Improvements to the Least Squares Monte Carlo Option Valuation Method

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This paper proposes several improvements to the least squares Monte Carlo (LSMC) option valuation method. We test different regression algorithms and suggest a variation to the estimation of the option continuation value, which reduces the execution time of the algorithm without any significant loss in accuracy. We test the choice of varying polynomial families with different number of basis functions and various variance reduction techniques, using a large sample of vanilla American options, and find that the use of low discrepancy sequences with Brownian bridges can increase substantially the accuracy of the simulation method. We also extend our analysis to the valuation of portfolios of compound and mutually exclusive options. For the latter, we also propose an improved algorithm which is faster and more accurate.

Key words: American options, real options, simulation, quasi Monte-Carlo methods

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1. Introduction

The valuation of finite maturity options with American-style exercise features can be intricate, with no generic closed form solution for even the most simple type of these contracts. Simulation, first introduced in option valuation problems by Boyle (1977), is one of the most powerful numerical techniques to value options. It is a very flexible valuation methodology and it can be shown to converge to the true option price. It is particularly relevant to solve problems of several state variables, since the convergence rate is not linked to the number of state variables, in contrast to the lattice methods, which suffer from the well known "*curse of dimensionality*" (Broadie and Glasserman 1997). Simulation can also easily accommodate different stochastic processes, multiple underlying assets, path dependence, and particular features of exotic options.

The major drawbacks of simulation are the high computation requirements, low speed, and also the difficulty in dealing with free boundary problems such as the American option valuation.[1] All of these limitations have been surmounted over the years, firstly by the rapid development of computers which now allows this technique to be used efficiently on personal computers or even on handheld computers, and also by recent developments in this field of study.

Longstaff and Schwartz (2001) suggested a method to value American options by simulation which estimates the continuation value by a least squares regression, designated as Least Squares Monte Carlo (henceforth referred as LSMC). This method has received much attention in the finance literature.

Despite its advantages the LSMC simulation method is not the most appropriate method to value American vanilla options since most of the binomials can produce more accurate results at a fraction of the computational time. Nevertheless, it is appropriate to value options where the binomial method renders itself unusable due to the curse of dimensionality, e.g. when discrete dividends are considered or with multi-state variables options, which are frequent for example in real options problems.

In the early stages of real options theory, valuation was, with few exceptions, confined to options for which financial options solutions could be applied. This was done mainly using few underlying assets (or state variables) and options with European features or American perpetual options.

[1] For many years it was considered in the literature that simulation could not be used to value American options efficiently.

Following the usual route, research has been closing the gap with reality. Investment projects are portfolios of options, frequently dependent upon several stochastic variables, which can be exercised several times before expiration, i.e. they are Bermudan or American options.

The value of a portfolio of options with interacting features can be significantly different from the sum of the individual options. Gamba (2003) proposed a model which decomposes complex multiple real options problems (with interacting options) into simple hierarchical sets of individual options. Extending the LSMC approach, this model deals also with American and Bermudan real options, which are frequent in capital budgeting projects. Among the possible interactions between real options, Gamba presents the following types: independent options, compound options, mutually exclusive options and switching problems.

We will test several methods that are able to improve the accuracy of the simulation valuation method of American vanilla options and portfolios of options. In the next section we will present the LSMC simulation method and an extension to Gamba's (2003) method to value portfolios of options and propose an improvement to the mutually exclusive options valuation algorithm. Section 3 details some computational aspects of the LSMC algorithm, presents alternative procedures for the estimation of the option continuation value which allows the use of faster regression algorithms, the benchmarks and the polynomial families which are used in this study. In section 4 several improvements to the LSMC will be described: simulation with Black-Scholes, control variates, method of moments and quasi random numbers. Section 5 presents the results of a detailed empirical analysis of the vanilla, compound and mutually exclusive options, analysing the accuracy of the improvements and the comparison among the polynomial families. Section 6 concludes.

2. The Least Squares Monte Carlo Simulation Method

The value of a finite lived American option ($\mathcal{O}$), in a risk neutral world, with maturity T is given by:

$$\mathcal{O} = \max_{\tau \leq T} E\left[e^{-r\tau}\Phi(K, S_\tau, \tau, \Psi_\tau)\right] \tag{1}$$

where the maximisation is over all stopping times τ (with $\tau \leq T$), and $E[\centerdot]$ denotes the risk neutral expectation conditional on the underlying asset price (S) at the time of the valuation. K is the option exercise price, and Φ is the payoff function defined as:

$$\Phi(K, S_t, t) = \max(\phi(K, S), \Psi_t) \tag{2}$$

where Ψ_t is the present value (at time t) of the continuation value, i.e. the expected payoff from keeping the option alive. At each stopping time, the option holder must decide whether or not to exercise the option. The option should be exercised if the continuation value is less than the value of immediate exercise.

To estimate the conditional expectation of the payoff from continuation, Longstaff and Schwartz (2001) suggest using the cross-sectional information in the simulation. Let $g(\omega) = E[\Psi|\omega]$ be the approximation to the conditional expectation of continuation, where ω represents the current state. $g_B(\omega)$ can be approximated by a least squares regression where the coefficients a_b are estimated using a set of basis functions in the underlying asset price:[2]

$$\hat{g}_B(\omega) = \sum_{b=0}^{B} \hat{a}_b p_b(x), \;\; \hat{a}_b \in \mathbb{R} \tag{3}$$

[2] Longstaff and Schwartz (2001) regress only paths that are in-the-money, since the exercise decision is only relevant when the option is in-the-money. Such a restricted regression allows not only a faster estimation of the regression coefficients but also improves the efficiency of the algorithm.

We can express the remaining part of this valuation method as:[3]

$$\hat{\mathcal{O}} = \max\left(\phi(K,S), \frac{1}{M}\sum_{i=1}^{M} e^{-r(\Delta t)}\varphi_i\right) \tag{4}$$

where $\Delta t = T/N$, and the value of each sample path option cash-flow φ_i at time $t=1$ is given by the algorithm presented in Figure 1:[4]

Figure 1 **Value of each sample path option cash-flow φ_i, for vanilla options**

for $j \leftarrow N$ **to** 1 **do**
  for $i \leftarrow 1$ **to** M **do**
    $\varphi_i \leftarrow \Phi_{i,t_j}$
  end
end

Φ_{i,t_j} is the option cash flow for the i^{th} path and time step t_j given by:

$$\Phi_{i,t_j} = \begin{cases} \phi(K,S_{i,t_j}) & \text{if} \quad \phi(K,S_{i,t_j}) > \Psi_{i,t_j} \\ e^{-r(\Delta t)}\Phi_{i,t_{j+1}} & \text{if} \quad \phi(K,S_{i,t_j}) \leq \Psi_{i,t_j} \end{cases} \tag{5}$$

$\phi(K,S_{i,t_j})$ is the option exercise value given by:

$$\phi(K,S_{i,t_j}) = \begin{cases} \max(S_{i,t_j} - K, 0) & \text{for a call option} \\ \max(K - S_{i,t_j}, 0) & \text{for a put option} \end{cases} \tag{6}$$

and, the estimated continuation value Ψ_{i,t_j}, is given by:

$$\Psi_{i,t_j} = \begin{cases} 0 & \text{for } j = N; \\ \hat{g}_{t_j,B}(\omega) & \text{for } j < N. \end{cases} \tag{7}$$

where $\hat{g}_{t_j,B}(\omega)$ is given by equation 3.

2.1. The Extended Least Squares Monte Carlo Method

Gamba (2003) extended the LSMC method to value portfolios of options, namely independent, compound and mutually exclusive options, as well as switching problems. As we are only going to analyse compound and mutually exclusive options, we will briefly summarize its valuation algorithm.

[3] Please refer to Longstaff and Schwartz (2001) for a more detailed description of the algorithm. Convergence of the algorithm, as the number of sample paths $M \to \infty$ and as the number of basis function increases to infinity ($B \to \infty$), has been analysed by Longstaff and Schwartz (2001), Clément, Lamberton, and Protter (2002), Stentoft (2004b), Tsitsiklis and Van Roy (2001) and Glasserman and Yu (2004).

[4] Our formulation does not require the same amount of memory as the original Longstaff and Schwartz (2001) specification, allowing simulations with larger number of sample paths which results in better accuracy even for computers with a relatively small RAM.

2.1.1. Compound Options. A portfolio of H compounded options with maturities T_h ($T_1 \leq T_2 \leq ... \leq T_H$) is valued applying the LSMC algorithm starting with the option with the longest maturity. Since the h^{th} option gives the right to exercise the subsequent option, its payoff must include the value of the $(h+1)^{\text{th}}$ option. The value of the h^{th} option($\mathcal{O}_h$) is given by:

$$\mathcal{O}_h(K, S_t, t) = \max_{t \leq \tau \leq T_h} E\left[e^{-r(\tau - t)} \Phi_h(K_h, S_\tau, \tau, \Psi_{h,\tau})\right] \tag{8}$$

where $\Psi_{h,\tau}$ is the continuation value, K_h the exercise price and Φ_h the payoff function.

We can express the value of the compound option as:

$$\hat{\mathcal{O}} = \max\left(\phi(K, S), \frac{1}{M} \sum_{i=1}^{M} e^{-r(\Delta t)} \varphi_{1,i}\right) \tag{9}$$

where $\Delta t = T_H / N$, and the value of each sample path option cash-flow $\varphi_{1,i}$ at time $t = 1$ is given by the algorithm presented in Figure 2.

