An Analysis of the True Notional Bond System Applied to the CBOT T-Bond Futures[*,†]

Ramzi Ben Abdallah[‡]
HEC Montréal, Canada

Michèle Breton
CREF and GERAD, HEC Montréal, Canada

June 3, 2007

Abstract

The main purpose of this paper is to apply the True Notional Bond System (TNBS) proposed by Oviedo (2006) for the theoretical pricing of the Chicago Board of Trade Treasury-bond futures, one of the most traded derivatives in the world. This system is proposed as an alternative to the current conversion factor system (CFS), whose poor performance is well known. In this paper, we price the CBOT T-bond futures as well as all its embedded delivery options and compare the corresponding results under the CFS in a stochastic interest rate framework. Our pricing procedure is an adaptation of the Dynamic Programming (DP) algorithm described in Ben-Abdallah et al. (2006), giving the value of the futures contract under the TNBS as a function of time and current short-term interest rate. Numerical illustrations, provided under the Vacisek and CIR models, show that the TNBS reduces dramatically the value of all the delivery options embedded in the CBOT T-bond futures.

JEL Classification: C61; C63; G12; G13.
Keywords: Futures, delivery options, dynamic programming.

[*]Preliminary version. Please do not quote without permission from author.

[†]Research supported by NSERC Canada and IFM2.

[‡]HEC Montréal, 3000 chemin de la Côte Sainte-Catherine, Montreal, Canada, H3T 2A7; Tel: (514) 340-6000, Ext 2469; E-mail: ramzi.ben-abdallah@hec.ca.

1 Introduction

A *futures contract* is an agreement between two investors traded on an exchange to sell or to buy an *underlying asset* at some given time in the future, called the *delivery date*, for a given price, called the *futures price*. By convention, at the time the futures is written (the *inception date*), the futures price is known and sets the value for both parties to zero. A futures contract is marked to market once a day to eliminate counterparty risk. Precisely, at the end of each trading day, the futures contract is rewritten at a new *settlement price*, that is, the closing futures price, and the difference with the last settlement futures price is substracted (*resp. added*) from the short (*resp. long*) trader account.

The *Treasury Bond futures* traded on the Chicago Board of Trade (the CBOT T-bond futures in the sequel) is the most actively traded and widely used futures contract in the United States, largely because of its ability to hedge long term interest rate risk. It calls for the delivery of $100,000 of a long-term governmental bond. The *notional* or *reference bond* is a bond with a 6% coupon rate and a maturity of 20 years. Delivery months are March, June, September and December.

Since the notional bond is a hypothetical bond that is generally not traded in the market place, the short has the option to choose which bond to deliver among a deliverable set fixed by the CBOT. The actual delivery day within the delivery month is also at the option of the short. These two delivery privileges offered to the short trader are known as the *choosing option* (or *quality option*) and the *timing option*.

The choosing option allows the delivery of any governmental bond with at least 15 years to maturity or earliest call. To make the delivery fair for both parties, the price received by the short trader is adjusted according to the quality of the T-bond delivered. This adjustment is made via a set of *conversion factors* defined by the CBOT as the prices of the eligible T-bonds at the first delivery date under the assumption that interest rates for all maturities equal 6% par annum, compounded semiannually. The T-bond actually delivered by the short trader is called the *cheapest-to-deliver* (CTD).

The timing option allows the short trader to deliver early within a delivery month according to special features, that is, the *delivery sequence* and the *end-of-month delivery rule*. The delivery sequence consists of three consecutive business days: The *position day*, the *notice day*, and the *delivery day*. During the position day, the short trader can declare his intention to deliver until up to 8:00 p.m., while the CBOT closes at 2:00 p.m. (Central

Standard Time). On the notice day, the short trader has until 5:00 p.m. to state which T-bond will be actually delivered. The delivery then takes place before 10:00 a.m. of the delivery day, against a payment based on the settlement price of the position day (adjusted according to the conversion factor). Finally, during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery, based on the last settlement price, remains possible according to the delivery sequence. The so-called *wild card play* (or *end-of-the day option* or *six hours option*) and the *end-of-month option* refer respectively to the timing option during the three day delivery sequence and to the end-of-month rule.

The modeling and measurement of the delivery options implicit in T-bond futures contracts has been extensively examined in the literature. In particular, the issue of the poor performance of the current conversion factor system (which is directly related to the value of the quality option) has been the subject of a substantial volume of research (see for instance Kane and Marcus (1984,1986), Jones (1985), Arak et al. (1986), Johnston and McConnel (1989) and Schulte and Violi (2001)). More recently, Oviedo (2006) proposes an alternative method for computing futures invoice prices called the *True Notional Bond System* (TNBS) aiming to improve the design of the T-bond futures by better achieving the objective of delivering same quality bonds. The author performs an empirical comparison of the losses resulting from delivering alternative bonds (other than the CTD) computed for each of the systems (the CFS and the TNBS). His results show that the average loss in the CFS is more than twice the one of the TNBS. However, the paper does not take into account all the special delivery features of the T-bond futures, namely the delivery sequence and the end-of-the month rule.

The main purpose of this paper is to obtain precise values of the quality and timing options under the TNBS and compare them with the corresponding under the CFS, in a stochastic interest rate framework. We assume that this rate moves according to a Markov diffusion process that is consistent with the no-arbitrage principle. Our pricing procedure is an adaptation of the Dynamic Programming (DP) algorithm described in Ben-Abdallah et al. (2006), giving the value of the futures contract under the TNBS as a function of time and current short-term interest rate.

The paper is organized as follows. In section 2, we first give a general description of the TNBS and then present the pricing model and the DP formulation for the value of the contract. Section 3 describes in details the numerical procedure. In section 4, we report on some numerical results obtained under the Vasicek (1977) and Cox, Ingersoll and Ross (1985) (hereafter CIR) models for the short rate process. Section 5 is a conclusion.

2 Model and DP formulation

2.1 Notation

We consider frictionless cash and T-bond futures markets in which trading takes place continuously. Denote

- $(c, M) \in \Theta$ an eligible T-bond with a principal of 1 dollar, a continuous coupon rate c, and a maturity M, where the set Θ of eligible bonds is known at the date the contract is written;
- $\{r_t\}$ a Markov process for the risk-free short-term interest rate;
- $\rho(r, t, \tau)$ the price at t of a zero-coupon bond maturing at $\tau \geq t$ when $r_t = r$ under the process $\{r_t\}$

$$\rho(r, t, \tau) = E\left[\exp(-\int_t^\tau r_u du) \mid r_t = r\right]; \tag{1}$$

- $p(t, c, M, r)$ the price at t of the eligible T-bond (c, M) when $r_t = r$,

$$p(t, c, M, r) = c\int_t^M \rho(r, t, u)du + \rho(r, t, M).$$

2.2 A general description of the True Notional Bond System

We recall here the description of the TNBS as proposed by Oviedo (2006). This system relies on the criterion of closeness of the futures invoice prices to spot market prices at expiration. In the TNBS, the futures invoice price is set as the present value of the remaining cash flows of the bond to be delivered, discounted at the yield to maturity implied by the settlement price.

In the TNBS, upon delivery, if the level of interest rates is $r_t = r$, the futures invoice price of the bond (c, M) is computed in two steps.

First, given the futures settlement price g^* for the T-bond futures at t, one can compute the yield to maturity of the notional bond, denoted here by r^B, that makes its price equal to g^*, that is

$$g^* = p\left(t, 6\%, 20, r^B\right). \tag{2}$$

Second, once the yield to maturity of the notional bond r^B is computed, the futures invoice price of the bond (c, M) is obtained by simply using this rate to discount its cash flows, that is, the futures invoice price is $p\left(t, c, M, r^B\right)$.

