Pricing VIX Futures on Affine Stochastic Volatility Models with Simultaneous Sate-Dependent Jumps both in the S&P 00 Price and Variance Processes: Evidence from Integrated Physical and Risk-Neutral Probability Measures

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The paper presents alternate stochastic variance models and develops closed-form solutions to the fair price of VIX futures. The derivation for theoretical futures prices is based upon the conditional moments of VIX squared on affine stochastic volatility models with simultaneous jumps both in the asset price and variance processes. An integrated analysis of spot and option prices, or equivalently integrated volatility and VIX time series, is proposed to estimate monthly-updated model parameters and the market prices of risks. Existing literature, however, has not provided the moment conditions of the total quadratic variation (except for the stochastic-volatility model), which are given in this study. The daily VIX futures prices in the subsequent month, since 19 May 2004, are adopted to test the futures price formulas. Our results show that state-dependent jumps in volatilities and prices simultaneously fare better in fitting short-dated VIX futures prices whereas stochastic volatility and random jumps in returns outperforms for medium- and long-lived futures. In addition, while both price and volatility jump-risk premia appear to be positive, the diffusive volatility-risk premium is found to be negative that is consistent with the negative correlation between volatility and index returns. Gauging by the magnitude of risk premia, the volatility jump-risk premium is found to be far more important than the one for price jumps in terms of the VIX futures valuation.

Key Words: VIX futures, affine stochastic volatility models, volatility jumps, integrated volatility, volatility and jump risk premia

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I. INTRODUCTION

In contrast to the implied volatility extracted from an option pricing model, the VIX (Volatility Index) uses a model-free formula to derive expected volatility directly from the prices of a weighted strip of S&P 500 index (SPX) options over a wide range of strike prices which incorporates information from the volatility skew. Thus, the VIX provides a more precise and robust measure of the market's expectation of near-term stock market volatility. This VIX calculation supplies a script for replicating the VIX with a static portfolio of S&P 500 options. This critical fact lays the foundation for tradable products based on the VIX, facilitating hedging and arbitrage of VIX derivatives. Chicago Board Options Exchange (CBOE) has launched VIX futures on 26 March 2004 and VIX options on 24 February 2006. These will be the first of an entire family of volatility products traded in exchanges. Since the underlying VIX, or equivalently the model-free implied volatility of SPX options, is not tradable, it is impossible to use the no-arbitrage relationship to derive the fair value of the VIX futures. This paper is aimed to conquer the pricing difficulty and provides closed-form solutions to the value of the VIX futures contract. In particular, the underlying of the VIX, i.e. the SPX, incorporates both diffusive stochastic volatility and simultaneous jumps in returns and volatility, which are more capable of fully capturing the empirical features of equity index returns or option prices (see, Andersen et al., 2001; Alizadeh et al., 2002; Duffie et al., 2000; Eraker et al., 2003; Eraker, 2004). For model parameter estimation, a joint model is specified, not only for the physical probability distribution that governs the random shocks observed in the spot index market, but also for the risk-neutral probability distribution that allows to compute VIX futures prices as expectations of discounted payoffs. Since these two distributions have to be equivalent, there exists a link between the two through an integral martingale representation that includes the innovations associated with the specific asset price processes and the risk premia associated with these sources of uncertainty. The main contributions of this paper are thus to propose closed-form solutions to the fair value of the VIX futures and to propose a methodology for an integrated analysis of spot and option prices, or equivalently integrated volatility and VIX. The market prices of risks are also estimated. Three specifications for the dynamics of the SPX prices are considered in the paper: affine diffusion with diffusive stochastic volatility model (SV) of Heston (1993); affine jump-diffusion with stochastic volatility and jumps in the asset price (SVJ) of Bates (1996); and the affine jump-diffusion with correlated jumps both in the asset price and stochastic volatility process (SVCJ) described by Duffie et al. (2000). The SVCJ class of models generalizes the models in Merton (1976), Heston (1993) and Bates (1996). Bates (2000) and Pan (2002) examine combined jump diffusion models with parameter estimation based upon joint options and returns, while Eraker et al. (2003) use returns data to investigate the performance of models with jumps in volatility and prices. Their results point toward models that include jumps to volatility. In response to these findings, Eraker (2004) uses joint options and returns data (an idea pursued in Chernov and Ghysels, 2000; and Pan, 2002) to investigate the performance of models with jumps in volatility and stock prices using the class of jump-in-volatility models proposed by Duffie et al. (2000) and an extension to allow for state-dependent jump frequency. A primary advantage of using joint options and returns is that risk premiums relating to volatility and jumps can be estimated. Eraker (2004) finds that while complex jump specifications (i.e. the stochastic volatility with state-dependent and correlated jumps; SVSCJ)[1] add little explanatory power in fitting options data,

[1] Allowing for volatility jumps, the stochastic volatility with state-dependent and correlated jumps (SVSCJ) specification generalizes the correlated jump model to allow the jump frequency to depend on volatility v_t, i.e. $\lambda_0+\lambda_1 v_t$, considered by Bates (1996), Pan (2002), Eraker et al. (2003) and Eraker

these models fare better in fitting options and returns data simultaneously. Garcia et al. (2006) use joint moments of integrated volatilities, constructed from high-frequency underlying returns, and the implied volatilities of Hull and White (1987) and Heston (1993) option pricing formulas, respectively, to estimate the volatility risk premium for the exchange rate futures options. They find that there are, in general, several large differences between the estimates based on futures returns only and those based on futures and option prices. It also seems that there is a large variation in the parameters across different time periods. Bollerslev et al. (2005) estimate the stochastic volatility risk premium by implementing the procedure of a generalized method of moments (GMM) with VIX and high-frequency five-minute-based integrated volatilities. They find that the extracted volatility risk premium helps predict future stock market returns. The VIX is known as an index based on model-free implied volatilities calculated from SPX option prices. Thus, Jiang and Tian (2005) extend Britten-Jones and Neuberger's (2000) model-free implied volatility under the diffusion assumption to asset price processes that include price jumps and implement it using observed option prices. Their results from the SPX options support that the model-free implied volatility is a more efficient forecast for future realized volatility than its predecessor (Black and Scholes' implied volatility).[2] Zhang and Zhu (2006) use historical VIX data to estimate parameters of a stochastic variance model and find that the model with parameters estimated from the whole period from 1990 to 2005 overprices the

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(2004).

[2] In 1993, the CBOE introduces the VIX that quickly becomes the benchmark for stock market volatility. The original VIX uses hypothetical 30-calendary-day at-the-money S&P 100 index (OEX) options data to compute an average of Black and Scholes' (1973) implied volatility with strike prices close to the current spot index level and maturities interpolated at about one month. In order to provide a more precise and robust measure of expected market volatility and to create a viable underlying index for tradable volatility products, the CBOE has changed the definition of VIX in September 2003. The CBOE has created an identical historical record for the new VIX dating back to 1990. The new VIX still measures the market's expectation of 30-calendary-day volatility, but it is based on the SPX option prices. VIX is based on real-time prices of options listed on the CBOE, and is designed to reflect investors' consensus view of future (30-day) expected stock market volatility. It is often referred to as the "investor fear gauge". Besides using options on the SPX rather than on the OEX, the new VIX changes its method of calculation to be model-free.

VIX futures by 16–44%. By using the parameters estimated from the recent one-year period, however, the discrepancy is dramatically reduced to 2–12%. Following the suggestions by the literature, this paper derives closed-form solutions to the fair price of the VIX futures by adopting the SVSCJ in Eraker (2004) for the SPX price dynamics. For the comparison purpose, its nested pricing models under the SV, SVJ and SVCJ specifications are also provided. In addition, model parameters are estimated based upon the joint data of the most recent one-month VIX data and the integrated volatilities calculated from high-frequency index returns. Existing literature, however, has not provided the moment conditions of the total quadratic variation, or equivalently total integrated volatility, for the SVJ, SVCJ and SVSCJ models, which are given in this study. The resulting parameter estimates are then adopted to investigate whether the subsequent month's futures price changes can be explained by our models in an efficient way. Our empirical findings are summarized as follows. The SVSCJ provides the best out-of-sample pricing fits for short-dated VIX futures, while the SVJ outperforms for medium- and long-dated futures. The diffusive price-risk premiums are found to be positive, while the premiums associated with the diffusive volatility shocks are negative for all models. The price jump-risk premium is found to be positive and the SVJ achieves the greatest value, followed by the SVCJ and SVSCJ. The volatility jump-risk premium is also found to be positive, however, the SVSCJ has a greater value than the SVCJ.

The rest of the paper is organized as follows: next two presents the general approach for pricing VIX futures; section three addresses the construction of integrated volatilities using high-frequency returns and their moment condition derivation; section four discusses alternate models for the index price dynamics upon which closed-form solutions for the fair value of the VIX futures are derived. Section five presents the data and econometric designs for model parameter estimation.

Section six examines in- and out-of-sample pricing fits to the VIX futures on our models; summary statistics for the market prices of risks are also included. Finally, section seven summarizes our empirical findings and points out the theoretical contributions on the literature to the valuation of the VIX futures.

II. VIX FUTURES VALUATION

A. Fair Price of VIX Futures

Since the VIX cannot be replicated by a portfolio of the SPX options and it is not a traded asset, one cannot use the no-arbitrage principle to obtain a simple relationship between VIX futures and the VIX as that of stock futures and stock price. Pricing VIX futures becomes an issue. Carr and Wu (2006) present that the price of the VIX futures has a lower bound and an upper bound. The lower bound is the forward volatility swap rate and the upper bound is the forward-starting variance swap rate. Zhang and Zhu (2006) choose a Heston-type stochastic-variance model (1993) and developed a numerical expression for VIX futures. Free parameters are estimated from market data over 1990–2005. This study instead demonstrates the VIX squared in terms of expected variance and derives closed-form solutions for the VIX futures, after the convexity adjustment. VIX futures are a futures contract on VIX and cash settlement equal to VIX. It is a pure play on implied volatility at settlement. When VIX compared to the SPX, VIX tends to rise as the SPX falls, while VIX tends to decline or remain constant as the SPX rises. When VIX compared to VIX futures, VIX futures move much more stable than spot VIX. The current futures price is the market's estimate of what the VIX index will be at settlement. In other words, VIX futures prices are based purely on expectation and thus there is no traditional futures-to-cash relationship, namely no cost-of-carry relationship. By Martingale pricing theory, the current VIX futures prices expiring at t can be computed under a risk-neutral probability measure Q as,

$$\mathrm{F}_0^{\mathrm{VIX}}(t) = \mathrm{E}_0^Q\left\{ \mathrm{VIX}_t \right\} \ . \tag{1}$$

Consider the VIX definition (converted to our notation) as specified in the CBOE white paper (CBOE 2003).

$$\mathrm{VIX}_t^2 = \frac{2}{\tau}\sum_i \frac{\Delta X_i}{X_i^2} Q_i(X_i) - \frac{1}{\tau}\left[\frac{F_t(T)}{X_0} - 1\right]^2 \tag{2}$$

where $\tau = T - t$ is the annualized 30-day period, Q_i is the price of the out-of-the money option with strike X_i and X_0 is the highest strike below the forward price $F_t(T)$. This formula can be recognized as a simple discretization of the fair value of the variance driven by the price-diffusion component, adjusted for the price-jump components ζ_1^* and ζ_2^*, if any, over $[t,T]$.

$$\mathrm{VIX}_t^2 \equiv \frac{\zeta_1^*}{\tau} \times \mathrm{E}_t^Q\left(\int_t^T \nu_u du\right) + \zeta_2^* = \frac{\zeta_1^*}{\tau} \times \mathrm{E}_t^Q\left(\upsilon_{t,T}^c\right) + \zeta_2^* \tag{3}$$

where $\upsilon_{t,T} = \int_t^T \mathrm{var}_u(d\ln S_u)$ is the total quadratic variation of asset log-price $\ln S$, consisting of the integrated volatility governing asset price diffusion, $\upsilon_{t,T}^c = \int_t^T \nu_u du$ with instantaneous variance ν, and the one attributed to the asset price jumps, $\upsilon_{t,T}^J = \int_t^T \nu_J du$, if any. Though variance in return jumps $\upsilon_{t,T}^J$ does not enter the construction of VIX squared explicitly, the information of jumps in returns has an influence on the VIX squared through the price-jump component adjustment ζ_1^* and ζ_2^*. Appendix A presents details for the VIX squared in terms of fair value of expected diffusive integrated variance.

B. Convexity Adjustment

As a theoretical expression of how the VIX futures is associated with expected variance, settlement is on the Wednesday before the third Friday based on the opening prices of options expiring in the following month. The VIX futures (strictly speaking VXB futures where the VXB is defined to be 10 times the VIX) settle bases on the square root of the value of the replicating strip (i.e. on volatility rather than variance) so there must be convexity adjustment for the difference between square root of expected variance and expected volatility, i.e. $\mathrm{F}_0^{\mathrm{VIX}}(t)=\mathrm{E}_0^Q\left(\sqrt{\mathrm{VIX}_t^2}\right)\leq\sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)}$. From the approximation of Brockhaus and Long (2000) and Bates (2006), who use the second order Taylor expansion for the square root of latent affine stochastic processes *x*, i.e. $\sqrt{x}$, the current VIX futures is worth theoretically

$$\mathrm{F}_0^{\mathrm{VIX}}(t)=\mathrm{E}_0^Q\left(\sqrt{\mathrm{VIX}_t^2}\right)\approx\sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)}-\frac{\mathrm{var}_0^Q(\mathrm{VIX}_t^2)}{8\times[\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}} \tag{4}$$

where $\mathrm{var}_0^Q(\mathrm{VIX}_t^2)/\{8\times[\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}\}$ is the convexity adjustment relevant to the VXB futures contract as of the valuation date 0. t is the settlement date of the VIX futures and $T=t+\tau$ is the expiration date of options in the strip. Thus, to calculate VIX futures we need both $\mathrm{E}_0^Q(\mathrm{VIX}_t^2)$ and $\mathrm{var}_0^Q(\mathrm{VIX}_t^2)$.

III. UNDERLYING MODELS AND CLOSED-FORM SOLUTIONS TO THE FAIR VALUE OF THE VIX FUTURES

Since VIX_t^2 is represented as the one-month expected average variance driven by diffusive and jump components over [*t*,*t*+30 days], different dynamics for the index price S_t will result in various expected average variance formulas and thus different theoretical VIX futures values.

A. SVSCJ Model

While a number of prior studies (see Bates, 1996; Bakshi et al., 1997; Andersen et al., 2002; Chernov et al., 2003) point out the importance of stochastic volatility and jumps in returns to equity price models, Andersen et al. (2001), Alizadeh et al. (2002), and Eraker et al. (2003) find the presence of an additional, rapidly moving factor driving conditional volatility, which, unlike jumps in returns, has a persistent component. Jumps in volatility provide such a factor. Together, this suggests a strong evidence for volatility driven by diffusive and jump components. Jump models, however, typically specify jumps to arrive with constant intensity. This assumption poses problem in explaining the tendency of large movements to cluster over time. Bates (2000), Pan (2002) and Eraker (2004) use the linear specification $\lambda_0 + \lambda_1 v_t$ for some non-negative constants λ_0 and λ_1, of jump-arrival intensity to allow jumps to arrive more frequently in high-volatility regimes. As a result, a state-dependent jump frequency allows for the possibility that when the market is more volatile, the jump-risk premia implicit option prices become higher. The stochastic volatility with state-dependent and correlated jumps (SVSCJ) is the most general model considered in this paper. The data-generating processes of $(\ln S_t, v_t)$ for the SVSCJ model are of the form,

$$
\begin{aligned}
d\ln S_t = \Big\{ & \; r-\delta-(\lambda_0^*+\lambda_1^*\nu_t)\,\kappa^*+\eta_S\nu_t-\frac{1}{2}\nu_t+(\lambda_0^*+\lambda_1^*\nu_t)(\mu_j^*+\rho_j\mu_\nu^*) \\
& +[(\lambda_0+\lambda_1\nu_t)(\mu_j+\rho_j\mu_\nu)-(\lambda_0^*+\lambda_1^*\nu_t)(\mu_j^*+\rho_j\mu_\nu^*)] \;\Big\}\, dt \\
& +\sqrt{\nu_t}\; d\omega_{S,t}+[z_S dN_t-(\lambda_0+\lambda_1\nu_t)(\mu_j+\rho_j\mu_\nu)dt]
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
d\nu_t &= \kappa_\nu(\theta_\nu-\nu_t)\,dt+\sigma_\nu\sqrt{\nu_t}\,d\omega_{\nu,t}+z_\nu dN_t \\
&= \kappa_\nu^*(\theta_\nu^*-\nu_t)\,dt+\eta_\nu\nu_t dt+\mu_\nu^*(\lambda_0^*+\lambda_1^*\nu_t)dt+[\mu_\nu(\lambda_0+\lambda_1\nu_t)-\mu_\nu^*(\lambda_0^*+\lambda_1^*\nu_t)]\,dt \\
&\quad +\sigma_\nu\sqrt{\nu_t}\,d\omega_{\nu,t}+\left[z_\nu dN_t-\mu_\nu(\lambda_0+\lambda_1\nu_t)\,dt\right]
\end{aligned} \tag{6}
$$

The instantaneous variance ν_t of asset log-prices is modeled as a combination of jumps in volatility and a one-factor square-root diffusive process that was originally proposed for finance by Cox et al. (1985). $\omega_{S,t}$ and $\omega_{\nu,t}$ are standard Brownian motions correlated by $\rho\, dt = corr(d\omega_{S,t}, d\omega_{\nu,t})$, which are independent of the compounded Poisson processes $z_S dN_t$ and $z_\nu dN_t$, respectively. N_t is a univariate Poisson (counting) process with state-dependent intensity $\lambda_0+\lambda_1\nu_t$. Jumps in volatility have an exponential distribution, $z_\nu \sim \exp(\mu_\nu)$,[3] and jumps in asset log-prices are normally distributed conditional on the realization of z_ν, formally $z_S | z_\nu \sim \mathrm{N}(\mu_j+\rho_j z_\nu, \sigma_j^2)$. Thus, z_S has mean $\mathrm{E}(z_S)=\mu_j+\rho_j\mu_\nu$, variance $\mathrm{var}(z_S)=\sigma_j^2+\rho_j^2\mu_\nu^2$, and correlated with z_ν by $\rho_j\mu_\nu/\sqrt{\sigma_j^2+\rho_j^2\mu_\nu^2}$.[4] The instantaneous covariance of $d\ln S_t$ and $d\nu_t$ is given by $\sigma_\nu\rho\nu_t dt+(\lambda_0+\lambda_1\nu_t)(\mu_j\mu_\nu+2\rho_j\mu_\nu^2)dt$, comprising the familiar leverage effect

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[3] The probability density function of z_ν following an exponential distribution has the form, $f(z_\nu)=(1/\mu_\nu)\exp(-z_\nu/\mu_\nu)$ with $z_\nu \geq 0$ and $\mu_\nu > 0$. The mean and variance for the exponential distribution are given by μ_ν and μ_ν^2.