Figure 2 **Value of each sample path option cash-flow $\varphi_{1,i}$ for the compound options**

for $j \leftarrow N$ **to** 1 **do**
  for $h \leftarrow H$ **to** 1 **do**
    for $i \leftarrow 1$ **to** M **do**
      $\varphi_{h,i} \leftarrow \Phi_{h,i,t_j}$
    end
  end
end

Φ_{h,i,t_j} is the h^{th} option cash flow for the i^{th} path and time step t_j given by:

$$\Phi_{h,i,t_j} = \begin{cases} 0 & \text{if} \quad t_j > T_h \\ \phi_h(K_h, S_{i,t_j}) & \text{if} \quad \phi_h(K_h, S_{i,t_j}) > \Psi_{h,i,t_j} \\ e^{-r(\Delta t)} \Phi_{h,i,t_{j+1}} & \text{if} \quad \phi_h(K_h, S_{i,t_j}) \leq \Psi_{h,i,t_j} \end{cases} \tag{10}$$

$\phi_h(K_h, S_{i,t_j})$ is the option exercise value given by:

$$\phi_h(K_h, S_{i,t_j}) = \begin{cases} \max(S_{i,t_j} + \hat{\mathcal{O}}_{h+1}(K_{h+1}, S_{i,t_j}) - K_h, 0) & \text{for a call option} \\ \max(K_h - S_{i,t_j} - \hat{\mathcal{O}}_{h+1}(K_{h+1}, S_{i,t_j}), 0) & \text{for a put option} \end{cases} \tag{11}$$

where $\hat{\mathcal{O}}_{h+1}(K_{h+1},, S_{i,t_j})$ is the estimated $(h+1)^{\text{th}}$ option value given by:

$$\hat{\mathcal{O}}_{h+1}(K_{h+1}, S_{i,t_j}) = \begin{cases} \max(S_{i,t_j} - K_{h+1}, \Psi_{h+1,i,t_j}) & \text{for a call option} \\ \max(K_{h+1} - S_{i,t_j}, \Psi_{h+1,i,t_j}) & \text{for a put option} \end{cases} \tag{12}$$

Ψ_{h,i,t_j} is the estimated continuation value given by:

$$\Psi_{h,i,t_j} = \begin{cases} 0 & \text{for } t_j = T_h; \\ \hat{g}_{h,t_j,B}(\omega) & \text{for } t_j < T_h. \end{cases} \tag{13}$$

where $\hat{g}_{t_j,B}(\omega)$ is given by equation 3.

Since we need the value of the subsequent options ($\hat{\mathcal{O}}_{h+1}$) for all the values of the underlying asset, it must be estimated using all paths.

2.1.2. Mutually Exclusive Options. An example of a mutually exclusive option, in a real options context, is the case of the choice between the expansion and abandon options. The decision has to be made, in a given time horizon T_H, meaning the choice of the best alternative. We have now a couple of control variables (τ, ζ), where τ is the optimal stopping time and $\zeta \in \{1, 2, ..., H\}$ the optimal (best) option. The value of the option to choose the best out of H options, is:

$$\mathcal{O}_0\left(S_t, t\right) = \max_{\tau \leq T, \zeta \in \{1,2,\ldots,H\}} E\left[e^{-r(\tau - t)}\mathcal{O}_\zeta(K_\zeta, S_\tau, \tau)\right] \tag{14}$$

We can express the value of the mutually exclusive option as:

$$\hat{\mathcal{O}}_0 = \max\left(\phi(K,S), \frac{1}{M}\sum_{i=1}^{M} e^{-r(\Delta t)}\varphi_{0,i}\right) \tag{15}$$

where $\Delta t = T_H/N$, and the value of each sample path option cash-flow $\varphi_{h,i}$ at time $t = 1$ is given by the algorithm presented in Figure 3.

Figure 3 Value of each sample path option cash-flow $\varphi_{h,i}$ for mutually exclusive options

for $j \leftarrow N$ **to** 1 **do**
  for $h \leftarrow H$ **to** 0 **do**
    for $i \leftarrow 1$ **to** M **do**
      $\varphi_{h,i} \leftarrow \Phi_{h,i,t_j}$
    end
  end
end

Φ_{h,i,t_j} is the h^{th} option cash flow for the i^{th} path and time step t_j given by equation 10, and $\phi_h(K_h, S_{i,t_j})$ is the option exercise value given by:

$$\phi_h(K_h, S_{i,t_j}) = \begin{cases} \max_{k\in\{1,2,\ldots,H\}} \hat{\mathcal{O}}_k(K_k, S_{i,t_j}) & \text{if} \quad h = 0 \text{ (mutually exclusive option)} \\ \max(S_{i,t_j} - K, 0) & \text{if} \quad h > 0 \text{ and for a call option} \\ \max(K - S_{i,t_j}, 0) & \text{if} \quad h > 0 \text{ and for a put option} \end{cases} \tag{16}$$

where $\hat{\mathcal{O}}_k(K_k, S_{i,t_j})$ is the estimated h^{th} option value given by:

$$\hat{\mathcal{O}}_k(K_k, S_{i,t_j}) = \begin{cases} \max(S_{i,t_j} - K_k, \Psi_{k,i,t_j}) & \text{for a call option} \\ \max(K_k - S_{i,t_j}, \Psi_{k,i,t_j}) & \text{for a put option} \end{cases} \tag{17}$$

Ψ_{k,i,t_j} is the estimated continuation value given by equation 13.

As we need the value of the subsequent options ($\hat{\mathcal{O}}_h$) for all values of the underlying asset, again it must be estimated using all paths.

2.1.3. Alternative Algorithm to Value Mutually Exclusive Options. We propose an alternative algorithm to value this type of options. "Since the decision about the option is irreversible... the choice is not made until the time to exercise the most favourable outcome has come" (Gamba 2003, p. 12). A mutually exclusive option can be valued using the LSMC algorithm to value single options (algorithm 1), replacing equation 6 by:

$$\phi(K, S_{i,t_j}) = \max_{h\in\{1,2,\ldots,H\}} \left\{\phi_h(K_h, S_{i,t_j})\right\} \tag{18}$$

where $\phi_h(K_h, S_{i,t_j})$ is the h^{th} option exercise value given by:

$$\phi_h(K_h, S_{i,t_j}) = \begin{cases} \max(S_{i,t_j} - K_h, 0) & \text{for a call option} \\ \max(K_h - S_{i,t_j}, 0) & \text{for a put option} \end{cases} \tag{19}$$

With this algorithm, we do not need to value each of the H individual options, which means that it is faster than Gamba's algorithm and produces more accurate valuations with a lower polynomial degree (as we will show in section 5).

3. Framework of analysis

3.1. Regression Algorithm

Although the LSMC method is very robust in relation to the choice of basis functions, this may cause some regression estimation problems since, for some polynomial families, the basis functions are highly correlated with each other. This will not affect the LSMC algorithm since it relies on the fitted value of the regression and not on the degree of correlation among the independent variables. But if this choice leads to a nearly singular matrix, then it is possible that some regression algorithms will give inaccurate numerical results for the estimated conditional expectation function (Longstaff and Schwartz 2001).

We tried two different least squares algorithms, LFIT and SVD, both of them suggested by Press, Teukolsky, Vetterling, and Flanney (1992). The SVD is a singular value decomposition algorithm which will produce better solutions when the choice of the polynomial family and the number of basis create an almost singular matrix.[5] The LFIT algorithm is considerably faster than the SVD, but there are instances where nearly singular matrices produce extremely poor approximations to the expectation function.[6] On our tests this occurs mainly when using the regression for time steps close to zero. On those instances the expected payoff function would yield very low values for any underlying asset price. In the next section we will present two novel procedures to estimate the continuation value that allows the use of a faster regression algorithm without comprising any significant loss of accuracy.

3.2. Alternative Estimation Procedures for the Option Continuation Value

We propose two corrections for this estimation problem when using the LFIT regression algorithm. In the first correction, designated here as Continuation value by Conditional Estimation (CONT-CE), at each time step we compare the estimated continuation value given by the least squares regression at that time step ($\hat{g}_{t,B}(\omega, t)$) with the present value of the continuation value estimated using the previous time step regression coefficients. If the current expected value is smaller than the present value of the previous time step, then we use the expected continuation value given by the parameters estimated in the previous time step. The CONT-CE procedure is then given by:

$$E[\Psi_i | S_{i,t_j}] = \max(\hat{g}_{t_j,B}(S_{i,t_j}), e^{-r\Delta t}\hat{g}_{t_{j+1},B}(S_{i,t_j})) \tag{20}$$

The second correction (Black-Scholes Continuation, henceforth CONT-BS), valid only for vanilla options, compares the estimated continuation value at each time step j with the value of a European option with the parameters from that simulation path and with time-to-maturity equal to the remainder of the life of the option ($t_N - t_j$). If the expected continuation value is less than the

[5] Both algorithms were adapted to take long double precision variables, which in our case allows 20 decimal places of precision both in calculations and also in storage. The TOL constant in the SVD algorithm was adjusted to $1.0E-15$. Otherwise the results would not be very accurate, although this increases the computational costs of the algorithm.

