ESTIMATION OF THE TERM STRUCTURE OF VOLATILITY FOR THE SPANISH PUBLIC DEBT MARKET

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Abstract. This paper tackles with the estimation of the Term Structure of Volatility (TSV) of interest rates included in the term structure of interest rates (TSIR) of the Spanish Public Debt Market. Analogously to the TSIR, the TSV illustrates the relationship of coupon-zero rate volatility to time to maturity. With that purpose, the orthogonal GARCH model introduced by Alexander and Chibumba (1997) and Alexander (2000) has been used. This method provides an accurate and efficient way of generating volatility term structures of interest rates that only requires the estimation of univariate GARCH models of the first few principal components of the system considered. The results of the empirical analysis carried out show some interesting features of the TSV of interest rates of the Spanish market. First, as expected interest rate volatility tends to decrease with time to maturity. Second, interest rate volatility has declined over the period of study. A number of recent improvements in monetary policymaking together with the very low level of interest rates during last years may have played a key role in this reduction of volatility.

Key words: Orthogonal GARCH, interest rate volatility, term structure of volatility, term structure of interest rates.

EFM classification codes: 550, 340.

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1. INTRODUCTION.

The analysis of the term structure of interest rates (hereafter TSIR) represents one of the topics of financial research with more development and richer results generated along the last two decades. The TSIR is important for various reasons. For example, it is central to obtain a better understanding of the transmission mechanism of monetary policy to the real economy. The term structure is also thought to contain useful information about future interest rates, inflation, and real economic activity. Furthermore, it contributes to price many financial assets and to design investment and hedging strategies.

One of the most fruitful lines of research in this field has been the one aimed to the description of the TSIR fluctuations through a reduced number of common factors (2 or 3 in most of the cases). In this sense, a first group of papers choose as common factors some specific interest rates corresponding to different maturities, the so-called key rates (see, for instance, Elton et al., 1990 for the U.S. market; and Navarro and Nave, 1997 and 2001, for the Spanish market). In contrast, a second group of papers in the literature identify those factors by using Principal Components Analysis (PCA, hereafter). Some of them are, for example, Litterman and Scheinkman (1991), Barber and Cooper (1996), Bliss (1997), and Abad and Benito (2006).

More recently, a new line of research focused on the estimation and analysis of the term structure of volatility (hereafter TSV) of interest rates included in the TSIR has appeared. Good examples of this approach can be seen, for example, in Alexander (2000), Benito and Novales (2005), or Perignon and Villa (2006). Analogously to the TSIR, the TSV illustrates the relationship of volatility of changes in zero-coupon interest rates to their time to maturity. The TSV of interest rates can be viewed as the covariance matrix of interest rate changes of different maturities included in the TSIR.

The accurate understanding of the TSV of interest rates has important practical implications from different points of view. On one hand, it is a critical issue in the context of risk management in order to accurately measure the market risk in fixed-income portfolios. In fact, Value at Risk (VaR) is the most widely used tool to measure market risk and is defined as the maximum amount that a portfolio of financial assets can loss with a given probability during a given time period. The RiskMetrics model developed by J.P. Morgan in 1996 has become a benchmark for calculating VaR. Its key element is a huge covariance matrix that captures the volatilities and correlations between the individual assets that make up the portfolio. Hence the knowledge of the TSV of interest rates can contribute enormously to facilitate the adequate management of market risk in portfolios with an important weight of fixed income securities with different maturities.

On the other hand, in the framework of monetary policy it is crucial to be able to know the transmission of uncertainty from the short-term interest rates to the long-term ones and vice versa in order to assess the effectiveness of the monetary policy measures adopted on firms' investments and on household's consumption decisions. Furthermore, the correct modelling and forecasting of interest rate volatility plays also a prominent role in other areas of finance such as pricing of interest rate derivatives, asset allocation, and portfolio selection.