[4] $\mathrm{corr}(z_S, z_\nu)=\mathrm{cov}(z_S, z_\nu)/\sigma_{z_S}\sigma_{z_\nu} = (\mathrm{E}z_S z_\nu - \mathrm{E}z_S \mathrm{E}z_\nu)/\sqrt{\sigma_j^2+\rho_j^2\mu_\nu^2}\,\mu_\nu = \rho_j\mu_\nu^2/\sqrt{\sigma_j^2+\rho_j^2\mu_\nu^2}\,\mu_\nu = \rho_j\mu_\nu/\sqrt{\sigma_j^2+\rho_j^2\mu_\nu^2}$.

(Black, 1976).[5] The risk-neutral price jump-size mean for dS/S is defined as $\kappa^* = \exp[(\mu_j^* + \rho_j \mu_v^*) + \sigma_j^2 / 2]/(1-\rho_j \mu_v^*) - 1$, satisfying the no-arbitrage condition. κ_v, θ_v and σ_v are the speed of adjustment, long-run mean and variance of v_t. While the presence of state-dependent jump frequency decreases the mean-reversion rate of the volatility dynamics from κ_v to $\kappa_v - \lambda_1 \mu_v$, the long-run mean increases from θ_v to $(\kappa_v \theta_v + \lambda_0 \mu_v)/(\kappa_v - \lambda_1 \mu_v)$. The dynamics of $(\ln S_t, v_t)$ under the risk-neutral probability measure Q are of the form,

$$\begin{aligned} d\ln S_t = &\left\{ r - \delta + \lambda_0^*(\mu_j^* + \rho_j \mu_v^* - \kappa^*) + [\lambda_1^*(\mu_j^* + \rho_j \mu_v^* - \kappa^*) - \frac{1}{2}] v_t \right\} dt \\ &+ \sqrt{v_t}\, d\omega_{S,t}^* + \left[z_S^* dN_t^* - (\mu_j^* + \rho_j \mu_v^*)(\lambda_0^* + \lambda_1^* v_t)\, dt \right] \end{aligned} \tag{7}$$

$$\begin{aligned} d v_t &= \kappa_v^*(\theta_v^* - v_t)\, dt + \sigma_v \sqrt{v_t}\, d\omega_{v,t}^* + z_v^* dN_t^* \\ &= (\kappa_v^* - \lambda_1^* \mu_v^*) \left(\frac{\kappa_v^* \theta_v^* + \lambda_0^* \mu_v^*}{\kappa_v^* - \lambda_1^* \mu_v^*} - v_t \right) dt + \sigma_v \sqrt{v_t}\, d\omega_{v,t}^* + [z_v^* dN_t^* - \mu_v^*(\lambda_0^* + \lambda_1^* v_t) dt] \end{aligned} \tag{8}$$

where the parameters with asterisk (*) denotes corresponding risk-neutral paramseters. Comparing the specification of the risk-neutral dynamics of $(\ln S_t, v_t)$ with that of the data-generating process, the market prices of different risk factors are obtained. Whereas η_v represents the market price of unit price risk, the SVSCJ model incorporates a premium for price jump-size uncertainty by $(\mu_j + \rho_j \mu_v) - (\mu_j^* + \rho_j \mu_v^*)$ and a premium for jump-timing risk by $(\lambda_0 + \lambda_1 v_t)/(\lambda_0^* + \lambda_1^* v_t)$. With this assumption, all price jump-risk premia will be absorbed by $(\lambda_0 + \lambda_1 v_t)(\mu_j + \rho_j \mu_v) - (\lambda_0^* + \lambda_1^* v_t) \times (\mu_j^* + \rho_j^* \mu_v^*)$. Similarly, the volatility jump-risk premium is given by

[5] $\mathrm{cov}(d\ln S_t, dv_t) = \mathrm{cov}(\sqrt{v_t}\, d\omega_{S,t} + z_S dN_t, \sigma_v \sqrt{v_t}\, d\omega_{v,t} + z_v dN_t) = \sigma_v \rho v_t dt + (\lambda_0 + \lambda_1 v_t)\, \mathrm{cov}(z_S, z_v)$
$= \sigma_v \rho v_t dt + (\lambda_0 + \lambda_1 v_t)(\mu_j \mu_v + 2\rho_j \mu_v^2)\, dt$.

$(\lambda_0 + \lambda_1 \nu_t)\mu_\nu - (\lambda_0^* + \lambda_1^* \nu_t)\mu_\nu^*$. Under the assumed framework, the total quadratic variation of the asset log-price $\upsilon_{t,T}$ consists of the price-diffusion variance $\upsilon_{t,T}^c$ and the price-jump variance $\upsilon_{t,T}^J$ shown as follows,

$$\begin{aligned}\upsilon_{t,T} &= \int_t^T \mathrm{var}_u(d\ln S_u) = \int_t^T \mathrm{var}_u(\sqrt{\nu_u}\, d\omega_{S,u}^*) + \int_t^T \mathrm{var}_u(z_S^* dN_u^*) \\ &= \int_t^T \nu_u du + \int_t^T \nu_J du = \upsilon_{t,T}^c + \upsilon_{t,T}^J\end{aligned}$$

where $\nu_J = (\lambda_0^* + \lambda_1^* \nu_t)\{(\mu_j^* + \rho_j \mu_\nu^*)^2 + [\sigma_j^2 + \rho_j^2 (\mu_\nu^*)^2]\}$. By applying Itô's lemma to $\exp[(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)t]\nu_t$, we have

$$\begin{aligned}d[e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)t}\nu_t] &= (\kappa_\nu^* \theta_\nu^* + \lambda_0^* \mu_\nu^*) e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)t} dt + \sigma_\nu e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)t} \sqrt{\nu_t}\, d\omega_{\nu,t}^* \\ &\quad + e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)t}\left[z_\nu^* dN_t^* - \mu_\nu^*(\lambda_0^* + \lambda_1^* \nu_t)dt\right]\end{aligned}$$

Now, we can compute ν_T for $T \geq t$,

$$\begin{aligned}\nu_T &= \alpha_{T-t,\mathrm{SVSCJ}}^* \times \nu_t + \beta_{T-t,\mathrm{SVSCJ}}^* + \sigma_\nu e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)T} \int_t^T e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)u} \sqrt{\nu_u}\, d\omega_{\nu,u}^* \\ &\quad + e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)T} \int_t^T e^{(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)u}\left[z_\nu^* dN_u^* - \mu_\nu^*(\lambda_0^* + \lambda_1^* \nu_u)du\right]\end{aligned} \tag{9}$$

where $\alpha_{T-t,\mathrm{SVSCJ}}^* = e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)}$ and $\beta_{T-t,\mathrm{SVSCJ}}^* = \left(\dfrac{\kappa_\nu^* \theta_\nu^* + \lambda_0^* \mu_\nu^*}{\kappa_\nu^* - \lambda_1^* \mu_\nu^*}\right)[1 - e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)}]$.

The mean and variance of the instantaneous variance ν in the SVSCJ model are thus

$$\mathrm{E}_t^Q(\nu_T) = \alpha_{T-t,\mathrm{SVSCJ}}^* \times \nu_t + \beta_{T-t,\mathrm{SVSCJ}}^* \tag{10}$$

$$\mathrm{var}_t^Q(\nu_T) = \mathrm{E}_t^Q[\nu_T - \mathrm{E}_t^Q(\nu_T)]^2 = C_{T-t,\mathrm{SVSCJ}}^* \times \nu_t + D_{T-t,\mathrm{SVSCJ}}^* \tag{11}$$

where $C_{T-t,\mathrm{SVSCJ}}^* = \dfrac{\sigma_\nu^2 + 2\lambda_1^*(\mu_\nu^*)^2}{\kappa_\nu^* - \lambda_1^* \mu_\nu^*}[e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)} - e^{-2(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)}]$

$$\begin{aligned}D_{T-t,\mathrm{SVSCJ}}^* &= \frac{[\sigma_\nu^2 + 2\lambda_1^*(\mu_\nu^*)^2]}{2(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)}\left(\frac{\kappa_\nu^* \theta_\nu^* + \lambda_0^* \mu_\nu^*}{\kappa_\nu^* - \lambda_1^* \mu_\nu^*}\right)[1 - e^{-(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)}]^2 \\ &\quad + \frac{\lambda_0^*(\mu_\nu^*)^2}{\kappa_\nu^* - \lambda_1^* \mu_\nu^*}[1 - e^{-2(\kappa_\nu^* - \lambda_1^* \mu_\nu^*)(T-t)}]\end{aligned}.$$

Therefore, the second moment of the instantaneous ν is computed as

$$
\begin{aligned}
\mathrm{E}_t^Q\left(\nu_T^2\right) &= \mathrm{var}_t^Q\left(\nu_T\right) + [\mathrm{E}_t^Q(\nu_T)]^2 \\
&= (\alpha^*_{T-t,\mathrm{SVSCJ}})^2 \times \nu_t^2 + (C^*_{T-t,\mathrm{SVSCJ}} + 2\alpha^*_{T-t,\mathrm{SVSCJ}} \times \beta^*_{T-t,\mathrm{SVSCJ}}) \times \nu_t \\
&\quad + [(\beta^*_{T-t,\mathrm{SVSCJ}})^2 + D^*_{T-t,\mathrm{SVSCJ}}]
\end{aligned} \tag{12}
$$

The expected quadratic variation of the log-price over $[t,T]$ is given by

$$
\begin{aligned}
\mathrm{E}_t^Q(\upsilon_{t,T}) &= \mathrm{E}_t^Q(\upsilon^c_{t,T}) + \mathrm{E}_t^Q(\upsilon^J_{t,T}) \\
&= \left\{ 1 + \lambda_1^*[(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2(\mu_\nu^*)^2] \right\} \times a^*_{T-t,\mathrm{SVSCJ}} \times \nu_t \\
&\quad + \left\{ 1 + \lambda_1^*[(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2(\mu_\nu^*)^2] \right\} \times b^*_{T-t,\mathrm{SVSCJ}} \\
&\quad + \lambda_0^*[(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2(\mu_\nu^*)^2](T-t)
\end{aligned} \tag{13}
$$

where

$$
a^*_{T-t,\mathrm{SVSCJ}} = \int_t^T \alpha^*_{u-t,\mathrm{SVSCJ}} = \frac{1 - e^{-(\kappa_\nu^* - \lambda_1^*\mu_\nu^*)(T-t)}}{\kappa_\nu^* - \lambda_1^*\mu_\nu^*}
$$

$$
b^*_{T-t,\mathrm{SVSCJ}} = \int_t^T \beta^*_{u-t,\mathrm{SVSCJ}} = \left(\frac{\kappa_\nu^*\theta_\nu^* + \lambda_0^*\mu_\nu^*}{\kappa_\nu^* - \lambda_1^*\mu_\nu^*}\right)\left[(T-t) - \left(\frac{1 - e^{-(\kappa_\nu^* - \lambda_1^*\mu_\nu^*)(T-t)}}{\kappa_\nu^* - \lambda_1^*\mu_\nu^*}\right)\right]
$$

$$
\mathrm{E}_t^Q\left(\upsilon^c_{t,T}\right) = a^*_{T-t,\mathrm{SVSCJ}} \times \nu_t + b^*_{T-t,\mathrm{SVSCJ}}
$$

$$
\begin{aligned}
\mathrm{E}_t^Q\left(\upsilon^J_{t,T}\right) &= \lambda_1^*[(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2(\mu_\nu^*)^2] \times a^*_{T-t,\mathrm{SVSCJ}} \times \nu_t \\
&\quad + [(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2(\mu_\nu^*)^2][\lambda_0^*(T-t) + \lambda_1^* b^*_{T-t,\mathrm{SVSCJ}}]
\end{aligned}
$$

From appendix (A.3), the VIX squared is expressed in terms of the expected variance attributed to the price-diffusion component by,

$$
\begin{aligned}
\mathrm{VIX}_t^2 &\equiv 2\lambda_0^*[\kappa^* - (\mu_j^* + \rho_j\mu_\nu^*)] + \frac{1}{\tau}\left\{1 + 2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j\mu_\nu^*)]\right\} \times \mathrm{E}_t^Q[\upsilon^c_{t,T}] \\
&= 2\lambda_0^*[\kappa^* - (\mu_j^* + \rho_j\mu_\nu^*)] + \frac{1 + 2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j\mu_\nu^*)]}{\tau} \times (a^*_{T-t,\mathrm{SVSCJ}} \times \nu_t + b^*_{T-t,\mathrm{SVSCJ}})
\end{aligned} \tag{14}
$$

where $\tau = T - t = 30/365$. The VIX squared is a linear function of the instantaneous variance. From equation (14), mean and variance of VIX_t^2 conditional on VIX_0^2 are computed as

$$\mathrm{E}_0^Q(\mathrm{VIX}_t^2) \equiv 2\lambda_0^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)] + \frac{1+2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)]}{\tau} \times (a_{T-t,\mathrm{SVSCJ}}^* \times \alpha_{t,\mathrm{SVSCJ}}^* \times v_0 + a_{T-t,\mathrm{SVSCJ}}^* \times \beta_{t,\mathrm{SVSCJ}}^* + b_{T-t,\mathrm{SVSCJ}}^*) \tag{15}$$

$$\mathrm{var}_0^Q(\mathrm{VIX}_t^2) \equiv \left\{\frac{1+2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)]}{\tau}\right\}^2 \times (a_{T-t,\mathrm{SVSCJ}}^*)^2 \times \left(C_{t,\mathrm{SVSCJ}}^* \times v_0 + D_{t,\mathrm{SVSCJ}}^*\right) \tag{16}$$

where $\alpha_{t,\mathrm{SVSCJ}}^* = e^{-(\kappa_v^* - \lambda_1^*\mu_v^*)t}$

$$\beta_{t,\mathrm{SVSCJ}}^* = \left(\frac{\kappa_v^*\theta_v^* + \lambda_0^*\mu_v^*}{\kappa_v^* - \lambda_1^*\mu_v^*}\right)[1 - e^{-(\kappa_v^* - \lambda_1^*\mu_v^*)t}]$$

$$C_{t,\mathrm{SVSCJ}}^* = \frac{\sigma_v^2 + 2\lambda_1^*(\mu_v^*)^2}{\kappa_v^* - \lambda_1^*\mu_v^*}[e^{-(\kappa_v^* - \lambda_1^*\mu_v^*)t} - e^{-2(\kappa_v^* - \lambda_1^*\mu_v^*)t}]$$

$$D_{t,\mathrm{SVSCJ}}^* = \frac{[\sigma_v^2 + 2\lambda_1^*(\mu_v^*)^2]}{2(\kappa_v^* - \lambda_1^*\mu_v^*)}\left(\frac{\kappa_v^*\theta_v^* + \lambda_0^*\mu_v^*}{\kappa_v^* - \lambda_1^*\mu_v^*}\right)[1 - e^{-(\kappa_v^* - \lambda_1^*\mu_v^*)t}]^2 + \frac{\lambda_0^*(\mu_v^*)^2}{\kappa_v^* - \lambda_1^*\mu_v^*}[1 - e^{-2(\kappa_v^* - \lambda_1^*\mu_v^*)t}]$$

$$v_0 \equiv \left(\frac{1}{a_{T-t,\mathrm{SVSCJ}}^*}\right)\left(\frac{\tau}{1+2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j\mu_v^*)]}\right) \times \mathrm{VIX}_0^2 - \left(\frac{1}{a_{T-t,\mathrm{SVSCJ}}^*}\right)\left\{\left(\frac{2\lambda_0^*[\kappa^* - (\mu_j^* + \rho_j\mu_v^*)] \times \tau}{1+2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j\mu_v^*)]}\right) + b_{T-t,\mathrm{SVSCJ}}^*\right\}$$

By substituting $\mathrm{E}_0^Q(\mathrm{VIX}_t^2)$ and $\mathrm{var}_0^Q(\mathrm{VIX}_t^2)$ in equation (4) with the ones above, the fair price of the VIX futures expiring at *t* under the SVSCJ model is obtained.

$$\mathrm{F}_0^{\mathrm{VIX}}(t \mid v_0, \tau, t; \{\eta_S, \kappa_v, \theta_v, \sigma_v, \rho, \eta_v, \lambda_0, \lambda_1, \mu_j, \sigma_j, \mu_v, \lambda_0^*, \lambda_1^*, \mu_j^*, \mu_v^*, \rho_j\}) \approx \sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)} - \frac{\mathrm{var}_0^Q(\mathrm{VIX}_t^2)}{8 \times [\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}} \tag{17}$$

B. SVCJ Model

In Duffie et al. (2000) and Eraker et al. (2003), the type of jumps in the SVCJ model are defined as simultaneous correlated jumps in ln*S* and v, with constant arrival intensity. Rockinger and Semenova (2005) propose empirical characteristic functions for the estimation on S&P 500 data for the affine stochastic volatility models with uncorrelated jumps both in the asset price and variance processes (SVJJ). Eraker et al. (2003), while using return data, argue that the SVCJ specification is a realistic model of equity indices (S&P 500 and Nasdaq) returns. These papers suggest that besides a stochastic volatility, jumps both in the mean and the volatility equation are relevant. With same symbolic definitions above, the SVCJ is the result of setting $\lambda_0 = \lambda_J$, $\lambda_0^* = \lambda_J^*$ and $\lambda_1 = \lambda_1^* = 0$ in the SVSCJ, i.e. replacing $\lambda_0 + \lambda_1 \nu_t$ $(\lambda_0^* + \lambda_1^* \nu_t)$ with λ_J (λ_J^*). The data-generating dynamics of $(\ln S_t, \nu_t)$ in the SVCJ model are thus assumed to be

$$
\begin{aligned}
d\ln S_t = &\Big\{ r - \delta - \lambda_J^* \kappa^* + \eta_S \nu_t - \frac{1}{2}\nu_t + \lambda_J^*(\mu_j^* + \rho_j^* \mu_\nu^*) \\
&+ \left[\lambda_J(\mu_j + \rho_j \mu_\nu) - \lambda_J^*(\mu_j^* + \rho_j^* \mu_\nu^*) \right] \Big\} dt + \sqrt{\nu_t}\, d\omega_{S,t} + z_S dN_t - \lambda_J(\mu_j + \rho_j \mu_\nu) dt
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
d\nu_t &= \kappa_\nu(\theta_\nu - \nu_t)\, dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t} + z_\nu dN_t \\
&= \kappa_\nu^*(\theta_\nu^* - \nu_t)\, dt + \eta_\nu \nu_t dt + \lambda_J^* \mu_\nu^* dt + (\lambda_J \mu_\nu - \lambda_J^* \mu_\nu^*)\, dt \\
&\quad + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t} + (z_\nu dN_t - \lambda_J \mu_\nu dt)
\end{aligned} \tag{19}
$$

where N_t is a univariate Poisson (counting) process with intensity λ_J. While the mean-reversion rate κ_ν is not affected by the jumps, the presence of jumps in the volatility dynamics changes the long-run mean of the variance process from θ_ν to $\theta_\nu + (\lambda_J \mu_\nu / \kappa_\nu)$. The corresponding risk-neutral processes are given by,

$$
d\ln S_t = [r - \delta + \lambda_J^*(\mu_\nu^* - \kappa^*) - \frac{1}{2}\nu_t]\, dt + \sqrt{\nu_t}\, d\omega_{S,t}^* + (z_S^* dN_t^* - \lambda_J^* \mu_\nu^* dt) \tag{20}
$$