[6] These instances directly increase with the number of basis functions considered.

European option value for that sample path and time, the continuation value is set equal to the European option value. The CONT-BS procedure is then given by:

$$E[\Psi_i|S_{i,t_j}] = \max(\hat{g}_{t,B}(S_{i,t_j}), BS(S_{i,t_j}, K, \sigma, \delta, r, t_N - t_j)) \tag{21}$$

where BS is the value of the European option given by the Black and Scholes (1973) formulae.

As we will show, both of these corrections perform very well for vanilla options, allowing for the use of the faster LFIT algorithm which in turn greatly increases the speed of the LSMC method.

3.3. The Benchmarks

All the results of the valuation of options by simulation methods presented will be compared with a benchmark value. For American vanilla options the benchmark is given by the binomial Black-Scholes with Richardson extrapolation tree (henceforth designated as BBSR) suggested by Broadie and Detemple (1996), along with the Cox, Ross, and Rubinstein (1979) parameters and 10,000 steps. For the compound and mutually exclusive options the Cox, Ross, and Rubinstein (1979) binomial method was used to compute the option values with 10,000 steps.

The option valuation error of each method is measured by the mean square relative error (MSRE) and by the root mean square relative error (RMSRE) which are given by:

$$\text{MSRE} = \frac{1}{M}\sum_{i=1}^{M} e_i^2 \tag{22}$$

$$\text{RMSRE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M} e_i^2} \tag{23}$$

where M is the number of sample options. The relative error is given by $e_i = \frac{\hat{\mathcal{O}}_i - \mathcal{O}_i}{\mathcal{O}_i}$, where $\mathcal{O}_i$ is the option benchmark price and $\hat{\mathcal{O}}_i$ is the estimated option price for a given method[7].

The test of Diebold and Mariano (1995) for the difference of two MSRE is used to compare the statistical significance of differences in accuracy of two option valuation alternatives.

3.4. Polynomial Families

As already mentioned the conditional expectation of the option continuation value $E[\Psi|\omega]$ can be represented as a linear function of the elements of a set of basis functions. Longstaff and Schwartz (2001) suggest the use of weighted Laguerre polynomials ($\mathcal{WL}_n(x)$ with n denoting the degree of the polynomial).

Longstaff and Schwartz (2001) mention that their results are robust to the choice of polynomial families, although they do not present any other results for the American put valuation problem besides the weighted Laguerre polynomial. Moreno and Navas (2003) and Stentoft (2004a) tested the effect of the choice of different polynomial families and also the number of polynomials in the option valuation accuracy for a small sample of options (the first considered only one option and the latter only three options). We will use, besides the weighted Laguerre polynomials, the same poynomial families as Moreno and Navas (2003): Powers ($\mathcal{W}_n(x)$), Legendre ($\mathcal{P}_n(x)$), Laguerre ($\mathcal{L}_n(x)$), Hermite-A ($\mathcal{H}_n(x)$), Hermite-B ($\mathcal{H}e_n(x)$), Chebyshev 1st kind A ($\mathcal{T}_n(x)$), Chebyshev 1st kind B ($\mathcal{C}_n(x)$), Chebyshev 1st kind C ($\mathcal{T}_n^*(x)$), Chebyshev 2nd kind A ($\mathcal{U}_n(x)$) and Chebyshev 2nd kind B ($\mathcal{S}_n(x)$).

[7] In order to avoid distortions, only options with estimated prices above or equal to 0.5 were used to compute the RMSRE, as is suggested by Broadie and Detemple (1996). All computations were performed, unless stated otherwise, with a personal computer with a dual Pentium 4 3Ghz processor, 2GB of RAM, and a Linux kernel. All programs were written in C and FORTRAN and compiled with the GNU GCC/G77 compiler.

All these polynomial families can be expressed as linear combinations of the others, therefore all of them should provide the same results if there are no numerical errors in the regression algorithm. Any significant numerical error can produce different results for different polynomial families.

4. Improvements to the Simulation Valuation Method

It is possible to decrease the valuation error of LSMC, but that requires the reduction of the standard error of the simulation estimates, which can be accomplished by increasing the number of paths or by using variance reduction procedures.[8]

One such technique is the antithetic variates method described by Boyle (1977), which corrects the first moment of the normal standard distribution.

We will use the introduction of a Black-Scholes valuation at the step before maturity in the same spirit of the binomial Black-Scholes method, antithetic variates and several other variance reduction techniques: control variates, method of moments and quasi-random numbers.

Some of these improvements, such as most of the low discrepancy sequences here considered (henceforth LDS, also known as quasi-Monte Carlo, or quasi-random numbers) have not so far been applied to the LSMC method.

With the same motivation as used in the binomial Black-Scholes method, suggested by Broadie and Detemple (1996), we will modify the LSMC method when valuing vanilla options, taking the value of a European option, with the same parameters as the American option (except for the maturity), as the option continuation value at the time before maturity. The value of the European option will be given by the Black and Scholes (1973) formulae.

The use of control variates to improve the results of option valuation by simulation was first suggested by Boyle (1977). We will test the valuation of American options by LSMC using control variates, with an optimal control ratio as suggested by Boyle, Broadie, and Glasserman (1997) and Tian and Burrage (2003)[9]. Despite the good results reported in the literature it is important to note that the successful use of this technique is not universal since it requires an accurate solution to a very similar and highly correlated problem as the one valued by simulation, which is not always available. When valuing American vanilla options the natural candidate will be the European option value given by the Black and Scholes (1973) formulae.

The method of moments variance reduction technique tries to adjust the random standard normal variates such that their moments match the statistical properties of the random variables, transforming the sample normal variates to match the first few moments of the underlying population[10].

It is possible to improve the convergence of the LSMC by using LDS or, as they are also referred, quasi-random sequences/numbers. These numbers are deterministic and altering their generation pattern can have negative effects on the simulation. Therefore, it is not advised to use them along with moment matching techniques or antithetic variates. Even the use of the Box and Muller (1958) or the improved Marsaglia and Bray (1964) polar method to generate the standard normal variates can break the low discrepancy of the sequence. Therefore, when using LDS, another method that preserves all the properties of the sequence must be used, such as the one proposed by Moro (1995).

[8] Over time several authors have proposed many improvements to the use of simulation on option valuation (e.g. Boyle (1977), Broadie and Glasserman (1997), Broadie, Glasserman, and Jain (1997)).

[9] In our implementation of this method, the optimal control ratio will be estimated using different sample paths than the ones used to estimate the option value.

[10] The resulting transformed variates are not going to be normally distributed or independent. Therefore the resulting option value estimated by simulation will be biased. Although for most financial applications this bias should be low, there is always the possibility where, in extreme circumstances, it could be very large. Since the variates are not going to be independent it is not possible to estimate the standard error of the estimated option value.

As we move to problems with higher dimensions, the advantages of using LDS decreases as the convergence rate falls, since the generated points are not so evenly dispersed in space, which means that it could take a large number of paths M to achieve the asymptotic level of discrepancy.

Paskov and Traub (1995) and Joy, Boyle, and Tan (1996) applied LDS to finance problems and reported that it produces a higher accuracy level at a lower computational cost, even when applied to relatively high dimensional problems. These results are in contrast to other studies such as those of Janse van Resenburg and Torrie (1993), Morokoff and Caflisch (1995), and Bratley, Fox, and Niederreiter (1992), which find that the higher convergence rates of LDS disappear or decrease in higher dimensional problems. These conflicting results may be attributed to the better behaviour of finance problems when compared to the functions used in the other numerical studies.

Recognising this, Moskowitz and Caflish (1996) suggest smoothing and dimension reduction techniques which can substantially improve the use of LDS under such conditions. One of the techniques available to reduce the effective dimension of the problem is the use of Brownian bridges (BB, henceforth).

Although many authors tested the performance of different LDS sequences for problems with different dimensions, their conclusions depend on the type of problem considered. Therefore, we will consider Halton (1960), Sobol (1967), Faure (1982) and Niederreiter (1988) sequences along with the use of BB, to select the one which produces the best results for the American option valuation problem.[11] To implement Sobol sequences we will use two different algorithms: one suggested by Bratley and Fox (1988), and the other the implementation proposed by Press, Teukolsky, Vetterling, and Flanney (1992) along with initialisation numbers suggested by Silva and Barbe (2003) which, according to the authors, produces better results for high dimensional problems than the original algorithm or even the initialisation numbers suggested by Jackel (2002)[12]

5. Empirical Results

5.1. Vanilla Options

This section presents empirical results of the afore mentioned variance reduction techniques for two samples of options: one with 20 American put options, which will be designated as small sample; and a larger sample of 5,000 (2,500 calls and 2,500 puts) American options, which is going to be used to test a smaller number of variance reduction techniques. For the small sample we conduct also a comparison of the polynomial families and regression methods.

5.1.1. Small Sample Results. We start by comparing the results of the regression techniques (LFIT and SVD) with different procedures to estimate the continuation value of options. We then compare the choice of different polynomial families when estimating the value of American vanilla put options. The impact of the choice of different numbers of basis functions against the choice of the number of paths will be also analyzed. This analysis will allow us to choose a regression technique and a procedure to estimate the continuation value of the option, along with the polynomial family and number of basis functions. Having done this we will proceed to test the accuracy effects of using control variates, moment matching techniques, and different low discrepancy sequences.