The CFS and the TNBS use the same inputs to compute the futures invoice price, namely the settlement futures price as well as the characteristics of the bond to be delivered. However, the functional form of the futures invoice price in the TNBS makes all the deliverable bonds equal for any level of flat yield curves, while in the CFS this is only achieved for the specific level of 6%.

2.3 Dynamic Programming formulation

To be consistent with the CBOT delivery rules, we consider a sequence of motoring dates t_m^h where the lower index $m = 0, ..., \overline{n}$ is computed in days from the date the contract is written and the upper index $h \in \{2, 5, 8\}$ indicates the time in hours within that day. Assuming that the contract is written at $t_0 = t_0^2$, we denote the marking to market dates by t_m^2 for $m = 0, \ldots, n$, where t_n represents the last trading date during the delivery month. We denote the delivery position dates by t_m^8 for $m = \underline{n}, \ldots, \overline{n}$, where $t_{\underline{n}}$ and $t_{\overline{n}}$ are respectively the first and the last date of the delivery month, $0 < \underline{n} < n < \overline{n}$. Finally, the delivery notice dates are denoted t_m^5 for $m = \underline{n} + 1, \ldots, \overline{n} + 1$.

Since under the TNBS equation (2) defines an equivalence between the settlement futures price and the implied yield r^B, we use this yield as a state variable. Our dynamic program is defined on the state space $\{(r, r^B) : r \geq 0, r^B \geq 0\}$ and determines the value of the contract for the short trader at each monitoring date, as a function of the spot interest rate at the current date and the yield to maturity of the notional bond as implied by the last settlement price, assuming that the short trader behaves optimally. A fair settlement price makes the value of the contract null for both parties at the settlement dates.

The contract is evaluated by backward recursion in three distinct periods: The end-of-the-month period, where no trading takes place, but delivery is still possible ($m = n, ..., \overline{n}$), the beginning of the delivery month where trading and delivery are both possible ($m = \underline{n}, ..., n$), and the period before the delivery month, where no action is taken by the short trader, but the settlement price is adjusted every day ($m = 0, ..., \underline{n}$).

2.3.1 End-of-the-month Period

The expected exercise value $v_m^e\left(r,r^B\right)$ at the delivery position date t_m^8 and the actual exercise value $v_m^a\left(r,r^B\right)$ for the short trader, for $m=n,...,\overline{n}$, are functions of the interest rate at the current date r, and the yield to maturity of the notional bond r^B as implied by the last settlement price. These values are expressed as follows:

$$v_m^e\left(r,r^B\right)=E\left[\left(v_m^a\left(r_{t_{m+1}^5},r^B\right)e^{-\int_{t_m^8}^{t_{m+1}^5}r_u du}\right)\mid r_{t_m^8}=r\right], \tag{3}$$

$$v_m^a\left(r,r^B\right)=\max_{(c,M)\in\Theta}\left\{p\left(t_{m+1}^5,c,M,r^B\right)-p\left(t_{m+1}^5,c,M,r\right)\right\}. \tag{4}$$

Otherwise, if the short trader decides not to deliver at t_m^8, for $m=n,...,\overline{n}$, the holding value $v_m^h\left(r,r^B\right)$ is computed by no-arbitrage to be the expected value of the future potentialities of the contract and given by (6) below. The short trader will of course issue an intention to deliver at $\left(t_m^8,r,r^B\right)$ if and only if

$$v_m^e(r,r^B)>v_m^h(r,r^B).$$

The value function for the short trader at t_m^8, for $m=n,...,\overline{n}$, is thus defined recursively by:

$$v_m^8\left(r,r^B\right)=\max\left\{v_m^e\left(r,r^B\right),v_m^h\left(r,r^B\right)\right\}, \tag{5}$$

$$v_m^h\left(r,r^B\right)=E\left[v_{m+1}^8\left(r_{t_{m+1}^8},r^B\right)e^{-\int_{t_m^8}^{t_{m+1}^8}r_u du}\mid r_{t_m^8}=r\right], \tag{6}$$

$$v_{\overline{n}}^8\left(r,r^B\right)=v_{\overline{n}}^e\left(r,r^B\right), \tag{7}$$

and the settlement value for the short trader at $(t_{m'}^2,r_{t_{m'}^2},r^B)$ is the expected discounted value at $t_{m'}^8$:

$$v_{m'}^2\left(r,r^B\right)=E\left[v_{m'}^8\left(r_{t_{m'}^8},r^B\right)e^{-\int_{t_{m'}^2}^{t_{m'}^8}r_u du}\mid r_{t_{m'}^2}=r\right], \tag{8}$$

where $m'=n$.

The fair settlement price $g_{t_n^2}^*\left(r\right)$ at t_n^2 should be selected so that the value to both parties is 0, taking into account the timing and quality options. To do so, the rate r^* is selected such that $v_n^2\left(r,r^*\right)=0$ for all r, thus obtaining a function $r^*\left(r\right)$ at t_n^2 and the fair settlement price $g_{t_n^2}^*\left(r\right)$ at t_n^2 is the price of the notional bond $(6\%,20)$ when its cash flows are discounted at the rate r^*.

2.3.2 Delivery Month

During the delivery month, the exercise value functions $v_m^e\left(r, r^B\right)$ and $v_m^a\left(r, r^B\right)$ at respectively t_m^8 and t_m^5, for $m = \underline{n}, ..., n-1$ remain the same as in the end-of-the-month period and are given by (3)-(4), while the holding value at t_m^8 accounts for the interim payment at the next marking-to-market date, that is,

$$\begin{aligned} v_m^h\left(r, r^B\right) &= E\left[\left(p\left(t_m^2, 6\%, 20, r^B\right) - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du}\right. \\ &\quad \left. + v_{m+1}^2\left(r_{t_{m+1}^2}\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r\right] \\ &= E\left[\left(p\left(t_m^2, 6\%, 20, r^B\right) - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r\right]. \end{aligned} \tag{9}$$

The value function at t_m^8 and t_m^2 is then given by (5) and by (8), with $m' = m$.

2.3.3 Initial Period

Within the time period $\left[t_0, t_{\underline{n}-1}^2\right]$, delivery is not possible, so that the value of the contract for the short trader only involves taking into account the interim payments in the marking to market account. The value function at t_m^2, for $m = 0, ..., \underline{n}-1$, is thus given by

$$v_m^2(r) = E\left[\left(g_m(r) - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r\right].$$

Therefore, the successive settlement prices can be obtained by the recursive relation

$$g_m^*(r) = \frac{E\left[g_{m+1}^*\left(r_{t_{m+1}^2}\right) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r\right]}{\rho(r, t_m^2, t_{m+1}^2)} \text{ for all } r,\ m = 0, ..., \underline{n}-1. \tag{10}$$

3 The Dynamic Programming procedure

Equations (3)-(10) define a dynamic program which can be used to find the fair settlement prices and the optimal timing and choosing strategies for the short trader by backward induction. In this section, we describe a numerical procedure to solve this dynamic program which does not admit a closed-form solution, even for the most simple case where the interest rate for all maturities is assumed to be constant. Two specific numerical problems must be addressed.

First, the optimization in (4) which consists in finding the CTD by solving the following expression at t_{m+1}^5

$$\max_{(c,M)\in\Theta}\left\{\left(c\int_t^M \rho\left(r^B,t,u\right)du+\rho\left(r^B,t,M\right)\right)\right.$$
$$\left.-\left(c\int_t^M \rho\left(r,t,u\right)du+\rho\left(r,t,M\right)\right)\right\}.$$

It is important to mention here that it can be easily shown that the optimal coupon is extremal and given by either $\underline{c}\equiv\min c$ whenever the short-term interest rate is less than the yield to maturity of the notional bond or $\overline{c}\equiv\max c$ in the opposite case. Since the set of eligible bonds is fixed, an intelligent enumeration of eligible bonds with extremal coupons will yield the optimal maturity and the value of $v_m^a\left(r,r^B\right)$.