$$
d\nu_t = \kappa_\nu^* \left(\frac{\kappa_\nu^* \theta_\nu^* + \lambda_J^* \mu_\nu^*}{\kappa_\nu^*} - \nu_t \right) dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t}^* + (z_\nu^* dN_t^* - \lambda_J^* \mu_\nu^* dt) \tag{21}
$$

where η_v and $\lambda_J \mu_v - \lambda_J^* \mu_v^*$ denote the risk premia relating to stochastic volatility and volatility jumps, respectively, whereas $\lambda_J \left(\mu_j + \rho_j \mu_v\right) - \lambda_J^* \left(\mu_j^* + \rho_j^* \mu_v^*\right)$ is the price-jump risk premium. $\kappa^* = \exp(\mu_j^* + \rho_j \mu_v^* + \sigma_j^2 / 2)/(1 - \rho_j \mu_v^*) - 1$ is the risk-neutral jump compensator of percentage price changes. Under the assumed framework, the total quadratic variation of the asset log-price $\upsilon_{t,T}$ still consists of price-diffusion variance $\upsilon_{t,T}^c$ and price-jump variance $\upsilon_{t,T}^J$. By applying Itô's lemma to $\exp(\kappa_v^* t)v_t$, we can compute v_T for $T \geq t$ and its first and second moments:

$$\mathrm{E}_t^Q\left(v_T\right) = \alpha_{T-t,\mathrm{SVCJ}}^* v_t + \beta_{T-t,\mathrm{SVCJ}}^* \tag{22}$$

$$\begin{aligned} \mathrm{E}_t^Q(v_T^2) = {} & (\alpha_{T-t,\mathrm{SVCJ}}^*)^2 \times v_t^2 + (C_{T-t,\mathrm{SVCJ}}^* + 2\alpha_{T-t,\mathrm{SVSCJ}}^* \times \beta_{T-t,\mathrm{SVCJ}}^*) \times v_t \\ & + [(\beta_{T-t,\mathrm{SVCJ}}^*)^2 + D_{T-t,\mathrm{SVCJ}}^*] \end{aligned} \tag{23}$$

where $\alpha_{T-t,\mathrm{SVCJ}}^* = e^{-\kappa_v^*(T-t)}$

$$\beta_{T-t,\mathrm{SVCJ}}^* = \left(\frac{\kappa_v^* \theta_v^* + \lambda_J^* \mu_v^*}{\kappa_v^*}\right)[1 - e^{-\kappa_v^*(T-t)}].$$

$$C_{T-t,\mathrm{SVCJ}}^* = \frac{\sigma_v^2}{\kappa_v^*}[e^{-\kappa_v^*(T-t)} - e^{-2\kappa_v^*(T-t)}]$$

$$D_{T-t,\mathrm{SVCJ}}^* = \frac{\sigma_v^2}{2\kappa_v^*}\left(\frac{\kappa_v^* \theta_v^* + \lambda_J^* \mu_v^*}{\kappa_v^*}\right)[1 - e^{-\kappa_v^*(T-t)}]^2 + \frac{\lambda_J^* (\mu_v^*)^2}{\kappa_v^*}[1 - e^{-2\kappa_v^*(T-t)}] \quad .$$

The expected total quadratic variation of the log-price is thus given by

$$\begin{aligned} \mathrm{E}_t^Q\left(\upsilon_{t,T}\right) &= \mathrm{E}_t^Q(\upsilon_{t,T}^c) + \mathrm{E}_t^Q(\upsilon_{t,T}^J) \\ &= a_{T-t,\mathrm{SVCJ}}^* \times v_t + b_{T-t,\mathrm{SVCJ}}^* + \lambda_J^*[(\mu_j^* + \rho_j \mu_v^*)^2 + \sigma_j^2 + \rho_j^2 (\mu_v^*)^2](T-t) \end{aligned} \tag{24}$$

where $a_{T-t,\mathrm{SVCJ}}^* = [1 - e^{-\kappa_v^*(T-t)}]/\kappa_v^*$

$$b_{T-t,\mathrm{SVCJ}}^* = \left(\frac{\kappa_v^* \theta_v^* + \lambda_J^* \mu_v^*}{\kappa_v^*}\right)\left\{(T-t) - \frac{[1 - e^{-\kappa_v^*(T-t)}]}{\kappa_v^*}\right\}$$

$$\mathrm{E}_t^Q\left(\upsilon_{t,T}^c\right) = a_{T-t,\mathrm{SVCJ}}^* \times \nu_t + b_{T-t,\mathrm{SVCJ}}^*$$

$$\mathrm{E}_t^Q\left(\upsilon_{t,T}^J\right) = \lambda_J^* [(\mu_j^* + \rho_j \mu_\nu^*)^2 + \sigma_j^2 + \rho_j^2 (\mu_\nu^*)^2](T-t)$$

From appendix (A.4), the VIX squared is expressed in terms of the expected variance attributed to the price-diffusion component by,

$$\mathrm{VIX}_t^2 \equiv 2\lambda_J^* [\kappa^* - (\mu_j^* + \rho_j \mu_\nu^*)] + \frac{1}{\tau}(a_{T-t,\mathrm{SVCJ}}^* \nu_t + b_{T-t,\mathrm{SVCJ}}^*) \tag{25}$$

where $\tau = T - t = 30/365$. The VIX squared is still a linear function of the instantaneous variance. From equation (25), mean and variance of VIX_t^2 conditional on VIX_0^2 are computed as

$$\begin{aligned} \mathrm{E}_0^Q(\mathrm{VIX}_t^2) \equiv\ & 2\lambda_J^* [\kappa^* - (\mu_j^* + \rho_j \mu_\nu^*)] \\ & + \frac{1}{\tau}\left(a_{T-t,\mathrm{SVCJ}}^* \alpha_{t,\mathrm{SVCJ}}^* \times \nu_0 + a_{T-t,\mathrm{SVCJ}}^* \beta_{t,\mathrm{SVCJ}}^* + b_{T-t,\mathrm{SVCJ}}^*\right) \end{aligned} \tag{26}$$

$$\mathrm{var}_0^Q(\mathrm{VIX}_t^2) = \left(\frac{a_{T-t,\mathrm{SVCJ}}^*}{\tau}\right)^2 (C_{t,\mathrm{SVCJ}}^* \times \nu_0 + D_{t,\mathrm{SVCJ}}^*) \tag{27}$$

where $\alpha_{t,\mathrm{SVCJ}}^* = e^{-\kappa_\nu^* t}$

$$\beta_{T-t,\mathrm{SVCJ}}^* = \left(\frac{\kappa_\nu^* \theta_\nu^* + \lambda_J^* \mu_\nu^*}{\kappa_\nu^*}\right)[1 - e^{-\kappa_\nu^* t}]$$

$$C_{t,\mathrm{SVCJ}}^* = \frac{\sigma_\nu^2}{\kappa_\nu^*}[e^{-\kappa_\nu^* t} - e^{-2\kappa_\nu^* t}]$$

$$D_{t,\mathrm{SVCJ}}^* = \frac{\sigma_\nu^2}{2\kappa_\nu^*}\left(\frac{\kappa_\nu^* \theta_\nu^* + \lambda_J^* \mu_\nu^*}{\kappa_\nu^*}\right)[1 - e^{-\kappa_\nu^* t}]^2 + \frac{\lambda_J^* (\mu_\nu^*)^2}{\kappa_\nu^*}[1 - e^{-2\kappa_\nu^* t}]$$

$$\nu_0 \equiv \frac{\tau}{a_{T-t,\mathrm{SVCJ}}^*} \times \mathrm{VIX}_0^2 - \frac{2\tau\lambda_J^* [\kappa^* - (\mu_j^* + \rho_j \mu_\nu^*)]}{a_{T-t,\mathrm{SVCJ}}^*} - \frac{b_{T-t,\mathrm{SVCJ}}^*}{a_{T-t,\mathrm{SVCJ}}^*} \quad .$$

By substituting $\mathrm{E}_0^Q(\mathrm{VIX}_t^2)$ and $\mathrm{var}_0^Q(\mathrm{VIX}_t^2)$ in equation (4) with the ones above, the fair price of the VIX futures expiring at *t* under the SVCJ model is obtained.

$$
\begin{aligned}
&\mathrm{F}_0^{\mathrm{VIX}}(t \mid \nu_0, \tau, t; \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu, \lambda_J, \mu_j, \sigma_j, \mu_\nu, \lambda_J^*, \mu_j^*, \mu_\nu^*, \rho_j\}) \\
&\approx \sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)} - \frac{\mathrm{var}_0^Q(\mathrm{VIX}_t^2)}{8 \times [\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}}
\end{aligned} \tag{28}
$$

C. SVJ Model

Given that the stochastic volatility model cannot explain the tail-fatness of the stock return distribution (Andersen et al., 1997), nor can it explain the smirkiness exhibited in the cross-sectional options data (Bates, 1996; Bakshi et al., 1997), the extension to include price jumps is well motivated. The stochastic volatility model with jumps in price only is a special case of the SVCJ obtained by letting $\lambda_J > 0$ and $\mu_\nu = 0$. It should be emphasized, however, that our main motivation is to study how such underlying processes are applied to the valuation of the VIX futures and, in particular, their role in reconciling the spot and option dynamics. The data-generating processes of $(\ln S_t, \nu_t)$ are expressed by,

$$
\begin{aligned}
d\ln S_t &= \left[r - \delta - \lambda_J^* \kappa^* + \eta_S \nu_t - \frac{1}{2}\nu_t + \lambda_J^* \mu_j^* + (\lambda_J \mu_j - \lambda_J^* \mu_j^*) \right] dt \\
&\quad + \sqrt{\nu_t}\, d\omega_{S,t} + (z_S dN_t - \lambda_J \mu_j)
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
d\nu_t &= \kappa_\nu (\theta_\nu - \nu_t)\, dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t} \\
&= \kappa_\nu^* (\theta_\nu^* - \nu_t)\, dt + \eta_\nu \nu_t dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t}
\end{aligned} \tag{30}
$$

where the price-jumps arrive at the exponential rate of $\lambda_J dt$ with the jump size, z_S, determined by the normal distribution $\mathrm{N}(\mu_j, \sigma_j^2)$. The corresponding risk-neutral processes then take the form,

$$
d\ln S_t = \left(r - \delta - \lambda_J^* \kappa^* - \frac{1}{2}\nu_t \right) dt + \sqrt{\nu_t}\, d\omega_{S,t}^* + z_S^* dN_t^* \tag{31}
$$

$$
d\nu_t = \kappa_\nu^* (\theta_\nu^* - \nu_t)\, dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu,t}^* \tag{32}
$$

Given the arrival of a jump event at time *t*, the risk-neutral stock price jumps from

S_{t^-} to $S_{t^-}\exp(z_S^*)$ and thus the mean relative jump size is $\kappa^* = \mathrm{E}[\exp(z_S^*)-1] = \exp(\mu_j^* + \sigma_j^2/2) - 1$. Thus, the term of $\lambda_J \mu_j - \lambda_J^* \mu_j^*$ in equation (29) denotes the price jump-risk premium. The diffusion variance ν_t is modeled by Heston's (1993) mean-reverting square-root process as specified, respectively, in equation (30) by a physical measure or in equation (32) under a risk-neutral measure. Under the assumed framework, the total quadratic variation in index returns can still be decomposed into the variance attributed to price diffusion and the one driven by price jumps. The first and second moments of the instantaneous variance ν in the SVJ model are

$$\mathrm{E}_t^Q(\nu_T) = \alpha^*_{T-t,\mathrm{SVJ}} \times \nu_t + \beta^*_{T-t,\mathrm{SVJ}} \tag{33}$$

$$\begin{aligned}\mathrm{E}_t^Q(\nu_T^2) &= (\alpha^*_{T-t,\mathrm{SVJ}})^2 \times \nu_t^2 + (C^*_{T-t,\mathrm{SVJ}} + 2\alpha^*_{T-t,\mathrm{SVJ}}\beta^*_{T-t,\mathrm{SVJ}}) \times \nu_t \\ &\quad + [(\beta^*_{T-t,\mathrm{SVJ}})^2 + D^*_{T-t,\mathrm{SVJ}}]\end{aligned} \tag{34}$$

where $\alpha^*_{T-t,\mathrm{SVJ}} = e^{-\kappa_\nu^*(T-t)}$

$\beta^*_{T-t,\mathrm{SVJ}} = \theta_\nu^*[1 - e^{-\kappa_\nu^*(T-t)}]$

$C^*_{T-t,\mathrm{SVJ}} = \dfrac{\sigma_\nu^2}{\kappa_\nu^*}[e^{-\kappa_\nu^*(T-t)} - e^{-2\kappa_\nu^*(T-t)}]$

$D^*_{T-t,\mathrm{SVJ}} = \dfrac{\sigma_\nu^2}{2\kappa_\nu^*}\theta_\nu^*[1 - e^{-\kappa_\nu^*(T-t)}]^2$.

The risk-neutral expected total quadratic variation of the log-price is computed as

$$\begin{aligned}\mathrm{E}_t^Q(\upsilon_{t,T}) &= \mathrm{E}_t^Q(\upsilon_{t,T}^c) + \mathrm{E}_t^Q(\upsilon_{t,T}^J) \\ &= a^*_{T-t,\mathrm{SVJ}} \times \nu_t + b^*_{T-t,\mathrm{SVJ}} + \lambda_J^*[(\mu_j^*)^2 + \sigma_j^2](T-t)\end{aligned} \tag{35}$$

where $a^*_{T-t,\mathrm{SVJ}} = \dfrac{[1 - e^{-\kappa_\nu^*(T-t)}]}{\kappa_\nu^*}$,

$b^*_{T-t,\mathrm{SVJ}} = \theta_\nu^*\left[(T-t) - \left(\dfrac{1 - e^{-\kappa_\nu^*(T-t)}}{\kappa_\nu^*}\right)\right]$,

$\mathrm{E}_t^Q(\upsilon_{t,T}^c) = a^*_{T-t,\mathrm{SVJ}}\nu_t + b^*_{T-t,\mathrm{SVJ}}$

$\mathrm{E}_t^Q(\upsilon_{t,T}^J) = \lambda_J^*[(\mu_j^*)^2 + \sigma_j^2](T-t)$.

From appendix (A.5), the VIX squared is expressed in terms of the expected variance attributed to the price-diffusion component by,

$$\mathrm{VIX}_t^2 \equiv 2\lambda_J^*(\kappa^* - \mu_j^*) + \frac{1}{\tau}(a_{T-t,\mathrm{SVJ}}^* v_t + b_{T-t,\mathrm{SVJ}}^*) \tag{36}$$

where $\tau = T - t = 30/365$ and same symbols are defined as before. The VIX squared is again a linear function of the instantaneous variance. Mean and variance of the VIX squared conditional on current instantaneous variance v_0 are computed as

$$\mathrm{E}_0^Q\left(\mathrm{VIX}_t^2\right) = 2\lambda_J^*(\kappa^* - \mu_j^*) + \frac{1}{\tau}(a_{T-t,\mathrm{SVJ}}^* \alpha_{t,\mathrm{SVJ}}^* \times v_0 + b_{T-t,\mathrm{SVJ}}^* + a_{T-t,\mathrm{SVJ}}^* \beta_{t,\mathrm{SVJ}}^*) \tag{37}$$

$$\mathrm{var}_0^Q(\mathrm{VIX}_t^2) = \left(\frac{a_{T-t,\mathrm{SVJ}}^*}{\tau}\right)^2 \left(C_{t,\mathrm{SVJ}}^* \times v_0 + D_{t,\mathrm{SVJ}}^*\right) \tag{38}$$

where $\alpha_{t,\mathrm{SVJ}}^* = e^{-\kappa_v^* t}$

$$\beta_{t,\mathrm{SVJ}}^* = \theta_v^*(1 - e^{-\kappa_v^* t})$$

$$C_{t,\mathrm{SVJ}}^* = \frac{\sigma_v^2}{\kappa_v^*}(e^{-\kappa_v^* t} - e^{-2\kappa_v^* t})$$

$$D_{t,\mathrm{SVJ}}^* = \frac{\sigma_v^2}{2\kappa_v^*}\theta_v^*(1 - e^{-\kappa_v^* t})^2$$

$$v_0 = \frac{\tau}{a_{T-t,\mathrm{SVJ}}^*}\mathrm{VIX}_0^2 - \frac{1}{a_{T-t,\mathrm{SVJ}}^*}[2\tau\lambda_J^*(\kappa^* - \mu_j^*) - b_{T-t,\mathrm{SVJ}}^*].$$

The fair price of the VIX futures expiring at *t* under the SVJ model is thus given by

$$\begin{aligned} &\mathrm{F}_0^{\mathrm{VIX}}(t \mid v_0, \tau, t; \{\eta_S, \kappa_v, \theta_v, \sigma_v, \rho, \eta_v, \lambda_J, \mu_j, \sigma_j, \lambda_J^*, \mu_j^*\}) \\ &\approx \sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)} - \frac{\mathrm{var}_0^Q(\mathrm{VIX}_t^2)}{8 \times [\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}} \end{aligned} \tag{39}$$

D. SV Model

The volatility specification of the SV model, introduced by Heston (1993), captures an important stylized feature of the stock return dynamics, i.e. stochastic volatility, and also allows the Brownian shocks to price *S* and variance v to be correlated with constant coefficient ρ, denoting the familiar leverage effect (Black, 1976). The SV model obtains as a special case of the general model in this paper with jumps restricted to zero ($z_S dN_t = z_v dN_t = 0$). It assumes the following data-generating process for log-prices,

$$d\ln S_t = (r - \delta + \eta_S v_t - \frac{1}{2} v_t)\, dt + \sqrt{v_t}\, d\omega_{S,t} \tag{40}$$

$$\begin{aligned} dv_t &= \kappa_v(\theta_v - v_t)\, dt + \sigma_v \sqrt{v_t}\, d\omega_{v,t} \\ &= \kappa_v^*(\theta_v^* - v_t)\, dt + \eta_v v_t dt + \sigma_v \sqrt{v_t}\, d\omega_{v,t} \end{aligned} \tag{41}$$

where η_S and η_v, respectively, refer to the price risk premium parameter and the volatility risk premium parameter of interest. The instantaneous variance v_t of log-prices is modeled as a one-factor square-root process that was originally proposed for finance by Cox et al. (1985). Under the risk-neutral probability measure *Q*, the logarithmic index price and its instantaneous variance are assumed to follow the diffusions,

$$d\ln S_t = (r - \delta - \frac{1}{2} v_t)\, dt + \sqrt{v_t}\, d\omega_{S,t}^* \tag{42}$$

$$dv_t = \kappa_v^*(\theta_v^* - v_t)\, dt + \sigma_v \sqrt{v_t}\, d\omega_{v,t}^* \tag{43}$$

Cox, Ingersoll and Ross (1985) show that the distribution of v_T conditional on v_t is a non-central chi-square, i.e. $\chi^2(2c^* v_t, 2q^* + 2, 2\varsigma_t^*)$ with $\varsigma_t^* = c^* v_t e^{-\kappa_v^*(T-t)}$, $c^* = 2\kappa_v^* / [\sigma_v^2 (1 - e^{-\kappa_v^*(T-t)})]$, $q^* = 2\kappa_v^* \theta_v^* / \sigma_v^2 - 1$, and the second and third arguments being the degrees of freedom and non-centrality parameters, respectively.[6] The first two conditional moments of the instantaneous variance v_T are the same as the ones in the SVJ model given by equations (33) and (34) above. The expected total quadratic variation of the log-price is presented as

$$\mathrm{E}_t^Q\left(\upsilon_{t,T}\right) = \mathrm{E}_t^Q\left(\upsilon_{t,T}^c\right) = a_{T-t,\mathrm{SV}}^* v_t + b_{T-t,\mathrm{SV}}^* \tag{44}$$

6 Its corresponding physical non-central chi-square distribution of v_T conditional on v_t is simply given by replacing the risk-neutral parameters κ_v^* and θ_v^* with κ_v and θ_v.

where $a^{*}_{T-t,\mathrm{SV}} = a^{*}_{T-t,\mathrm{SVJ}}$ and $b^{*}_{T-t,\mathrm{SV}} = b^{*}_{T-t,\mathrm{SVJ}}$.