Table 1 shows the results of the valuation of 20 American put options with the same set of parameters considered in the literature and also used by Longstaff and Schwartz (2001).[13] The

[11] As suggested by Boyle, Broadie, and Glasserman (1997) we will ignore the first 64 numbers of the sequence, as this will produce better results, especially when considering a small number of simulation paths.

[12] The procedures used to generate the Faure and Niederreiter sequences were adapted from the FORTRAN code available in `http://www.qmcmc.com`.

[13] We considered 20 options resulting from the combination of the following parameters: $S = \{36, 38, 40, 42, 44\}$, $\sigma = \{0.2, 0.4\}$, and $T = \{1, 2\}$. The strike price for all options was considered to be $K = 40$, the risk-free interest rate $r = 5\%$, and the dividend yield (δ) considered is zero.

Table 1 LSMC American put option valuation with Monte Carlo sequences.

				LFIT and CONT-CE			SVD and CONT-LS		
S	σ	T	Bin.	Sim.	se	Abs. error	Sim.	se	Abs. error
36	0.2	1	4.486675	4.470381	(0.005960)	| 0.016293 |	4.470802	(0.005958)	| 0.015873 |
36	0.2	2	4.848308	4.847661	(0.006808)	| 0.000647 |	4.846443	(0.006811)	| 0.001865 |
36	0.4	1	7.108984	7.110657	(0.008375)	| 0.001673 |	7.111465	(0.008392)	| 0.002481 |
36	0.4	2	8.514188	8.515901	(0.010484)	| 0.001713 |	8.515352	(0.010488)	| 0.001164 |
38	0.2	1	3.257198	3.241319	(0.004670)	| 0.015879 |	3.240969	(0.004670)	| 0.016230 |
38	0.2	2	3.751382	3.738031	(0.005921)	| 0.013352 |	3.738822	(0.005922)	| 0.012561 |
38	0.4	1	6.154592	6.151349	(0.007930)	| 0.003243 |	6.151966	(0.007927)	| 0.002627 |
38	0.4	2	7.674908	7.670174	(0.009701)	| 0.004735 |	7.671729	(0.009708)	| 0.003179 |
40	0.2	1	2.319576	2.321144	(0.005227)	| 0.001569 |	2.319964	(0.005221)	| 0.000388 |
40	0.2	2	2.889953	2.882488	(0.006441)	| 0.007465 |	2.883264	(0.006439)	| 0.006690 |
40	0.4	1	5.318296	5.307354	(0.008912)	| 0.010943 |	5.307129	(0.008886)	| 0.011167 |
40	0.4	2	6.923462	6.924590	(0.009958)	| 0.001128 |	6.922486	(0.009932)	| 0.000975 |
42	0.2	1	1.621158	1.609214	(0.005773)	| 0.011944 |	1.609764	(0.005774)	| 0.011394 |
42	0.2	2	2.216724	2.209274	(0.006921)	| 0.007451 |	2.209522	(0.006918)	| 0.007202 |
42	0.4	1	4.588164	4.577236	(0.009941)	| 0.010928 |	4.577656	(0.009939)	| 0.010509 |
42	0.4	2	6.250238	6.245262	(0.010768)	| 0.004976 |	6.245670	(0.010774)	| 0.004568 |
44	0.2	1	1.112963	1.114994	(0.005618)	| 0.002031 |	1.115263	(0.005618)	| 0.002299 |
44	0.2	2	1.693332	1.686344	(0.006972)	| 0.006988 |	1.686295	(0.006970)	| 0.007037 |
44	0.4	1	3.952790	3.926583	(0.010599)	| 0.026207 |	3.927136	(0.010598)	| 0.025654 |
44	0.4	2	5.646732	5.637643	(0.011732)	| 0.009089 |	5.638750	(0.011728)	| 0.007982 |
RMSRE					0.003167			0.003085	

Comparison of the binomial and simulation valuation techniques to value American style options. In the comparison the strike price of the put is 40, the interest rate is 0.06, and the underlying stock price S, the volatility σ, and the time-to-maturity of the option T are as indicated in the table. The European option values are based on the Black-Scholes closed form formulae. The binomial value of the option was obtained with a 10,000 steps BBSR tree. The simulation was performed with the following options: number of exercisable times of the option per year: 50; number of paths: 100,000 (50,000 plus 50,000 antithetic); Weighted Laguerre polynomial with 3 basis functions. The random number generator routine was re-initialised for every batch of call and put options, with the seed: 12345. "Bin." stands for Binomial, "Sim." stands for Simulation, "se" stands for standard error and "Abs. error" stands for Absolute error of the option value given by LSMC method.

simulation was performed with 100,000 (50,000 plus 50,000 antithetic) paths and with 50 time steps per year, and three weighted Laguerre polynomials.

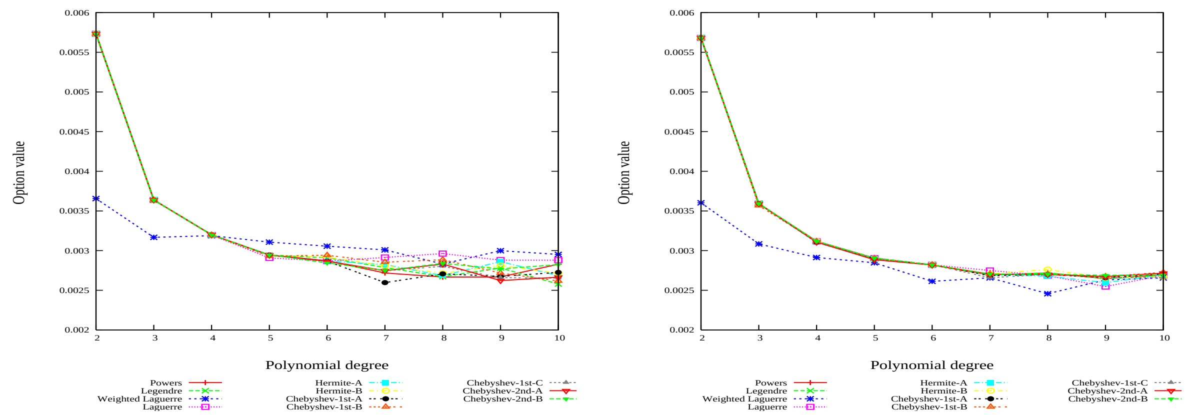
Table 1 features both the conditional continuation (CONT-CE) described earlier, along with the regression procedure LFIT, and the results of the regression routine SVD.[14] The results are very similar, with an RMSRE for the put options sample of 0.003167 for the LFIT with CONT-CE and an RMSRE of 0.003085 for the SVD.[15] Whereas the accuracy of the two regression procedures and continuation estimation procedures are very similar, the time required to compute them is quite different, being generally 50% more computationally expensive to use SVD.

We will consider all the polynomial families mentioned before, and analyse the effect of the choice on the accuracy of the LSMC method, using the same sample of options. Figure 4 shows the RMSRE obtained with different numbers of basis functions. We present results using LFIT and CONT-CE as well as the SVD regression algorithms. Table 2 shows the DM tests for equal MSRE given by the two algorithms. For the large majority of polynomial families and basis functions the results given by the LFIT with CONT-CE estimation and the SVD regression methods are not statistically

[14] The table does not show, but the differences between the CONT-CE and CONT-BS conditional continuation estimation procedures are usually minor (and few are statistically significant), but CONT-CE is faster to implement. Therefore, we only show results for the CONT-CE estimation method.

[15] The DM test for equal MSRE is statistically significant at a 5% level.

Figure 4 Accuracy comparison of LSMC methods using different basis functions: vanilla put options.



(a) LFIT with CONT-CE (b) SVD with CONT-LS

Note. Simulation parameters are the same as in Table 1.

Table 2 DM tests for equal MSRE given by the LFIT regression method with CONT-CE estimation and SVD regression method.

	Number of polynomials								
Polynomial	2	3	4	5	6	7	8	9	10
$\mathcal{W}_n(x)$	0.595**	0.341	0.511	0.237	0.298	0.136	-0.278	0.113	0.726
$\mathcal{P}_n(x)$	0.595**	0.341	0.519	0.209	0.507	0.517	-0.254	0.741	-0.550
$\mathcal{WL}_n(x)$	0.384	0.509**	1.686	1.544*	2.514***	1.995***	1.983***	2.068***	1.663
$\mathcal{L}_n(x)$	0.595**	0.342	0.504	0.024	0.271	0.928	1.577**	1.808***	1.056
$\mathcal{H}_n(x)$	0.592**	0.341	0.536	0.280**	0.269**	0.692	0.043	1.447**	-0.067
$\mathcal{H}e_n(x)$	0.595**	0.341	0.522	0.237	0.555	0.567	-0.259	0.844***	0.241
$\mathcal{T}_n(x)$	0.582**	0.341	0.597	0.267*	0.261	-0.465	0.047	0.089	0.109
$\mathcal{C}_n(x)$	0.595**	0.523	0.505	0.159	0.697	0.847	1.026	0.227	-0.585
$\mathcal{T}^*_n(x)$	0.595**	0.341	0.515	0.232	0.259	0.482	0.553	-0.021	-0.028
$\mathcal{U}_n(x)$	0.595**	0.342	0.583	0.338***	0.279	0.246	0.731	-0.272	-0.274
$\mathcal{S}_n(x)$	0.569**	0.324	0.515	0.243*	0.147	0.236	0.756	0.439	0.707

The table shows the DM test statistic (unit: $1E-06$). ***p-value$<1\%$; **p-value$<5\%$; *p-value$<10\%$. $\mathcal{W}_n(x)$ – Powers; $\mathcal{P}_n(x)$ – Legendre; $\mathcal{WL}_n(x)$ – Weighted Laguerre; $\mathcal{L}_n(x)$ – Laguerre; $\mathcal{H}_n(x)$ – Hermite-A; $\mathcal{H}e_n(x)$ – Hermite-B; $\mathcal{T}_n(x)$ – Chebyshev 1st kind A; $\mathcal{C}_n(x)$ – Chebyshev 1^{st} kind B; $\mathcal{T}^*_n(x)$ – Chebyshev 1^{st} kind C; $\mathcal{U}_n(x)$ – Chebyshev 2^{nd} kind A; $\mathcal{S}_n(x)$ – Chebyshev 2^{nd} kind B.

different at 5% level. The exceptions are mostly for regressions with only 2 polynomials and for the weighted Laguerre polynomial family where the MSRE is lower, and statistically significant, when using the SVD regression method.