The second problem is the computation of the expectations in (3), (6), (8), (9) and (10) of functions which are analytically intractable. To solve it, we compute expectations of linear finite elements interpolation functions over a finite discretization grid.

Let $\mathcal{G}=\{a_1,\ldots,a_q\}$ be a grid defined on the set of interest rates, with the convention that $a_0=-\infty$ and $a_{q+1}=+\infty$. Given a function $h:\mathcal{G}\to\mathbb{R}$, the interpolation function $\widehat{h}:\mathbb{R}\to\mathbb{R}$ is given by:

$$\widehat{h}\left(r\right)=\sum_{i=0}^{q}\left(\alpha_i+\beta_i r\right)I\left(a_i\leq r<a_{i+1}\right),\text{ for all }r\in\mathbb{R},$$

where the function I is the indicator function and the coefficients α_i and β_i are obtained by matching $\widehat{h}$ and h on $\mathcal{G}$, that is

$$\alpha_i=\frac{a_{i+1}h(a_i)-a_ih(a_{i+1})}{a_{i+1}-a_i},$$
$$\beta_i=\frac{h(a_{i+1})-h(a_i)}{a_{i+1}-a_i},i=1,...q-1,$$

and $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\alpha_q = \alpha_{q-1}$, $\beta_q = \beta_{q-1}$.

The expected value at t and $r_t = a_k$ of a future payoff $\widehat{h}$ at τ is then given by:

$$\begin{aligned}
\widetilde{h}(t,\tau,a_k) &\equiv E\left[\widehat{h}(r_\tau)\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= E\left[\sum_{i=0}^{q} (\alpha_i + \beta_i r_\tau)\, I(a_i \le r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= \sum_{i=0}^{q} \alpha_i A_{k,i}^{\tau-t} + \beta_i B_{k,i}^{\tau-t} \text{ for all } a_k \in \mathcal{G},
\end{aligned}$$

where $A_{k,i}^{\tau-t}$ and $B_{k,i}^{\tau-t}$ are defined as the *transition parameters* and given by the following expressions:

$$A_{k,i}^{\tau-t} \equiv E\left[I(a_i \le r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \tag{11}$$

and

$$B_{k,i}^{\tau-t} \equiv E\left[r_\tau I(a_i \le r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right], \tag{12}$$

where $t_0 \le t \le \tau$, $k = 0, \ldots, q$, and $i = 0, \ldots, q$.

We assume that these transition parameters and the discount factor $\rho(r,t,\tau)$ can be obtained with precision from the dynamics of $\{r_t,\ t \ge t_0\}$. Notice that for several dynamics of the interest rates, closed-form solutions exist for the transition parameters and discount factor, as discussed in Ben-Ameur et al (2007). Examples include Vasicek (1977), CIR (1985), and Hull and White (1990). Closed form formulas for the transition parameters and discount factor for the Vasicek and in the CIR model are recalled in the Appendix.

The algorithm consists in solving the dynamic program (3)-(10) by backward induction from the last delivery position date $t_{\frac{8}{n}}$ on the grid $\mathcal{G}$, which is the grid used for both the spot interest rate and the yield to maturity of the notional bond.

We start by finding the CTD and the actual exercise values for the short trader at the notice dates on all the points of $\mathcal{G} \times \mathcal{G}$, that is, for all possible interest rates and for all possible futures prices represented by the yield to maturity of the notional bond. We obtain, on each point of the interest rates grid, a vector of actual exercise values associated to the grid of yields to maturity r^B. For each yield to maturity, a linear interpolation yields a continuous function of the interest rate. The expected exercise values are then obtained at the position date. These are compared with the holding values, which are known on the two-dimensional grid points. The optimal value function at the notice date is then interpolated and the expectation is computed between either two successive notice dates (during the end-of-the-month period) or the last settlement and current notice dates (during the delivery month). This yields the value for the short trader at the settlement date as a function of the interest rate and the yield to maturity of the notional bond, which is null for a fair settlement price. At every settlement date, a simple research on the grid of r^B's for a given interest rate yields the yield to maturity of the notional bond $r^*(r)$ that makes the settlement value null for that interest rate. This optimal rate is then used to discount the notional bond's cash flows, thus obtaining the fair futures price for all interest rates. These are then interpolated, and their expected values computed, in order to compute the holding value.

The detailed algorithm is provided in the Appendix.

4 Numerical Illustration

In our numerical experiments, the finite set of deliverable bonds contains 62 bonds with maturity ranging from 15 to 30 years in steps of 6 months. Since only the bonds with extremal coupon rates are optimal to deliver, we consider only two coupon rates corresponding to the highest and lowest coupon rates in the current CBOT set of deliverable bonds, namely $\overline{c} = 7.625\%$ and $\underline{c} = 4.5\%$. The inception date is chosen to be three months prior to the first day of the delivery month.

We apply our dynamic programming procedure to obtain futures prices at the inception date under both the Vasicek and CIR term structure models, using the closed-form formulas (13)-(15) or (16)-(18) for the discount factor and transition parameters. Table 1 below gives the (risk neutral) parameter values used in the numerical experiments, where $\overline{r}$ is the long-term mean, κ is the mean reversion speed and σ is the volatility of the short-term interest rate. These parameters are those of Shoji and Osaki (1996) who estimate these models using the 1-month U.S. Treasury Bill rate over the period 1964-1992.

Table 1: Input data

	Vasicek	CIR
$\overline{r}$	0.062098	0.061677
κ	0.565888	0.545788
σ	0.025416	0.091471

The interest rates grid points $a_1, ..., a_q$ are selected to be equally spaced with $a_0 = -\infty$, $a_1 = \overline{r} - 8d$, $a_q = \overline{r} + 8d$ for the Vasicek model, while $a_1 = \max(0; \overline{r} - 8d)$ and $a_q = \overline{r} + 8d$ for the CIR model where $d = \sqrt{\frac{\sigma^2}{2\kappa}(1 - \exp(-0.5\kappa))}$ for the Vasicek model and $d = \sqrt{\overline{r}\frac{\sigma^2}{\kappa}(\exp(-0.25\kappa) - \exp(-0.5\kappa)) + \overline{r}\frac{\sigma^2}{2\kappa}(1 - \exp(-0.25\kappa))^2}$ for the CIR model. The number of grid points is 600.

We first give the precise definition adopted here for the timing option. In fact, definitions of implicit delivery options are not uniform throughout the literature and one must be cautious in comparing results across studies. In this paper, the *timing option* gives the short trader the right to deliver on any day during the delivery month. The delivery month is divided into two periods; in the first 15 business days, the futures market is open, while in the last 7 business days, it is closed.

We disentangle the individual effects of each implicit option by pricing various futures contracts embedding different combinations of these options. We thus price four different futures contracts, namely the *straight futures contract* (F1) (offering no options at all which corresponds to the case where the short declares his intention to deliver on the first position day and is allowed to deliver only one bond assumed here to be the notional bond), the contract offering the quality option alone (F2) (without timing option), the contract offering only the timing option (F3) and the full contract (F4) (offering all the embedded delivery options). The computation of these four prices allows us to price each option alone as well as in the presence of the other option. For instance, the value of the quality option in the presence of the timing option is computed as the difference (F3)-(F4) between the price of the contract offering to the seller only the timing option and the full contract. Similarly, the difference (F1)-(F2) is the value of the quality option without timing, (F1)-(F3) is the value of the timing option without quality and finally (F2)-(F4) is the value of the timing option when the quality option is offered to the short trader[1]. The price of all the embedded

[1] It is important to mention here that the way we price the timing option allows us to

options is obtained by comparing the price of the straight contract with the price of the contract with all embedded options.

