

Performance Gauging in Discrete Time

Using a Luenberger Portfolio Productivity Indicator

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January 15, 2008

Abstract

This paper proposes a pragmatic, discrete time indicator to gauge the performance of portfolios over time. Integrating the shortage function (Luenberger 1995) into a Luenberger portfolio productivity indicator (Chambers 2002), this study estimates the changes in the relative positions of portfolios with respect to the traditional Markowitz mean-variance efficient frontier, as well as the eventual shifts of this frontier over time. Based on the analysis of local changes relative to these mean-variance and higher moment (in casu, mean-variance-skewness) frontiers, this methodology allows to neatly separate between on the one hand performance changes due to portfolio strategies and on the other hand performance changes due to the market evolution. This methodology is empirically illustrated using mimicking portfolio approach Fama and French (1996, 1997) using US monthly data from January 1931 to August 2007.

JEL: C43, G11.

Keywords: shortage function, mean-variance, mean-variance-skewness, efficient portfolios, Luenberger portfolio productivity indicator.

1 Introduction

It is unthinkable to gauge the performance of portfolio management solely by examining return levels. Since Markowitz (1952) groundbreaking work, every investor knows that risk must equally be considered. This dual objective of maximizing returns and minimizing risks turns performance evaluation into a complicated and controversial task. Indeed, no method that is currently available in the literature seems

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to be universally approved. There is an ever growing literature on this topic in traditional investment contexts (for surveys, see Galagedera (2003), Nitzsche, Cuthbertson, and O’Sullivan (2008) or Le Sourd (2007)), as well as in the specific context of hedge fund management (for instance, Géhin (2004) or Eling and Schuhmacher (2007)), and even a meta-literature criticizing these methods as well (see, for example, Amenc and Le Sourd (2005)).

Performance appraisal is linked to the theory of optimal investment choices, i.e., to the ability of investors to manage assets so as to maximize a utility function (i.e., a non-linear function based on a set of various moments characterizing the portfolios’ return distributions). In other words, performance evaluation analyzes the efficiency of an investment at least in terms of a traditional return-risk relationship. It is often assumed in this context that all investors have similar behaviors towards these dimensions (representative agent paradigm). The risk parameters in the utility function depend upon various parameters like investor’s objectives, preferences, time horizon,.... These simplifications are acceptable in cases where aggregate results suffice, but they are simply problematic in other cases. We argue later on that the methodology proposed in this paper allows for heterogeneity among investors and therefore answers quite a few of these issues.

We explicitly restrict this contribution to the traditional mean-variance model and a more recently introduced mean-variance-skewness framework (see Briec, Kerstens, and Jokung (2007)) for performance evaluation, ignoring any further higher moments (e.g., kurtosis). On the one hand, while the mean-variance approach is still a popular reference for practitioners and academics alike, its restrictive nature may lead to erroneous weights in portfolio selection. While some proposals are around allowing investors to maximise a utility function including higher moments (see, for example, Chunnachinda, Dandapani, Hamid, and Prakash (1997) or Jondeau and Rockinger (2003)), the empirical evidence provides mixed support at best. Nevertheless, enlarging the classical framework with a mean-variance-skewness model is a potentially interesting improvement for fund managers. On the other hand, the method developed in this research can be easily extended to consider all higher moments (Briec, Kerstens, and Jokung (2007)), at least if one accepts a complexification in optimisation algorithms and an increase in computational time alike.

Recently, a new approach has been proposed in the investment literature by Cantaluppi and Hug (2000) that directly measures the performance of a portfolio by reference to its maximum potential on the (ex ante or ex post) portfolio frontier. Their proposal is in fact intimately related to some explicit efficiency measures transposed from production theory into the context of portfolio benchmarking by Morey and Morey (1999). Informally speaking, their first measure computes the maximum mean return expansion while the risk is fixed at its current level, while an alternative risk contraction function measures the maximum proportionate reduction of risk while fixing the mean-return level.¹ These approaches are

¹Cantaluppi and Hug (2000) talk similarly about ‘return loss’ and ‘surplus risk’.

generalized by Bricc, Kerstens, and Lesourd (2004) who integrate the shortage function (Luenberger (1995)) as an efficiency measure into the mean-variance model and also develop a dual framework to assess the degree of satisfaction of investors preferences. Similar to developments in other fields, this leads to a decomposition of portfolio performance into allocative and portfolio efficiency. The advantage is that this shortage function is compatible with general investor preferences and that it can be extended to higher dimensional spaces (e.g., the mean-variance-skewness space (see Bricc, Kerstens, and Jokung (2007))).

This paper tackles the problem of tracing the performance of portfolios in discrete time with respect to the ever changing portfolio frontiers by borrowing from recent developments in the theory of productivity indices (see Diewert (2005) for a review). Employing the shortage function, a Luenberger portfolio productivity indicator (Chambers (2002)) is introduced that allows for the estimation of the relative positions of portfolios with respect to changes in the efficient frontier, and that offers an accurate local measure of the eventual shifts of this frontier over time. The proposed methodology for fund performance appraisal in discrete time is therefore founded in a well-established theoretical framework. We show later on that the Luenberger portfolio productivity indicator and especially its decomposition provide an excellent measurement tool to reconsider the traditional performance attribution question: what is the individual contribution of fund managers to portfolio performance and what is due to changes in the financial market.

The next section is devoted to a brief presentation of the relevant literature concerning portfolio performance evaluation and the more recently introduced efficiency measures operating relative to the portfolio frontier. Section 3 introduces the basic theoretical building blocks for the analysis. In particular, it introduces the shortage function as proposed by Luenberger (1992), studies its axiomatic properties, and the link between the shortage function and the direct and indirect mean-variance utility functions. Thereafter, the Luenberger portfolio productivity indicator and its decomposition are presented. Section 4 presents some technical and strategic aspects of the empirical procedures and discusses the choice of data set. Empirical results are provided in Section 5. Conclusions and issues for future work are summarized in the final section.

2 Performance Measurement in Investment: A Brief Review

2.1 Traditional Performance Measures

An enormous and ever growing amount of literature about portfolio performance evaluation derives more or less directly from the initial works of Markowitz (1952) and the founders of "Modern Portfolio Theory" with the development of asset pricing theories such as the CAPM (Tobin (1958), Sharpe (1964), Lintner

(1965) and Mossin (1966)). During these early years, performance appraisal evolved from total-risk foundations (e.g., the standard deviation or variance of returns) to performance indexes where the returns in excess of the risk-free rate are matched with some risk measure. Among these early contributions, two classics are on the one hand the Sharpe (1966) ratio and on the other hand the Treynor (1965) ratio. These ratios gauge performance without any benchmark (Le Sourd (2007)). More recently, these indicators have taken benefit from the development of value at risk (VaR) techniques (especially in the hedge fund industry context: see Gregoriou and Gueyie (2003)). Another popular performance indicator is Jensen's α (Jensen (1968)), whereby performance is measured by the excess return over the equilibrium reward calculated with the CAPM. Since it does use a benchmark, it is more relative in nature (see Le Sourd (2007)).

This early tradition has received a wide variety of criticisms because of the supposed weaknesses of the underlying equilibrium models on which performance indicators were build and the implicit assumption that financial asset returns are normally, independently and identically distributed, among others.²

The first series of objections touches upon several issues. One is the irrelevance of unconditional performance evaluation: investors are supposed to form expectations about returns irrespective of their expectations over the states of the economy, which may lead to various distortions in performance levels or stability. It has meanwhile been acknowledged that agents use information to condition their expectations (see Fama and French (1989)), making unconditional evaluation techniques rather irrelevant. More generally, the question of the benchmark choice is also clearly central in portfolio performance gauging, especially when funds have different management styles. When the reference point is inappropriate, then the measure is biased (Grinblatt and Titman 1994). For instance, the evaluated portfolio might be over-rated if the benchmark is inefficient (see Roll (1978)). One potential solution consists in improving the model by which expected returns are calculated (using APT or multi-factor models, such as Fama and French (1992, 1993)) or to obtain performance evaluation independent from the market model (for instance, Cornell (1979) or Grinblatt and Titman (1993)). For example, the Fama and French (1993) three-factor model has been developed because the CAPM approach proved to perform poorly in explaining realized returns.³

Another series of problems in relation with the underlying equilibrium models comes from the non-stability of risk-free rates or the volatility of betas. In these cases, performance evaluation is clearly biased because equilibrium returns are misevaluated or simply because a constant beta is irrelevant. These issues have been recognized by Jensen (1972) or Dybvig and Ross (1985). One answer is to allow for time-varying betas in equilibrium models (e.g., Shanken (1990) or Ferson and Schadt (1996)). Another answer to this

²See, for instance, the debate around CAPM in Fama and French (2004).

³In this model, R_i , the return of fund i , is explained by a combination of market risk factor in excess of the risk free rate and two additional factors, respectively *size risk* (measured as the difference of the returns between a portfolio composed of small firms and one composed of big firms) and *value risk* (measured as the difference between the returns of two portfolios, one composed of firms with high book-to-market ratios and one with low book-to-market ratios).

issue and the corresponding misvaluation of Jensen's alpha has been proposed by Grinblatt and Titman (1989).