It is possible to conclude that, with more than 2 polynomial terms, the results obtained with LFIT and CONT-CE are almost identical to the results given by SVD. All the families produce practically the same identical results, with the exception of the weighted Laguerre polynomial, which produces better results than the others with the SVD procedure. With more than five terms there are some differences in using LFIT and CONT-CE or SVD, with the latter producing better results than the former. But those differences are not always statistically significant. This can be noted by the different results given by the polynomial families and also by the small increase of the RMSRE when increasing the number of polynomials, contrary to what would be expected.

Table 3 presents the DM tests of equal MSRE given by different polynomial families with 5 polynomial degrees and LFIT regression method with CONT-CE estimation. Using 5 polynomial terms the differences in accuracy given by the different polynomial families are very small, and in only 6 out of 54 occasions are these differences statistically significant.

Table 3 DM tests of equal MSRE given by different polynomial families with 5 polynomial degrees and LFIT regression method with CONT-CE estimation.

	Polynomial family									
	$\mathcal{P}_n(x)$	$\mathcal{WL}_n(x)$	$\mathcal{L}_n(x)$	$\mathcal{H}_n(x)$	$\mathcal{H}e_n(x)$	$\mathcal{T}_n(x)$	$\mathcal{C}_n(x)$	$\mathcal{T}^*_n(x)$	$\mathcal{U}_n(x)$	$\mathcal{S}_n(x)$
$\mathcal{W}_n(x)$	0.006	-0.979	0.213	0.008	0.026	-0.004	0.078**	0.006	0.006	-0.006
$\mathcal{P}_n(x)$	—	-0.099	0.021	0.001	0.012	-0.010***	0.072*	0	0	-0.001
$\mathcal{WL}_n(x)$	—	—	1.192***	0.987	1.005	0.975	1.057*	0.985	0.985	0.974
$\mathcal{L}_n(x)$	—	—	—	-0.206	-0.187	-0.217	-0.136	-0.208	-0.208	-0.219
$\mathcal{H}_n(x)$	—	—	—	—	0.016	-0.012	0.070	-0.002	-0.002	-0.013
$\mathcal{H}e_n(x)$	—	—	—	—	—	-0.030	0.052	-0.020	-0.020	-0.032
$\mathcal{T}_n(x)$	—	—	—	—	—	—	0.082**	0.010***	0.010***	-0.001
$\mathcal{C}_n(x)$	—	—	—	—	—	—	—	-0.072*	-0.072*	-0.083*
$\mathcal{T}^*_n(x)$	—	—	—	—	—	—	—	—	0	-0.011
$\mathcal{U}_n(x)$	—	—	—	—	—	—	—	—	—	-0.011

The table shows the DM test statistic (unit: $1E-06$). ***p-value< 1%; **p-value< 5%; *p-value< 10%. Please refer to Table 2 for a description of the polynomials.

It is not strange to note that some of the polynomials present the same results, as it is possible to show that all of them can be expressed as linear combinations of the others. Moreno and Navas (2003) also note that the coefficients of each of these polynomials with respect to the Powers functions form a non-singular matrix. Thus the span generated by each of them is the same as the one generated by the Powers function. Therefore the LSMC should provide identical option prices for all the above mentioned basis functions.

For a fixed number of terms it is clear that our results are almost identical for each family considered, and are closer to what is expected than the results obtained by Moreno and Navas (2003), where the differences were attributed to numerical errors.

The weighted Laguerre polynomials show the best results for a small number of basis functions when using LFIT and CONT-CE, and for almost all the considered degrees in the SVD regression procedure (Figure 4). For three polynomial degrees, the weighted Laguerre family suggested by Longstaff and Schwartz (2001) has the best accuracy for our sample of put options, despite the small differences within polynomial families. This is not entirely surprising as the weighted Laguerre polynomials have, by construction, one more term than the other polynomial families.

If the degree of accuracy given by each family is extremely similar, the choice of basis functions should be determined by the time required to compute the polynomials. Therefore, Powers ($\mathcal{W}_n(x)$) is our polynomial family of choice, since it is the fastest to compute, with a gain of 30% to 40% in computation time.

It is important to stress that the sample size is small, and that we are performing the analysis using always the same number of paths. Glasserman and Yu (2004) analysed the convergence of the LSMC method, and concluded that the number of paths should increase as the number of basis functions also increases.

This can be formally assessed, following Stentoft (2004a), by the following regression:

$$\ln \mathrm{RMSRE} = \alpha + \beta_B \ln B + \beta_M \ln M \tag{24}$$

where B is the number of polynomials, and M is, as before, the number of paths. We computed the RMSRE for the same 20 options sample for 2 to 10 polynomial degrees of the Powers polynomial family and, for each number of basis functions, for 10,000 to 100,000 paths with antithetic variates (with steps of 10,000 paths).

So far we have analyzed the impact of: choosing different regression techniques with varying estimation procedures to compute the option continuation value; the choice of polynomial families and number of basis functions; and increasing the number of basis functions without increasing the number of simulation paths. Summing up, it is possible to use a faster regression technique (LFIT)

Table 4 Regression results of the RMSRE against the number of polynomials and the number of paths.

	LFIT with CONT-CE		SVD	
Coefficients	Estimate	p-values	Estimate	p-value
α	-0.21522	0.527	0.19518	0.585
β_B	-0.32658	0.000	-0.33534	0.000
β_M	-0.44922	0.000	-0.48617	0.000
Adjusted R^2	0.770		0.777	

B is the polynomial degree and M is the number of paths

Table 5 DM tests of equal MSRE given by different number of polynomials.

Number of polynomials						
3 vs 2	4 vs 3	5 vs 4	6 vs 5	7 vs 6	8 vs 7	9 vs 8
13.180***	0.531***	0.366**	0.210**	-0.073	-0.169	-0.267

The table shows the DM test statistic (unit: $1E-06$). ***p-value$<1\%$; **p-value$<5\%$; *p-value$<10\%$. Simulation parameters are the same as in Table 1.

if we combine it with a conditional estimation of the continuation value procedure (CONT-CE). Further, the choice of polynomial families is irrelevant to the accuracy of the LSMC method, but can have great impact on the time needed to compute the option values, so we recommend the use of Powers polynomial family. Finally it is only possible to increase accuracy by increasing the number of basis functions if we also increase the number of simulation paths. We found that when using Powers polynomial family with 100,000 paths and antithetic variates there is not much to gain in increasing the number of basis above 5.

Having done this we will proceed to test the accuracy effects of using control variates, moment matching techniques, and different low discrepancy sequences.

Table 4 has the results of such regression. The reduction in the RMSRE can be achieved by increasing both the number of paths and also the number of polynomial degrees. Similar results are obtained when using LDS.

For the sample tested there is no major improvement in increasing the number of basis functions above 5. Table 5 has the DM tests results for equal MSRE given by the LSMC method with consecutive numbers of basis functions. Increasing from 5 to 6 basis functions results in a statistically significant but very small improvement. Increasing the number of polynomials above 6 results in improvements which are not statistically significant.

Table 6 shows the results of the previously mentioned improvements to the simulation technique for the same sample of 20 put options.

The simulation with Black-Scholes method, described in section 4, does not provide results which are consistently more accurate than the LSMC method. Since the LSMC with Black-Scholes method is more time consuming and does not produce results which are significantly more accurate, we will concentrate the discussion of the results on the LSMC method without Black-Scholes in the last step.

The use of control variates reduces the RMSRE, but these reductions are only statistically significant, at a 5% level, for the LSMC with Halton, Niederreiter, Bratley and Fox (1988), and Silva and Barbe (2003) LDS sequences. However these improvements come with an increase of the computational cost.

Table 7 presents the DM test results for equal MSRE given by the LSMC method using different variance reduction techniques. The use of antithetic variates improves the results (RMSRE), but the

Table 6 Comparison of different variance reduction methods to value American put options.