4.1 Option Prices

We now report the prices of the quality and timing options at the inception date for levels of interest rates ranging from 1% to 16%. Parameter values are those given in Table 1.

Figure 1 compares the values of the embedded quality option (with and without the timing option) as a function of the interest rate for the Vasicek and CIR dynamics.

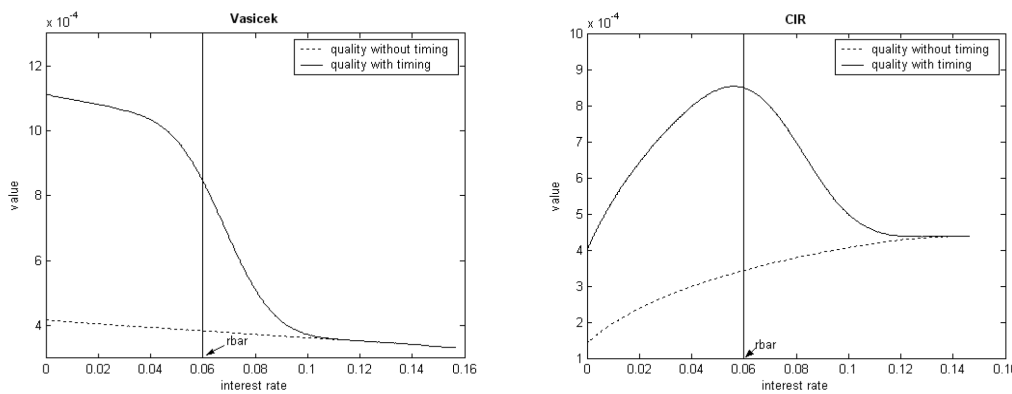


Figure 1: Quality option values vs interest rates

Without the timing option, the quality option is worth an average 0.0374 percentage points of par (ppp) for the Vasicek model while this value is 0.0346 ppp for the CIR model. When the timing option is embedded in the contract, we notice a considerable increase in the value of the quality option which is worth an average 0.0663 ppp (Vasicek) and 0.0615 ppp (CIR). The fact that the quality option is more valuable in the presence of the timing option is due to switches in the cheapest-to-deliver bond that could occur during the delivery month. We can also notice from Figure 1 that the shape of the curve representing the quality option with respect to interest rates is different for the two interest rate models considered here (downward sloping for Vasicek and humped for CIR). This shape depends essentially on

value the option to deliver late in the delivery month. Some other papers in the literature assume that delivery is only possible on the last day of the delivery month and thus define the timing option as the option to deliver early in the month.

the convexity of the cheapest-to-deliver bonds (corresponding to different levels of interest rates) with respect to the notional bond. Furthermore, we observe, for both models, that the presence of the timing option has no big impact on the value of the quality option for high levels of interest rates.

In Figure 2, we report on the evolution, during the delivery month, of the quality option values under both the Vasicek and the CIR dynamics. We observe that the quality option does not have a specific relation with respect to time to maturity. In fact, we can see for the CIR dynamic that the relation of the quality option with elapsed time is decreasing for low interest rates, increasing for higher levels, while the curve is flat for very high interest rates.

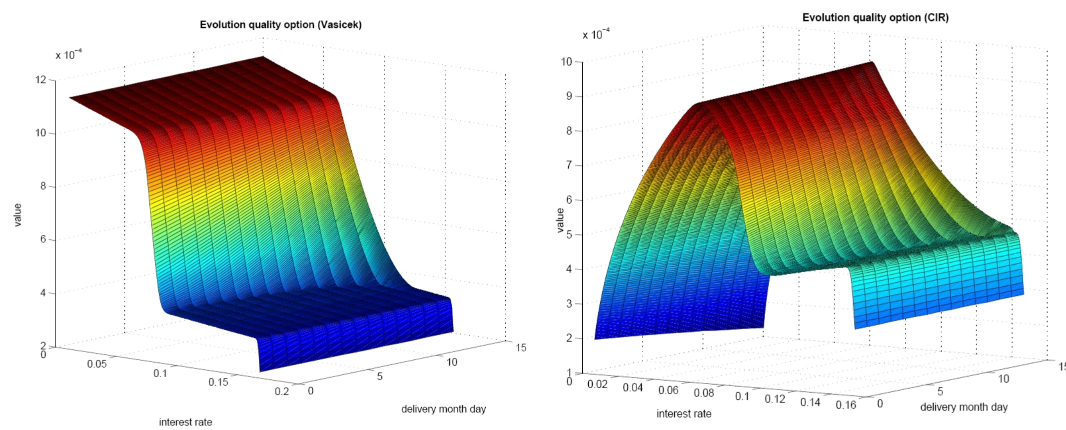


Figure 2: Evolution of the value of the quality option through the delivery month

Figure 3 plots the value of the timing option (with and without the quality option) for both dynamics considered here.

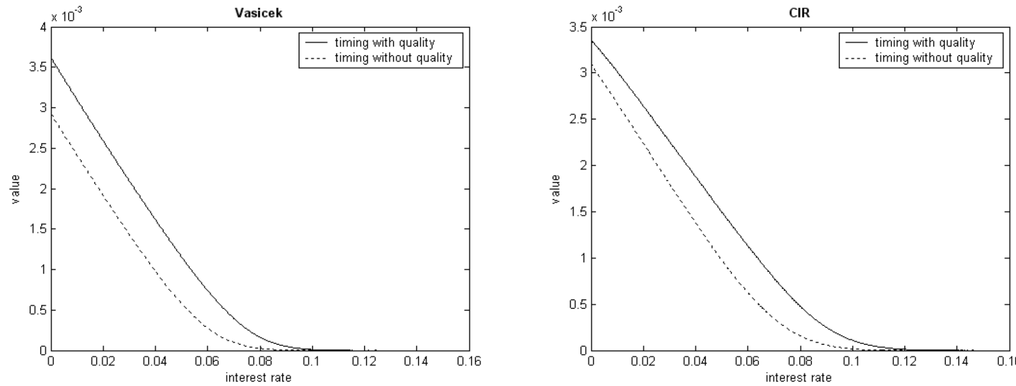


Figure 3: Timing option values vs interest rates

Without the quality option, the timing option is worth an average 0.0583 ppp (Vasicek) and 0.0804 ppp (CIR) while when the quality option is offered to the seller, the timing option is more valuable and is worth an average 0.0872 ppp (Vasicek) and 0.11 ppp (CIR). Again, and for the same reason we discussed earlier, the timing option is more valuable in the presence of the quality option. It is important to notice here that the value of the timing option can exceed the value of the quality option. In addition, the timing option is observed to be negatively related to interest rates and this can be easily explained by the fact that since we are valuing the option to deliver late, when interest rates increase and reach a high level, the short trader can invest the proceeds of early exercise at higher rates. Moreover, the value of the timing option is zero for high levels of interest rates and this is consistent with the behavior of the quality option for same levels of interest rates observed in Figure 1.

4.2 Comparison of the options values under the CFS and the TNBS

We compare here the average values of the quality and timing options obtained under the conversion factor system and the TNBS.

Results are reported in Table 2 for both interest rate models. We can notice that, for the Vasicek model, the average value of the quality option in the CFS is nearly 6 times the one of the TNBS. For example, if we consider a $100,000 futures contract, then the quality option is worth $390 under the CFS while this value is dramatically reduced to $66.3 under the TNBS. The ratio is 5.2 for the CIR dynamic. For details about the valuation of the quality option under the CFS, we refer to Ben-Abdallah et al. (2006).

Table 2 also shows that the TNBS reduces considerably the value of the timing option which is nearly 3 times higher in the CFS.