The estimation of the TSV of interest rates constitutes a problem with higher dimension than the estimation of the TSIR, since the number of parameters to be estimated is much bigger. For this reason, the representation of the TSV dynamics through a small number of factors becomes to this point an issue of special relevance, since it could permit a great reduction in the dimension of the original system of interest rates. In this context, a body of research based on the use of factor ARCH models for estimation of large covariance matrices has recently grown. The basic feature of these models is that the co-movements of the different financial assets are driven by a number of common factors that evolve according to an ARCH-GARCH process. As Christiansen (1999) points out, variations of the factor ARCH model have been put forward by a number of researchers, and the main difference between the models lies in their assumptions about the common factors. For example, Engle et al. (1990) present a factor ARCH model related with Arbitrage Pricing Theory of Ross (1976) where the excess returns of financial assets are driven by a number of common unobservable factors whose conditional variances are described by a GARCH-M specification. More recently, the orthogonal GARCH model proposed by Alexander and Chibumba (1997), and Alexander (2000) uses principal component analysis to generate a number of orthogonal factors that can each be modelled as univariate GARCH processes. In fact, the orthogonal GARCH model can be thought as an extension of the factor GARCH model of Engle et al. (1990) to a multifactor model with orthogonal factors.

The contribution of this paper is the application of the orthogonal GARCH model for estimating the TSV of interest rates of the Spanish public debt market using historical time series data of interest rates. The orthogonal GARCH model works particularly well in highly correlated systems, such as term structures of interest rates. This method provides an accurate and efficient way of generating mean-reverting volatility term structures of interest rates that only requires the estimation of univariate GARCH models of the first few principal components of the system of interest rates considered.

The rest of the paper is organised as follows. The next section presents the methodology used. Section 3 describes the data and Section 4 shows the results obtained with the empirical analysis. Finally, Section 5 points out the main conclusions obtained.

2. METHODOLOGY

Since their introduction by Engle (1982) and Bollerslev (1986), the GARCH (generalized autoregressive conditionally heteroskedastic) models have been used extensively both in academia and by practitioners to estimate the volatility of financial variables. The success of GARCH models can be largely attributed to their ability to capture several stylised facts of financial data, such as time-varying volatility, persistence and clustering of volatility, and asymmetric reactions to positive and negative shocks of equal magnitude.

Given the empirical success of GARCH processes in the modelling of univariate volatility and since it is now widely accepted that financial volatilities move together over time across assets and markets, a natural extension has been the use of multivariate GARCH models to measure the dynamic volatilities and correlations of large dimension systems. However, the multivariate GARCH models present many difficulties with estimation, mainly caused by the exponential increase in the number of parameters with the dimension of the system and the restrictions required by the positive definiteness of the covariance matrix. As one solution to these problems, several factor GARCH models based on the assumption that there are few common factors whose time-varying variances drive the whole covariance matrix of the system have been introduced in the literature (see, for example, Engle et al., 1990; or Diebold and Nerlove, 1989).

In this context, the orthogonal GARCH model proposed by Alexander and Chibumba (1997) and Alexander (2000) lies within the realm of the factor GARCH models proposed by Engle et al. (1990). Specifically, the orthogonal GARCH model combines the GARCH methodology and the principal component analysis. Its central idea is to use principal component analysis to generate a number of orthogonal factors that can each be treated in a univariate GARCH framework. The orthogonal GARCH model reduces the dimensionality of estimating the conditional covariance matrix by generating *kxk* GARCH covariances from *m* univariate GARCH models, where *k* is the number of variables in the system and *m* is the number of principal components actually used. This allows us to capture the *k*-dimension of the system by estimating *m* univariate GARCH models of the principal components of the larger original system. The gain in efficiency clearly depends on the extent to which $m<k$.

As Alexander (2000) points out, the orthogonal GARCH method has several advantages over direct multivariate GARCH. On the one hand, it permits to reduce the dimension problems that arise when estimating covariance matrices in large systems of assets, particularly when the assets are highly correlated. On the other, it gives us the possibility of cutting out any noise in the data that would otherwise make correlation estimates unstable. Finally, this method allows us to generate estimates for volatilities and correlations of variables in the system even when data are sparse and unreliable, for example in illiquid markets.