Another source of problems in performance evaluation is the non-Gaussian nature of stock returns due to dynamic trading strategies.⁴ Problematic here is the underestimation of risk in performance appraisal. With asymmetric distributions or fat tails, performance gauging must take into account moments of order higher than 2 (like skewness, kurtosis or even beyond: see Ang and Chua (1979)). A ratio to account for higher moments has been proposed by Sortino and der Meer (1991). More recently, various attempts to consider this issue have been formulated (see for example Stutzer (2000)): some of these derive from VaR (Gregoriou and Gueyie (2003)), some are extensions of the Sharpe ratio (Madan and McPhail 2000) or the Sortino ratio (Kaplan and Knowles 2004). Others propose generalized methods such as the Omega measure (see Keating and Shadwick (2002) and Kazemi, Schneeweis, and Gupta (2003)). In relation to model specification issues in a non-normal world, Harvey and Siddique (2000) have proposed to incorporate a conditional skewness measure to take into account the necessary reward for systematic skewness in funds returns.

Many of these traditional performance measures are frequently associated with a prominent question in the investment industry, namely performance attribution. While it is in blatant contradiction with CAPM theory, performance appraisal is linked to *stock picking*⁵ and *market timing*⁶ (see Henriksson and Merton (1981) or Merton (1981)). In other terms, the investment industry is always looking for tools to trace good fund managers that can regularly exploit market anomalies and that could pick stocks in the market to obtain an α that is significantly different from 0 and manage their portfolios' betas dynamically.

Summing up this brief literature review, the standard approaches to investment performance appraisal may appear unsatisfactory with respect to at least three generic shortcomings: (i) they may yield under- or over-estimations because of the selection of an inappropriate benchmark or equilibrium models for expected returns, (ii) they may be biased due to the non-normal nature of return distributions or unknown utility functions for investors when higher moments have to be considered, and (iii) they may be unstable because of the dependency of the measure with the time-frame in which it is computed. One could also add that these measures usually rely upon other strong assumptions, such as the uniqueness of investor's preferences. The rather recent frontier-based measures may be a solution for some of these shortcomings.

⁴Hedge funds are especially concerned by this issue.

⁵The ability of fund managers to choose good assets obtaining excess risk-adjusted rates of return.

⁶The ability of fund managers to correctly anticipate market turning points.

2.2 Frontier-Based Efficiency Measures

Frontier-based measures of fund performance have gained some limited popularity since the late nineties.⁷ One of the seminal articles in the finance literature is the work of Cantaluppi and Hug (2000) who propose an 'efficiency ratio' in relation to the mean-variance (Markowitz) efficient frontier.⁸ In their search for a more universal approach to portfolio performance measurement, they contest the relative nature of most current proposals that define performance with respect to some other supposedly relevant portfolio or index. Instead, they suggest looking for the maximum performance that could have been achieved by a given portfolio relative to a relevant portfolio frontier, i.e., a frontier resulting from a particular choice of investment universe and satisfying any additional constraints imposed on the investor. Basically, it is a matter of utilizing the traditional ex ante computation of optimal portfolios in an ex post fashion. Ex ante, one first selects the investment universe; then one determines the investment horizon with corresponding estimates for future returns, risks, and correlations for the asset universe; and finally one computes an efficient frontier based on these estimates and the investment restrictions. This same process can be executed ex post to benchmark portfolios: computations are then simply performed with historical rather than expected values. Since a portfolio manager that ex ante would have had perfect foresight could have invested in a frontier optimal portfolio, the ex post efficient frontier provides a natural benchmark for performance gauging and Cantaluppi and Hug (2000) informally present both a 'return loss' and a 'surplus risk' efficiency measure.

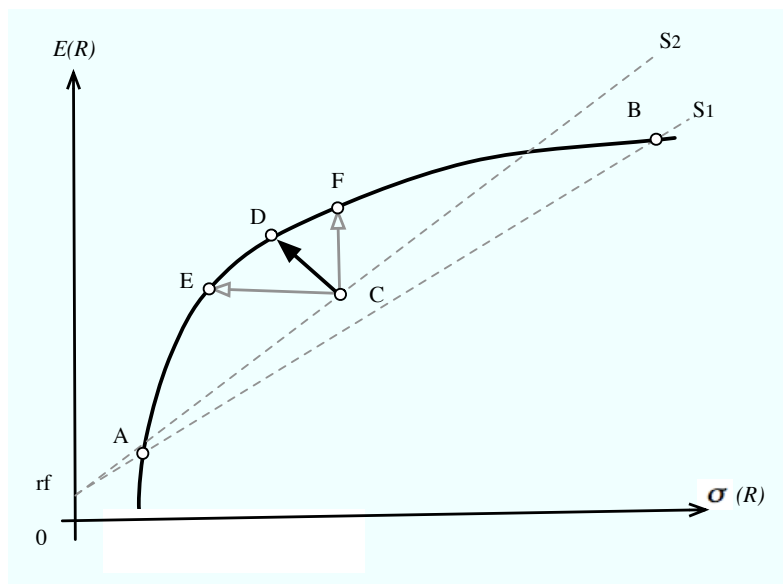


Figure 1: Sharpe Ratio vs. Efficiency Measures

We can illustrate this basic point with Figure 1 (very much in the spirit of Cantaluppi and

⁷For a survey of this burgeoning literature: see Galagedera (2003).

⁸As stated by these authors, this is not strictly speaking a new method since it has been employed by, e.g., Kandel and Stambaugh (1995) as well.

Hug (2000)) which compares the Sharpe ratio and the efficiency ratio idea. This figure is drawn in the mean-standard deviation space and depicts three portfolios A, B, and C with respect to a common portfolio frontier. Starting with the Sharpe ratio, it is immediately apparent that portfolio C enjoys a higher Sharpe ratio compared to portfolios A and B, despite the fact that the latter portfolios are part of the Markowitz frontier while portfolio C is not. To remedy this problem, the 'efficiency ratio' approach suggests measuring the inefficiency of portfolio C using either a 'return loss' efficiency measure (vertical projection line) or a 'surplus risk' efficiency measure (horizontal projection line).

Sengupta (1989) is to our knowledge the first author to transpose the idea of an efficiency measure into a traditional Markowitz portfolio frontier context. Morey and Morey (1999) are the first to give a precise formal definition of the 'return loss'⁹ and 'surplus risk' efficiency measures proposed by Cantaluppi and Hug (2000).¹⁰ In the same vein, Briec, Kerstens, and Lesourd (2004) are the first to develop a link between portfolio efficiency measures and mean-variance utility, which leads them to propose an efficiency measure that simultaneously seeks to improve the return and to reduce the variance of a given portfolio.¹¹ In Figure 1, this leads -intuitively speaking- to the projection of portfolio C into a diagonal direction towards the Markowitz frontier. Theoretically, these contributions bring portfolio theory in line with developments in production theory and elsewhere in micro-economics, where distance functions are proven concepts related to efficiency measures that allow to develop dual relations with economic (e.g., mean-variance utility) support functions.

More or less independently, a variety of authors have been transposing efficiency measures, that are related to distance functions (i.e., functional representations of choice sets), from production theory into finance. This literature employs mathematical programming techniques to estimate non-parametric frontiers of choice sets (e.g., technologies) and positions any observation with respect to the boundary of these choice sets.¹² This has sometimes been accompanied with the utilization of production frontiers to rate, for instance, the performance of mutual funds along a multitude of dimensions (rather than mean and variance solely). To the best of our knowledge, the seminal article of Murthi, Choi, and Desai (1997) is a case in point. These authors employ return as a desirable output to be increased and risk and a series of transaction costs as an input to be reduced and measure the performance of each mutual fund with respect to a piecewise linear frontier (rather than a traditional non-linear portfolio frontier). More recently, Choi and Murthi (2001) employ a similar framework and compare the resulting efficiency measures to the traditional Sharpe ratio. The same idea has been employed in the context of asset selection, whereby changes in stock performance are related to changes in production or operating efficiency (see the seminal articles of Alam and Sickles (1998) and Chu and Lim (1998)). Preliminary

⁹This is also the approach taken by Sengupta (1989)

¹⁰Even though Morey and Morey (1999) seem unaware of these authors.

¹¹A generalization of the same approach into mean-variance-skewness space is developed in Briec, Kerstens, and Jokung (2007).

¹²This approach is often referred to with the moniker Data Envelopment Analysis (DEA).

results seem to suggest that changes in productive efficiency are at least partially translated into changes in stock prices (see also Edirisinghe and Zhang (2007) for a further development).