Table 2: Comparison of options values under the CFS and the TNBS

	Value of the quality option with timing (ppp)		Value of the timing option with quality (ppp)	
	Vasicek	CIR	Vasicek	CIR
CFS	0.39	0.32	0.25	0.32
TNBS	0.0663	0.0615	0.0872	0.11
ratio	5.88	5.2	2.86	2.9

Figures 4 and 5 compare, respectively, the quality and the timing options under both systems for both dynamics.

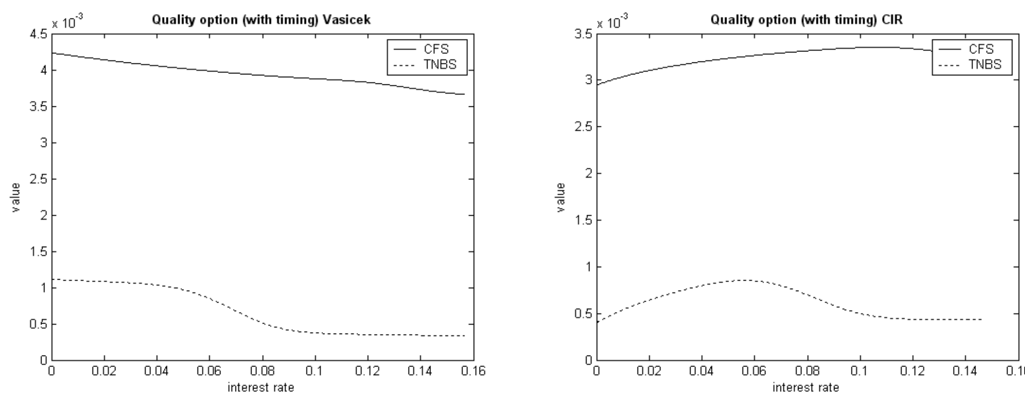


Figure 4: Comparison of the quality option under the CFS and the TNBS

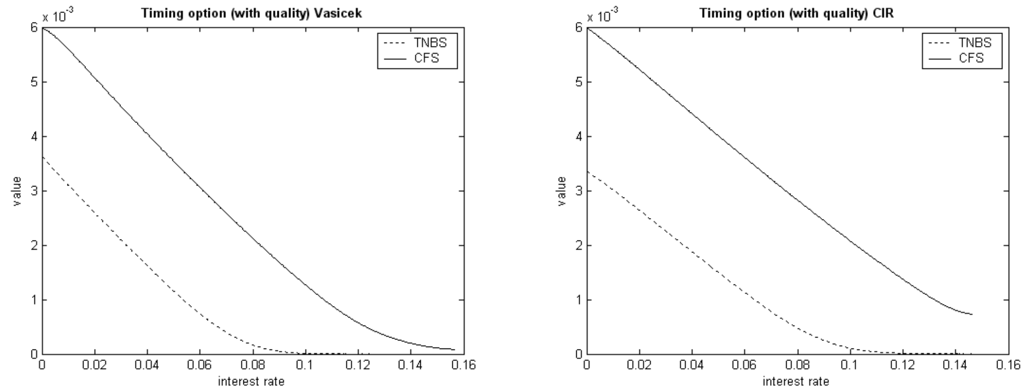


Figure 5: Comparison of the timing option under the CFS and the TNBS

4.3 Optimal delivery strategy

We present here an example of the optimal delivery decision and the associated change in the CTD for the day 15 of the delivery month under the Vasicek model and for various combinations of the two state variables r and r^B. This illustration corresponds to the parameters given in Table 1. The decision is represented by a binary variable equal to 1 if delivery is optimal and 0 otherwise and the CTD is identified by the pair (c, M). Table 3 reports on the optimal decision for small variations of interest rates ranging between 5.7% and 6.3%. Figure 6 summarizes the delivery strategy for all combinations of our two state variables. According to Figure 6, it is not optimal to deliver for some combinations of interest rates and the cheapest-to-deliver bond is (4.5%,30) whenever the yield to maturity of the notional bond is greater than the the short-term interest rate while the CTD switches to the bond (7.625%,30) in the opposite case. The impact of a change of the input parameters on the optimal delivery strategy has also been studied and results show that we can have all combinations of extremal coupon rates and maturities for the CTD while the decision is always similar to that reported in Figure 6. It is worthwhile mentioning that bonds with both maximal coupon rates and maturities or both minimal coupon rates and maturities may be optimal to deliver (which is not the case for deterministic rates).

Table 3: Optimal delivery strategy on the day 15 of the DM (Vasicek)

	$r^B(\%)$						
$r(\%)$	5.7	5.8	5.9	6	6.1	6.2	6.3
5.7	0	0	0	0	1	1	1
5.8	0	0	0	0	0	1	1
5.9	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0
6.1	1	0	0	0	0	0	0
6.2	1	1	0	0	0	0	0
6.3	1	1	1	0	0	0	0

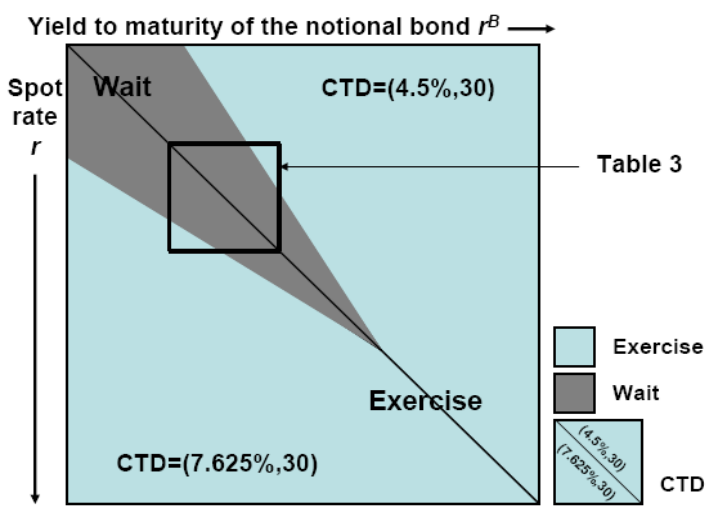


Figure 6: Optimal delivery strategy (decision and CTD)

4.4 Sensitivity of option prices

In this last section, we perform sensitivity analyses of the option values to the parameters of the interest rate models. We present here the results for the Vasicek model. Results about the impact of a change in the parameters on the options values under the CIR model are similar and given in figures 10, 11 and 12 in the Appendix.

Figure 7 presents the impact of a variation of the mean reversion speed when $\overline{r} = 0.06$ and $\sigma = 0.02$.

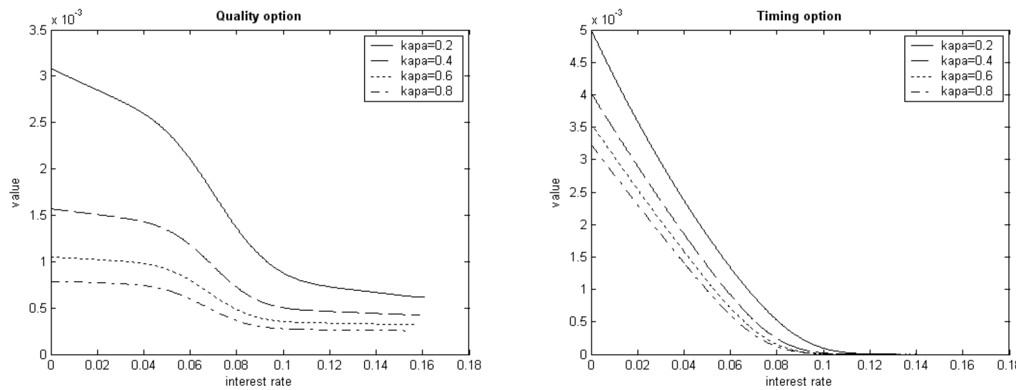


Figure 7: Options values sensitivities to kappa (Vasicek)

We observe that both the quality and timing options values are negatively affected by the mean reversion speed. This is consistent with the observation of Chen et al. (1999) about the quality option value in the Japanese futures market under the Hull and White dynamics. The authors argue that an increase in the mean reversion rate, which determines the relative volatilities of long and short rates, dampens out short term rate movements quickly and therefore reduces the long term volatility, which is positively related to the quality option value.