The application of the orthogonal GARCH model to estimate the TSV of interest rates from the TSIR original data involves splitting the empirical analysis in the following three stages. First of all, a principal component analysis has been carried out on the time series of changes in spot interest rates corresponding to *k* different maturities. This analysis permits us to keep *m* principal components (where $m<k$) containing most of the information about the spot rates changes' variability. These *m* principal components constitute the key market factors that represent the most important sources of information, and all the rest of the variation is ascribed to noise.

According to equation (1), linear combinations of these *m* principal components does permit generate most of the information regarding the *k* original variables.

$$\Delta ir_{jt} = \theta_{1j} \cdot Z_{1t} + \theta_{2j} \cdot Z_{2t} + \cdots + \theta_{mj} \cdot Z_{mt} + \gamma_j \qquad (1)$$

where:

Δir_{jt} indicates the change at time *t* in the spot interest rate corresponding to a maturity *j*;

Z_i represents the i^{th}-principal component of the system of interest rate changes considered;

γ_j is a constant term, specific for every Δir_j, indicating the error assumed in cutting out the *k-m* remaining principal components.[1]

Second, the time-varying conditional variances of the first *m* principal components are obtained using an univariate standard GARCH(1,1) process. The GARCH(1,1) model defines the conditional mean and variance of the i^{th}-principal component at time *t* respectively as:

$$Z_{it} = \mu + \delta \cdot Z_{it-1} + \varepsilon_{it} \tag{2}$$

$$\mathrm{var}(Z_{it}) = \omega + \alpha \cdot \varepsilon_{it-1}^2 + \beta \cdot \mathrm{var}(Z_{it-1}) \tag{3}$$

where $\mathrm{var}(Z_{it})$ denotes the conditional variance of the component principal Z_i obtained from the estimated GARCH(1,1) where $\omega > 0,\ \alpha, \beta \geq 0$, and $\alpha + \beta < 1$.

This simple GARCH model effectively captures volatility clustering and provides convergent forecasts to the long-term average level of volatility. The coefficient α measures the intensity of reaction of volatility to yesterday's unexpected return ε_{t-1}^2, and the coefficient β measures the persistence in volatility.

Third, the conditional variances of the changes in spot interest rates can be obtained as a linear combination of the conditional variances of the first *m* principal components. Since the principal components are, by definition, orthogonal to each other, we no longer need to measure the covariances, substantially reducing the number of parameters to be estimated. Therefore, the conditional variance of the interest rate variations can be calculated following equation (4):

$$\mathrm{var}(\Delta ir_{jt}) = \theta_{1j}^2 \cdot \mathrm{var}(Z_{1t}) + \theta_{2j}^2 \cdot \mathrm{var}\ (Z_{2t}) + \cdots + \theta_{mj}^2 \cdot \mathrm{var}\ (Z_{mt}) \tag{4}$$

Analogously, and assuming that errors γ_j and γ_h are not correlated, the conditional covariance between any two changes in interest rates of different maturity can be calculated from (5) since the *m* principal components obtained are orthogonal:

[1] Provided that *k* is the number of variables considered in the analysis, every one of them can be fully described by a linear combination of the *k* principal components. However, reducing the dimension of the problem by keeping only the *m* principal components which have more information necessarily implies to lose part of that information.

$$\begin{aligned}
&\text{cov}(\Delta ir_{jt}, \Delta ir_{ht}) = \\
&\qquad = \text{cov}\left(\theta_{1j} \cdot Z_{1t} + \theta_{2j} \cdot Z_{2t} + \cdots + \theta_{mj} \cdot Z_{mt} + \gamma_{jt}, \theta_{1h} \cdot Z_{1t} + \theta_{2h} \cdot Z_{2t} + \cdots + \cdot\theta_{mh} \cdot Z_{mt} + \gamma_{ht}\right) \\
&\qquad = \theta_{1j} \cdot \theta_{1h} \cdot \text{var}(Z_{1t}) + \theta_{2j} \cdot \theta_{2h} \cdot \text{var}(Z_{2t}) + \cdots + \theta_{mj} \cdot \theta_{mh} \cdot \text{var}(Z_{mt})
\end{aligned} \tag{5}$$