Therefore, without pretending to settle these issues once and for all, it is possible to state that frontier-based portfolio benchmarking methods at least partially remedy some of the generic shortcomings of traditional performance measures mentioned earlier: (i) they select an appropriate benchmark in terms of the ex post portfolio frontier, and (ii) they can be perfectly generalized to higher moments in case of non-normal return distributions. It remains to be seen how they behave under extensive stress testing. This contribution aims to remedy to some extent the third defect, i.e., the instability of performance measures because of the dependency of these measures with respect to the time-frame in which these are computed. We resolve this at least partially by defining a portfolio productivity indicator based upon general efficiency measures that allows tracking the evolution in financial markets in discrete time. This is -to the best of our knowledge- the first contribution drawing upon index theory to resolve practical portfolio benchmarking issues.

3 Static Portfolio Frontiers and Their Evolution in Discrete Time

3.1 Static Portfolio Frontiers: The Shortage Function as Efficiency Measure

To introduce some basic notation and definitions, consider the problem of selecting a portfolio from n financial assets at time period t . Let R_1^t, \dots, R_n^t be random returns of assets $1, \dots, n$ in period t . For each time period t , each of these assets is defined through some expected return $E(R_i^t)$ for $1, \dots, n$. Furthermore, returns of assets i and j are correlated, so that the variance-covariance matrix for time period t (Ω^t) is defined as $\Omega_{i,j}^t = Cov(R_i^t, R_j^t)$ for $i, j \in \{1, \dots, n\}$.

Notice that by adding the skewness-coskewness tensor, the extension to the mean-variance-skewness frontier is rather straightforward. Indeed, the shortage function is compatible with general investor preferences (favoring uneven moments and disliking even moments). Thus, in the mean-variance-skewness space a shortage function is capable to look simultaneously for reductions in risk and augmentations in return and skewness. In view of the familiarity of the traditional mean-variance frontier notion and for reasons of space, the formal analysis is limited to the mean-variance case, while the interested reader is referred to Briec, Kerstens, and Jokung (2007) for details on the use of the shortage function relative to the mean-variance-skewness frontier.

A portfolio $x^t = (x_1^t, \dots, x_n^t)$ is simply a vector of weights specified over these n financial assets that sums to unity $\left(\sum_{i=1, \dots, n} x_i^t = 1 \right)$. If shorting is impossible, then these weights must satisfy non-negativity

conditions ($x_i \geq 0$). The return of portfolio x^t is: $R^t(x) = \sum_{i=1, \dots, n} x_i^t R_i^t$. Therefore, the expected return of portfolio x^t is $E(R^t(x)) = \sum_{i=1, \dots, n} x_i^t E(R_i^t)$ and its variance is $V(R^t(x)) = \sum_{i,j} x_i^t x_j^t Cov(R_i^t, R_j^t)$.

Therefore, the set of admissible portfolios can be written in general as¹³:

$$\mathfrak{S}^t = \left\{ x^t \in \mathbb{R}^n : \sum_{i=1, \dots, n} x_i^t = 1, x^t \geq 0 \right\}. \quad (3.1)$$

Following the seminal approach by Markowitz (1952), one can define at time period t the mean-variance representation of the set \mathfrak{N}^t of portfolios as:

$$\mathfrak{N}^t = \{(V(R^t(x^t)), E(R^t(x^t))) ; x^t \in \mathfrak{S}\} \quad (3.2)$$

Since such a representation cannot be used for quadratic programming because the subset \mathfrak{N}^t is non-convex (see, e.g., (Luenberger 1998)), the above set is extended by defining a mean-variance (portfolio) representation set through:

$$\mathfrak{R}^t = \{\mathfrak{N}^t + (R_+ \times (-R_+))\} \cap R_+^2 \quad (3.3)$$

Briec, Kerstens, and Lesourd (2004) show that it is useful to rewrite the above subset as follows:

$$\mathfrak{R}^t = \{(V', E') \in R_+^2 ; \exists x^t \in \mathfrak{S}, (-V', E') \leq (-V(R(x^t)), E(R(x^t)))\} \quad (3.4)$$

The addition of the cone is necessary for the definition of a sort of "free-disposal hull" of the mean variance representation of feasible portfolios and is compatible with the definition in Markowitz (1952). Markowitz (1952) proposes a general algorithm to evaluate the above mean-variance efficient frontier.

Before generalizing the well-known Markowitz's approach, we introduce the shortage function, a concept introduced by Luenberger (1992, 1995) in a production theory context where it measures the distance between some point of the production possibility set and the Pareto frontier. Before we introduce this function formally, it is of interest to focus on the basic properties of the subset \mathfrak{R}^t on which we define the shortage function below. Briec, Kerstens, and Lesourd (2004) have shown that \mathfrak{R}^t is convex, closed and satisfies a free disposal assumption. These properties of the representation set allow defining an efficiency measure in the context of Markowitz portfolio theory. We now introduce for time period t the shortage function defined by:

Definition 3.1 *The function defined as:*

$$S_g(x^t) = \max \{ \delta ; (V(R^t(x^t)) - \delta g_V^t, E(R^t(x^t)) + \delta g_E^t) \in \mathfrak{R} \}$$

¹³This set of admissible portfolios can be easily adapted for additional constraints (e.g., transaction costs) that can be written as linear functions of asset weights (Pogue (1970)): see Briec, Kerstens, and Lesourd (2004).

is the shortage function at time period t for portfolio x^t in the direction of vector $g^t = (-g_V^t, g_E^t)$.

Notice that the direction where one looks for efficiency improvement depends on time. The purpose of this time-dependent direction g^t is to cater for the potentially changing preferences of the investor over time. The shortage function looks for improvements in the direction of both an increased mean return and a reduced risk. The pertinence of the shortage function as a portfolio management efficiency indicator results from its properties. In particular, this indicator characterizes the Markowitz frontier, is weakly monotonic and continuous on \mathfrak{X}^t , and generalizes the Morey and Morey (1999) approaches who look either for return expansions or risk reductions only.¹⁴

Markowitz (1952) also proposed an optimisation program in a dual, mean-variance utility based framework to determine the portfolio corresponding to a given degree of risk aversion. Such a portfolio maximizes a mean-variance utility function defined by:

$$U_{(\rho, \mu)}(x^t) = \mu E(R^t(x^t)) - \rho V(R^t(x^t)) \quad (3.5)$$

where $\mu \geq 0$ and $\rho \geq 0$. The quadratic optimisation program may simply be written as follows:

$$\begin{aligned} \max \quad & U_{(\rho, \mu)}(x^t) = \mu E(R^t(x^t)) - \rho V(R^t(x^t)) \\ \text{s.t.} \quad & \sum_{i=1, \dots, n} x_i^t = 1, x_i^t \geq 0 \end{aligned} \quad (3.6)$$

where the ratio $\varphi = \rho/\mu \in [0, +\infty]$ stands for risk aversion.

To provide now a dual interpretation of the shortage function, we first need to define the mean-variance indirect utility function.

Definition 3.2 *The function $U^{t, *}(\rho, \mu) = \max U_{(\rho, \mu)}(x^t)$ defined in expression (3.6) is called the indirect mean-variance utility function at time period t .*

Thus, the support function of the Markowitz frontier is given by the indirect utility function $U^{t, *}(\rho, \mu)$.

From the duality result by Luenberger (1995), who connected expenditure and shortage functions, the shortage function can be derived from the indirect mean-variance utility function, and conversely through a dual pair of relationships. Following this dual relation between shortage function and mean-variance utility function, it is also possible to disentangle between various efficiency notions when evaluating potential improvements in portfolios. By analogy with other domains in economics, Bricc, Kerstens, and Lesourd (2004) distinguish formally between (i) Portfolio efficiency (*PE*), (ii) Allocative efficiency (*AE*), and (iii) Overall efficiency (*OE*). For reasons of space and given that the empirical application ignores the

¹⁴See Bricc, Kerstens, and Lesourd (2004) for details.

utility approach, we provide the intuition behind this taxonomy, but refer the reader to Briec, Kerstens, and Lesourd (2004) for technical details.

Starting from a portfolio under evaluation, Portfolio efficiency (PE) guarantees only reaching a point on the Markowitz frontier using the shortage function. However, this point need not necessarily coincide with a portfolio maximizing the investor's indirect mean-variance utility function. Starting again from a portfolio under evaluation, it is possible to define another efficiency measure that guarantees reaching the point on the Markowitz frontier maximizing the mean-variance utility function. For this purpose, Overall efficiency (OE) is the ratio between (i) the difference between indirect mean-variance utility and the value of the mean-variance utility function for the evaluated portfolio and (ii) a normalisation based on the direction vector. Finally, since the Overall efficiency (OE) notion is clearly more demanding than the Portfolio efficiency (PE) concept, one can define a residual notion of Allocative efficiency (AE) which is simply the difference between Overall efficiency (OE) and Portfolio efficiency (PE). Thus, Allocative efficiency (AE) measures the needed portfolio reallocation along the portfolio frontier to achieve the maximum of the indirect mean-variance utility function.