	RMSRE				DM statistic			
	No CV		CV		No CV vs CV		No BS vs BS	
Method	No BS	BS	No BS	BS	No BS	BS	No CV	CV
LSMC	0.003534	0.003449	0.002574	0.002539	5.866*	5.450	0.062	0.150
LSMC using ANTI	0.002945	0.002938	0.002216	0.002258	3.762	3.536	-0.359**	0.142***
LSMC Halton LDS	0.000836	0.000897	0.000848	0.000913	-0.020***	-0.029***	0.131***	0.120***
LSMC Faure LDS	0.004422	0.004379	0.004413	0.004370	0.073	0.078	-0.384	-0.385
LSMC Niederreiter LDS	0.001680	0.001639	0.001661	0.001620	0.064***	0.006***	-0.295	-0.288
LSMC Sobol BF LDS	0.001218	0.001218	0.001206	0.001205	0.029**	0.031***	-0.063	-0.065
LSMC Sobol SB LDS	0.001830	0.001895	0.001811	0.001877	0.070***	0.069***	-0.123***	-0.122***
LSMC MM1	0.002448	0.002441	0.002550	0.002527	-0.510	-0.427	0.290**	-0.146
LSMC MM2	0.002406	0.002357	0.002544	0.002605	-0.682	-1.231	0.075	-0.643

DM test statistic (unit: $1E-06$). ***p-value< 1%; **p-value< 5%; *p-value< 10%. Simulation used 100,000 paths, and 50 time steps (N) per year, LFIT regression algorithm, POWERS polynomial family with 5 basis functions, and the Moro (1995) normal variates. L'Ecuyer and Touzin (2000) MRG31k3p uniform random variates generator was used with the seed 12345. All the low discrepancy sequences (LDS) were used with Brownian bridges (BB). The optimal control variate ratio c^* was estimated with $20,000$ paths. CV stands for control variates, BS for Black-Scholes valuation at the step before maturity, ANTI for antithetic variates, BF for Bratley and Fox (1988), SB for Silva and Barbe (2003), MM1 for one moment matching and MM2 for two moment matching.

Table 7 DM tests of equal MSRE given by different variance reduction methods.

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Sobol SB LDS
(1) LSMC	3.821	6.498**	6.702**	11.79***	-7.058***	9.670***	10.101***	9.144***
(2) LSMC using AV	—	2.677	2.881	7.972***	-10.088**	5.849*	7.188***	5.323**
(3) LSMC MM1	—	—	0.204	-5.296***	13.56***	-3.172***	-4.511***	-2.646**
(4) LSMC MM2	—	—	—	-5.091***	13.760***	-2.968***	-4.307***	-2.441**
(5) LSMC Halton LDS	—	—	—	—	-18.85***	-2.123***	-0.784	-2.650***
(6) LSMC Faure LDS	—	—	—	—	—	16.730***	18.070***	16.200***
(7) LSMC Niederreiter LDS	—	—	—	—	—	—	1.339	-0.527
(8) LSMC Sobol BF LDS	—	—	—	—	—	—	—	-1.865***

DM test statistic (unit: $1E-06$). ***p-value< 1%; **p-value< 5%; *p-value< 10%. Simulation parameters and legend are the same as in Table 6.

improvement is not statistically significant, while the use of moment matching variance reduction technique is effective. The use of two moments matching does not translate into much more accuracy than the one moment matching technique.

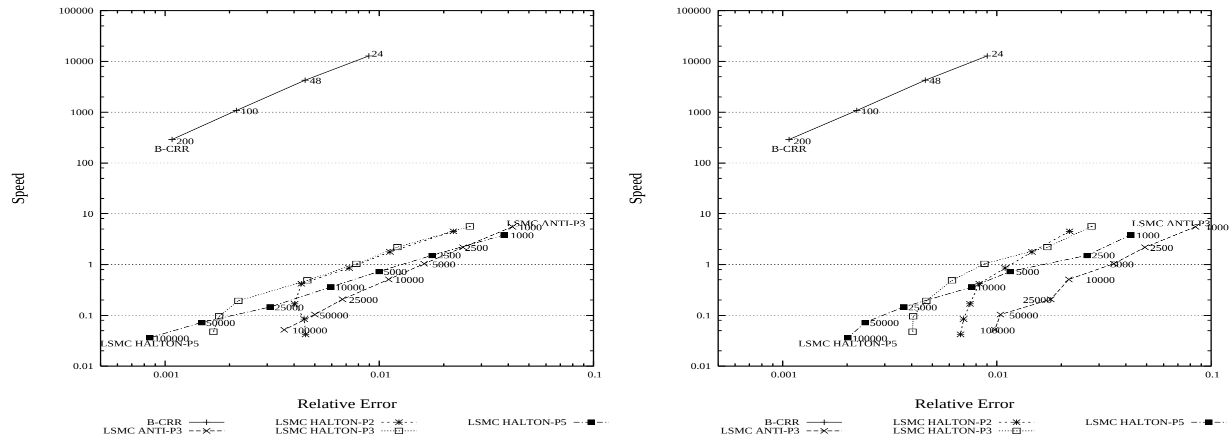
With the exception of the Faure LDS, the use of quasi-Monte Carlo techniques along with BB produces a much more accurate result than all the other variance reduction techniques. The smallest RMSRE is achieved by Halton sequences (0.000836). The MSRE produced by LSMC with Halton LDS is smaller than all the other methods tested and these differences are almost always statistically significant.

Despite being the most time consuming, and contrary to Boyle, Broadie, and Glasserman (1997), Halton sequences with BB produce very good results for our option valuation problem. Although, as Papageorgiou (2002) demonstrates, BB cannot be seen as a panacea to effectively reducing the number of dimensions of all problems. For the problem of valuing American options by LSMC simulation is a very effective technique. This LDS method will be used on the larger sample of options.

5.1.2. Large Sample Results. The 5,000 American options sample was constructed with randomly generated parameters. We are going to use the same distribution of parameters to draw the random sample values as the ones suggested by Broadie and Detemple (1996)[16]. We selected

[16] Sample volatility (σ) is distributed uniformly between 0.1 and 0.6, time-to-maturity (T) is uniform between 0.1 and 1.0 year with probability 0.75, and uniform between 1.0 and 5.0 years with probability 0.25. The asset price (S)

Figure 5 Speed/accuracy comparison of several simulation and binomial methods: large sample.



(a) Puts (b) Calls

Note. The graph is in a log-log scale. Relative error is the RMSRE computed using expression 23. Only options with value above 0.5 were considered. Speed is measured by options values calculated per second. The benchmark value of the American options was obtained using a BBSR binomial tree with 10,000 steps. Numbers beside some methods stand for the number of time steps considered in the case of binomial lattices, and paths for simulation. ANTI stands for antithetic variates.

$2,500$ complete random sets of parameters and computed the value of American call and put options.

Computation speed is measured by the number of options computed per second. This measure is clearly influenced by hardware, the operating system used, the program language, the code implementation, the compiler and the compilation options. Although the absolute computation time is relevant, we are mainly interested in the relative speed/accuracy trade-off of the various methods. Therefore, to allow this comparison, the results were obtained under the same conditions.

The results are shown on Figure 5 for put and call options samples. We considered LSMC simulation with antithetic variates with Powers polynomial familya and 3 basis functions, Halton LDS with Brownian Bridges, with the Powers polynomial family and 2, 3 and 5 basis functions.

The smallest RMSRE is achieved with the LSMC with Halton LDS and 5 basis functions with 100,000 paths: 0.0008474 and 0.0020143 for the put and call options sample respectively. For the put options, this compares to a RMSRE of 0.0036006 given by the original algorithm LSMC with antithetic variates and 3 basis functions suggested by Longstaff and Schwartz (2001). The results for the put options sample are better than the results of a binomial tree with CRR parameters and 200 steps. For the call options sample the results are comparable to a 100 steps binomial tree with CRR parameters. LSMC values more precisely American put options than American call options.

As expected when simulating with a small number of paths, more accurate results are achieved with a small number of basis functions.

The use of a higher number of basis functions, along with Halton LDS can substantially improve the results[17]. The original LSMC algorithm has a RMSRE which is 4.2 times higher than the LSMC Halton-P5 procedure. The use of 5 basis functions and the Halton LDS is only 1.43 times slower than the original 3 basis LSMC algorithm with pseudo-random numbers.

follows a uniform distribution between 70 and 130. The dividend rate (δ) is uniform between 0.0 and 0.1. The riskless interest rate (r) is uniform between 0.0 and 0.1, with probability 0.8, and with probability 0.2 equal to 0.0. Because only the ratio between asset price and exercise price (K) is of interest, since relative errors do not change if S and K are scaled by the same factor, the exercise price is fixed to 100.0.

[17] The DM tests when comparing the LSMC methods shown in the graphs and, for the LSMC method with Halton LDS and 5 powers basis, when comparing the number of paths are all statistically significant.