3. DATA

Term structure of interest rates (TSIR) has been estimated using data from the Spanish Public Debt market by using a correction on the Nelson and Siegel (1987) method in order to avoid errors in the estimation of short-term spot interest rates. [2] [3]

The sample period extends from January 1996 to December 2006, spanning a time interval in which Spanish interest rates varied substantially within an overall downward trend. Since we use daily estimations of TSIR, a number of 2782 data of spot interest rates for eleven different maturities have been employed in our analysis. Specifically, we have used the usual spot interest rates considered by RiskMetrics for the Spanish market: one, three and six month interest rates, and one, two, three, four, five, seven, nine, and ten year interest rates. Notice that in order to do an adequate risk management, the key variable in the analysis is the volatility of interest rate *changes*, so first order differences of the spot interest rates have been calculated and those changes have been taken as our dependent variable. Additionally, this permits us to work with stationary variables, since interest rate levels are variables integrated of first order.

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[2] The authors would like to thank our colleagues A. Diaz and E. Navarro from the University of Castilla-La Mancha (UCLM) their acceptance to use their TSIR estimated from the Spanish market for the period 1996-2006 as a database for this paper.

[3] A detailed explanation of this correction on the Nelson and Siegel method can be found in A. Diaz's website (http://www.uclm.es/area/aef/Etti.asp).

4. EMPIRICAL RESULTS

This section encloses the main results of the estimation of the TSV of interest rates in the Spanish Public Debt Market using the orthogonal GARCH model. First of all, results of principal component analysis on the system of daily changes in interest rates are presented. Table 1 shows that the first three principal components are sufficient to explain more than 93 per cent of total variation of the system of interest rate changes. In particular, the first principal component (Z_1) helps to explain more than 47% of the total variation over the period of study. The addition of a second principal component (Z_2) contributes to increase that percentage up to almost 75% and the sum of the third principal component (Z_3) does permit to explain more than 93% of the variance of the system.[4]

Table 2 presents the factor loadings of the first three principal components. The first principal component is highly and positively correlated with all interest rate changes and can be interpreted as a parallel shift of the term structure, which means that all interest rates move in the same direction and by a similar amount, so inducing a roughly parallel shift in the TSIR specially in the medium term maturities, where the coefficients are more similar. The second principal component represents the tilt of the TSIR: the factor loadings have positive values for short term interest rate changes and negative values for medium and long term interest rate changes. Thus, an upward movement in this second component induces a change in the slope of the most part of the TSIR, since short maturities move up whereas long maturities move down. Finally, the factor loadings on the third component are positive for very short rates, but decreasing and becoming negative for the medium-term rates, and then increasing and becoming positive again for the longer maturities. According to this, the third principal component can be interpreted as a curvature or convexity factor.

Therefore, the results on the number of retained principal components and their interpretation are widely consistent with those of several empirical studies for the US

[4] Even though these cumulative percentages could seem low at first sight, they are in fact totally in line with those obtained by Alexander (2000) for the UK zero-coupon yields for the sample period January 1992–March 1995. Additionally, notice that we are dealing not with the interest yield or zero coupon rates as most of the papers where this technique is applied on interest rate curves, but with their first order differences. When the variables considered are the zero-coupon rates in levels, the cumulative percentages of total variation explained by the three principal components are 96.76%, 99.19%, and 99.96%, respectively, in line with the results obtained for the Spanish market by other authors for former sample periods.

and the European interest rate markets. See, for example, Litterman and Scheinkman (1991) for the US, Alexander (2000) for UK, and Benito and Novales (2005) for Spain.