In a given time period this whole approach can be illustrated in the mean-variance space in Figure 2. The shortage function looks for improvements in the direction of both an increased mean return and a reduced risk. For instance, the inefficient portfolio A is projected onto the mean-variance frontier at point B. Furthermore, given knowledge about the investor's risk-aversion, one can establish the ideal point on the portfolio frontier conforming to his/her preferences (i.e., the tangency point of the mean-variance utility function and the Markowitz frontier). In Figure 2, point D maximizes the direct utility function. To illustrate the above decomposition starting from the portfolio denoted by point A, we now immediately see that $OE = |CA|$, $PE = |BA|$, and $AE = |CB|$.

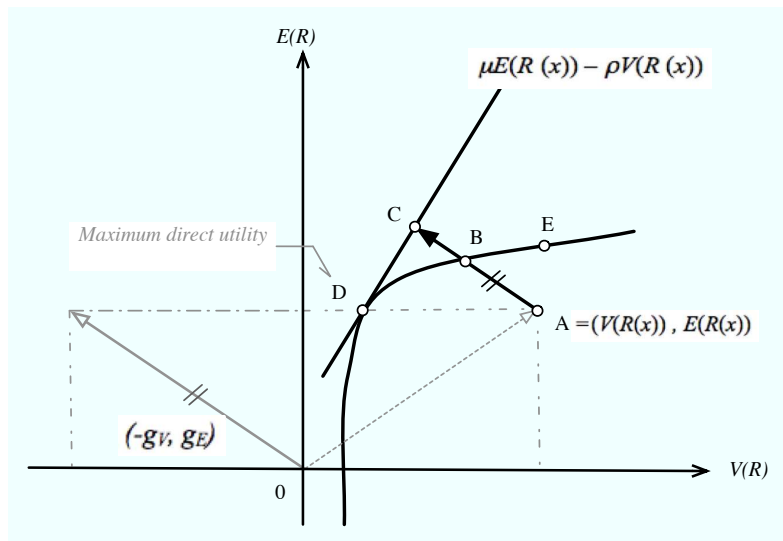


Figure 2: Shortage Function & OE Decomposition

3.2 Portfolio Performance Change in Discrete Time: A Luenberger Portfolio Productivity Indicator

This subsection is concerned with the dynamic study of portfolio performance in discrete time utilizing the shortage function. Using a recent Luenberger productivity indicator based on some combinations of shortage functions (see Chambers (2002)), our new proposal applies this Luenberger indicator to measuring dynamic portfolio performance.

Compared to the representation set at time period b , the shortage function of a portfolio observed at time period a is:

$$S_g^b(x^a) = \max_{\delta} \{ \delta \geq 0; (V(R^a(x^a)) - \delta g_V^a, E(R^a(x^a)) + \delta g_E^a) \in \mathfrak{R}^b \}, \quad (3.7)$$

where $(a, b) \in \{t, t+1\} \times \{t, t+1\}$. Remark that $E(R^a(x^a))$ stands for the expected return of portfolio x^a calculated at time period a , and an analogous interpretation applies to the covariance matrix.

The difference derived from expressions (3.7) between two periods at $a = t$ and $a = t+1$, given a representation set at $b = t$ yields:

$$\Delta_t = S_g^t(x^t) - S_g^t(x^{t+1}). \quad (3.8)$$

Considering the representation set at $b = t+1$, we can compute a similar indicator:

$$\Delta_{t+1} = S_g^{t+1}(x^t) - S_g^{t+1}(x^{t+1}). \quad (3.9)$$

To avoid a choice between time periods, it is natural (see, for instance, Chambers (2002)) to take the arithmetic mean of the two indicators defined above to obtain the following discrete time Luenberger portfolio productivity indicator of performance change:

$$L(x^t, x^{t+1}) = \frac{1}{2}(\Delta_t + \Delta_{t+1}), \quad (3.10)$$

which is the portfolio analogue of a Luenberger productivity indicator.¹⁵

This performance change can be equivalently written:

$$L(x^t, x^{t+1}) = S_g^t(x^t) - S_g^{t+1}(x^{t+1}) + \frac{1}{2} [(S_g^{t+1}(x^{t+1}) - S_g^t(x^{t+1})) + (S_g^{t+1}(x^t) - S_g^t(x^t))] \quad (3.11)$$

¹⁵Notice that the Luenberger productivity indicator does not satisfy circularity in this formulation. There are various ways to make it circular. Furthermore, following Diewert (2005), observe that indexes are based on ratios, while indicators are based on differences. Ratio and difference approaches to index numbers differ in terms of basic properties of practical significance: e.g., (i) ratios are unit invariant, differences are not, (ii) differences are invariant to changes in origin, ratios are not, (iii) ratios have difficulties handling zeros, differences have not, etc. In general, a variety of well-known issues in index theory (see, e.g., Diewert (2005)) can probably shed light on some new problems that may crop up when transposing index numbers into portfolio theory.

where the difference outside the brackets measures the efficiency change of the shortage function between periods t and $t + 1$:

$$EFFCH = S_g^t(x^t) - S_g^{t+1}(x^{t+1}), \quad (3.12)$$

and the arithmetic mean of the two differences inside the brackets captures the shift in portfolio performance between the two periods evaluated at the portfolio composition in $t + 1$ and at the portfolio composition in t :

$$FRCH = \frac{1}{2} [(S_g^{t+1}(x^{t+1}) - S_g^t(x^{t+1})) + (S_g^{t+1}(x^t) - S_g^t(x^t))]. \quad (3.13)$$

Hence, the above equation decomposes portfolio performance change into two components: one representing efficiency change relative to a moving portfolio frontier ($EFFCH$), another indicating the average change in the portfolio frontier itself ($FRFCH$). This decomposition offers a measurement framework for financial market performance gauging because: on the one hand, efficiency change ($EFFCH$) captures the performance of the fund managers over time relative to a shifting portfolio frontier, and on the other hand portfolio frontier change ($FRFCH$) indicates how the financial market itself has locally changed over time and enlarges or reduces the opportunities available to investors. When the Luenberger indicator of portfolio performance change $L(x^t, x^{t+1})$ or any of its components ($EFFCH$ or $FRCH$) is positive (negative), then portfolio performance increases (decreases) between the two time periods considered.

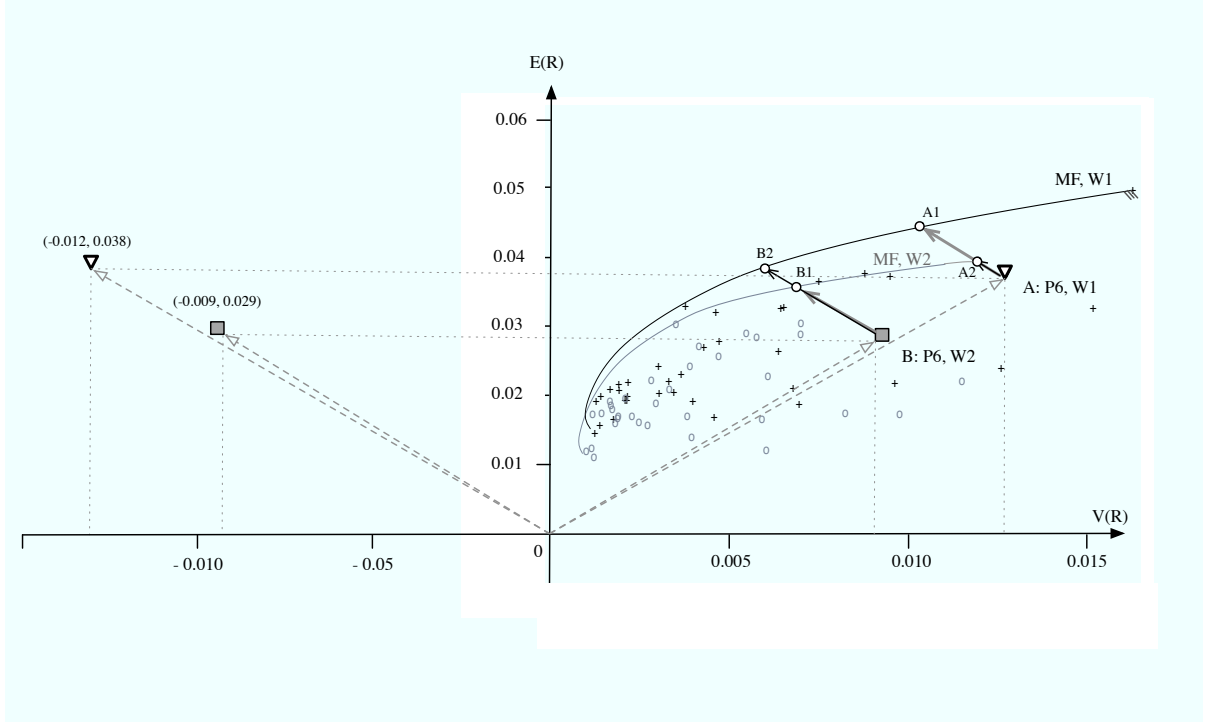


Figure 3: Luenberger Portfolio Productivity Indicator & Its Decomposition: Portfolio Nr. 6