From appendix (A.6), the VIX squared is expressed in terms of the expected diffusive variance by,

$$\mathrm{VIX}_t^2 \equiv \frac{1}{\tau}\mathrm{E}_t^Q(\upsilon_{t,T}^c) = \frac{1}{\tau}(a^{*}_{T-t,\mathrm{SV}}\nu_t + b^{*}_{T-t,\mathrm{SV}}) \tag{45}$$

where $\tau = T - t = 30/365$ and same symbols are defined as before. The VIX squared is again a linear function of the instantaneous variance. Mean and variance of the VIX squared conditional on current instantaneous variance ν_0 are computed as

$$\mathrm{E}_0^Q(\mathrm{VIX}_t^2) = \frac{1}{\tau}(a^{*}_{T-t,\mathrm{SV}}\alpha^{*}_{t,\mathrm{SV}} \times \nu_0 + a^{*}_{T-t,\mathrm{SV}}\beta^{*}_{t,\mathrm{SV}} + b^{*}_{T-t,\mathrm{SV}}) \tag{46}$$

$$\mathrm{var}_0^Q(\mathrm{VIX}_t^2) = \left(\frac{a^{*}_{T-t.\mathrm{SV}}}{\tau}\right)^2 (C^{*}_{t,\mathrm{SV}} \times \nu_0 + D^{*}_{t,\mathrm{SV}}) \tag{47}$$

where $\alpha^{*}_{t,\mathrm{SV}} = \alpha^{*}_{t,\mathrm{SVJ}}$, $\beta^{*}_{t,\mathrm{SV}} = \beta^{*}_{t,\mathrm{SVJ}}$, $C^{*}_{t,\mathrm{SV}} = C^{*}_{t,\mathrm{SVJ}}$, $D^{*}_{t,\mathrm{SV}} = D^{*}_{t,\mathrm{SVJ}}$ and

$$\nu_0 = \frac{\tau}{a^{*}_{T-t,\mathrm{SV}}}\mathrm{VIX}_0^2 - \frac{b^{*}_{T-t,\mathrm{SV}}}{a^{*}_{T-t,\mathrm{SV}}} .$$

By substituting $\mathrm{E}_0^Q(\mathrm{VIX}_t^2)$ and $\mathrm{var}_0^Q(\mathrm{VIX}_t^2)$ in equation (4) with the ones in equations (46) and (47), the fair price of the VIX futures expiring at *t* under the SV model is obtained.

$$\begin{aligned} &\mathrm{F}_0^{\mathrm{VIX}}(t \mid \nu_0, \tau, t; \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu\}) \\ &\approx \sqrt{\mathrm{E}_0^Q(\mathrm{VIX}_t^2)} - \frac{\mathrm{var}_0^Q(\mathrm{VIX}_t^2)}{8 \times [\mathrm{E}_0^Q(\mathrm{VIX}_t^2)]^{3/2}} \end{aligned} \tag{48}$$

IV. THEORETICAL RELATIONSHIP BETWEEN INTEGRATED VOLATILITY AND VIX

In order to obtain more informative parameter estimates and the risk premia relating to volatility and jumps, our empirical results presented are based on estimates obtained from joint spot and option prices, or equivalently integrated volatility and VIX in this paper.[7] The theoretical relationship between integrated volatility and VIX is discussed in this section. By the theory of quadratic variation (see, e.g., Andersen et al., 2002),

$$\lim_{N\to\infty}\sum_{i=1}^{2^N}\left[\ln S_{t+i/2^N(T-t)}-\ln S_{t+(i-1)/2^N(T-t)}\right]^2 \xrightarrow{a.s.} \int_t^T \mathrm{var}_u^P(d\ln S_u;\Phi)\,du=\int_t^T \nu_u(\Phi)\,du+\int_t^T \nu_J(\Phi)\,du \equiv \upsilon_{t,T}^c+\upsilon_{t,T}^J=\upsilon_{t,T} \tag{49}$$

where $\nu_t(\cdot)$ is the point-in-time price-diffusion variance that is latent and its consistent estimation through filtering is complicated by a host of market microstructure complications. $\upsilon_{t,T}^c=\int_t^T \nu_u(\Phi)du$ presents the integrated variance from time t to T, consisting of the variance driven by price-diffusion and volatility-jump components (if any). $\upsilon_{t,T}$ denotes the realized variance from time t to T computed by summing the squared high-frequency returns over the $[t,T]$ time-interval. It is important to recognize that, in the presence of price jumps, the quadratic variation $\upsilon_{t,T}$ comprises two components, $\upsilon_{t,T}^c$ and $\upsilon_{t,T}^J$ with $\nu_J(\cdot)$ being the point-in-time price-jump variance and $\upsilon_{t,T}^J=\int_t^T \nu_J(\Phi)du$.[8] Φ represents the model parameters governing the index log-price dynamics under the physical probability measure P that are restricted to lie within some compact set containing the

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[7] The usage of joint data of underlying returns and option prices for the model parameter estimation and the extraction of related risk premia was an idea pursued in Chernov and Ghysels (2000) and Pan (2002).

[8] Hence, in implementing the moment conditions involving $\upsilon_{t,T}^c$, the following substitutions are also required:

$$\begin{aligned}
&\mathrm{E}_t(\upsilon_{t,T}^c)=\mathrm{E}_t(\upsilon_{t,T})-\mathrm{E}_t(\upsilon_{t,T}^J)\\
&\mathrm{E}_t(\upsilon_{t,T}^c)^2=\mathrm{E}_t(\upsilon_{t,T})^2+\mathrm{E}_t(\upsilon_{t,T}^J)^2-2\mathrm{E}_t(\upsilon_{t,T})\mathrm{E}_t(\upsilon_{t,T}^J)\quad.\\
&\mathrm{E}_t(\upsilon_{t,T}^c\ln S_T)=\mathrm{E}_t(\upsilon_{t,T}\ln S_T)-\mathrm{E}_t(\upsilon_{t,T}^J\ln S_T)
\end{aligned}$$

true parameters of the process, say Φ_0. In other words, by summing increasingly finer sampled squared high-frequency returns, it is possible to obtain increasingly more accurate estimates of the realized volatility of the process. Following Aït-Sahalia et al. (2003), the daily realized volatility of the spot index is measured by the sum of squared 5-minute intraday returns and the squared close-to-open return. To correct for the bias in estimated realized volatility due to autocorrelation intraday returns, this study adopts a correction method suggested, in various forms, by French et al. (1987), Zhou (1996), and Hansen and Lunde (2006). In this correction method, the annualized realized variance over the period [t, $t+\tau$] is calculated as:

$$\upsilon_{t,t+\tau} = \frac{1}{\tau}\sum_{i=1}^{n} R_i^2 + \frac{2}{\tau}\sum_{h=1}^{m}\left(\frac{n}{n-h}\right)\sum_{i=1}^{n-h} R_i R_{t+h} \tag{50}$$

where R_i is the index return during the i-th interval, n is the total number of intervals in the period, and m is the number of correction terms included. Similar to the findings of Jiang and Tian (2005),[9] this study uses equation (50) with one correction term (i.e., m=1) to calculate the daily volatility using five-minute SPX returns. This is because 5-minute returns in our sample, 17 March 2004–18 April 2006,[10] have a first-order autocorrelation of 0.0344 while higher-order autocorrelations are much smaller. Conditional moments for the integrated volatility for the stochastic volatility (SV) model have previously been derived by Bollerslev and Zhou (2002), Meddahi (2002), Andersen et al. (2004), Bollerslev et al. (2005), and Garcia et al. (2006). Meanwhile, the derivation of the operational moment conditions of $\upsilon_{t,T}^{c}$ implied by the stochastic volatility with price jumps (SVJ) model structure is provided by Bollerslev and Zhou (2002). Existing literature, however, has

[9] Jiang and Tian (2005) adopt equation (6) with one correction term (i.e., m=1) to calculate the daily volatility using five-minute SPX returns due to a first-order autocorrelation of 0.31 in their sample period, June 1988–December 1994.

[10] Since our parameter estimation task involves the usage of lag 30-day integrated volatility, the data period mentioned here is one-month earlier than the period for model parameter estimation starting on 21 April 2004.

not provided the moment conditions of the total quadratic variation $\upsilon_{t,T}$ for the SVJ, SVCJ and SVSCJ models, which are required for our parameter estimation purpose and are given in the appendix B.

Using option prices, it is also possible to construct a model-free measure of the risk-neutral expectation of the integrated volatility. In particular, VIX at time *t* denotes the time-*t* implied volatility measure computed as a weighted average, or integral, of a continuum of 30-day maturity options. As formally shown by Britten-Jones and Neuberger (2000) for the stochastic-volatility model, the model-free implied volatility, or equivalently the VIX in this study, equals the risk-neutral expectation of the integrated volatility, adjusted for the price-jump components ζ_1^* and ζ_2^* as shown in equation (3), i.e. $\mathrm{VIX}_t^2 \equiv \zeta_1^* \mathrm{E}_t^Q(\upsilon_{t,T}^c)/\tau + \zeta_2^*$ with $\tau = T - t = 30/365$. Appendix A provides details of the derivation for the values of ζ_1^* and ζ_2^* under alternate index log-price stochastic processes. Substituting for $\mathrm{E}_t^Q(\upsilon_{t,T}^c)$ in equation (3) by $a_{T-t}^* v_t + b_{T-t}^*$, it follows $v_t \equiv \tau\,\mathrm{VIX}_t^2/(a_{T-t}^*\zeta_1^*) - b_{T-t}^*/a_{T-t}^* - \tau\zeta_2^*/(a_{T-t}^*\zeta_1^*)$. Combining these results, it now becomes to directly and analytically link the expectation of the realized volatility under the risk-neutral dynamics with the expectation of the realized volatility under the physical probability measure.

$$\mathrm{E}_t^P(\upsilon_{t,T}) = \mathrm{U}_{T-t}\mathrm{VIX}_t^2 + \mathrm{V}_{T-t} \tag{51}$$

$$\mathrm{E}_t^P(\upsilon_{t,T}^2) = \mathrm{X}_{T-t}\mathrm{VIX}_t^4 + \mathrm{Y}_{T-t}\mathrm{VIX}_t^2 + \mathrm{Z}_{T-t} \tag{52}$$

where for the SVSCJ model, as an example,

$$\mathrm{U}_{T-t,\mathrm{SVSCJ}} = \{1 + \lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\} \times \frac{a_{T-t,\mathrm{SVSCJ}}}{a_{T-t,\mathrm{SVSCJ}}^*}\frac{\tau}{\zeta_{1,\mathrm{SVSCJ}}^*}$$

$$
\begin{aligned}
\mathrm{V}_{T-t,\mathrm{SVSCJ}} &= \lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2](T-t) \\
&\quad + \{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times\begin{pmatrix} b_{T-t,\mathrm{SVSCJ}}-\dfrac{a_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}}b^*_{T-t,\mathrm{SVSCJ}} \\ -\dfrac{a_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}}\dfrac{\tau\zeta^*_{2,\mathrm{SVSCJ}}}{\zeta^*_{1,\mathrm{SVSCJ}}}\end{pmatrix}
\end{aligned}
$$

$$
\mathrm{X}_{T-t,\mathrm{SVSCJ}} = \{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2\times\left(\frac{\tau\, a_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}}\right)^2
$$

$$
\begin{aligned}
\mathrm{Y}_{T-t,\mathrm{SVSCJ}} &= \{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2 \\
&\qquad \times(\mathrm{A}_{T-t,\mathrm{SVSCJ}}+2a_{T-t,\mathrm{SVSCJ}}b_{T-t,\mathrm{SVSCJ}})\times\frac{\tau}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}} \\
&\quad + 2\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2](T-t) \\
&\qquad \times\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times\frac{\tau\, a_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}} \\
&\quad - \{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2\times\frac{2\tau\, a^2_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}} \\
&\qquad \times\left(\frac{b^*_{T-t,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}}+\frac{\tau\zeta^*_{2,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Z}_{T-t,\mathrm{SVSCJ}} &= \{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2 \\
&\qquad \times\left[\mathrm{B}_{T-t,\mathrm{SVSCJ}}+a^2_{T-t,\mathrm{SVSCJ}}\left(\frac{b^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}+\tau\zeta^*_{2,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}}\right)^2\right] \\
&- \left\{\begin{aligned} &\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2 \\ &\quad\times(\mathrm{A}_{T-t,\mathrm{SVSCJ}}+2a_{T-t,\mathrm{SVSCJ}}b_{T-t,\mathrm{SVSCJ}}) \\ &+2\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2](T-t) \\ &\quad\times\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{T-t,\mathrm{SVSCJ}} \end{aligned}\right\}\times\left(\frac{b^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}+\tau\zeta^*_{2,\mathrm{SVSCJ}}}{a^*_{T-t,\mathrm{SVSCJ}}\zeta^*_{1,\mathrm{SVSCJ}}}\right) \\
&+ \left\{\begin{aligned} &\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2](T-t) \\ &+\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times b_{T-t,\mathrm{SVSCJ}} \end{aligned}\right\}^2
\end{aligned}
$$

where $\tau = T-t = 30/365$. These equations, in conjunction with the moment restrictions of integrated volatility under the physical measure, provide the necessary identification of the risk premium parameters. While Bollerslev and Zhou (2004) and Bollerslev et al. (2005) have provided expressions for equation (51) under the SV specification, the rest function forms of equations (51) and (52) for the SVSCJ, for the

SVCJ by restricting $\lambda_0 = \lambda_J$, $\lambda_0^* = \lambda_J^*$ and $\lambda_1 = \lambda_1^* = 0$, and for the SVJ by further imposing $\mu_v = \mu_v^* = 0$, are given here.

Further, the time to expiration, *T*, in the VIX calculation is measured in minutes rather than in days, shown as follows,

$$T = \frac{\mathrm{M}_{\text{Current day}} + \mathrm{M}_{\text{Settlement day}} + \mathrm{M}_{\text{Other days}}}{\text{Minutes in a year}} \tag{53}$$

where

$\mathrm{M}_{\text{Current day}}$ = # of minutes remaining until midnight of the current day
$\mathrm{M}_{\text{Settlement day}}$ = # of minutes from midnight until 8:30 a.m. on SPX settlement day
$\mathrm{M}_{\text{Other days}}$ = Total # of minutes in the days between current day and settlement day

Therefore, using daily closing VIX data is equivalent to using daily closing time as the time of the calculation. The beginning and ending time of the 30-day realized volatility calculation, $\upsilon_{t,t+30\,\text{days}}$, is thus assumed to be the closing time at *t* (Chicago time) and the closing time at *t*+30 calendar days, respectively.

V. DATA AND MODEL ESTIMATION

A. *Historical VIX Time Series and VIX Futures Contracts*

Parameters of our VIX futures pricing models are estimated from joint options, or equivalently the VIX, and stock markets data, or equivalently integrated volatility calculated from five-minute intraday index returns over 30-day horizon. Price data on VIX futures and contemporaneous VIX levels came from the transaction records provided by the Chicago Futures Exchange (CFE), whereas the intraday SPX returns are from Chicago Merchant Exchange (CME). Model parameters are updated to remain the same within one-month window, i.e. the period between settlement dates of each month. The settlement date for VIX futures is the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the month in which the contract expires. Since the first date available from CFE for the settlement value of VIX futures is 19 May 2004, the common sample period for parameter estimation is chosen from 21 April 2004 to 18 April 2006, in total 502 trading dates. Daily settlement VIX futures prices from 19 May 2004 to 16 May 2006, resulting in 1956 observations and spanning 27 expirations, are adopted to test the validity of our pricing models. Table 1 summarizes the descriptive statistics for integrated volatility, VIX time series and daily VIX futures settlement prices. There is a tendency for integrated volatility to be lower than VIX and for VIX to be lower than its futures price. The integrated volatility is measured under the physical probability measure, whereas the prices of VIX and VIX futures are the resulting integration of investors' subjective probabilities across states and the adjustment for investors' risk aversion. Given the stylized characteristics of SPX options and VIX futures being derivatives and thus useful for the hedge purposes, the derivatives price-implied volatility would contain the negative volatility risk premium. In other words, the more exposure to volatility risk the investors face, the more expensive the volatility derivatives or higher value of implied volatility. Thus, it is not surprising for integrated volatility to be lower than either VIX or VIX futures prices. The second phenomenon indicates that the more vega risk exposure is, the higher price of the VIX futures will be. For investors with portfolios, for example, having great risk exposure to SPX option's implied volatility, e.g. a vega-position option writer, they would like to pay more to purchase the VIX futures as hedge. Compared to the SPX index, VIX tends to rise as the SPX falls and tends to decline or remain constant as the SPX rises. Given the existed evidence of a negative correlation between VIX and SPX index, the value of his portfolio reduces dramatically at the state of high VIX or equivalently at the state of low SPX. Thus, investors will put more weights on the risk-aversion adjustment factor at high VIX or low SPX. The resulting risk-neutral distribution across SPX levels, or equivalently the product of subjective probabilities towards future VIX and risk-aversion adjustment factors, implicit in VIX futures prices is more negatively skewed than the one given by VIX time series, or equivalently the SPX option traders. In other words, the component of expected SPX returns attributed to volatility risk given by VIX futures traders is positive and stronger than the one given by SPX option traders. The phenomenon reveals that most of the VIX futures traders during our sample period are volatility hedgers. This explanation justifies the tendency of higher VIX futures prices relative to its underlying VIX.

Table 1 Descriptive statistics of 30-day realized volatility, VIX and daily settlement prices of VIX futures across maturities

The 30-day realized volatility, $\upsilon^{1/2}(t,t+30)$, presents the total quadratic variation over [t,t+30 days], measured by the sum of autocorrelation-adjusted squared 5-minute intraday index returns and the squared close-to-open return. VIX and VIX daily settlement prices are divided by 100 and 1000, respectively, to denote the volatility level. The Sample period for estimating monthly-updated model parameters jointly using integrated volatility and VIX time series is 21 April 2004–18 April 2006, in total 502 trading days, while the data period for testing our VIX futures formulas is 19 May 2004–16 May 2006, resulting in 1956 observations and spanning 27 expirations.