5.2. Portfolios of Real Options

We now test the accuracy of the LSMC method to value American compound and mutually exclusive real options. We use similar examples as in Gamba (2003). The value of a business, or asset, is assumed to follow a geometric Brownian motion:

$$dS_t = (r - \delta) S_t dt + \sigma S_t dW \tag{25}$$

5.2.1. Option Parameters. We use two sets of compound options:

1. The first set is a call option on another call option:
   - option to defer the investment until T_1 (years): investing by paying K_1, we get e_1 part of the business and the option to expand ($\mathcal{O}_2$). The payoff of this option is $\phi_1(K_1, S_t) = \max\{e_1 S_t - K_1 + \mathcal{O}_2(K_2, S_t), 0\}$.
   - option to expand: it can be exercised once the previous option is exercised and until T_2. With an additional capital expenditure of K_2, the firm gets the remaining part of the asset ($e_2 = 1 - e_1$). The payoff of this option is: $\phi_2(K_2, S_t) = \max\{e_2 S_t - K_2, 0\}$
2. The second set is an alternative strategy, similar to a call on a put option:
   - option to defer the investment until T_1 (years): investing by paying $K = K_1 + K_2$, we get all the business and the option to contract the scale of the project ($\mathcal{O}_2$). The payoff of this option is $\phi_1(K, S_t) = \max\{S_t - K + \mathcal{O}_2(K_2, S_t), 0\}$.
   - option to contract the scale of the project: it can be exercised once the previous option is exercised and until T_2. Part of the initial investment can be recovered ($X = K_2$), reducing the scale to k part of the business. The payoff is therefore: $\phi_2(X, S_t) = \max(X - kS_t, 0)$

The parameters of the compound options under analysis are: $K_1 = K_2 = X = 80$, $e_1 = e_2 = k = 0.5$, $T_1 = T - 2$, $T_2 = T$, $S_0 = \{100, 110\}$, $\delta = \{0.03, 0.05\}$, $\sigma = \{0.2, 0.3\}$, $T = \{4, 5\}$ years, and $r = 0.05$, which means a sample of 16 options for each set.

For the mutually exclusive options, let us assume that a firm already in the business with e_1 share of market, has the following options[18]:

- option to expand: it can be exercised until T_1, with a capital expenditure of K_1; the firm gets the remaining market share ($e_2 = 1 - e_1$) as outcome. The payoff of this option is $\phi_1(K_1, S_t) = \max\{e_2 S_t - K_1, 0\}$

or

- option to abandon: alternatively to the previous strategy, the project can be abandoned, until T_2, with an outcome of K_2. The payoff of this option is $\phi_2(K_2, S_t) = \max\{K_2 - e_1 S_t, 0\}$

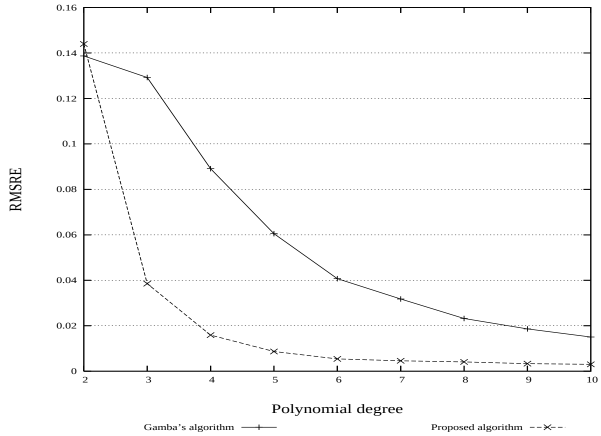
To analyse the case of the best of two call options, we also consider an alternative expansion strategy, which can be exercised until T_3, with an additional capital expenditure of K_3 and an outcome of e_3 share of the market. The payoff is $\phi_3(K_3, S_t) = \max\{e_3 S_t - K_3, 0\}$.

The parameters for this example are: $S_0 = \{90, 100, 110\}$, $\delta = \{0.05, 0.10\}$, $\sigma = \{0.2, 0.4\}$, $T = \{3, 5\}$, $K_1 = 50$, $K_2 = 30$, $K_3 = 35$, $e_1 = 0.5$, $e_2 = 0.5$, $e_3 = 0.4$, $T_1 = T_2 = T_3 = T$, $r = 0.05$. With these parameters our sample has 24 options of the best of two call type, and another 24 of the best of a call and put type.

When valuing real options, computational time is not as important as for the valuation of financial options. This, along with the numerical inaccuracies of the LFIT regression algorithm, which cannot be surmounted with the use of the CONT-CE continuation estimation, makes the use of the SVD algorithm the only feasible alternative.

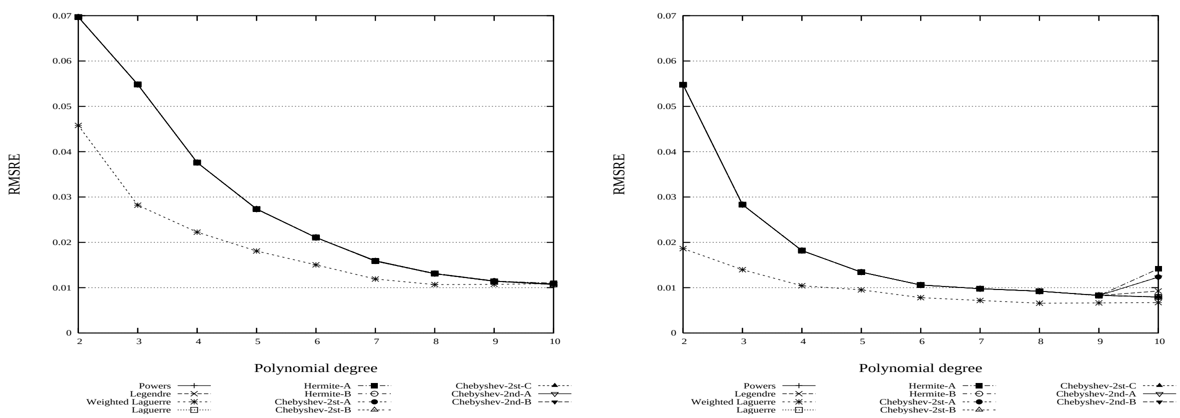
[18] Contrary to what is used by ourselves, the example of a mutually exclusive option presented in Gamba (2003) is not a plain mutually exclusive option: it is a compound option on a mutually exclusive option. If we intend to assess the accuracy of the method to value mutually exclusive options, we should test it independently of other features.

Figure 6 Accuracy of the proposed algorithm when valuing mutually exclusive American options.



Note. The bechmark value of the American option was obtained using the binomial method with Cox, Ross, and Rubinstein (1979) parameters and 10,000 steps. Simulation was done with 100,000 paths and antithetic variates without Brownian bridges. The random number generator routine of L'Ecuyer and Touzin (2000) was re-initialised for every batch of strategies, with the seed 12345. Moro (1995) normal variates were used. The regression was performed using SVD algorithm with all the paths and Powers polynomial functions.

Figure 7 Accuracy of the LSMC method when valuing American compound and mutually exclusive options using different basis functions.



(a) Compound options (b) Mutually exclusive options

Note. Please refer to note of the Figure 6

5.2.2. Comparison of the Algorithms to Value Mutually Exclusive Options. We start our analysis by comparing our algorthim with the one proposed by Gamba (2003). Figure 6 presents the results of the comparison between the two algorithms and their convergence when we increase the number of basis functions. The alternative algorithm that we propose has a statistically significant higher accuracy, particularly when valuing the best of two expansion options. All the DM tests for equal MSRE for polynomial degrees higher than 2 are statiscally significant at a 0.1% level.[19]

5.2.3. Comparison of the Polynomial Families. The eleven polynomial families are compared in the valuation of the 32 compound options and 48 mutually exclusive options (Figure 7, Tables 8 and 9). With the exception of the weighted Laguerre polynomials, the results of the

[19] Similar results are obtained with different simulation methods.

Table 8 DM tests of equal MSRE given by different polynomial families: portfolios of options.

Option	2	3	4	5	6	7	8	9	10
	Number of polynomials								
Compound									
$\mathcal{WL}_n$ vs $\mathcal{W}_n$	-2764.8***	-2215.2***	-925.6***	-420.1***	-215.7***	-112.0***	-56.5***	-16.7	16.3
$\mathcal{WL}_n$ vs $\mathcal{W}_{n+1}$	-907.7**	-629.5***	-250.4***	-114.8**	-27.3	-28.2*	-17.7	-1.4	—
Mut. exclusive									
$\mathcal{WL}_n$ vs $\mathcal{W}_n$	-2658.9***	-605.4***	-223.7***	-90.0***	-50.0***	-44.3***	-42.2***	-25.3**	-16.5**
$\mathcal{WL}_n$ vs $\mathcal{W}_{n+1}$	-456.0***	-135.1***	-72.4***	-20.4*	-34.8***	-33.7***	-25.9***	-18.4**	—

The table shows the DM test statistic (unit: $1E-06$). ***p-value$<$ 1%; **p-value$<$ 5%; *p-value$<$ 10%. $\mathcal{W}_n$ – Powers; $\mathcal{WL}_n$ – Weighted Laguerre. Simulation parameters as in Figure 6.

Table 9 DM tests of equal MSRE given by different number of polynomials of the weighted Laguerre family: portfolios of options.

Option	3 vs 2	4 vs 3	5 vs 4	6 vs 5	7 vs 6	8 vs 7	9 vs 8	10 vs 9
	Number of polynomials							
Compound	-709.5***	-358.4***	-214.6***	-131.2**	-55.2*	-17.4	-27.2	-36.8*
Mutually exclusive	-149.4***	-88.6***	-17.6*	-29.5***	-9.5**	-8.6**	0.6	2.0

The table shows the DM test statistic (unit: $1E-06$). ***p-value$<$ 1%; **p-value$<$ 5%; *p-value$<$ 10%. Simulation parameters as in Figure 6.

different families are almost identical. Weighted Laguerre polynomials produce more accurate estimates, particularly for lower polynomial degrees. Even if we consider the extra term and compare with a degree of $n+1$ of the other polynomial families, they still outperform the others up to five polynomials (Table 8).