Figure 3 illustrates the above performance indicator with $g = (-g_V^a, g_E^a)$. In this figure, we illustrate the Luenberger indicator and its decomposition with the help of a certain portfolio 6 (drawn from the empirical analysis: see details below). The two overlapping windows $W1$ and $W2$ range respectively from 1934/01 till 1937/01 and 1934/02 till 1937/02. Figure 3 plots two Markowitz frontiers computed with the returns in the sample over $W1$ and $W2$. Portfolios are plotted using crosses and dots, except $P6$ that is once plotted with a black triangle in $W1$ and once with a gray square in $W2$. Arrows indicate the respective distances towards the frontiers in both periods ($SH11, SH12, SH21$, and $SH22$ correspond respectively to the measures $S_g^t(x^t), S_g^{t+1}(x^t), S_g^t(x^{t+1})$, and $S_g^{t+1}(x^{t+1})$ defined before). Table 1 accounts for the differences between the variation of one classical performance measure (Sharpe) and the Luenberger indicator and its components. To obtain $EFFCH = 0.0320$, it suffices to compute: $0.3795 - 0.3475$. Clearly, this portfolio has moved closer to the portfolio frontier over time yielding a positive $EFFCH$. Computing the $FRCH = -0.0729$ requires the following calculations: $0.5((0.3475 - 0.4191) + (0.3053 - 0.3795))$. The negative number simply reflects the productivity decrease due to the inward shift of the portfolio frontier around portfolio 6. Notice that this inward shift of the portfolio frontier is not a global phenomenon: it does not affect the lower risk-return combinations. The Luenberger indicator is simply the sum of these two components: in this case, the improvement of the $EFFCH$ is overruled by the local deterioration of the $FRCH$ and we end up with a negative portfolio frontier productivity change.

Table 1: Numerical Example: Decomposition of Luenberger Productivity Indicator

Shortage Functions			Luenberger Decomposition	
$S_g^t(x^t)$	(SH11)	0.3795	$EFFCH$	0.0320
$S_g^{t+1}(x^t)$	(SH12)	0.3053	$FRCH$	-0.0729
$S_g^t(x^{t+1})$	(SH21)	0.4191	$L(x^t, x^{t+1})$	-0.0409
$S_g^{t+1}(x^{t+1})$	(SH22)	0.3475		

Turning to computational matters, the representation set \mathfrak{R} (3.3) is used to directly compute the various shortage functions and thus the Luenberger indicators by recourse to standard quadratic optimisation methods. Assume a sample of m portfolios $x^{1,t}, x^{2,t}, \dots, x^{m,t}$ are observed over a given finite time horizon $t = 1, \dots, T$. Now, consider a specific portfolio $x^{k,t}$ for $k \in \{1, \dots, m\}$ at time period t whose performance needs gauging. To calculate the Luenberger performance indicator, the four different shortage functions composing it must be computed by solving a simple quadratic program. To solve for $S_g^t(x^t)$, the following

basic quadratic program must be computed:

$$\begin{aligned}
& \max \delta & (3.14) \\
& s.t. E(R^t(y^{k,t})) + \delta g_R \leq \sum_{i=1, \dots, n} x_i^t E(R_i^t) \\
& V(R^t(y^{k,t})) - \delta g_V \geq \sum_{i,j} \Omega_{i,j}^t x_i^t x_j^t \\
& \sum_{i=1, \dots, n} x_i^t = 1, x_i^t \geq 0, \delta \geq 0, i = 1, \dots, n.
\end{aligned}$$

where δ and x^t are decision variables to be optimized over the portfolio set. This program is computed for each portfolio with respect to its portfolio set at period t and period $t+1$. For the latter computation, one simply replaces the left-hand side of the first two constraints by the return and risk of the evaluated portfolio in period $t+1$ to end up with $S_g^t(x^{t+1})$. To compute the remaining two shortage functions, one proceeds as follows. To obtain $S_g^{t+1}(x^{t+1})$, all that is needed is to replace the superscript t by $t+1$ everywhere in the above quadratic program. $S_g^{t+1}(x^t)$ results when replacing the left-hand side of the first two constraints by the return and risk of the evaluated portfolio in period t while the portfolio set remains fixed at period $t+1$ (like for $S_g^{t+1}(x^{t+1})$).

We add two remarks on computational issues. First, while in principle several options are available for the choice of the direction vector (see Bricc, Kerstens, and Lesourd (2004) for details), we opt here to employ the observation under evaluation itself ($g^a = (-g_V^a, g_E^a) = (-V(R^a(x^a)), E(R^a(x^a)))$). In this case, the shortage function measures the maximum percentage of simultaneous risk reduction and expected return augmentation. Second, it is well known that in certain cases the shortage function is not well-defined and achieves a value of infinity (e.g., Luenberger (1995)).¹⁶ Focusing on the choice of direction vector, Bricc and Kerstens (2007) show that the shortage function, one of the most general distance functions available in the literature so far, may not achieve its distance in the general case where a point need not be part of technology and where the direction vector can take any value. As a consequence, the feasibility of the Luenberger productivity indicator can in general not be guaranteed. Apart from reporting any eventual infeasibilities, that contribution shows that there is no easy solution in general. Notice that the efficiency measures proposed by Morey and Morey (1999), as special cases of the shortage function approach, are more vulnerable to the infeasibility issue. Its incidence in a portfolio context has never been reported.

Finally, though the Luenberger indicator is not based on a utility approach, it is important to realize that the performance changes traced over time do reflect gains and losses in utility. Using the previously mentioned duality result, (3.8) and (3.9) can be directly expressed with respect to the differential of the

¹⁶This is related to the property of determinateness in index theory which can be loosely stated as requiring that an index remains well-defined when any of its arguments is not.

mean-variance utility function. This can be shown introducing the adjusted risk aversion function (see Brier, Kerstens, and Lesourd (2004)):

$$(\rho^t, \mu^t)(x^t) = \arg \min \{U^t * (\rho, \mu) - U_{(\rho, \mu)}(x^t); \mu g_E + \rho g_V = 1, \mu \geq 0, \rho \geq 0\}, \quad (3.15)$$

that implicitly characterizes the agent's risk aversion.¹⁷ It can be useful to calculate the differential of the shortage function to express (3.8) and (3.9) with respect to the mean-variance utility function. Brier, Kerstens, and Lesourd (2004) show that:

$$\frac{\partial S_g(x^t)}{\partial x^t} = \frac{\partial U_{(\rho^t, \mu^t)(x^t)}(x^t)}{\partial x^t} = (\mu^t I - 2\rho^t \Omega) R^t \quad (3.16)$$

$$\left(\frac{\partial S_g(x^t)}{\partial R^t(x^t)}, \frac{\partial S_g(x^t)}{\partial V(R^t(x^t))} \right) = (\rho^t, -\mu^t)(x^t) \quad (3.17)$$

where R^t denotes the vector of the expected asset returns. From equation (3.16) the performance changes can be now be rewritten as:

$$\Delta_t = \int_{x^{t+1}}^{x^t} \frac{\partial S_g^t(x)}{\partial x} .dx = \int_{x^{t+1}}^{x^t} \frac{\partial U_{(\rho, \mu)^t(x)}(x)}{\partial x} .dx \quad (3.18)$$

and

$$\Delta_{t+1} = \int_{x^{t+1}}^{x^t} \frac{\partial S_g^{t+1}(x)}{\partial x} .dx = \int_{x^{t+1}}^{x^t} \frac{\partial U_{(\rho, \mu)^{t+1}(x)}(x)}{\partial x} .dx \quad (3.19)$$

The above expression shows that the performance change can be expressed with respect to the variation of the direct utility function, but weighted with the adjusted risk-aversion function.

4 Research Methodology: Implementation Strategy and Data

For the purpose of illustrating how the Luenberger indicator and its components can serve to track individual fund managers' performance, we opt for using a mimicking portfolio approach (Fama and French (1996)). This mimicking portfolio approach employs portfolios categorized on some variable or combination of variables of interest (e.g., Fama and French (1996) form portfolios on firm size and book-to-market equity, while Fama and French (1997) do the same on industry). In our case, we employ portfolios formed on specific factors or styles. To compose these portfolios and compute the corresponding value-weighted monthly returns, the underlying universe of financial assets is restricted to all stocks listed

¹⁷Another advantage of the use of the shortage function and its dual relation to mean-variance utility is that its shadow prices reveal the risk aversion characterizing the underlying portfolio. In a mean-variance-skewness model, the shortage function's shadow prices reveal risk aversion and prudence. This adjusted risk aversion function is similar to the adjusted price function defined by Luenberger (1995) in a consumer context.

on the main North American stock markets (in particular, NYSE, AMEX and NASDAQ).

This data set has four important characteristics: (i) the asset universe is common to all portfolios and available over a long time period, (ii) the portfolios are not handled by real fund managers over a certain relatively short time span, but represent a variety of management styles that could have been implemented over a long run by some idealized portfolio owner, (iii) the value-weighted and non-optimised nature of these portfolios potentially allows for a wide scope of inefficiencies, and (iv) these portfolios have a known unit of time (i.e., a month), since they are re-composed each month or each several months depending on factors or styles. Notice that, by contrast, real world funds have the disadvantage of having no natural time unit (e.g., the frequency of rescheduling is (i) hard to infer precisely from mission statements, (ii) can vary slightly over time, and (iii) need not coincide across funds).