	$\upsilon^{1/2}(t,t+30)$	VIX_T	Daily VIX Futures Settlement Price			
			All	<60 days	60–180	>180 days
Obs.	502	502	1956	793	727	436
Mean	0.0954	0.1348	0.1533	0.1425	0.1580	0.1650
Median	0.0915	0.1316	0.1513	0.1386	0.1523	0.1603
Std. Dev.	0.0137	0.0194	0.0218	0.0206	0.0202	0.0170
Minimum	0.0769	0.1023	0.1017	0.1017	0.1253	0.1352
Maximum	0.1429	0.1996	0.2195	0.2066	0.2149	0.2195
Skewness	1.2477	0.7911	0.7390	0.9640	1.2408	1.5248
Kurtosis	4.1257	3.2665	3.4544	3.3636	3.7012	5.1418

B. Estimation Procedure and Conditional Moment Restrictions

It has been especially difficult to estimate the continuous-time stochastic volatility models that are best suited for pricing derivatives. Our jump-related models create additional latent state variable of asset prices jump and time-varying jump intensity. Previous estimation approaches include analytically tractable specifications such as Gaussian and Regime-switching specifications, GMM approaches based on analytic moment conditions, Bates' (2006) AML (approximate maximum likelihood), and simulation-based approaches such as Gallant and Tauchen's (2002) EMM (Efficient Method of Moments), or Jacquier et al.'s (1994) MCMC (Monte Carlo Markov Chain) approach. Bates (2006) provides an overview of parameter estimation of continuous-time models. Since the latent stochastic volatility can be inferred from observed VIX data, we focus primarily on parameter estimation and thus moment-based approaches provide an adequate framework. In addition, the analytical solutions for the conditional moments in alternate models under physical and risk-neutral probability measures set the stage for the construction of a GMM-type estimator. The efficiency of the resulting estimator defined from these conditions depends upon the particular choice of instruments (Hansen, 1985; Hansen et al., 1988; and Gallant and Tauchen, 1996). This paper simply arguments the first and second moments with lag-one and lag-one squared counterparts as well as the cross moment, resulting in the following moments $f_t(\Phi)$. By construction $\mathrm{E}_t[f_t(\Phi)] = 0$ using numerical minimization[11] and joint data of VIX time series and integrated volatilities, the estimates of model parameters are obtained.

[11] Although the GMM procedure is the standard procedure to operationally implement $\mathrm{E}_t[f_t(\Phi)] = 0$ in the literature, this paper instead adopts numerical minimization that is common in derivatives studies.

$$
\begin{aligned}
& f_t(\Phi) = \\
& \upsilon_{t+\Delta,t+2\Delta} - \mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}) \\
& \upsilon_{t+\Delta,t+2\Delta}\upsilon_{t-\Delta,t} - \mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}\upsilon_{t-\Delta,t}) \\
& \upsilon_{t+\Delta,t+2\Delta}\upsilon_{t-\Delta,t}^2 - \mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}\upsilon_{t-\Delta,t}^2) \\
& \upsilon_{t+\Delta,t+2\Delta}^2 - \mathrm{E}_t^P[\upsilon_{t+\Delta,t+2\Delta}^2] \\
& \upsilon_{t+\Delta,t+2\Delta}^2\upsilon_{t-\Delta,t} - \mathrm{E}_t^P[\upsilon_{t+\Delta,t+2\Delta}^2\upsilon_{t-\Delta,t}] \\
& \upsilon_{t+\Delta,t+2\Delta}^2\upsilon_{t-\Delta,t}^2 - \mathrm{E}_t^P[\upsilon_{t+\Delta,t+2\Delta}^2\upsilon_{t-\Delta,t}^2] \\
& \ln S_{t+\Delta} - \mathrm{E}_t^P[\ln S_{t+\Delta}] \\
& \ln S_{t+\Delta}\ln S_{t-\Delta} - \mathrm{E}_t^P[\ln S_{t+\Delta}\ln S_{t-\Delta}] \\
& \ln S_{t+\Delta}\ln S_{t-\Delta}^2 - \mathrm{E}_t^P[\ln S_{t+\Delta}\ln S_{t-\Delta}^2] \\
& \ln S_{t+\Delta} \times \left\{ \frac{\upsilon_{t+\Delta,t+2\Delta} - \lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta}{\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times a_{\Delta,\mathrm{SVSCJ}}} - \frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}} \right\} \\
& \qquad - \mathrm{E}_t^P\left[\ln S_{t+\Delta} \times \left\{ \frac{\upsilon_{t+\Delta,t+2\Delta} - \lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta}{\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times a_{\Delta,\mathrm{SVSCJ}}} - \frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}} \right\} \right] \\
& \upsilon_{t,T} - \mathrm{U}_{T-t}\mathrm{VIX}_t^2 - \mathrm{V}_{T-t} \\
& \upsilon_{t,T}\times\upsilon_{t-(T-t),t} - \mathrm{U}_{T-t}\mathrm{VIX}_t^2\times\upsilon_{t-(T-t),t} - \mathrm{V}_{T-t}\times\upsilon_{t-(T-t),t} \\
& \upsilon_{t,T}\times\upsilon_{t-(T-t),t}^2 - \mathrm{U}_{T-t}\mathrm{VIX}_t^2\times\upsilon_{t-(T-t),t}^2 - \mathrm{V}_{T-t}\times\upsilon_{t-(T-t),t}^2 \\
& \upsilon_{t,T}^2 - \mathrm{X}_{T-t}\mathrm{VIX}_t^4 - \mathrm{Y}_{T-t}\mathrm{VIX}_t^2 - \mathrm{Z}_{T-t} \\
& \upsilon_{t,T}^2\times\upsilon_{t-(T-t),t} - \mathrm{X}_{T-t}\mathrm{VIX}_t^4\times\upsilon_{t-(T-t),t} - \mathrm{Y}_{T-t}\mathrm{VIX}_t^2\times\upsilon_{t-(T-t),t} - \mathrm{Z}_{T-t}\times\upsilon_{t-(T-t),t} \\
& \upsilon_{t,T}^2\times\upsilon_{t-(T-t),t}^2 - \mathrm{X}_{T-t}\mathrm{VIX}_t^4\times\upsilon_{t-(T-t),t}^2 - \mathrm{Y}_{T-t}\mathrm{VIX}_t^2\times\upsilon_{t-(T-t),t}^2 - \mathrm{Z}_{T-t}\times\upsilon_{t-(T-t),t}^2
\end{aligned}
\tag{54}
$$

where $\tau = T - t = 30/365$. Structure parameters of alternate models are given by,

$$\Phi_{\mathrm{SV}} = \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu\}$$

$$\Phi_{\mathrm{SVJ}} = \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu, \lambda_J, \mu_j, \sigma_j, \lambda_J^*, \mu_j^*\}$$

$$\Phi_{\mathrm{SVCJ}} = \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu, \lambda_J, \mu_j, \sigma_j, \mu_\nu, \lambda_J^*, \mu_j^*, \mu_\nu^*, \rho_j\}$$

$$\Phi_{\mathrm{SVSCJ}} = \{\eta_S, \kappa_\nu, \theta_\nu, \sigma_\nu, \rho, \eta_\nu, \lambda_0, \lambda_1, \mu_j, \sigma_j, \mu_\nu, \lambda_0^*, \lambda_1^*, \mu_j^*, \mu_\nu^*, \rho_j\}.$$

The closed-form solutions to the operational moments in equation (54) are given by,

$$\mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}) = \alpha_{\Delta,\mathrm{SVSCJ}} \times \upsilon_{t,t+\Delta} + (1-\alpha_{\Delta,\mathrm{SVSCJ}})\lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\Delta$$
$$+ \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\} \times \beta_{\Delta,\mathrm{SVSCJ}}\,\Delta$$

$$\mathrm{E}_t^P(\upsilon^2_{t+\Delta,t+2\Delta}) = \mathrm{H}_{\Delta,\mathrm{SVSCJ}} \times \upsilon^2_{t,t+\Delta} + \mathrm{I}_{\Delta,\mathrm{SVSCJ}} \times \upsilon_{t,t+\Delta} + \mathrm{J}_{\Delta,\mathrm{SVSCJ}}$$

$$\mathrm{E}_t^P(\ln S_{t+\Delta}) = \ln S_t + \frac{[\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]}{\{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}} \times \upsilon_{t,t+\Delta}$$
$$+ [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v)]\Delta$$
$$- \frac{[\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)] \times \lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\Delta}{\{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}}$$

$$\mathrm{E}_t^P\left\{\ln S_{t+\Delta} \times \left\{\frac{\upsilon_{t+\Delta,t+2\Delta} - \lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\Delta}{\{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\} \times a_{\Delta,\mathrm{SVSCJ}}} - \frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}}\right\}\right\}$$
$$= \mathrm{N}_{\Delta,\mathrm{SVSCJ}} \times \upsilon^2_{t,t+\Delta} + \mathrm{O}_{\Delta,\mathrm{SVSCJ}} \times \upsilon_{t,t+\Delta} + \mathrm{P}_{\Delta,\mathrm{SVSCJ}} \times \ln S_t + \mathrm{Q}_{\Delta,\mathrm{SVSCJ}}$$

where the values of $\mathrm{H}_{\Delta,\mathrm{SVSCJ}}$, $\mathrm{I}_{\Delta,\mathrm{SVSCJ}}$ and $\mathrm{J}_{\Delta,\mathrm{SVSCJ}}$ can be found in (B.12), while $\mathrm{N}_{\Delta,\mathrm{SVSCJ}}$, $\mathrm{O}_{\Delta,\mathrm{SVSCJ}}$, $\mathrm{P}_{\Delta,\mathrm{SVSCJ}}$, and $\mathrm{Q}_{\Delta,\mathrm{SVSCJ}}$ are derived in (B.14). Similarly, these solutions are obtained for the SVCJ by restricting $\lambda_0 = \lambda_J$, $\lambda_0^* = \lambda_J^*$ and $\lambda_1 = \lambda_1^* = 0$, for the SVJ by further imposing $\mu_v = \mu_v^* = 0$, and for the SV by additionally setting $\lambda_J = \lambda_J^* = 0$.

VI. EMPIRICAL RESULTS

Summary statistics for parameter estimation results and out-of-sample pricing fits for VIX futures are shown in Tables 2 and 3, respectively. The market prices of diffusive and jump risks are reported in Table 4.

A. *In-Sample Pricing Fit*

Physical and risk-neutral parameter estimates from alternate models implicit in the joint VIX and integrated volatilities along with in-sample mean absolute errors are shown in Table 2. The in-sample mean absolute pricing error (*MAE*) was considerably and consistently smaller under the SVSCJ (*MAE*=4.3785) than the ones under the SVCJ (*MAE*=4.6388), the SVJ (*MAE*=4.7589) and the SV (*MAE*=5.5574). There are several interesting features of the parameter estimates in Table 2. The long-term mean of the volatility process after taking into account volatility jumps, if any, $(\kappa_v\theta_v + \lambda_0\mu_v)/(\kappa_v - \lambda_1\mu_v)$ is 0.0371, 0.0279, 0.0714 and 0.0700 for the SV, SVJ, SVCJ and SVSCJ, while the long-run mean under the risk-neutral measure are 0.0389, 0.0310, 0.0573 and 0.0314, respectively. The difference between these parameters across the two measures is the market prices associated with total volatility change risk. Eraker (2004) uses the MCMC approach along with the joint SPX option and return data over January 1, 1987–December 31, 1990 to estimate model parameters. In contrast, Eraker et al. (2003) use only daily SPX returns over January 2, 1980–December 31, 1999 to perform their estimation. While the comparable total long-run variance means, $(\kappa_v\theta_v + \lambda_0\mu_v)/(\kappa_v - \lambda_1\mu_v)$, in Eraker (2004) are 0.0487, 0.0416, 0.0377 and –0.0003 across models, Eraker et al. (2003) find 0.0228, 0.0205, 0.0230 and 0.0240 instead.

The speed of mean reversion, $\kappa_v - \lambda_1\mu_v$, when taken into volatility jumps, is 6.8977, 9.4814, 8.5660 and 9.1504 for the SV, SVJ, SVCJ and SVSCJ, while their risk-neutral counterparts are 6.6516, 8.8939, 7.9864 and 9.0062. The speed of mean reversion can be compared with the risk-neutral values of 1.49 and 2.45 or 1.15 and 2.03 for the SV and SVJ found by Bates (2000) for S&P 500 futures options for the period 1988 to 1993 and Bakshi et al. (1997) for S&P 500 index options over the period 1988 to 1991. Based on daily SPX returns over 1953-1996, Andersen et al.'s (2002) EMM-based estimates are 3.93 and 3.7 for the SV and SVJ, while Bates' (2006) AML-based estimates are 5.94 and 4.38. In addition, the value of 3.29 for the SV is found by Nandi (1998) for S&P 500 index options over the period 1991 to 1992. The diffusive volatility risk premia, $\eta_v v_t$, are estimated to be negative across all models. The consistently negative estimates of η_v are –0.2460, –0.5875, –0.5797 and –0.3133 for the SV, SVJ, SVCJ and SVSCJ, indicating a substantial negative premium for diffusive volatility risk, consistent with the negative correlation between volatility and index returns, i.e. ρ being –0.46, –0.72, –0.52 and –0.27 across models. The figures of ρ for the SV and SVJ can be compared to (–0.57, –0.55) and (–0.58, –0.61) in Bates (2000) and Bates (2006), –0.64 and –0.57 in Bakshi et al. (1997), –0.60 and –0.62 in Andersen et al. (2002), and –0.79 for the SV in Nandi (1998). The joint values of (η_v, ρ) suggest that investors are averse to changes in diffusive volatility.

Next, the instantaneous covariance of Brownian increments in index returns and volatilities, i.e. $\sigma_v \rho v_t + (\lambda_0 + \lambda_1 v_t)(\mu_j \mu_v + 2\rho_j \mu_v^2)$, is on average –0.0012, –0.0030, –0.1570 and 0.0329 for the SV, SVJ, SVCJ and SVSCJ, when taken into account price and volatility jumps if any. Their risk-neutral counterparts for the SVCJ and SVSCJ are downward to the values of –0.3251 and 0.0005, respectively. This positive instantaneous covariance for the SVSCJ is caused by a relatively greater portion attributed to the jump shocks than the one due to diffusive shocks.

The variation coefficient under the respective probability measures, i.e.

$\sqrt{2\mu_v^2\lambda_0 / v_t + \sigma_v^2 + 2\mu_v^2\lambda_1}$ and $\sqrt{2(\mu_v^*)^2\lambda_0^* / v_t + \sigma_v^2 + 2(\mu_v^*)^2\lambda_1^*}$, determines how fat-tailed the (risk-neutral) distribution is and thereby the relative values of deep OTM options versus near-the-money options. The values of {0.3169, 0.4215, 10.2839, 11.3183} under *P* and {0.3169, 0.4215, 4.9006, 1.1691} under *Q* for the SV, SVJ, SVCJ and SVSCJ are found in this study compares with σ_v values of 0.742 or 0.315 for the SV and 0.378 or 0.244 for the SVJ in Bates (2000) or Bates (2006), 0.39 for the SV and 0.38 for the SVJ in Bakshi et al. (1997), 0.197 for the SV and 0.184 for the SVJ in Andersen et al. (2002), and 0.26 for the SV in Nandi (1998).

The jumps under the SVJ occur frequently, with λ_J =2.4223 and λ_J^* =3.1575, compared to the one under the SVCJ with λ_J =0.7852 and λ_J^* =0.7912 or the one under the SVSCJ with $(\lambda_0, \lambda_1) = (0.5951, 0.5369)$ and $(\lambda_0^*, \lambda_1^*) = (2.0047, 0.2262)$. The estimates of the mean jump sizes, μ_j and μ_j^*, are negative for all models and the SVJ achieves the greatest values in absolute magnitudes. The SVCJ and SVSCJ models have simultaneous correlated jumps in spot prices and volatility. The correlation, ρ_j, is found to be positive for both models and has greater value for the SVSCJ. The volatility jump-size is estimated to be $(\mu_v, \mu_v^*) = (0.7335, 0.4659)$ for the SVSCJ and $(\mu_v, \mu_v^*) = (0.5785, 0.4128)$ for the SVCJ. The *t* values for the parameter estimates of the price and volatility processes are significantly different from zero at the 5% significant level.

Table 2 Parameter Estimates and In-Sample Pricing Fits

This table shows the parameter estimates of SV, SVJ, SVCJ and SVSCJ models and their in-sample mean absolute pricing errors (MAE). The figures reported here are the averages over 24 non-overlapping estimation months from 21 April 2004 to 18 April 2006. The *t*-statistics calculated from the Newey-West (1987) heteroscedasticity and autocorrelation consistent standard errors are given in parentheses. ** and * denote statistical significance at 1% and 5% levels, respectively.

Parameters	SV	SVJ	SVCJ	SVSCJ
Sample size	24	24	24	24
MAE	5.5574	4.7589	4.6452	4.3785
η_S	0.5523**	0.5302**	0.5293**	0.5413**
	(6.7867)	(2.8422)	(3.6870)	(3.0529)
κ_v	6.8977**	9.4814**	8.4928**	9.3676**
	(4.7651)	(2.6767)	(7.1222)	(6.5577)
θ_v	0.0514**	0.0292**	0.0290**	0.0197*
	(9.3658)	(5.5443)	(2.5209)	(2.9084)
σ_v	0.3183**	0.4084**	0.3759**	0.1125**
	(2.6982)	(3.40287)	(5.9960)	(6.1823)
ρ	–0.4641**	–0.7212**	–0.4708**	–0.2712**
	(–8.6914)	(–3.6819)	(–7.9043)	(–5.3926)
η_v	–0.2460**	–0.5875**	–0.6385**	–0.3133**
	(–5.6678)	(–2.6116)	(–5.3140)	(–3.1039)
λ_0 or λ_J		2.4223*	0.8008**	0.5951**
		(2.4223)	(7.8518)	(8.4847)
λ_1				0.5369**
				(9.0157)
μ_j		–0.5912**	–0.0725**	–0.1543**
		(–5.9118)	(–6.9754)	(–5.9298)
σ_j		0.3704**	0.0028*	0.0811*
		(3.7039)	(2.2570)	(2.2128)
μ_v			0.5669**	0.7335**
			(5.7850)	(6.5548)
ρ_j			–0.1591*	0.3251*
			(2.3959)	(2.4224)
λ_0^* or λ_J^*		3.1575**	0.7328**	2.0047**
		(3.1575)	(7.9124)	(4.0386)
λ_1^*				0.2262**
				(5.4496)
μ_j^*		–0.4991**	–0.0704*	–0.1983**
		(–4.9913)	(–2.0516)	(–4.7909)
μ_v^*			0.3181**	0.4659**
			(4.1276)	(3.7838)

B. Out-of-Sample Pricing Fit

While a more complicated model will generally lead to a better in-sample fit, it will not necessarily perform better in out-of-sample pricing because any overfitting will be penalized. To test whether the additional parameters of the volatility models are economically informative for VIX futures pricing, this section provides a comparison of out-of-sample pricing. Three measures of goodness of fit were then employed to assess the out-of-sample pricing performance of the VIX futures pricing models on the SV, SVJ, SVCJ and SVSCJ specifications. These were the root mean squared pricing error (RMSE), the mean percentage pricing error (PE), and the mean absolute pricing error (MAE). Table 3 reports RMSE, PE, and MAE values for several categories according to time to expiration. Out of maturity combinations reported in Table 3, RMSE and MAE were lower for the SVSCJ (SVJ) model for the short-term (medium- and long-term) futures contracts. Thus, improvement was generated for short-dated VIX futures under the SVSCJ. From the panel of PE values, in contrast to the SV's results, the SVJ and SVCJ models substantially overpriced the short-term and medium-term VIX futures and underpriced the long-dated VIX futures. The PE values showed that the SVSCJ pricing errors across maturities were less than zero, indicating an overpricing fit for VIX futures.