Figure 8 presents the impact of a variation of the long term mean when $\kappa = 0.56$ and $\sigma = 0.02$. We observe a negative relation between the quality option and the long term mean for low and high levels of interest rates. This relation becomes positive for intermediate levels. The timing option is however observed to be always positively affected by the long term mean.

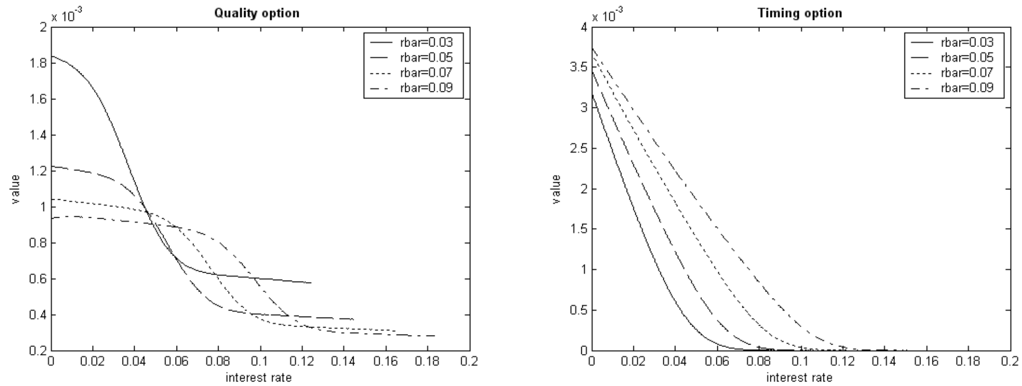


Figure 8: Options values sensitivities to rbar (Vasicek)

Figure 9 presents the impact of a variation in volatility when $\overline{r} = 0.06$ and $\kappa = 0.56$. As expected, the relation between options values and volatility is observed to be positive. Chen et al. (1999) find similar results for the quality option in the Japanese futures market.

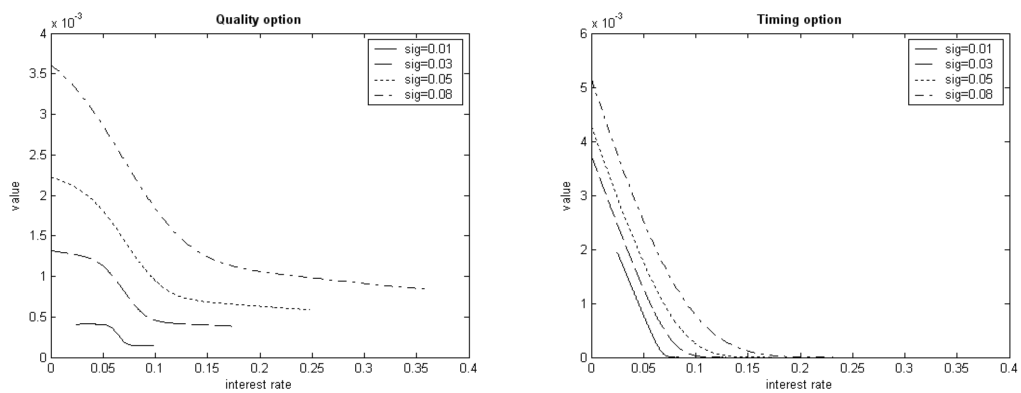


Figure 9: Options values sensitivities to sigma (Vasicek)

5 Conclusion

In this paper, we propose an efficient numerical method for pricing CBOT T-bond futures when the TNBS is the system adopted to obtain same quality bonds. This method takes into account all the inter-dependent embedded delivery options. We price the contract and the delivery options in a stochastic interest rate framework. Numerical illustrations, provided here under the Vasicek and CIR models, show that the TNBS dramatically reduces the value of the quality and timing options.

6 Appendix

6.1 Transition parameters

We give, for the Vasicek (1977) and CIR (1985) models, the closed-form formulas for the transition parameters $A_{k,i}^{\delta}$ and $B_{k,i}^{\delta}$ defined respectively in (11) and (12) as well as for the discount factor $\rho(r, t, t+\delta)$ defined in (1). For proofs and more details about the derivation of these closed-forms we refer to Ben-Ameur et al (2007).

6.1.1 The Vasicek model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma dB_t, \quad \text{for } t \geq 0,$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, κ is the mean reversion speed, $\overline{r}$ is the long term mean and σ is the volatility.

The discount factor and the transition parameters are then given by

$$\rho(r, t, t+\delta) = \exp(-\mu_2(r, \delta) + \sigma_2^2(\delta)/2), \tag{13}$$

$$A_{k,i}^{\delta} = e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)} \left[\Phi(x_{k,i}) - \Phi(x_{k,i-1})\right], \tag{14}$$

and

$$\begin{aligned} B_{k,i}^{\delta} \quad = \quad & e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)} \left[(\mu_1(a_k, \delta) - \sigma_{12}(\delta))(\Phi(x_{k,i}) - \Phi(x_{k,i-1}))\right. \\ & \left. -\sigma_1(\delta)\,(e^{-x_{k,i}^2} - e^{-x_{k,i-1}^2})/\sqrt{2\pi}\right], \end{aligned} \tag{15}$$

where

$$\begin{aligned} x_{k,i} &= (a_i - \mu_1(a_k, \delta) + \sigma_{12}(\delta))/\sigma_1 \text{ for } i = 0, ..., q, \\ x_{k,-1} &= -\infty \end{aligned}$$

and Φ is the standard normal distribution function.

6.1.2 The CIR model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma\sqrt{r_t}dB_t, \quad \text{for } t \geq 0.$$

For the CIR model, the discount factor and the transition parameters are given by

$$\rho(r, t, t+\delta) = \exp(X(\delta, 1, 0) - rY(\delta, 1, 0)), \tag{16}$$

$$A^{\delta}_{k,i} = \rho(a_k, t, t+\delta)\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2}\frac{(\lambda_\kappa/2)^u}{u!}\left[F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta})\right], \tag{17}$$

and

$$\begin{aligned} B^{\delta}_{k,i} \quad = \quad & \rho(a_k, t, t+\delta)\eta\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2}\frac{(\lambda_\kappa/2)^u}{u!}\left[-2(a_{i+1}f_{d+2u}(\frac{a_{i+1}}{\eta}) - a_i f_{d+2u}(\frac{a_i}{\eta})\right. \\ & \left. +(d+2u)(F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta}))\right], \end{aligned} \tag{18}$$

where F_{d+2u} and f_{d+2u} are the distribution and the density functions of a chi-square random variable with $d + 2u$ degrees of freedom,

$$\begin{aligned} X(\delta, \omega, \upsilon) &= \frac{2\kappa\overline{r}}{\sigma^2}\log\left[\frac{2\gamma(\omega)e^{(\gamma(\omega)+\kappa)\delta/2}}{(\upsilon\sigma^2 + \gamma(\omega) + \kappa)(e^{\gamma(\omega)\delta} - 1) + 2\gamma(\omega)}\right], \\ Y(\delta, \omega, \upsilon) &= \frac{\upsilon(\gamma(\omega) + \kappa + e^{\gamma(\omega)\delta}(\gamma(\omega) - \kappa)) + 2\omega(e^{\gamma(\omega)\delta} - 1)}{(\upsilon\sigma^2 + \gamma(\omega) + \kappa)(e^{\gamma(\omega)\delta} - 1) + 2\gamma(\omega)}, \\ \gamma(\omega) &= \sqrt{\kappa^2 + 2\omega\sigma^2}, \\ \gamma &= \sqrt{\kappa^2 + 2\sigma^2}, \\ \eta &= \frac{\sigma^2 e^{\gamma\delta} - 1}{2(\gamma + \kappa)(e^{\gamma\delta} - 1) + 2\gamma}, \\ d &= \frac{4\kappa\overline{r}}{\sigma^2} \text{ and} \\ \lambda_k &= \frac{8\gamma^2 e^{\gamma\delta}a_k}{\sigma^2\left[(\gamma + \kappa)(e^{\gamma\delta} - 1) + 2\gamma\right](e^{\gamma\delta} - 1)}. \end{aligned}$$

