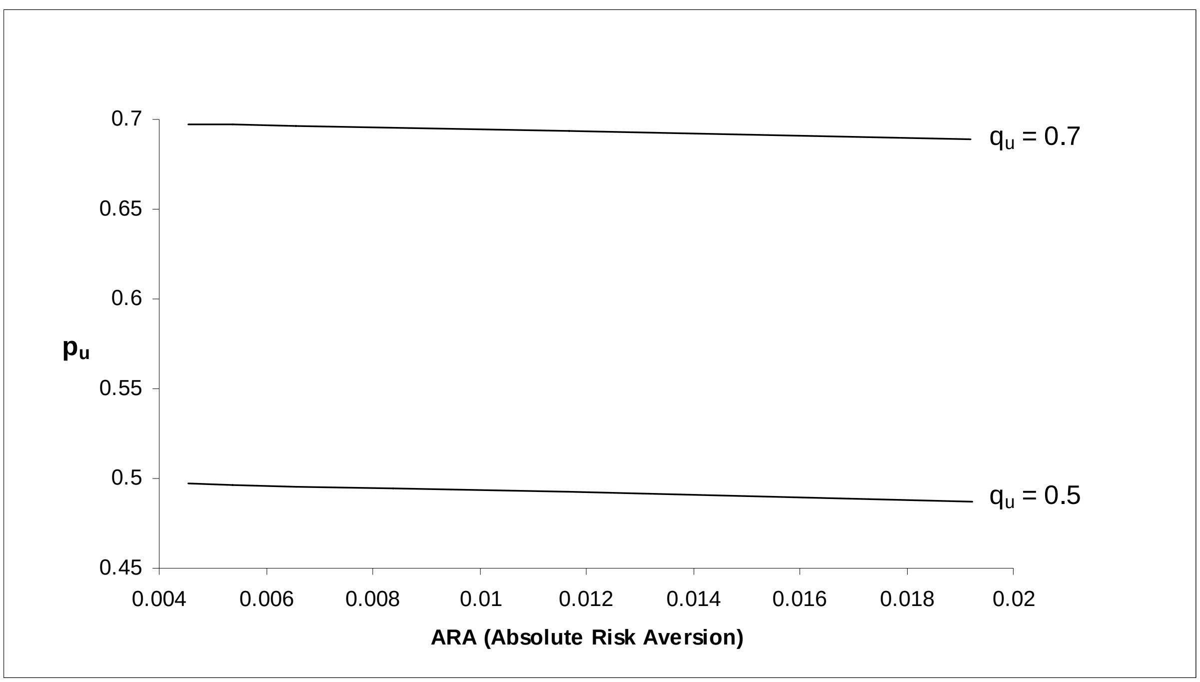

Figure 1: Changes in risk-adjusted probabilities relative to changes in absolute risk aversion.

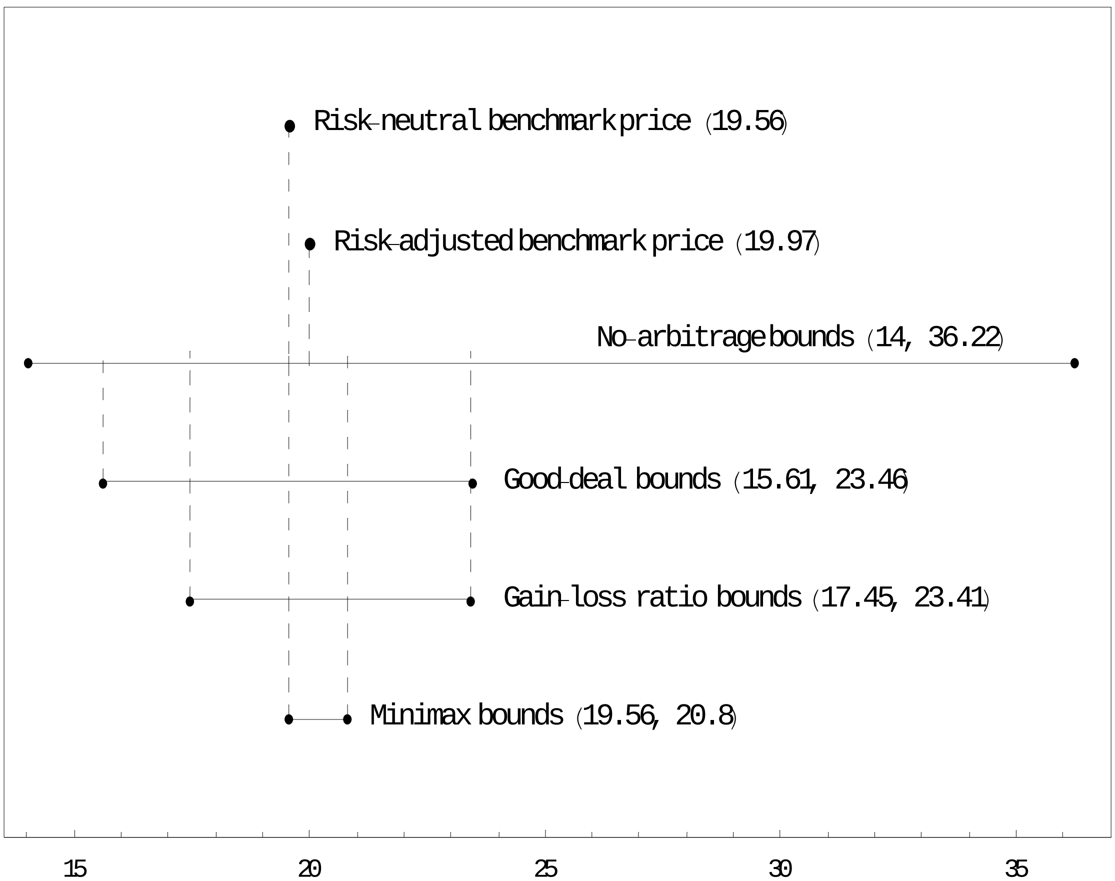

Figure 2: Benchmark prices with alternative bounds.