Table 3 Out-of-Sample Root Mean Squared Pricing Errors (RMSE), Percentage Pricing Errors (PE), and Mean Absolute Pricing Errors (MAE)

		Days to Expiration		
Pricing errors		<60	60–180	>180
RMSE	SV	4.6125	5.0469	20.5714
	SVJ	4.2026	4.3946	5.5327
	SVCJ	4.9342	5.3136	6.2425
	SVSCJ	4.0424	4.6157	5.9926
PE	SV	0.0036	0.0071	–0.0879
	SVJ	–0.0133	–0.0302	0.0022
	SVCJ	–0.0224	–0.0367	0.0092
	SVSCJ	–0.0055	–0.0324	–0.0003
MAE	SV	3.5545	4.1741	14.5512
	SVJ	3.5064	4.1358	4.0024
	SVCJ	3.9794	5.0210	4.2364
	SVSCJ	3.4141	4.4326	4.5087

C. Market Prices of Price and Volatility Risks

This section focuses on the risk premiums that price four important sources of risks: diffusive price shocks, price jumps, diffusive volatility shocks and volatility jumps. Premia for the "conventional" return risks (Brownian price shocks) are parameterized by $\eta_S v_t$ for a constant coefficient η_S. This is similar to the risk-return trade-off in a CAPM framework. Premia for "volatility" risks, on the other hand, are not as transparent, since volatility is not directly traded as an asset. Because volatility is, itself, volatile, options may reflect an additional volatility risk premium. Volatility risk is priced via the extra term $\eta_v v_t$ in the risk-neutral dynamics of v_t. For a negative coefficient η_v, the time-*t* instantaneous mean growth rate of the volatility process v_t is, therefore, $\eta_v v_t$ higher under the risk-neutral measure *Q* than under the data-generating measure *P*. Since option prices respond positively to the volatility of the underlying price in this model, option prices are increasing in $-\eta_v$.

The market prices of risks are implicit in the dynamics of (ln*S*, ν) under the joint distribution associated with the risk-neutral measure *Q* and the data-generating measure *P*. Comparing the specification of the risk-neutral dynamics of (ln*S*, ν) with that of the data-generating process, one can obtain an intuitive understanding of how different risk factors are priced. Consequently, the instantaneous risk-neutral expected rate of index returns is $r-\delta-(\lambda_0^*+\lambda_1^* v_t)\kappa^* - v_t/2+(\lambda_0^*+\lambda_1^* v_t)(\mu_j^*+\rho_j\mu_v^*)$, while $\kappa_v^*(\theta_v^*-v_t)+\mu_v^*(\lambda_0^*+\lambda_1^* v_t)$ for the volatility process in the SVSCJ model. Focusing first on the time-*t* instantaneous risk premium associated with the diffusive price shock is $\eta_S v_t$, while that associated with the volatility shock is $\eta_v v_t$. Similarly, the time-*t* expected excess index return compensating for the jump risk whenever the underlying price jumps is $(\lambda_0+\lambda_1 v_t)(\mu_j+\rho_j\mu_v)-(\lambda_0^*+\lambda_1^* v_t)(\mu_j^*+\rho_j\mu_v^*)$, while the form of volatility jump-risk premia is $(\lambda_0+\lambda_1 v_t)\mu_v-(\lambda_0^*+\lambda_1^* v_t)\mu_v^*$ if the volatility jumps. The diffusive and jump risk premiums associated with price and volatility risks for the SV, SVJ and SVCJ models are obtained by restricting

$\lambda_0 = \lambda_0^* = \lambda_1 = \lambda_1^* = \rho_j = \mu_v = \mu_j = \mu_j^* = 0$, $\lambda_0 = \lambda_J, \lambda_0^* = \lambda_J^*, \lambda_1 = \lambda_1^* = \rho_j = \mu_v = 0$ and $\lambda_0 = \lambda_J, \lambda_0^* = \lambda_J^*, \lambda_1 = \lambda_1^* = 0$, respectively. Table 4 reports related results. We find the diffusive price-risk premia are remarkably similar across models, ranging from 0.0036 for the SVCJ to 0.0043 for the SV. The diffusive volatility-risk premia are on average negative and the SVCJ and SVJ have strong negative diffusive volatility-risk premium of –0.0044 and –0.0042, respectively, compared to the SV of –0.0026 and the SVSCJ of –0.0021. The results are consistent with the relative magnitudes of parameter estimates η_S and η_v across models in Table 2. Associated with the market prices of jump risks, the SVJ achieves the greatest price jump-risk premia of 0.1794, followed by the SVCJ with the value of 0.0507 and the SVSCJ with the value of 0.0011. In contrast, the SVSCJ has the greater volatility jump-risk premium of 0.2510 than the SVCJ with the value 0.1410. In summary, the diffusive and jump risk premia vary across models. In terms of volatility shocks, the SVSCJ has the greatest volatility jump risk premium, while the SVCJ has the strongest negative diffusive volatility-risk premium. Thus, the existence of volatility jumps is related to the estimation of volatility risk premia. For the price shocks, the SVJ has the greatest price jump-risk premium and the SV achieves the greatest diffusive price-risk premium.

Table 4 Market Prices of Diffusive and Jump Risks Implicit in the Joint VIX and Integrated Volatility

The diffusive price-risk premium is $\eta_S v_t$, while that associated with the volatility shock is $\eta_v v_t$. Similarly, the price jump-risk premium that prices the logarithmic price changes $d\ln S$ is $(\lambda_0 + \lambda_1 v_t)(\mu_j + \rho_j \mu_v) - (\lambda_0^* + \lambda_1^* v_t)(\mu_j^* + \rho_j \mu_v^*)$ for the SVSCJ, $\lambda_J(\mu_j + \rho_j \mu_v) - \lambda_J^*(\mu_j^* + \rho_j \mu_v^*)$ for the SVCJ, and $\lambda_J \mu_j - \lambda_J^* \mu_j^*$ for the SVJ. The form of volatility jump-risk premia associated with the volatility change dv is $(\lambda_0 + \lambda_1 v_t)\mu_v - (\lambda_0^* + \lambda_1^* v_t)\mu_v^*$ for the SVSCJ, and $\lambda_J \mu_v - \lambda_J^* \mu_v^*$ for the SVCJ. The instantaneous variance v_t is computed as $v_t = (\tau / a_{T-t}^* \zeta_1^*)\mathrm{VIX}_T^2 - \tau \zeta_2^* / a_{T-t}^* \zeta_1^* - b_{T-t}^* / a_{T-t}^*$ where $\zeta_1^* = 1 + 2\lambda_1^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)]$ and $\zeta_2^* = 2\lambda_0^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)]$ for the SVSCJ, $\zeta_1^* = 1$ and $\zeta_2^* = 2\lambda_J^*[\kappa^* - (\mu_j^* + \rho_j \mu_v^*)]$ for the SVCJ, $\zeta_1^* = 1$ and $\zeta_2^* = 2\lambda_J^*(\kappa^* - \mu_j^*)$ for the SVJ, and $\zeta_1^* = 1$ and $\zeta_2^* = 0$ for the SV. For the SVSCJ, $a_{T-t}^* = \{1 - \exp[-(\kappa_v^* - \lambda_1^* \mu_v^*)(T-t)]\} / (\kappa_v^* - \lambda_1^* \mu_v^*)$ and $b_{T-t}^* = (\kappa_v^* \theta_v^* + \lambda_0^* \mu_v^*)[(T-t) - a_{T-t}^*] / (\kappa_v^* - \lambda_1^* \mu_v^*)$. The values of a_{T-t}^* and b_{T-t}^* for the SVCJ are obtained by setting $\lambda_0^* = \lambda_J^*$ and $\lambda_1^* = 0$, while further restricting $\mu_v^* = 0$ we have a_{T-t}^* and b_{T-t}^* for both SVJ and SV. The sample covers the period of 21 April 2004–18 April 2006, in total 502 trading days.

Risk premiums and models		$d\ln S$	dv
Diffusive risk premium	SV	0.0043	–0.0026
	SVJ	0.0039	–0.0042
	SVCJ	0.0036	–0.0044
	SVSCJ	0.0041	–0.0021
Jump risk premium	SV	—	—
	SVJ	0.1794	—
	SVCJ	0.0507	0.1410
	SVSCJ	0.0011	0.2510

VII. CONCLUSION

The contributions of this paper to existing literature are (i) to propose closed-form solutions to the fair value of the VIX futures under alternate affine diffusion-jump stochastic volatility processes; (ii) to propose a methodology for an integrated analysis of spot and option prices, or equivalently integrated volatility and VIX; (iii) closed-form moment conditions for the total quadratic variations of index returns, i.e. our integrated volatilities, are derived for model estimation; and (iv) the market prices of risks are estimated. Four models are specified in this paper including the ones with jumps in returns and in volatility simultaneously. For the VIX futures valuation, our empirical results indicate that both of these jump components are important. The model with only diffusive stochastic volatility and jumps in returns, i.e. SVJ, outperforms for medium- and long-dated futures contracts, while additionally including a state-dependent component driving the conditional volatility of returns, which is rapidly moving, i.e. the SVSCJ, can further reduce out-of-sample pricing errors for short-dated futures. Parameters and risk premia also have important impacts on VIX futures prices. To obtain accurate estimates, this study follows the work of Chernov and Ghysels (2000), Eraker (2004), and Pan (2002) to use joint spot and option prices, or equivalently integrated volatilities and VIX, to perform our empirical work. The diffusive price-risk premiums are found to be positive, while the premiums associated with the diffusive volatility shocks are negative for all models. The price jump-risk premium is found to be positive and the SVJ achieves the greatest value, followed by the SVCJ and SVSCJ. The volatility jump-risk premium is also found to be positive, however, the SVSCJ has a greater value than the SVCJ. Overall, our results, on one hand, support the claim that a model with stochastic volatility and correlated state-dependent random jumps both in underlying returns and volatility is a better model for short-dated VIX futures. On the other hand, a model with stochastic volatility and random jumps a better alternative to other models for medium- and long-term VIX futures.

APPENDIX

A. VIX Squared in Terms of Fair Value of Price-Diffusion Variance

Breeden and Litzenberger (1978) demonstrate that the risk-neutral probability density function of the stock price S_T at time T is given by,

$$p^*(S_T,T;S_t,t) = \left.\frac{\partial^2\tilde{C}(S_t,X,t,T)}{\partial X^2}\right|_{X=S_T} = \left.\frac{\partial^2\tilde{P}(S_t,X,t,T)}{\partial X^2}\right|_{X=S_T}$$

where $\tilde{C}$ and $\tilde{P}$ represent undiscounted call and put prices respectively. The value of a claim with a generalized terminal payoff $g(S_T)$ is then calculated as,

$$\mathrm{E}^Q[g(S_T)\,|\,S_t] = \int_0^\infty dX\ p^*(X,T;S_t,t)g(X) = \int_0^F dX\,\frac{\partial^2\tilde{P}}{\partial X^2}g(X) + \int_F^\infty dX\,\frac{\partial^2\tilde{C}}{\partial X^2}g(X)$$

where $F = S_t e^{(r-\delta)(T-t)}$ denoting the forward price of the stock with a risk-free interest rate r and a dividend yield δ, and Q being the risk-neutral probability measure. Integrating by parts twice and using the put-call parity relation $\tilde{C} - \tilde{P} = F - X$ give,

$$
\begin{aligned}
\mathrm{E}^Q[g(S_T)\,|\,S_t] &= \frac{\partial \tilde{P}}{\partial X} g(X)\Big|_0^F - \int_0^F dX\, \frac{\partial \tilde{P}}{\partial X} g'(X) + \frac{\partial \tilde{C}}{\partial X} g(X)\Big|_F^\infty - \int_F^\infty dX\, \frac{\partial \tilde{C}}{\partial X} g'(X) \\
&= g(F) - \int_0^F dX\, \frac{\partial \tilde{P}}{\partial X} g'(X) + \int_F^\infty dX\, \frac{\partial \tilde{C}}{\partial X} g'(X) \\
&= g(F) - \tilde{P}(X) g'(X)\Big|_0^F + \int_0^F dX\, \tilde{P}(X) g''(X) \\
&\quad - \tilde{C}(X) g'(X)\Big|_F^\infty + \int_F^\infty dX\, \tilde{C}(X) g''(X) \\
&= g(F) + \int_0^F dX\, \tilde{P}(X) g''(X) + \int_F^\infty dX\, \tilde{C}(X) g''(X)
\end{aligned}
\tag{A.1}
$$

By letting $t \to T$ in equation (A.1), any European-style twice differentiable payoff may be replicated using a portfolio of European options with strikes from 0 to ∞ with the weight of each option equal to the second derivative of the payoff at the strike price of the option. This portfolio of European options is a static hedge because the weight of an option with a particular strike depends only on the strike price and the form of the payoff function and not on time or the level of the stock price. Note that equation (A.1) is completely model-independent. Now consider a log contract, $\ln(S_T / F)$. Then $g''(X) = -S_T^{-2}\big|_{S_T = X}$ and it follows from equation (A.1) that

$$
\mathrm{E}_t^Q\left[\ln\left(\frac{S_T}{F}\right)\right] = -\int_0^F \frac{dX}{X^2} \tilde{P}(X) - \int_F^\infty \frac{dX}{X^2} \tilde{C}(X) \quad .
$$

By the definition of VIX squared, we have

$$
\mathrm{VIX}_t^2 \equiv \frac{2}{\tau} \times \left[\int_0^F \frac{dX}{X^2} \tilde{P}(X) + \int_F^\infty \frac{dX}{X^2} \tilde{C}(X)\right] = -\frac{2}{\tau} \mathrm{E}_t^Q\left[\ln\left(\frac{S_T}{F}\right)\right] \tag{A.2}
$$

where $\tau = T - t = 30/365$. Thus, VIX squared can be expressed in terms of the risk-neutral expectation of the log contract. Different dynamics for the index price S_t will result in various expressions for VIX squared. The stochastic volatility with state-dependent and correlated jumps (SVSCJ) in both index returns and volatility is the most general process considered in this paper. The dynamics of $(\ln S_t, \nu_t)$ under the risk-neutral measure Q are of the form,

$$
\begin{aligned}
d \ln S_t &= \left[r - \delta - \left(\lambda_0^* + \lambda_1^* \nu_t\right)\kappa^* - \frac{1}{2}\nu_t\right] dt + \sqrt{\nu_t}\, d\omega_{S,t}^* + z_S^* dN_t^* \\
d\nu_t &= \kappa_\nu^*\left(\theta_\nu^* - \nu_t\right) dt + \sigma_\nu \sqrt{\nu_t}\, d\omega_{\nu,t}^* + z_\nu^* dN_t^*
\end{aligned}
$$

where the correlated Brownian motions $d\omega^*_{S,t}$ and $d\omega^*_{v,t}$ are independent, respectively, of the compounded Poisson processes $z^*_S dN^*_t$ and $z^*_v dN^*_t$. The jumps arrive at the exponential rate of $(\lambda^*_0 + \lambda^*_1 v_t)dt$ with jumps in volatility driven by an exponential distribution, $z^*_v \sim \exp\left(\mu^*_v\right)$, and jumps in asset log-prices normally distributed conditional on the realization of z^*_v, formally $z^*_S \left| z^*_v \sim N\left(\mu^*_j + \rho_j z^*_v, \sigma^2_j\right)\right.$. κ^* is the price-jump size mean for the percentage price change . The diffusive component of v_t is governed by Heston's (1993) stochastic volatility process. Applying Itô's Lemma to $\ln(S_T / F)$ under the SVSCJ and comparing with equation (A.2), we have

$$
\begin{aligned}
\mathrm{VIX}^2_t &\equiv -\frac{2}{\tau}\times \mathrm{E}^Q_t\left[\ln\left(\frac{S_T}{F}\right)\right] = 2(r-\delta) - \frac{2}{\tau}\mathrm{E}^Q_t\left(\ln S_T - \ln S_t\right) \\
&= -\frac{2}{\tau}\mathrm{E}^Q_t\left\{\int_t^T [-\frac{1}{2}v_u du - (\lambda^*_0 + \lambda^*_1 v_u)\kappa^* du + (\lambda^*_0 + \lambda^*_1 v_u)(\mu^*_j + \rho_j \mu^*_v)du]\right\} \\
&= 2\lambda^*_0[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)] + \frac{1}{\tau}\left\{1 + 2\lambda^*_1[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)]\right\}\times \mathrm{E}^Q_t\left(\int_t^T v_u du\right) \qquad \text{(A.3)} \\
&= 2\lambda^*_0[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)] + \frac{1}{\tau}\left\{1 + 2\lambda^*_1[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)]\right\}\times \mathrm{E}^Q_t(\upsilon^c_{t,T}) \\
&= 2\lambda^*_0[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)] + \frac{1}{\tau}\left\{1 + 2\lambda^*_1[\kappa^* - (\mu^*_j + \rho_j \mu^*_v)]\right\}\times \mathrm{E}^Q_t(\upsilon_{t,T} - \upsilon^J_{t,T})
\end{aligned}
$$

where $\upsilon_{t,T} = \int_t^T \mathrm{var}_u(d\ln S_u)$ is the total quadratic variation of $\ln S$ over the period $[t,T]$ that reconciles the spirit of integrated variance constructed in the literature. It consists of the price-diffusion part, $\upsilon^c_{t,T} = \int_t^T v_u du$, and the price-jump part, $\upsilon^J_{t,T} = \int_t^T v_J du$ where $\mathrm{E}^Q_t(v_J) = (\lambda^*_0 + \lambda^*_J v_t)[(\mu^*_j + \rho_j \mu^*_v)^2 + \sigma^2_j + \rho^2_j(\mu^*_v)^2]dt$.[12] Thus, the fair value of total quadratic variation of $\ln S$ after subtracting the price-jump component and also adjusting for price-jump mean is explicitly given by the value of an infinite strip of European options in a completely model-independent way. Note that the correlation between Brownian shocks in index price and volatility,

[12] The realized variance of ln*S* over [*t*,*T*] is calculated as $\mathrm{var}_t\left(\int_t^T d\ln S_u\right) = \mathrm{var}_t\left(\ln S_T - \ln S_t\right)$, which is a constant and thus different from the random variable-type quadratic variation of ln*S* over [*t*,*T*], i.e. $\upsilon_{t,T} = \int_t^T \mathrm{var}_u(d\ln S_u)$.

i.e. $\rho\, dt = \mathrm{corr}(d\omega^*_{S,t}, d\omega^*_{\nu,t})$, does not enter the VIX formula explicitly.