6.2 Futures pricing algorithm

1. **Initialization:**

   Define $\mathcal{G}$. Set $\widetilde{v}^h_{\overline{n}}\left(r, r^B\right) = 0$ for all $r, r^B \in \mathcal{G} \times \mathcal{G}$.

2. **Step 1:** (end-of-the-month, $m = n, ..., \overline{n}$)

   2.1 Set $j = 1$.

   2.1.1 Set $m = \overline{n}$.

   2.1.2 Compute $v^a_m\left(r, r^B_j\right)$ at (m, r, r^B_j) for all $r \in \mathcal{G}$.

   2.1.3 Interpolate $v^a_m(r, r^B_j)$, setting $h(r) = v^a_m(r, r^B_j)$, $r \in \mathcal{G}$, yielding

   $$\widehat{v}^a_m\left(r, r^B_j\right) = \widehat{h}(r),$$

   and compute the expectation of $\widehat{h}(r) = \widehat{v}^a_m\left(r, r^B_j\right)$ at $t = t^8_m$ and $\tau = t^5_{m+1}$ for all $r \in \mathcal{G}$, yielding

   $$\widetilde{v}^e_m\left(r, r^B_j\right) = \widetilde{h}\left(t^8_m, t^5_{m+1}, r\right) \quad \text{for all } r \in \mathcal{G}.$$

   2.1.4 Compute

   $$\widetilde{v}_m\left(r, r^B_j\right) = \max\left[\widetilde{v}^e_m\left(r, r^B_j\right), \widetilde{v}^h_m\left(r, r^B_j\right)\right] \quad \text{for all } r \in \mathcal{G},$$

   and interpolate $h(r) = \widetilde{v}_m(r, r^B_j)$, $r \in \mathcal{G}$, yielding

   $$\widehat{v}_m\left(r, r^B_j\right) = \widehat{h}(r).$$

   2.1.5 While $m \geq n$, compute the expectation of $\widehat{h}(r) = \widehat{v}_m\left(r, r^B_j\right)$ at $t = t^8_{m-1}$ and $\tau = t^8_m$ for all $r \in \mathcal{G}$, yielding

   $$\widetilde{v}^h_{m-1}\left(r, r^B_j\right) = \widetilde{h}\left(t^8_{m-1}, t^8_m, r\right) \quad \text{for all } r \in \mathcal{G},$$

   set $m = m - 1$ and go to step 2.1.2,
   Else, compute the expectation of $\widehat{h}(r) = \widehat{v}_n\left(r, r^B_j\right)$ at $t = t^2_n$ and $\tau = t^8_n$, yielding $\widetilde{v}^2_n\left(r, r^B_j\right) = \widetilde{h}\left(t^2_n, t^8_n, r\right)$.

   2.1.6 While $j < q$, set $j = j + 1$ and go to step 2.1.1.

   2.2 Find using linear interpolation the function $r^*(r)$ such that $\widetilde{v}^2_n\left(r, r^*\right) = 0$ and compute, for all $r \in \mathcal{G}$, the futures price $\widetilde{g}^*_n\left(r\right)$ as the price of the notional bond if its yield to maturity is $r^*(r)$.

2.3 Interpolate $h(r) = \widetilde{g}_n^*(r)$, $r \in \mathcal{G}$, yielding

$$\widehat{g}_n^*(r) = \widehat{h}(r),$$

and compute the expectation of $\widehat{h}(r)$ at $t = t_{n-1}^8$ and $\tau = t_n^2$, for all $r \in \mathcal{G}$, yielding

$$\widetilde{g}_{n-1}^8(r) = \widetilde{h}\left(t_{n-1}^8, t_n^2, r\right) \quad \text{for all} \quad r \in \mathcal{G}.$$

3. **Step 2** (delivery month, $m = \underline{n}, ..., n-1$)

3.1 Set $m = n - 1$.

3.2 Set $j = 1$.

3.2.1 Compute $v_m^a\left(r, r_j^B\right)$ for all $r \in \mathcal{G}$.

3.2.2 Interpolate and compute expectations at $t = t_m^8$ and $\tau = t_{m+1}^5$ as in step 2.1.3, setting $h(r) = \widetilde{v}_m^a(r, r_j^B)$, $r \in \mathcal{G}$, yielding $\widetilde{v}_m^e\left(r, r_j^B\right) = \widetilde{h}\left(t_m^8, t_{m+1}^5, r\right)$ for all $r \in \mathcal{G}$.

3.2.3 Using (9), compute

$$\widetilde{v}_m^h\left(r, r_j^B\right) = p\left(t_m^2, 6\%, 20, r_j^B\right)\rho\left(r, t_m^8, t_{m+1}^2\right) - \widetilde{g}_m^8(r) \quad \text{for all } r \in \mathcal{G}.$$

3.2.4 Compute

$$\widetilde{v}_m\left(r, r_j^B\right) = \max\left[\widetilde{v}_m^e\left(r, r_j^B\right), \widetilde{v}_m^h\left(r, r_j^B\right)\right] \quad \text{for all } r \in \mathcal{G},$$

and interpolate as in step 2.1.4, setting $h(r) = \widetilde{v}_m(r, r_j^B)$, $r \in \mathcal{G}$, yielding $\widehat{v}_m\left(r, r_j^B\right) = \widehat{h}(r)$.

3.2.5 Compute the expectation of $\widehat{h}(r) = \widehat{v}_m\left(r, r_j^B\right)$ at $t = t_m^2$ and $\tau = t_m^8$ yielding $\widetilde{v}_m^2\left(r, r_j^B\right) = \widetilde{h}\left(t_m^2, t_m^8, r\right)$, for all $r \in \mathcal{G}$.

3.2.6 While $j < q$, set $j = j + 1$ and go to step 3.2.1.

3.3 Find as in step 2.2 the function $r^*(r)$ such that $\widetilde{v}_m^2(r, r^*) = 0$ and compute the futures price $\widetilde{g}_m^*(r)$, for all $r \in \mathcal{G}$, as the price of the notional bond if its yield to maturity is $r^*(r)$.

3.4 Interpolate as in step 2.3, setting $h(r) = \widetilde{g}_m^*(r)$, $r \in \mathcal{G}$, yielding $\widehat{g}_m^*(r) = \widehat{h}(r)$.

3.5 While $m \geq \underline{n}$, compute the expectation of $\widehat{h}(r)$ at $t = t_{m-1}^8$ and $\tau = t_m^2$, for all $r \in \mathcal{G}$, yielding $\widetilde{g}_{m-1}^8(r) = \widetilde{h}\left(t_{m-1}^8, t_m^2, r\right)$ for all $r \in \mathcal{G}$, set $m = m-1$ and go to step 3.2.
Else, compute the expectation of $\widehat{h}(r)$ at $t = t_{m-1}^2$ and $\tau = t_m^2$, for all $r \in \mathcal{G}$, yielding $\widetilde{g}_{m-1}^2(r) = \widetilde{h}\left(t_{m-1}^2, t_m^2, r\right)$ for all $r \in \mathcal{G}$.