To test the capabilities of our new methodology for tracking these inefficiencies, we compute the performance of these idealised funds over a series of sliding time windows with respect to a common fund frontier composed of all selected mimicking portfolios. Since the reallocation of assets within the sample of portfolios is at least partly asynchronous, the resulting heterogeneity in portfolio performance under idealised circumstances forms a perfect level playing field to assess the long run success of certain portfolio management strategies conditioned on styles or factors. In particular, this framework allows to highlight two interesting perspectives in the empirical part of this research that are specific to our methodological choices.

First, we can compare these portfolios in terms of the Luenberger indicator and its decomposition over a very long time period and under identical circumstances and contrast it to more traditional performance appraisal tools. Borrowing from the existing literature we define first the Sharpe and Sortino ratios to evaluate the mean-variance respectively the mean-variance-skewness models employed. The Sharpe (1966) ratio is defined as follows:

$$S = \frac{E_{R_P} - r_f}{\sigma_{R_P}} \quad (4.1)$$

whereby R_P refers to the portfolio return and r_f to the risk-free rate. The Sortino ratio (Sortino and van der Meer (1991)) is defined as follows:

$$Sort = \frac{R_p - MAR}{(\int_{-\inf}^{MAR} (MAR - x)^2 f(x) dx)^{1/2}} \quad (4.2)$$

where MAR is the minimum acceptable return and $f(x)$ is the probability density function of returns. The denominator is the downside risk or target semi-deviation (or the square root of the target semi-variance). The latter indicator is already suitable for markets with non-normal distributions.

We can now define their respective variations in discrete time to have a traditional analogue to the Luenberger portfolio productivity indicator.¹⁸ To be explicit, the change in Sharpe ratio is defined as

¹⁸We are unaware of any definitions of these variations on the Sharpe and Sortino ratios in the literature.

follows:

$$\Delta_t S = S_{t+1} - S_t, \quad (4.3)$$

while the change in Sortino ratio is defined as:

$$\Delta_t \text{Sort} = \text{Sort}_{t+1} - \text{Sort}_t. \quad (4.4)$$

In line with the Luenberger indicator, these definitions follow the difference approach to index numbers.

Second, the decomposition of the Luenberger indicator provides in our view a unique tool for the long run assessment of the relative success of implementing different portfolio strategies (e.g., based on various styles, factors, etc.). In particular, we believe that the efficiency change component (*EFFCH*) provides an alternative, but particularly suitable measurement tool to detect the eventual ability of fund managers for stock picking and market timing, since the performance measurement is not contaminated by the change in the financial market (i.e., it is separated from the frontier change (*FRFCH*)).

In particular, we use a data set made available by K. French consisting in series of monthly returns from January 1931 to August 2007 for 36 value-weighted (hence, potentially non-optimal) portfolios denoted P_1, P_2, \dots, P_{36} and formed on specific factors or styles.¹⁹ With a given set of N portfolios, the minimal size for the time window is $N + 1$: hence, all computations have been performed with the same time window of 37 months. The sliding tick for this window is one month. Therefore, since we dispose of 920 months in the data set, we end up with 883 time windows.²⁰ We also use a 3-month T-Bill as reference for the risk-free rate. These data have been obtained from the Federal Reserve Board and are only available since January 1934. Consequently, changes in the traditional ratios ((4.3) and (4.4)) can only be computed from January 1937 onwards. This difference in availability only affects the comparisons between these traditional measures and the Luenberger portfolio productivity indicator. Furthermore, these risk-free rates of returns were annualized and have been converted to a monthly basis.

Thus, given that all 36 portfolios must be evaluated with 4 different shortage functions over 883 time windows, we end up with 127,152 computations in total for the mean-variance model and an equal amount for the mean-variance-skewness model. Recall that in the case of the mean-variance(-skewness) model, each portfolio is projected using a shortage function simultaneously looking for return (and skewness) augmentation and risk reduction. Notice the computational advantage of using efficiency measures, since it would be extremely difficult to compare 883 complete Markowitz frontiers with one another (while ignoring the impossibility to do anything similar in the mean-variance-skewness case). The proposed approach only needs the projections of these 26 portfolios in each time window (the rest of the Markowitz or mean-variance-skewness frontiers can be safely ignored). Notice furthermore that the incidence of the

¹⁹Complete information can be found on the web pages of K. French: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/

²⁰The first time window ranges over the interval [01/1931, 01/1934] and the last one over the interval [08/2004, 08/2007].

infeasibility problem mentioned before turns out to be rather minor: we observe infeasibilities for only 165 (i.e., 0.519 % (= 165/(883.36)) and 201 (i.e., 0.632 % (= 201/(883.36)) portfolios in the mean-variance respectively the mean-variance-skewness model. Thus, the problem seems to be rather small in this data base.

The following list provides essential information on how these portfolios have been composed:

- (1) Fama-French Benchmark (P1–P6):** These portfolios combine stocks with respect to two main characteristics. The first one is their book-to-market ratio (BTM). On this basis, 3 categories are established (Growth, Neutral and Value portfolios). The second characteristic is the size of the firm proxied by its market equity (ME). Mixing these categories results in 6 profiles (see table 2)²¹.

Table 2: Portfolio Profile

		30% Smallest BTM	In-Between BTM	30% Biggest BTM
Below size	median	Buy + Growth Firms (P1)	Buy + Neutral Firms (P2)	Buy + Value Firms (P3)
Above size	median	Sell + Growth Firms (P4)	Sell + Neutral Firms (P5)	Sell + Value Firms (P6)

Note: Breakpoints for each category are computed over the NYSE data, although each portfolio combines stocks from NYSE, AMEX and NASDAQ.

- (2) Size (P7–P11):** Five portfolios (one per quintile) based on firms' size composing each portfolio. Size is proxied by market equity. For instance, P7 is based on the 20% smallest firms listed on the NYSE, the AMEX, and the NASDAQ while P11 draws on the 20% biggest firms.
- (3) Growth (P12–P16):** Same logic as for size-based portfolios, but book-to-market (BTM) serves as a proxy for growth opportunity. In other words, P12 is a portfolio composed of the smallest firms while P16 combines the biggest ones.
- (4) Dividend Yield (P17–P21):** Ibidem, with dividend yields (DY) replacing ME or BTM.
- (5) Momentum (P22):** We focus here on the more typical momentum portfolio. For each month t , stocks are included in this portfolio provided (i) they are ranked in the 10% most performing stocks in terms of return at the end of the previous month ($t - 1$) and (ii) they were already listed one year and a month before ($t - 13$).²² Some investors believe that well performing stocks in the past will deliver the well performing stocks of the future, whence they play a "momentum strategy".

²¹It could be interesting to compare these with the Morningstar classification system which is based on the same criteria. ME is divided in 3 categories, therefore each fund receives a pictogram indicating its synthetic position in a 3×3 matrix. In this research we only have a 3×2 matrix

²²P22 corresponds to the last quintile portfolio computed by K. French in his data file `MomentumPortfolio`

- (6) Short Term Reversal (P23):** Similarly to the Momentum portfolio (P22). P23 is a typical short term reversal portfolio. For each month t , stocks are included in this portfolio provided (i) they have been ranked in the 10% least performing stocks in terms of return at the end of the previous month ($t - 1$) and (ii) they were already listed one month before ($t - 2$).²³ Contrary to the beliefs of momentum traders, short term reversal investors think that returns inevitably tend to revert to the mean over time. Therefore, it is worth buying poorly performing stocks to benefit from their possible appreciation in the short term.
- (7) Long Term Reversal (P24):** Same logic as for P23, but stocks are now picked (i) on the basis of their poor performance observed in $t - 13$ and (ii) provided they were listed five years before ($t - 61$).²⁴
- (8) Industry Portfolios (P25–P36):** Portfolios are based on all stocks listed on NYSE, AMEX and NASDAQ with respect to their four-digit SIC. These portfolios simply aim at mimicking industry returns. These are coded by a number ranging from 25 to 36 corresponding to (i) Non Durable Goods, (ii) Durable Goods, (iii) Manufactured Goods, (iv) Energy, (v) Chemicals, (vi) Business Equipment, (vii) Telecommunication, (viii) Utils, (ix) Shops, (x) Health, (xi) Money, and (xii) Others.

5 Empirical Results

This section scrutinises these portfolios in terms of their mean-variance (MV) and mean-variance-skewness (MVS) Luenberger portfolio productivity indicators and also compares these to the Δ Sharpe respectively Δ Sortino ratios (see equations (4.3) and (4.4)).