As a result of changing the state-dependent jump arrival frequency, $\lambda_0^* + \lambda_1^* \nu_t$, into a constant, λ_J^*, i.e. restricting $\lambda_0^* = \lambda_J^*$ and $\lambda_1^* = 0$ in the SVSCJ, the VIX squared for the SVCJ model becomes

$$\mathrm{VIX}_t^2 \equiv 2\lambda_J^*[\kappa^* - (\mu_j^* + \rho_j \mu_\nu^*)] + \frac{1}{\tau}\mathrm{E}_t^Q(\upsilon_{t,T}^c) \tag{A.4}$$

where $\upsilon_{t,T}^c = \upsilon_{t,T} - \upsilon_{t,T}^J$ and $\upsilon_{t,T}^J = \lambda_J^*[(\mu_j^* + \rho_j \mu_\nu^*)^2 + (\sigma_j^2 + \rho_j^2 \mu_\nu^2)]\tau$.

Further, the SVJ model with the volatility driven only by diffusive component leads to the VIX squared shown as follows,

$$\mathrm{VIX}_t^2 \equiv 2\lambda_J^*(\kappa^* - \mu_j^*) + \frac{1}{\tau}\mathrm{E}_t^Q(\upsilon_{t,T}^c) \tag{A.5}$$

where $\upsilon_{t,T}^c = \upsilon_{t,T} - \upsilon_{t,T}^J$ and $\upsilon_{t,T}^J = \lambda_J^*[(\mu_j^*)^2 + \sigma_j^2]\tau$.

Finally, the VIX squared under diffusive dynamics of ln*S* with mean-reverting square root stochastic volatility (the SV model) is given as the total quadratic variation of ln*S*, i.e. $\upsilon_{t,T}$, which is fully contributed by the smooth part, $\upsilon_{t,T}^c$.

$$\mathrm{VIX}_t^2 \equiv \frac{1}{\tau}\mathrm{E}_t^Q(\upsilon_{t,T}^c) = \frac{1}{\tau}\mathrm{E}_t^Q(\upsilon_{t,T}) \tag{A.6}$$

B. Conditional Moments of Total Quadratic Variation of Log-Price

B.1 Conditional Mean

The first and second conditional moments of the point-in-time volatility, under the physical probability measure *P*, driven by the price-diffusion component for the SVSCJ model satisfy,

$$\mathrm{E}_t^P(\nu_T) = \alpha_{T-t,\mathrm{SVSCJ}}\nu_t + \beta_{T-t,\mathrm{SVSCJ}} \tag{B.1}$$

$$\begin{aligned}\mathrm{E}_t^P(\nu_T^2) = {} & \alpha_{T-t,\mathrm{SVSCJ}}^2 \nu_t^2 + (C_{T-t,\mathrm{SVSCJ}} + 2\alpha_{T-t,\mathrm{SVSCJ}}\beta_{T-t,\mathrm{SVSCJ}})\nu_t \\ & + \beta_{T-t,\mathrm{SVSCJ}}^2 + D_{T-t,\mathrm{SVSCJ}}\end{aligned} \tag{B.2}$$

$$\begin{aligned}\mathrm{E}_t^P(\ln S_T) &= \ln S_t + [r-\delta-\lambda_0^*\kappa^* + \lambda_0(\mu_j+\rho_j\mu_\nu)](T-t)\\ &\quad + [\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j+\rho_j\mu_\nu)][a_{T-t,\text{SVSCJ}}\nu_t + b_{T-t,\text{SVSCJ}}]\end{aligned} \tag{B.3}$$

where $\alpha_{T-t,\text{SVSCJ}} = e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}$

$$\beta_{T-t,\text{SVSCJ}} = \left(\frac{\kappa_\nu\theta_\nu+\lambda_0\mu_\nu}{\kappa_\nu-\lambda_1\mu_\nu}\right)[1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}]$$

$$D_{T-t,\text{SVSCJ}} = \frac{(\sigma_\nu^2+2\lambda_1\mu_\nu^2)}{2(\kappa_\nu-\lambda_1\mu_\nu)}\left(\frac{\kappa_\nu\theta_\nu+\lambda_0\mu_\nu}{\kappa_\nu-\lambda_1\mu_\nu}\right)[1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}]^2 + \frac{\lambda_0\mu_\nu^2}{\kappa_\nu-\lambda_1\mu_\nu}[1-e^{-2(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}]$$

$$C_{T-t,\text{SVSCJ}} = \frac{(\sigma_\nu^2+2\lambda_1\mu_\nu^2)}{(\kappa_\nu-\lambda_1\mu_\nu)}[e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)} - e^{-2(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}]$$

$$a_{T-t,\text{SVSCJ}} = \frac{1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}}{\kappa_\nu-\lambda_1\mu_\nu}$$

$$b_{T-t,\text{SVSCJ}} = \left(\frac{\kappa_\nu\theta_\nu+\lambda_0\mu_\nu}{\kappa_\nu-\lambda_1\mu_\nu}\right)\left[(T-t)-\left(\frac{1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}}{\kappa_\nu-\lambda_1\mu_\nu}\right)\right]$$

The conditional mean of the total quadratic variation of the log-price for the SVSCJ model satisfies,

$$\begin{aligned}\mathrm{E}_t^P(\upsilon_{t,T}) &= \mathrm{E}_t^P(\upsilon_{t,T}^c) + \mathrm{E}_t^P(\upsilon_{t,T}^J)\\ &= \lambda_0[(\mu_j+\rho_j\mu_\nu)^2 + \sigma_j^2 + \rho_j^2\mu_\nu^2](T-t)\\ &\quad + \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times(a_{T-t,\text{SVSCJ}}\nu_t + b_{T-t,\text{SVSCJ}})\end{aligned} \tag{B.4}$$

where $a_{T-t,\text{SVSCJ}} = \dfrac{1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}}{\kappa_\nu-\lambda_1\mu_\nu}$

$$b_{T-t,\text{SVSCJ}} = \left(\frac{\kappa_\nu\theta_\nu+\lambda_0\mu_\nu}{\kappa_\nu-\lambda_1\mu_\nu}\right)\left[(T-t)-\left(\frac{1-e^{-(\kappa_\nu-\lambda_1\mu_\nu)(T-t)}}{\kappa_\nu-\lambda_1\mu_\nu}\right)\right] \quad .$$

Focusing on the one-day horizon, i.e. $\Delta = 1/365$, it follows that,

$$
\begin{aligned}
\mathrm{E}_t^P[\mathrm{E}_{t+\Delta}^P(\upsilon_{t+\Delta,t+2\Delta})] &= \lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
&\quad +\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times[a_{\Delta,\mathrm{SVSCJ}}\mathrm{E}_t^P(\nu_{t+\Delta})+b_{\Delta,\mathrm{SVSCJ}}] \\
&= \alpha_{\Delta,\mathrm{SVSCJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta})+(1-\alpha_{\Delta,\mathrm{SVSCJ}})\times\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
&\quad +\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times\beta_{\Delta,\mathrm{SVSCJ}}\,\Delta
\end{aligned}
$$

By the Law of Iterated Expectations or reduction in information sets (Meddahi and Renault, 2004),

$$
\begin{aligned}
&\mathrm{E}_t^P[\mathrm{E}_{t+\Delta}^P(\upsilon_{t+\Delta,t+2\Delta})] = \mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}) \\
&= \alpha_{\Delta,\mathrm{SVSCJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta})+(1-\alpha_{\Delta,\mathrm{SVSCJ}})\times\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
&\quad +\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}\times\beta_{\Delta,\mathrm{SVSCJ}}\,\Delta
\end{aligned}
\tag{B.5}
$$

By setting $\lambda_0=\lambda_J$ and $\lambda_1=0$, we have the conditional mean of total quadratic variation of log-price for the SVCJ,

$$
\begin{aligned}
\mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}) &= \alpha_{\Delta,\mathrm{SVCJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta}) \\
&\quad +(1-\alpha_{\Delta,\mathrm{SVCJ}})\times\lambda_J[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta+\beta_{\Delta,\mathrm{SVCJ}}\,\Delta
\end{aligned}
\tag{B.6}
$$

where $\alpha_{\Delta,\mathrm{SVCJ}}=e^{-\kappa_\nu\Delta}$ and $\beta_{\Delta,\mathrm{SVCJ}}=\left(\dfrac{\kappa_\nu\theta_\nu+\lambda_J\mu_\nu}{\kappa_\nu}\right)(1-e^{-\kappa_\nu\Delta})$.

Further additionally restricting $\mu_\nu=0$ in (B.6), the conditional mean of integrated volatility for the SV and SVJ becomes,

$$
\begin{aligned}
\mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}) &= \alpha_{\Delta,\mathrm{SV\ or\ SVJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta}) \\
&\quad +(1-\alpha_{\Delta,\mathrm{SV\ or\ SVJ}})\times\lambda_J(\mu_j^2+\sigma_j^2)\Delta+\beta_{\Delta,\mathrm{SV\ or\ SVJ}}\,\Delta
\end{aligned}
\tag{B.7}
$$

where $\alpha_{\Delta,\mathrm{SV\ or\ SVJ}}=e^{-\kappa_\nu\Delta}$ and $\beta_{\Delta,\mathrm{SV\ or\ SVJ}}=\theta_\nu(1-e^{-\kappa_\nu\Delta})$.

Replacing ν_t in (B.3) with (B.4), we have

$$\mathrm{E}_t^P(\ln S_{t+\Delta}) = \ln S_t + \frac{[\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]}{\{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}} \times \mathrm{E}_t^P(\upsilon_{t,t+\Delta})$$
$$+ [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v)]\Delta \tag{B.8}$$
$$- \frac{[\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)] \times \lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\Delta}{\{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}}$$

B.2 Conditional Variance of Integrated Volatility

The stochastic differential equation for $\mathrm{E}_t^P(\upsilon_{t,T})$ could be generated as a function of ν_t by applying Itô's lemma to the affine equation in the SVSCJ,

$$d\mathrm{E}_t^P(\upsilon_{t,T}) = \frac{\partial \mathrm{E}_t^P(\upsilon_{t,T})}{\partial t} dt + \frac{\partial \mathrm{E}_t^P(\upsilon_{t,T})}{\partial \nu} d\nu_t$$
$$= -\left\{\lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2] + \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\} \times \nu_t\right\} dt$$
$$+ \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}$$
$$\times \left\{a_{T-t,\mathrm{SVSCJ}}\sigma_v\sqrt{\nu_t}\,d\omega_{v,t} + a_{T-t,\mathrm{SVSCJ}}\,[z_v dN_t - \mu_v(\lambda_0 + \lambda_1\nu_t)dt]\right\} \tag{B.9}$$

Now fix the upper limit *T*, and let the lower limit *t* be time-varying. The Itô integral implied by (B.7) then takes the form,

$$\mathrm{E}_T^P(\upsilon_{T,T}) = \mathrm{E}_t^P(\upsilon_{t,T}) - \upsilon_{t,T}$$
$$+ \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}$$
$$\times \left\{\sigma_v \int_t^T a_{T-u,\mathrm{SVSCJ}}\sqrt{\nu_u}\,d\omega_{v,u} + \int_t^T a_{T-u,\mathrm{SVSCJ}}[z_v dN_u - \mu_v(\lambda_0 + \lambda_1\nu_u)du]\right\}$$

where

$$\upsilon_{t,T} = \lambda_0[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2](T-t) + \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\} \times \upsilon_{t,T}^c$$

Since $\mathrm{E}_T^P(\upsilon_{T,T}) = 0$, which implies that

$$\upsilon_{t,T} - \mathrm{E}_t^P(\upsilon_{t,T}) = \{1+\lambda_1[(\mu_j + \rho_j\mu_v)^2 + \sigma_j^2 + \rho_j^2\mu_v^2]\}$$
$$\times \left\{\sigma_v \int_t^T a_{T-u,\mathrm{SVSCJ}}\sqrt{\nu_u}\,d\omega_{v,u} + \int_t^T a_{T-u,\mathrm{SVSCJ}}[z_v dN_u - \mu_v(\lambda_0 + \lambda_1\nu_u)du]\right\}$$

By standard arguments and the substitution of equation (B.1), we have

$$
\begin{aligned}
\mathrm{var}_t^P(\upsilon_{t,T}) &= \mathrm{E}_t^P[\upsilon_{t,T} - \mathrm{E}_t^P(\upsilon_{t,T})]^2 \\
&= \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times \left\{A_{T-t,\mathrm{SVSCJ}} \times \nu_t + B_{T-t,\mathrm{SVSCJ}}\right\}
\end{aligned} \tag{B.10}
$$

where

$$
A_{T-t,\mathrm{SVSCJ}} = \frac{(\sigma_\nu^2+2\mu_\nu^2\lambda_1)}{(\kappa_\nu-\lambda_1\mu_\nu)^2}\left[\frac{1}{(\kappa_\nu-\lambda_1\mu_\nu)} - 2\alpha_{T-t,\mathrm{SVSCJ}} \times (T-t) - \frac{\alpha_{T-t,\mathrm{SVSCJ}}^2}{(\kappa_\nu-\lambda_1\mu_\nu)}\right]
$$

$$
\begin{aligned}
B_{T-t,\mathrm{SVSCJ}} &= 2\mu_\nu^2\lambda_0\left[\frac{(T-t)}{(\kappa_\nu-\lambda_1\mu_\nu)^2} - \frac{(\alpha_{T-t,\mathrm{SVSCJ}}^2 - 4\alpha_{T-t,\mathrm{SVSCJ}}+3)}{2(\kappa_\nu-\lambda_1\mu_\nu)^3}\right] \\
&+ \frac{(\sigma_\nu^2+2\mu_\nu^2\lambda_1)}{(\kappa_\nu-\lambda_1\mu_\nu)^2}\left(\frac{\kappa_\nu\theta_\nu+\lambda_0\mu_\nu}{\kappa_\nu-\lambda_1\mu_\nu}\right)\times\left[(T-t)(1+2\alpha_{T-t,\mathrm{SVSCJ}}) + \frac{(\alpha_{T-t,\mathrm{SVSCJ}}^2+4\alpha_{T-t,\mathrm{SVSCJ}}-5)}{2(\kappa_\nu-\lambda_1\mu_\nu)}\right]
\end{aligned}
$$

Focusing on the one-day horizon, (B.10) and (B.4) imply that

$$
\begin{aligned}
&\mathrm{E}_t^P(\upsilon_{t,t+\Delta}^2) = \mathrm{var}_t^P(\upsilon_{t,t+\Delta}) + [\mathrm{E}_t^P(\upsilon_{t,t+\Delta})]^2 \\
&= \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times a_{\Delta,\mathrm{SVSCJ}}^2 \times \nu_t^2 \\
&+ \left\{\begin{array}{l}
\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times (A_{\Delta,\mathrm{SVSCJ}} + 2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}}) \\
+ 2\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
\quad \times \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \times a_{\Delta,\mathrm{SVSCJ}}
\end{array}\right\} \times \nu_t \\
&+ \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times B_{\Delta,\mathrm{SVSCJ}} \\
&+ \left\{\begin{array}{l}
\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
+ \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \times b_{\Delta,\mathrm{SVSCJ}}
\end{array}\right\}^2
\end{aligned} \tag{B.11}
$$

Leading the arguments by one period and applying the Law of Iterated Expectation produces,

$$
\begin{aligned}
&\mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}^2) = \mathrm{E}_t^P[\mathrm{E}_{t+\Delta}^P(\upsilon_{t+\Delta,t+2\Delta}^2)] \\
&= \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times a_{\Delta,\mathrm{SVSCJ}}^2 \times \mathrm{E}_t^P(\nu_{t+\Delta}^2) \\
&+ \left\{\begin{array}{l}
\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times (A_{\Delta,\mathrm{SVSCJ}} + 2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}}) \\
+ 2\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
\quad \times \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \times a_{\Delta,\mathrm{SVSCJ}}
\end{array}\right\} \times \mathrm{E}_t^P(\nu_{t+\Delta}) \\
&+ \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \times B_{\Delta,\mathrm{SVSCJ}} \\
&+ \left\{\begin{array}{l}
\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta \\
+ \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \times b_{\Delta,\mathrm{SVSCJ}}
\end{array}\right\}^2
\end{aligned}
$$

Now substituting for $\mathrm{E}_t^P(\nu_{t+\Delta})$ by equation (B.1) and $\mathrm{E}_t^P(\nu_{t+\Delta}^2)$ by equation (B.2), and reversely substituting out ν_t^2 by equation (B.11) and ν_t by equation (B.4), it follows that

$$\mathrm{E}_t^P(\upsilon_{t+\Delta,t+2\Delta}^2) = \mathrm{H}_{\Delta,\mathrm{SVSCJ}} \times \mathrm{E}_t^P(\upsilon_{t,t+\Delta}^2) + \mathrm{I}_{\Delta,\mathrm{SVSCJ}} \times \mathrm{E}_t^P(\upsilon_{t,t+\Delta}) + \mathrm{J}_{\Delta,\mathrm{SVSCJ}} \tag{B.12}$$

where

$$\mathrm{H}_{\Delta,\mathrm{SVSCJ}} = \alpha_{\Delta,\mathrm{SVSCJ}}^2$$

$$\begin{aligned}\mathrm{I}_{\Delta,\mathrm{SVSCJ}} = & \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \\ & \times\frac{1}{a_{\Delta,\mathrm{SVSCJ}}}\times\begin{bmatrix} a_{\Delta,\mathrm{SVSCJ}}^2\times(C_{\Delta,\mathrm{SVSCJ}}+2\alpha_{\Delta,\mathrm{SVSCJ}}\beta_{\Delta,\mathrm{SVSCJ}}) \\ +(\alpha_{\Delta,\mathrm{SVSCJ}}-\alpha_{\Delta,\mathrm{SVSCJ}}^2)\times(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\end{bmatrix} \\ & +2\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta\times(\alpha_{\Delta,\mathrm{SVSCJ}}-\alpha_{\Delta,\mathrm{SVSCJ}}^2)\end{aligned}$$

$$\begin{aligned}& \mathrm{J}_{\Delta,\mathrm{SVSCJ}} \\ & = \{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\}^2 \\ & \quad \times\begin{bmatrix} -\frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}}\times\begin{bmatrix} a_{\Delta,\mathrm{SVSCJ}}^2(C_{\Delta,\mathrm{SVSCJ}}+2\alpha_{\Delta,\mathrm{SVSCJ}}\beta_{\Delta,\mathrm{SVSCJ}}) \\ +(\alpha_{\Delta,\mathrm{SVSCJ}}-\alpha_{\Delta,\mathrm{SVSCJ}}^2)(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\end{bmatrix} \\ +(1-\alpha_{\Delta,\mathrm{SVSCJ}}^2)(B_{\Delta,\mathrm{SVSCJ}}+b_{\Delta,\mathrm{SVSCJ}}^2) \\ +a_{\Delta,\mathrm{SVSCJ}}^2(\beta_{\Delta,\mathrm{SVSCJ}}^2+D_{\Delta,\mathrm{SVSCJ}}) \\ +\beta_{\Delta,\mathrm{SVSCJ}}(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\end{bmatrix} \\ & +\{\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta\}^2\times(1-\alpha_{\Delta,\mathrm{SVSCJ}})^2 \\ & +\lambda_0[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\Delta\times\{1+\lambda_1[(\mu_j+\rho_j\mu_\nu)^2+\sigma_j^2+\rho_j^2\mu_\nu^2]\} \\ & \quad \times\begin{bmatrix} 2b_{\Delta,\mathrm{SVSCJ}}(1-\alpha_{\Delta,\mathrm{SVSCJ}})+2a_{\Delta,\mathrm{SVSCJ}}\beta_{\Delta,\mathrm{SVSCJ}} \\ -\frac{1}{a_{\Delta,\mathrm{SVSCJ}}}\times\begin{bmatrix} a_{\Delta,\mathrm{SVSCJ}}^2(C_{\Delta,\mathrm{SVSCJ}}+2\alpha_{\Delta,\mathrm{SVSCJ}}\beta_{\Delta,\mathrm{SVSCJ}}) \\ +(\alpha_{\Delta,\mathrm{SVSCJ}}-\alpha_{\Delta,\mathrm{SVSCJ}}^2)(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\end{bmatrix}\end{bmatrix}\end{aligned}$$

By setting $\lambda_0 = \lambda_J$ and $\lambda_1 = 0$ in equation (B.12), we have the conditional variance of integrated volatility for the SVCJ model. Additionally restricting $\mu_\nu = 0$ and further $\lambda_J = 0$ in the SVCJ model, $\mathrm{E}_t^P[(\upsilon_{t+\Delta,t+2\Delta}^c)^2]$ for the SVJ and SV models, respectively, are obtained.