4. **Step 3** (before the delivery month, $m = -1, \ldots, \underline{n}-1$)

4.1 Set $m = \underline{n} - 1$.

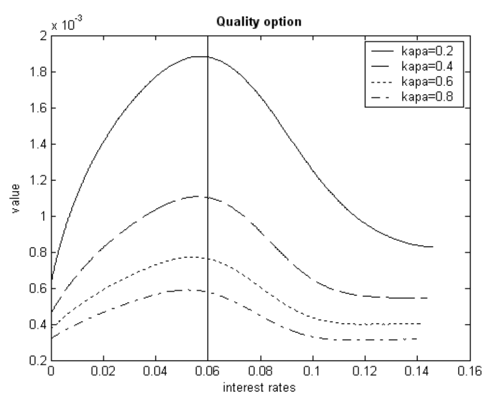
4.2 Using (10), compute

$$\widetilde{g}_m^*(r) = \frac{\widetilde{g}_m^2(r)}{\rho(r, t_m^2, t_{m+1}^2)}.$$

4.3 Interpolate setting $h(r) = \widetilde{g}_m^*(r)$, $r \in \mathcal{G}$, yielding $\widehat{g}_m^*(r) = \widehat{h}(r)$.
Compute the expectation of $\widehat{h}(r)$ at $t = t_{m-1}^2$ and $\tau = t_m^2$, for all $r \in \mathcal{G}$, yielding $\widetilde{g}_{m-1}^2(r) = \widetilde{h}\left(t_{m-1}^2, t_m^2, r\right)$ for all $r \in \mathcal{G}$.

4.4 While $m \geq -1$, set $m = m - 1$ and go to step 4.2.

6.3 Sensitivity of option prices under the CIR model



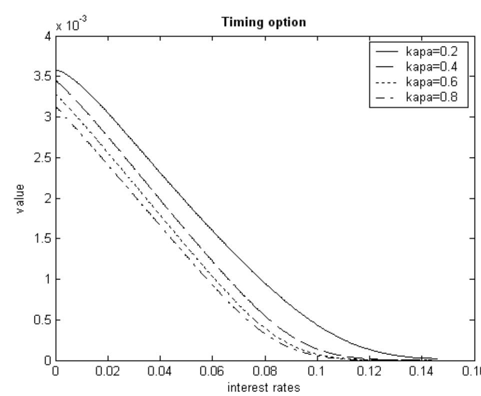


Figure 10: Options values sensitivities to kappa (CIR)

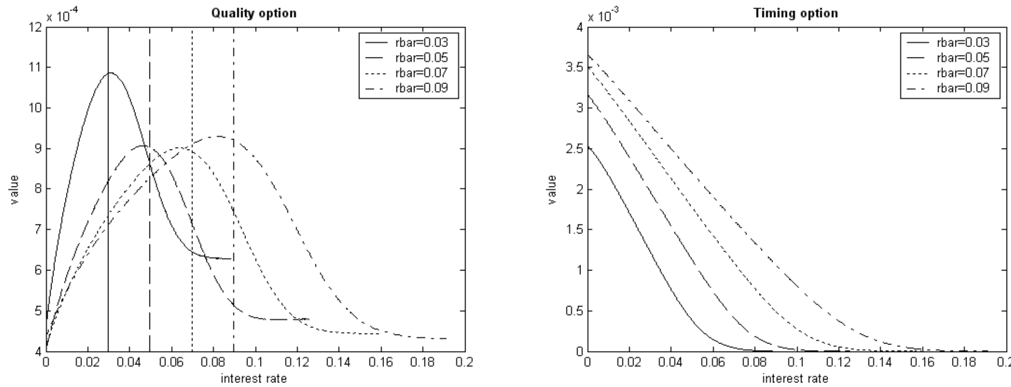


Figure 11: Options values sensitivities to rbar (CIR)

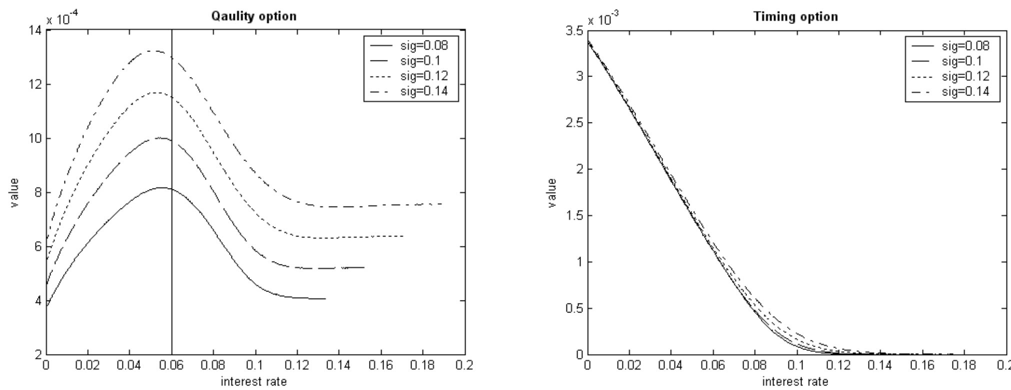


Figure 12: Options values sensitivities to sigma (CIR)

References

[1] Arak, M., L.S. Goodman, and S. Ross, "The Cheapest to Deliver Bond on the Treasury Bond Futures Contract," *Advances in Futures and Options Research*, 1 (1986), 49-74.

[2] Ben-Abdallah, R., H. Ben-Ameur, and M. Breton, "Pricing CBOT Treasury Bond Futures," *Les Cahiers du GERAD*, G-2006-77 (2006).

[3] Ben-Ameur, H., M. Breton, L. Karoui, and P. L'Écuyer, Pricing Call and Put Options Embedded in Bonds, to appear in *Journal of Economic Dynamics and Control*.

[4] Chen, R.R., J.H. Chou, and B.H. Lin, "Pricing the Quality Option in Japanese Government Bond Futures," *Applied Financial Economics*, 9 (1999), 51-65.

[5] Cox, J.C, J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985), 385-407.

[6] Hull, J.C. and A. White, "Pricing Interest-Rate Derivative Securities," *Review of Financial Studies*, 3 (1990), 573-592.

[7] Johnston, E.T. and J.J. McConnel, "Requiem for A Market: An Analysis of the Rise and Fall of a Financial Futures Contract," *Review of Financial Studies*, 2 (1989), 1-23.

[8] Jones, S., "Conversion Factor Risk in Treasury Bond Futures: A Comment," *Journal of Futures Markets*, 5 (1985), 115-119.

[9] Kane, A. and A.J. Marcus, "Conversion Factor Risk and Hedging in the Treasury Bond Futures Market," *Journal of Futures Markets*, 4 (1984), 55-64.

[10] Kane, A. and A.J. Marcus, "The Quality Option in the Treasury Bond Futures Market: An Empirical Assessment," *Journal of Futures Markets*, 6 (1986a), 231-248.

[11] Oviedo, R.A., "Improving the Design of Treasury-Bond Futures Contracts," *The Journal of Business*, 79 (2006), 1293-1315.

[12] Schulte, W. and R. Violi, "Interactions Between Cash and Derivative Bond Markets: Some Evidence for the Euro Area," *BIS Papers*, 5 (2001), 67-112.

[13] Shoji, I. and T. Osaki, "A Statistical Comparison of the Short-Term Interest Rate Models for Japan, U.S., and Germany," *Financial Engineering and the Japanese Markets*, 3 (1996), 263-275.

[14] Vasicek, O., "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977), 177-188.