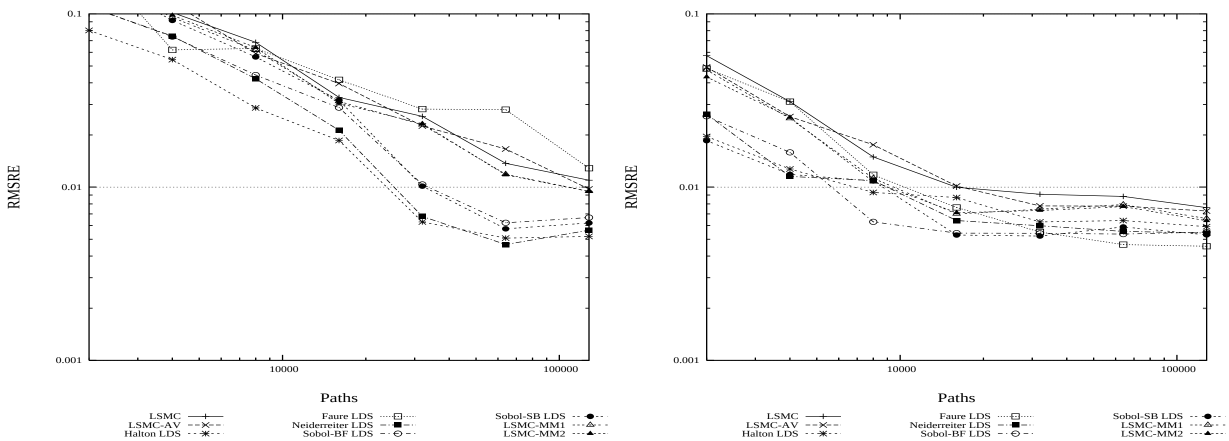
The results show that the LSMC estimates converge by increasing the number of basis functions, although at a decreasing rate (Table 9). While for the compound options significant accuracy improvements stop at 6 polynomials, for the mutually exclusive options that number is eight. It is also interesting to note that, even using the SVD regression method, numerical errors of the least squares algorithm are perceived for ten terms in some of the polynomial families for the mutually exclusive options.

5.2.4. Comparison of the Variance Reduction Techniques. The comparison of the various variance reduction techniques with different number of paths is presented in Figure 8 and Tables 10 and 11. For the compound options, another 32 options were analysed in order to have a more reliable empirical assessment, within a reasonable time, adding alternative values to e_1, e_2 and k. In the first example, we consider the possibility of the initial investment giving only 25% of the business, leaving the remaining 75% for the expansion phase ($e_1 = 0.25; e_2 = 0.75$). In the second example, we added the possibility of total abandonment ($k = 1$ and $X = K$).

By increasing the number of paths the accuracy improves notably, with DM statistics significant for almost all the methods, at 5% level, until 64,000 paths for the compound options and 16,000 paths for the mutually exclusive options[20]. The comparison of the alternative simulation methods (Tables 10 and 11) shows that the LDS can improve significantly for almost all of them, the accuracy of the simulation, with the exception of the Faure LDS for the compound options. The LDS performance seems to be dependent on the option and the number of paths used. The use of antithetic variates does not improve accuracy, while moment matching methods are effective, although not as effective as the LDS.

[20] The paths used increase from $2^1 * 1000$ to $2^7 * 1000$ paths. DM statistics are available upon request.

Figure 8 Accuracy of the LSMC method for the valuation of American compound and mutually exclusive options using different simulation methods.



(a) Compound options (b) Mutually exclusive options

Note. The simulation was done with Brownian bridges for the LDS methods; 6 and 8 weighted Laguerre polynomials were used for the compound and mutually exclusive options, respectively. Other simulations parameters are the same as in Figure 6. SIM: Simulation with pseudo-random numbers; AV - antithetic variates; BF - Bratley and Fox (1988); SB - Silva and Barbe (2003); MM1 - Moment matching method (1st moment); MM2 - Moment matching method (1st and 2nd moments).

Table 10 DM tests of equal MSRE given by different variance reduction methods: compound options.

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Sobol SB LDS
(1) LSMC	88.4	-47.7*	-51.4**	598.7***	-163.2***	-166.9***	-150.2***	-155.3***
(2) LSMC using AV	—	-134.8*	-138.6*	511.5***	-250.3***	-254.1***	-237.4***	-242.5***
(3) LSMC MM1	—	—	-3.7	-646.3***	115.5***	119.2***	102.5***	107.7***
(4) LSMC MM2	—	—	—	-650.0***	111.8***	115.5***	98.8***	103.9***
(5) LSMC Faure LDS	—	—	—	—	-761.8***	-765.5***	-748.8***	-754.0***
(6) LSMC Halton LDS	—	—	—	—	—	-3.7	13.0	7.8
(7) LSMC Niederreiter LDS	—	—	—	—	—	—	16.7	11.6
(8) LSMC Sobol BF LDS	—	—	—	—	—	—	—	-4.7

DM test statistic (unit: $1E-06$). ***p-value< 1%; **p-value< 5%; *p-value< 10%. BF stands for Bratley and Fox (1988), SB for Silva and Barbe (2003), MM1 for one moment matching and MM2 for two moment matching. Simulation was done with 64000 paths and remaining parameters as in Figure 8.

Table 11 DM tests of equal MSRE given by different variance reduction methods: mutually exclusive options.

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Sobol SB LDS
(1) LSMC	0.4	-51.2**	-51.4**	-41.9	-25.6	-59.5**	-71.5***	-72.5***
(2) LSMC using AV	—	-51.6**	-51.8**	-42.3*	-26.0	-59.8***	-71.9***	-72.9***
(3) LSMC MM1	—	—	-0.2	9.3	25.6	-8.3	-20.3*	-21.3**
(4) LSMC MM2	—	—	—	9.5	25.8	-80.2	-20.0*	-21.1**
(5) LSMC Faure LDS	—	—	—	—	16.3	-17.5	-29.6**	-30.6***
(6) LSMC Halton LDS	—	—	—	—	—	-33.9***	-45.9***	-46.9***
(7) LSMC Niederreiter LDS	—	—	—	—	—	—	-12.0*	-13.1
(8) LSMC Sobol BF LDS	—	—	—	—	—	—	—	-1.1

DM test statistic (unit: $1E-06$). ***p-value< 1%; **p-value< 5%; *p-value< 10%. BF stands for Bratley and Fox (1988), SB for Silva and Barbe (2003), MM1 for one moment matching and MM2 for two moment matching. Simulation was done with 16000 paths and remaining parameters as in Figure 8.

6. Concluding Remarks

We tested the choice of the polynomial family, number of basis functions, and number of paths on the accuracy of the LSMC American option valuation method when valuing vanilla and portfolios of options.

We show that any of the polynomial families tested provide almost identical results, with weighted Laguerre having a slight advantage over the others, mainly for portfolios of options. If this choice is not relevant in terms of accuracy, it does have an impact in terms of computation time. Therefore we recommend, when time is relevant, the use of the Powers polynomial family which is one of the fastest to compute.

As for increasing the number of basis functions, we conclude that after a certain number, accuracy will only be improved if the number of paths are also increased. The number of basis functions needed to have an accurate valuation depends on the type of option.

For the vanilla options, we also suggest and test two different procedures to estimate the continuation value of the American option: conditional estimation (CONT-CE) and Black-Scholes (CONT-BS). Our tests show that the use of CONT-CE can significantly reduce the computational burden of the algorithm by allowing the use of a faster regression algorithm without any significant loss of accuracy.

We also tested the accuracy of several variance reduction techniques: simulation with Black-Scholes and control variates for the vanilla options, and method of moments and quasi-random numbers for all the options. We show, for a large sample of 2,500 vanilla options, that the use of Halton LDS along with BB with powers family, five basis functions and 100,000 paths can reduce the RMSRE 4.2 times with an increase of only 1.43 times the computation time, when comparing with LSMC procedure with Powers polynomials and three basis functions. This improvement in accuracy is particularly noteworthy if we take into account that we are using the CONT-CE method to estimate the continuation value which in turn allows the use of a faster regression algorithm. Without this continuation estimation procedure a more sophisticate regression algorithm has to be used. In the case of the SVD regression algorithm the computation time would increase by 50%.

For all the options analysed, LDS, with very few exceptions with the Faure LDS, provides better results with fewer paths, while the use of antithetic variates does not improve the LSMC.

We have also proposed an alternative algorithm to value mutually exclusive options, which compares favorably with the algorithm of Gamba (2003), converging faster with the number of basis functions.

The analysis done in this paper assumes the existence of a fair benchmark for the option value. However, simulation methods outperform other valuation methods more significantly in the cases where such benchmarks are not available. In this case, a careful numerical analysis is required, using different simulation methods, regressors and paths to determine the accuracy of the valuation.

Acknowledgments

This work was supported by the Portuguese Foundation for Science and Technology - Project POCTI/GES/46947/2002. We would like thank Andrea Gamba, Dean Paxson, José Moreira, Mark Shackleton, Martin Martens and Stephen Taylor for helpful comments. All remaining errors are the sole responsibility of the authors.

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