B.3 Cross Moment Condition of Leverage Effect

Apply Itô's lemma to $\ln S_t \times v_t$, express the product as a stochastic differential equation, and take the conditional expectation of $\ln S_T \times v_T$ under the SVSCJ,

$$
\begin{aligned}
&\mathrm{E}_t^P(\ln S_T \times v_T) \\
&= -(\kappa_v - \lambda_1\mu_v)\times\int_t^T \mathrm{E}_t^P(\ln S_u \times v_u)\,du + \ln S_t \times v_t + (\kappa_v\theta_v + \lambda_0\mu_v)\times\int_t^T \mathrm{E}_t^P(\ln S_u)du \\
&\quad + [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v) + \sigma_v\rho]\times\int_t^T \mathrm{E}_t^P(v_u)du \\
&\quad + [\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]\times\int_t^T \mathrm{E}_t^P(v_u^2)\,du + \int_t^T \frac{\mathrm{E}_t^P(z_S dN_u \times z_v dN_u)}{du}du
\end{aligned}
$$

Interchanging the two integration operators and taking derivatives of both sides with respect to the upper time limit, then yields the first-order linear ordinary differential equation,

$$
\begin{aligned}
\frac{d\mathrm{E}_t^P(\ln S_s \times v_s)}{ds} &= -(\kappa_v - \lambda_1\mu_v)\times \mathrm{E}_t^P(\ln S_s \times v_s) + (\kappa_v\theta_v + \lambda_0\mu_v)\times \mathrm{E}_t^P(\ln S_s) \\
&\quad + [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v) + \sigma_v\rho]\times \mathrm{E}_t^P(v_s) \\
&\quad + [\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]\times \mathrm{E}_t^P(v_s^2) + \frac{\mathrm{E}_t^P(z_S dN_s \times z_v dN_s)}{ds}
\end{aligned}
$$

By applying Itô's lemma to $\exp[(\kappa_v - \lambda_1\mu_v)\,s]\times \mathrm{E}_t^P(\ln S_s \times v_s)$, we have

$$
\begin{aligned}
d[e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(\ln S_s \times v_s)] &= (\kappa_v\theta_v + \lambda_0\mu_v)\times e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(\ln S_s)ds \\
&\quad + [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v) + \sigma_v\rho]\times e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(v_s)ds \\
&\quad + [\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]\times e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(v_s^2)ds \\
&\quad + e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(z_S dN_s \times z_v dN_s)
\end{aligned}
$$

Now, we can compute $\mathrm{E}_t^P(\ln S_T \times v_T)$ for $T \geq t$,

$$
\begin{aligned}
&\mathrm{E}_t^P(\ln S_T \times v_T) \\
&= e^{-(\kappa_v-\lambda_1\mu_v)(T-t)}\times(\ln S_t \times v_t) + (\kappa_v\theta_v + \lambda_0\mu_v)e^{-(\kappa_v-\lambda_1\mu_v)T}\times\int_t^T e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(\ln S_s)ds \\
&\quad + [r - \delta - \lambda_0^*\kappa^* + \lambda_0(\mu_j + \rho_j\mu_v) + \sigma_v\rho]e^{-(\kappa_v-\lambda_1\mu_v)T}\int_t^T e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(v_s)ds \\
&\quad + [\eta_S - \lambda_1^*\kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j\mu_v)]e^{-(\kappa_v-\lambda_1\mu_v)T}\int_t^T e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(v_s^2)ds \\
&\quad + e^{-(\kappa_v-\lambda_1\mu_v)T}\int_t^T e^{(\kappa_v-\lambda_1\mu_v)s}\mathrm{E}_t^P(z_S dN_s \times z_v dN_s)
\end{aligned}
$$

Substituting existing solutions for $\mathrm{E}_t^P(\ln S_T)$, $\mathrm{E}_t^P(\nu_T)$, $\mathrm{E}_t^P(\nu_T^2)$ and $\mathrm{E}_t^P(z_S dN_T \times z_\nu dN_T)$, the solution is given by,

$$\begin{aligned}\mathrm{E}_t^P(\ln S_T \times \nu_T) = {}& \alpha_{T-t,\mathrm{SVSCJ}} \times (\ln S_t \times \nu_t) + (\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu) a_{T-t,\mathrm{SVSCJ}} \times \ln S_t \\ & + \mathrm{K}_{T-t,\mathrm{SVSCJ}} \times \nu_t^2 + \mathrm{L}_{T-t,\mathrm{SVSCJ}} \times \nu_t + \mathrm{M}_{T-t,\mathrm{SVSCJ}}\end{aligned} \tag{B.13}$$

where

$$\mathrm{K}_{T-t,\mathrm{SVSCJ}} = [\eta_S - \lambda_1^* \kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j \mu_\nu)] \frac{C_{T-t,\mathrm{SVSCJ}}}{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)}$$

$$\begin{aligned}&\mathrm{L}_{T-t,\mathrm{SVSCJ}} = \\ &[r - \delta - \lambda_0^* \kappa^* + \lambda_0(\mu_j + \rho_j \mu_\nu) + \sigma_\nu \rho + \lambda_1(\mu_j \mu_\nu + 2\rho_j \mu_\nu^2)] \times \alpha_{T-t,\mathrm{SVSCJ}}(T-t) \\ &+ [\eta_S - \lambda_1^* \kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j \mu_\nu)] \\ &\qquad \times \left\{ \begin{aligned} & \alpha_{T-t,\mathrm{SVSCJ}}(T-t) \left[\frac{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)}{(\kappa_\nu - \lambda_1 \mu_\nu)} + \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) \right] - \frac{C_{T-t,\mathrm{SVSCJ}}}{(\kappa_\nu - \lambda_1 \mu_\nu)} \\ & + \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) \left[a_{T-t,\mathrm{SVSCJ}} - \frac{2C_{T-t,\mathrm{SVSCJ}}}{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)} \right] \end{aligned} \right\}\end{aligned}$$

$$\begin{aligned}&\mathrm{M}_{T-t,\mathrm{SVSCJ}} = \\ &[r - \delta - \lambda_0^* \kappa^* + \lambda_0(\mu_j + \rho_j \mu_\nu) + \sigma_\nu \rho + \lambda_1(\mu_j \mu_\nu + 2\rho_j \mu_\nu^2)] \\ &\qquad \times \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) [a_{T-t,\mathrm{SVSCJ}} - \alpha_{T-t,\mathrm{SVSCJ}}(T-t)] \\ &+ [\eta_S - \lambda_1^* \kappa^* - \frac{1}{2} + \lambda_1(\mu_j + \rho_j \mu_\nu)] \\ &\qquad \times \left\{ \begin{aligned} & \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right)^2 \times \left[a_{T-t,\mathrm{SVSCJ}} + \frac{C_{T-t,\mathrm{SVSCJ}}}{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)} - \alpha_{T-t,\mathrm{SVSCJ}}(T-t) \right] \\ & + \frac{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)}{2(\kappa_\nu - \lambda_1 \mu_\nu)} \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) \times \left[a_{T-t,\mathrm{SVSCJ}} - 2\alpha_{T-t,\mathrm{SVSCJ}}(T-t) \right] \\ & + \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) \frac{1}{(\kappa_\nu - \lambda_1 \mu_\nu)} \left(\frac{1}{2} C_{T-t,\mathrm{SVSCJ}} - \beta_{T-t,\mathrm{SVSCJ}} \right) \\ & + \frac{\lambda_0 \mu_\nu^2}{\kappa_\nu - \lambda_1 \mu_\nu} \times \left[a_{T-t,\mathrm{SVSCJ}} - \frac{C_{T-t,\mathrm{SVSCJ}}}{(\sigma_\nu^2 + 2\lambda_1 \mu_\nu^2)} \right] + \left(\frac{\kappa_\nu \theta_\nu + \lambda_0 \mu_\nu}{\kappa_\nu - \lambda_1 \mu_\nu} \right) b_{T-t,\mathrm{SVSCJ}} \end{aligned} \right\} \\ &+ [r - \delta - \lambda_0^* \kappa^* + \lambda_0(\mu_j + \rho_j \mu_\nu)] b_{T-t,\mathrm{SVSCJ}} \\ &+ \lambda_0(\mu_j \mu_\nu + 2\rho_j \mu_\nu^2) a_{T-t,\mathrm{SVSCJ}}\end{aligned}$$

Focusing on the unit time-interval, or $T = t + \Delta$, and using the earlier result that

$\mathrm{E}_t^P(\upsilon_{t,t+\Delta})$ in (B.4), $\mathrm{E}_t^P(\upsilon_{t,t+\Delta}^2)$ in (B.11) and the Law of Iterated Expectations, the cross moment condition of (B.13), which is directly implementable, becomes

$$\begin{aligned}
&\mathrm{E}_t^P\left[\ln S_{t+\Delta}\times\left\{\frac{\upsilon_{t+\Delta,t+2\Delta}-\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\Delta}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{\Delta,\mathrm{SVSCJ}}}-\frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}}\right\}\right] \\
&=\mathrm{N}_{\Delta,\mathrm{SVSCJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta}^2)+\mathrm{O}_{\Delta,\mathrm{SVSCJ}}\times\mathrm{E}_t^P(\upsilon_{t,t+\Delta})+\mathrm{P}_{\Delta,\mathrm{SVSCJ}}\times\ln S_t+\mathrm{Q}_{\Delta,\mathrm{SVSCJ}}
\end{aligned}\tag{B.14}$$

where

$$\mathrm{N}_{\Delta,\mathrm{SVSCJ}}=\frac{\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2\times a_{\Delta,\mathrm{SVSCJ}}^2}$$

$$\begin{aligned}
\mathrm{O}_{\Delta,\mathrm{SVSCJ}}=&\frac{(\alpha_{\Delta,\mathrm{SVSCJ}}\times\ln S_t+\mathrm{L}_{\Delta,\mathrm{SVSCJ}})}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{\Delta,\mathrm{SVSCJ}}} \\
&-\frac{(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{\Delta,\mathrm{SVSCJ}}^3} \\
&-\frac{2\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\,\Delta\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2\times a_{\Delta,\mathrm{SVSCJ}}^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_{\Delta,\mathrm{SVSCJ}}=&(\kappa_v\theta_v+\lambda_0\mu_v)a_{\Delta,\mathrm{SVSCJ}}-\frac{b_{\Delta,\mathrm{SVSCJ}}\alpha_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}} \\
&-\frac{\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\Delta\times\alpha_{\Delta,\mathrm{SVSCJ}}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{\Delta,\mathrm{SVSCJ}}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Q}_{\Delta,\mathrm{SVSCJ}}=&\mathrm{M}_{\Delta,\mathrm{SVSCJ}}-\frac{b_{\Delta,\mathrm{SVSCJ}}\times\mathrm{L}_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}}-\mathrm{K}_{\Delta,\mathrm{SVSCJ}}\left(\frac{b_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}}\right)^2-\frac{\mathrm{B}_{\Delta,\mathrm{SVSCJ}}\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}^2} \\
&+\frac{\left\{\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\,\Delta\right\}^2\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}^2\times a_{\Delta,\mathrm{SVSCJ}}^2} \\
&+\frac{(A_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\times b_{\Delta,\mathrm{SVSCJ}}\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}}{a_{\Delta,\mathrm{SVSCJ}}^3} \\
&+\frac{\lambda_0[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\Delta\times\left\{(\mathrm{A}_{\Delta,\mathrm{SVSCJ}}+2a_{\Delta,\mathrm{SVSCJ}}b_{\Delta,\mathrm{SVSCJ}})\times\mathrm{K}_{\Delta,\mathrm{SVSCJ}}-a_{\Delta,\mathrm{SVSCJ}}^2\mathrm{L}_{\Delta,\mathrm{SVSCJ}}\right\}}{\{1+\lambda_1[(\mu_j+\rho_j\mu_v)^2+\sigma_j^2+\rho_j^2\mu_v^2]\}\times a_{\Delta,\mathrm{SVSCJ}}^3}
\end{aligned}$$

By setting $\lambda_0=\lambda_J$ and $\lambda_1=0$ in equation (B.14), we have the cross moment of total quadratic variation for the SVCJ model. Additionally restricting $\mu_v=0$ and further $\lambda_J=0$ in the SVCJ model, cross moments for the SVJ and SV models, respectively, are obtained.

BIBLIOGRAPHY

Aït-Sahalia, Yacine, Per A. Mykland, and Lan Zhang, 2003, How often to sample a continuous-time process in the presence of market microstructure noise, *Review of Financial Studies* 18, 351–416.

Andersen, Torben G., Luca Benzoni, and Jesper Lund, 2002, Towards an empirical foundation for continuous-time equity return models, *Journal of Finance* 57, 1239–1284.

Andersen, Torben G. and Tim Bollerslev, 1997, Intraday periodicity and volatility persistence in financial markets, *Journal of Empirical Finance* 4, 115–158.

Alizadeh, Sassan, Michael Brandt, and Francis X. Diebold, 2002, Range-based estimation of stochastic volatility models, *Journal of Finance* 57, 1047–1091.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Heiko Ebens, 2001, The distribution of realized stock return volatility, *Journal of Financial Economics* 61, 43–76.

Andersen, Torben G., Tim Bollerslev, and Nour Meddahi, 2004, Analytical evaluation of volatility forecasts, *International Economic Review* 45, 1079–1110.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical performance of alternative option pricing models, *Journal of Finance* 52, 2003–2049.

Bates, David S., 1996, Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options, *Review of Financial Studies* 9, 69–107.

Bates, David S., 2000, Post-‘87 Crash fears in S&P 500 futures options, *Journal of Econometrics* 94, 181–238.

Bates, David S., 2006, Maximum likelihood estimation of latent affine processes, *Review of Financial Studies* 19, 909–965.

Black, Fischer, 1976, Studies of stock price volatility changes, *Proceedings of the 1976 Meetings of the Business and Economics Statistics Section*, *American Statistical Association*, 177–181.

Black, Fischer and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–659.

Bollerslev, Tim, Michael Gibson and Hao Zhou, 2005, Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities, Working paper, Department of Economics, Duke University.

Bollerslev, Tim and Hao Zhou, 2002, Estimating stochastic volatility diffusion using conditional moments of integrated volatility, *Journal of Econometrics* 109, 33–65.

Bollerslev, Tim and Hao Zhou, 2004, Volatility puzzles: A simple framework for gauging return-volatility regressions, *Journal of Econometrics* 50, 1457–1498.

Breeden, Douglas and Robert Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business* 51, 621–651.

Britten-Jones, Mark and Anthony Neuberger, 2000, Option price, implied price processes, and stochastic volatility, *Journal of Finance* 2, 839–866.

Brockhaus, O. and D. Long, 2000, Volatility swaps made simple, *Risk* 13, 92–95.

Carr, Peter and Liuren Wu, 2006, A tale of two indices, *Journal of Derivatives* 13, 13–29.

Chicago Board Options Exchange, 2003, VIX — CBOE Volatility Index, http://www.cboe.com/micro/vix/vixwhite.pdf.

Chernov, Mikhail, and Eric Ghysels, 2000, Towards a united approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation, *Journal of Financial Economics* 56, 407–458.

Chernov, Mikhail, Eric Ghysels, A. Ronald Gallant, and George Tauchen, 2003, Alternative models for stock price dynamics, *Journal of Econometrics* 116, 225–257.

Cox, John C, Jonathan E Ingersoll JR, and Stephen A Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Duffie, Darrell, Jun Pan and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

Eraker, BjØrn, 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, *Journal of Finance* 59, 1367–1403.

Eraker, BjØrn, Michael Johannes, and Nicholas Polson, 2003, The impact of jumps in volatility and returns, *Journal of Finance* 53, 1269–1300.

French, Kenneth R., G. William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics* 19, 3–30.

Gallant, A Ronald and George Tauchen, 1996, Which moments to match?, *Econometric Theory* 12, 657–681.

Gallant, A Ronald and George Tauchen, 2002, Simulated score methods and indirect inference for continuous-time models, University of North Carolina.

Garcia, René, Marc-André Lewis, Sergio Pastorello and Éric Renault, 2006, Estimation of objective and risk-neutral distributions based on moments of integrated volatility, Working paper, *CIRANO and CIREQ*.

Hansen, Lars Peter, 1985, A method for calculating bounds on the asymptotic covariance matrices of generalized methods estimators, *Journal of Econometric* 30, 203–238.

Hansen, Lars Peter, John C. Heaton, and Masao Ogaki, 1988, Efficiency bounds implied by multiperiod conditional moment restrictions, *Journal of the American Statistical Association* 83, 863–871.

Hansen, Peter R. and Asger Lunde, 2006, Realized variance and market microstructure noise, *Journal of Business & Economic Statistics* 24, 127–161.

Heston, Steven, 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327–343.

Hull, John C. and Alan White, 1987, The pricing of options on assets with stochastic volatility, *Journal of Finance* 42, 281–300.

Jacquier, Eric, Nicholas G. Polson and Peter E. Rossi, 1994, Bayesian analysis of stochastic volatility models, *Journal of Business and Economic Statistics* 12, 371–389.

Jiang, George J. and Yisong S. Tian, 2005, The model-free implied volatility and its information content, *Review of Financial Studies* 18, 1305–1342.

Meddahi, Nour, 2002, Theoretical comparison between integrated and realized volatility, *Journal of Applied Econometrics* 17, 479–508.

Meddahi, Nour and Eric Renault, 2004, Temporal aggregation of volatility models, *Journal of Econometrics* 119, 355–379.

Merton, Robert, 1976, Option pricing when the underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125–144.

Nandi, Saikat, 1998, How important is the correlation between returns and volatility in a stochastic volatility model? Empirical evidence from pricing and hedging in the S&P 500 index options market, *Journal of Banking and Finance* 22, 589–610.

Newey, Whitney K. and Kenneth. D. West, 1987, A simple positive definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–705.

Pan, Jun, 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study, *Journal of Financial Economics* 63, 3–50.

Rockinger, Michael and Maria Semenova, 2005, Estimation of Jump-diffusion processes via empirical characteristic functions, Working paper, CEPR.

Zhang, Jin E. and Yingzi Zhu, 2006, VIX futures, *Journal of Futures Markets* 26, 521–531.

Zhou, Bin, 1996, High-frequency data and volatility in foreign-exchange rates, *Journal of Business & Economic Statistics* 14, 45–52.