A first part of the analysis consists in searching for a common ground in the information provided by this Luenberger productivity indicator and its counterpart traditional performance measures. The idea is to identify whether or not these two categories of performance gauges provide similar results. Rank correlations are computed over the period 02/1937 to 08/2007 (for data availability reasons) between: on the one hand, in MV space $L(x^t, x^{t+1})$ and the Δ Sharpe ratio; and on the other hand, in MVS space between $L(x^t, x^{t+1})$ and the Δ Sortino ratio. To impose minimal assumptions, these correlations are evaluated by a Spearman rho test. Results are presented in Table 3. Notice that we only report significant results throughout this section.

In Table 3, one observes that for about 50% of portfolios the Luenberger productivity indicator is positively correlated with the Δ Sharpe ratio in the MV model. However, this result is not uniformly

²³P23 corresponds to the first quintile portfolio computed by K. French in his data file `Short-TermReversalPortfolio`

²⁴P24 corresponds to the first quintile portfolio computed by K. French in his data file `Short-TermReversalPortfolio`

Table 3: Portfolios with Significant Correlations between $L(x^t, x^{t+1})$ and Δ Sharpe (MV) resp. Δ Sortino (MVS)

Portfolio N	Mean–Variance		Mean–Variance–Skewness	
	ρ	p-value	ρ	p-value
1	0.0573	0.0958*	–	–
4	0.1579	0.0000***	0.1495	0.0000***
5	0.1130	0.0011***	0.1457	0.0000***
6	0.0988	0.0137**	0.0911	0.0088***
7	0.1765	0.0000***	0.1327	0.0001***
8	0.1222	0.0004***	0.1303	0.0001***
9	0.0967	0.0049***	0.1115	0.0012***
10	0.0686	0.0460**	–	–
12	0.0677	0.0490**	0.0782	0.0229**
16	0.0661	0.0557*	0.0695	0.0436**
17	0.0643	0.0619*	–	–
18	0.0814	0.0179**	0.0724	0.0359**
23	–	–	0.0730	0.034**
24	–	–	0.0760	0.0275**
25	0.0661	0.0547*	–	–
26	0.0695	0.0456**	–	–
27	0.0655	0.0568*	0.0765	0.0264**
32	–	–	0.0590	0.0888*
33	0.0765	0.0271**	0.0595	0.0852*
35	0.0618	0.0733*	–	–
36	0.0874	0.0112**	–	–

Note: Spearman Correlation coefficient with $H_0: \rho = 0$.
*, ** and *** signs represent 10%, 5% respectively 1% thresholds.

observed across the 8 portfolio families. For instance, rank correlations are strongest for the families 1 and 2 followed by 8 (i.e., Fama French Benchmark, Size, and Industry portfolios). By contrast, Short Term and Long Term Reversal as well as Momentum portfolios do not appear at all in this table. In the MVS world, less portfolios are significantly correlated. This last result is probably linked to two reasons: (i) the portfolio mimicking approach is fundamentally a non-optimised diversification strategy geared towards a MV framework, and (ii) the variation on the Sortino ratio does not offer an equally theoretically founded performance measure compared to the Luenberger indicator, which builds upon the shortage function that is suitable to characterize MVS portfolio sets.

Keeping in mind that traditional measures are unable to distinguish the contribution of portfolio managers to the performance evolution, while the Luenberger portfolio productivity indicator and its decomposition allow for such a distinction, we now try to test the relevance of the said decomposition. Two questions are considered at this point: (i) is the evolution of $L(x^t, x^{t+1})$, $EFFCH$ and $FRCH$ due to mere chance, and (ii) do the series of $L(x^t, x^{t+1})$, $EFFCH$ and $FRCH$ have a mean that is different from zero? While the first question is concerned with the detection of any significant influence of portfolio managers on the Luenberger and its components, the second question focuses on the size of any eventual effect. Obviously, positive improvements in $EFFCH$ could indicate some expertise among some portfolio

managers (at least over short periods of time) to push portfolios towards the moving portfolio frontier target, while a negative result could point to their failure to do so.

To answer the first question, we utilize a Wald-Wolfowitz run test. Results are proposed in Table 4. Looking at the decomposition first, a first major result is that most portfolios exhibit non-random *EFFCH* series in both MV and MVS models. By contrast, *FRCH* appears to be almost completely random, as could be expected from efficient market theory. Second, the Luenberger indicator $L(x^t, x^{t+1})$ is mainly non-random for the two first portfolio families. In particular, Fama French Benchmark portfolios 3 (not in MV), 4, 5 and 6 (i.e., mainly those that are above the median size, whatever their position in terms of BTM) and portfolios 7, 8, 9 and 10 (not in MVS) (i.e., P7 to P9 are portfolios composed within the subset of the 60% smallest firms).

The second question is answered using a Wilcoxon test for differences. Over the whole time period, we cannot report any portfolio that has non-zero performance indicators except P31 (a significant $L(x^t, x^{t+1})$ in MV) and P23 (a significant *EFFCH* in MVS). Of course, this is in line with the efficient market hypothesis as well, since it would be hard to imagine that the portfolio mimicking approach could generate and sustain superior results over such a long run. In a sufficiently short time horizon, one can imagine that some portfolios (e.g., styles, etc.) may have performed well because, for a variety of reasons, their profile fits into some market niche favoured by the economy. Therefore, we look at the short run. Fixing an arbitrary period consisting of the last ten years in the data base, the Wilcoxon test is recomputed and results are reported in Table 5.

While no portfolio gets a significant *EFFCH* in MV or in MVS (except P35), quite a few obtain non-zero $L(x^t, x^{t+1})$ and *FRCH*. Notice that not a single portfolio obtains a non-zero Δ Sharpe or Δ Sortino ratio over the same time span. These portfolios obtain a significant Luenberger not because of any capability from the idealized manager, but simply due to changes in the market that temporarily and locally favour certain “niches” in the portfolio set. Combining this information with the result regarding the first question, one can conjecture that the non-random *EFFCH* found there must be caused by some coincidentally favourable circumstances situated in some sub-period(s) different from the last ten years.

Finally, knowing that non-zero performance is at best only observable in the short-term, we wonder whether there is any time-dependency within these indicator-based performance results within the same ten year time period. In Tables 6, 7 and 8, we report a first-order autocorrelation regression for *EFFCH*, *FRCH* respectively the Luenberger indicator for both MV and MVS models. Since few portfolios reveal non-zero short-term performance, we anticipate finding few if any significant AR(1) processes. For the *EFFCH* component, Table 6 shows that there is a negative persistence for most of the portfolios. Thus, any non-zero performance in these non-optimised mimicking portfolios cannot be sustained over time. For the *FRCH* component and $L(x^t, x^{t+1})$, the Tables 7 and 8 contain more or less the same portfolios and indicate that most of these portfolios enjoy rather a positive persistence. The latter results probably

simply reflect the fact that market cycles cover a time span substantially larger than the monthly tick size for the sliding windows in our analysis.

6 Conclusions

The main objective of this contribution has been to introduce a general method for measuring the evolution of portfolio efficiency over time inspired by developments in index theory. Benchmarking portfolios by simultaneously looking for risk contraction and mean-return (and skewness) augmentation in the mean-variance (mean-variance-skewness) model using the shortage function framework, we have defined a new Luenberger discrete time portfolio productivity indicator. The cardinal virtues of this approach can be summarized as follows: (i) it does not require the complete estimation of the efficient frontier and tracing its evolution over time, but simply projects the portfolios on the relevant part of the frontier with the shortage function using non-parametric envelopment methods to obtain an easily interpretable efficiency measure and an ensuing productivity indicator; (ii) the decomposition of the Luenberger portfolio productivity indicator distinguishes between the efficiency change (*EFFCH*) and the portfolio frontier change (*FRCH*). While the latter component measures the local changes in the frontier movements induced by market volatility, the former can in principle capture the efficiency changes attributable to the investor or portfolio manager. This efficiency change component (*EFFCH*) allows testing in an alternative, but conceptually promising way the eventual ability of fund managers to generate a superior performance, since this measurement is not contaminated by any changes in the financial market itself.

A simple empirical application on a limited sample of investment portfolios has illustrated the computational feasibility of this general framework in both the mean-variance and mean-variance-skewness frameworks. Given the mimicking portfolio approach adopted and the long time period available, we have been able to shed some light on the question of the relative performance of implementing different portfolio strategies (e.g., based on various styles, factors, etc.). Summarizing some key empirical results, the Luenberger portfolio productivity indicator is correlated with its counterpart traditional performance measures in both mean-variance and mean-variance-skewness frameworks. Furthermore, most portfolios exhibit non-random *EFFCH* series in both mean-variance and mean-variance-skewness models, while *FRCH* series are almost completely random. Additionally, the *EFFCH* series does almost never yield a non-zero performance. By contrast, the *FRCH* component of some portfolios can in the short run be significantly different from zero, because the market coincidentally seems to create favourable circumstances. Overall, these results are perfectly concordant with efficient market theory and are probably driven by the mimicking portfolio approach which relies in the selected data base on non-optimised rules. Nevertheless, this new framework opens up possibilities to systematically attribute performance and quantify

any eventual individual fund manager performance.

Obviously, the current work has some limitations. One restriction in the current analysis is that it does not account for transaction costs, but assumes that portfolios can be reshuffled in every time period to remain in track with the evolving portfolio frontiers. This can in principle be overcome at the cost of complexifying the analysis slightly. However, we do not anticipate any fundamental problem in extending the proposed Luenberger indicator, since all extensions of basic portfolio models could in principle be fitted into the basic shortage function models. On the positive side, as already pointed out in the text, extensions to higher moments are straightforward (following Briec, Kerstens, and Jokung (2007)).

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Table 4: Run Tests for the Luenberger Indicator and Its Components (whole sample)

Portfolio	Mean-Variance						Mean-Variance-Skewness					
	<i>EFFCH</i>		<i>FRCH</i>		$L(x^t, x^{t+1})$		<i>EFFCH</i>		<i>FRCH</i>		$L(x^t, x^{t+1})$	
	Z	p-value	Z	p-value	Z	p-value	Z	p-value	Z	p-value	Z	p-value
3	-	-	-	-	-	-	-1.115	0.2648	-1.17	0.2419	-1.716	0.0862*
4	-3.000	0.0027***	-1.448	0.1477	-3.337	0.0008***	-2.180	0.0292**	-0.820	0.4117	-3.468	0.0005***
5	-2.789	0.0053***	-1.116	0.2645	-4.634	0.0000***	-1.649	0.0991*	-1.550	0.1212	-5.031	0.0000***
6	-3.764	0.0002***	-1.689	0.0912*	-3.045	0.0023***	-4.476	0.0000***	-1.680	0.0929***	-2.690	0.0071***
7	-6.503	0.0000***	-2.063	0.0391**	-4.354	0.0000***	-8.303	0.0000***	-2.294	0.0218**	-3.915	0.0001***
8	-2.668	0.0076***	-1.284	0.199	-3.169	0.0015***	-1.811	0.0701*	-1.274	0.2029	-4.067	0.0000***
9	-1.517	0.1293	-1.034	0.301	-2.655	0.0079***	-0.772	0.4399	-1.314	0.1887	-2.798	0.0051***
10	-1.716	0.0861*	-0.027	0.9783	-2.727	0.0064***	-	-	-	-	-	-
14	-	-	-	-	-	-	-1.986	0.0470**	-0.096	0.9232	-0.096	0.9232
15	-2.140	0.0323**	0.372	0.7097	1.314	0.1891	-2.188	0.0287**	1.352	0.1765	1.238	0.2159
16	-2.059	0.0395**	0.204	0.8388	-0.966	0.3341	-4.860	0.0000***	-0.776	0.4377	-1.816	0.0693*
17	-1.137	0.2554	0.086	0.9314	-2.221	0.0263**	-1.964	0.0495**	-0.120	0.9048	-1.057	0.2906
20	-2.256	0.0241**	-0.676	0.4992	-1.515	0.1298	-2.724	0.0064***	1.225	0.2207	-1.107	0.2681
22	-4.495	0.0000***	-0.961	0.3366	-0.919	0.3582	-6.263	0.0000***	-0.805	0.4206	-0.732	0.4642
23	-2.473	0.0134**	-1.700	0.089*	-0.769	0.4421	-6.686	0.0000***	-1.204	0.2288	-0.422	0.673
24	-2.230	0.0258**	-0.683	0.4949	-1.747	0.0806**	-7.022	0.0000***	1.223	0.2212	-0.716	0.4742
25	-2.765	0.0057***	-1.421	0.1554	-0.745	0.4562	-2.237	0.0253**	0.738	0.4603	0.774	0.4389
26	-2.199	0.0278**	-0.023	0.982	0.104	0.9174	-6.850	0.0000***	-1.655	0.0979*	-1.643	0.1003
28	-3.410	0.0006***	-0.467	0.6402	0.273	0.7849	-8.426	0.0000***	-0.977	0.3287	0.220	0.8256
29	-1.683	0.0924*	0.371	0.7108	1.449	0.1474	-1.869	0.0617**	0.453	0.6503	1.387	0.1654
30	-2.800	0.0051***	-0.334	0.7388	-0.595	0.5519	-5.298	0.0000***	0.409	0.6824	0.304	0.7608
31	-3.740	0.0002***	-0.354	0.7235	-0.564	0.5727	-7.750	0.0000***	-0.554	0.5793	-0.975	0.3294
32	-	-	-	-	-	-	-5.008	0.0000***	-0.235	0.8139	-0.492	0.623
33	-2.978	0.0029***	-1.416	0.1568	-0.576	0.5649	-3.486	0.0005***	-0.337	0.7364	-0.601	0.5478
34	-3.444	0.0006***	0.081	0.9354	-1.307	0.1912	-6.388	0.0000***	0.728	0.4665	0.524	0.6007
32	-	-	-	-	-	-	-1.716	0.0861*	1.498	0.1342	-0.675	0.4998
36	-1.383	0.1667	-1.651	0.0988*	-1.773	0.0763*	-	-	-	-	-	-

Note: Wald-Wolfowitz run test with H_0 : Series X follows a random process.

*, ** and *** signs represent 10%, 5% respectively 1% thresholds.

Table 5: Wilcoxon Tests for the Luenberger Indicator and its Components (last ten years)

Portfolio	Mean-Variance					
	<i>EFFCH</i>		<i>FRCH</i>		$L(x^t, x^{t+1})$	
	W	p-value	W	p-value	W	p-value
1	-	-	-	-	0.0028	0.0643*
2	-	-	0.0058	0.0824*	-	-
11	-	-	0.0047	0.0967*	0.0059	0.0422**
12	-	-	-	-	0.0006	0.0994*
19	-	-	0.0075	0.0311**	0.0133	0.0367**
20	-	-	0.0091	0.0148**	-	-
25	-	-	0.0077	0.0203**	0.0123	0.0607*
29	-	-	0.0108	0.0054***	0.0145	0.0178**
31	-	-	-	-	-0.0002	0.0521*
36	-	-	0.0065	0.051*	-	-
Mean-Variance-Skewness						
1	-	-	0.0108	0.0967*	0.0119	0.065*
19	-	-	0.0101	0.0698*	0.0164	0.0685*
25	-	-	0.0277	0.0517*	-	-
28	-	-	-0.0192	0.0580*	-0.0186	0.0889*
29	-	-	0.0534	0.0841*	0.0571	0.0961*
34	-	-	-	-	0.0161	0.0967*
35	0.0023	0.0736*	-	-	-	-

Note: Wilcoxon test with H_0 : Value is not different of 0.

*, ** and *** signs represent 10%, 5% respectively 1% thresholds.

Table 6: AR(1) Model for *EFFCH*

Portfolio	<i>EFFCH</i>			
	Mean-Variance		Mean-Variance-Skewness	
	ρ	p-value	ρ	p-value
1	-	-	-0.3416	0.0016***
15	0.1952	0.0728*	-	-
18	-	-	-0.2162	0.0466**
19	0.2352	0.033**	-0.1946	0.0779*
23	-0.1892	0.0826*	-0.3515	0.001***
24	-	-	-0.3593	0.0008***
25	-	-	-0.4731	0.0000***
27	-	-	-0.2778	0.0123**
28	-	-	-0.3025	0.0059***
29	0.2900	0.009***	-	-
35	-0.1834	0.093*	-	-

Note: *, ** and *** signs represent 10%, 5% respectively 1% thresholds.

Table 7: AR(1) Model for $FRCH$

Portfolio	$FRCH$			
	Mean-Variance		Mean-Variance-Skewness	
	ρ	p-value	ρ	p-value
1	0.2176	0.0487**	–	–
3	0.2462	0.0233**	0.2733	0.014**
4	0.1933	0.0764*	–	–
7	–	–	0.2221	0.0412**
12	0.2448	0.0259**	–	–
16	0.2059	0.0582*	0.2181	0.0447**
17	0.2502	0.022**	0.2456	0.0257**
22	0.2223	0.0409**	–	–
30	0.3165	0.0032**	–	–
31	0.2923	0.0069***	–	–
32	0.2075	0.0565*	0.2073	0.0570**
33	0.2039	0.0633*	–	–
34	–	–	–0.2888	0.0418**
36	0.2312	0.0341**	–	–

Note: *, ** and *** signs represent 10%, 5% respectively 1% thresholds.

Table 8: AR(1) Model for $L(x^t, x^{t+1})$

Portfolio	$L(x^t, x^{t+1})$			
	Mean-Variance		Mean-Variance-Skewness	
	ρ	p-value	ρ	p-value
1	0.4218	0.0001***	–	–
3	0.2461	0.0233**	–	–
7	0.2533	0.0201**	0.2385	0.0322**
10	0.2276	0.0376**	–	–
11	0.3391	0.0018***	–	–
12	0.3222	0.003***	–	–
17	0.2546	0.0193**	0.1827	0.0965*
25	–0.3278	0.0041***	–0.3686	0.0007***
31	0.2555	0.0181**	–	–

Note: *, ** and *** signs represent 10%, 5% respectively 1% thresholds.