Table 1. Principal Component Analysis

Proportion of variance explained by the first three principal components

Component	Eigenvalue	Cumulative Proportion
Z_1	5.218	47.342%
Z_2	2.935	74.118%
Z_3	2.092	93.136%

Table 2: Factor loadings of the principal component analysis

Maturity	Z_1	Z_2	Z_3
1 month	.214	.779	.474
3 months	.309	.892	.285
6 months	.434	.854	.048
1 year	.564	.595	-.368
2 years	.758	.130	-.610
3 years	.847	-.113	-.500
4 years	.897	-.231	-.281
5 years	.917	-.292	-.038
7 years	.858	-.328	.383
9 years	.707	-.305	.607
10 years	.630	-.286	.656

Once principal component analysis has been performed and the time series of the first three principal components have been generated from the matrix of factor score coefficients,[5] the next step has been to estimate univariate GARCH(1,1) models for each of the principal components. The results obtained are shown in Table 3.

[5] The transpose of this matrix is the inverse of the component loading matrix when keeping all the principal components.

Table 3: GARCH (1,1) models of the first three principal components

	Mean equation		Variance equation		
	μ	δ	ϖ	α	β
Z_1	0.0136 (0.0170)	0.1716*** (0.0205)	0.0073 (0.0074)	0.0488** (0.0245)	0.9441*** (0.0309)
Z_2	0.04266*** (0.0117)	-0.2826*** (0.0254)	0.0138** (0.0058)	0.2023*** (0.0486)	0.7976*** (0.0486)
Z_3	-0.0431* (0.0249)	-0.2979*** (0.0845)	0.0464** (0.0200)	0.2977*** (0.0636)	0.7022*** (0.0636)

Standard error values of estimates are in brackets. *, **, and *** indicate significance at the 10, 5, and 1 percent levels, respectively.

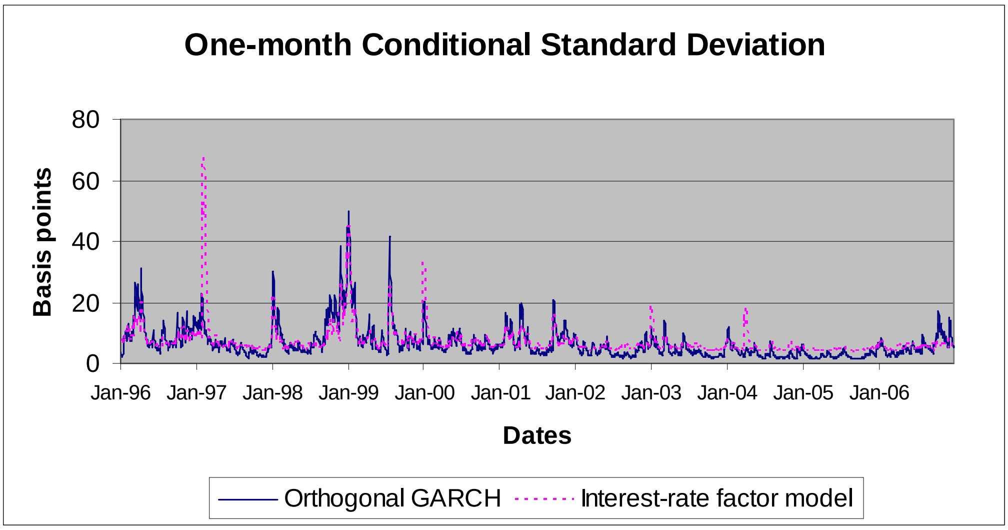
As it can be seen, coefficient α, which measures the intensity of reaction of volatility to the previous period unexpected market return ε_{it-1}^{2}, and coefficient β, which measures the persistence in volatility are both clearly significant at usual levels of significance for the three principal components. Note, however, that the first principal component has a lower market reaction but higher persistence than the other two principal components.

Finally, the full covariance matrix for interest rate changes of different maturities can be constructed from the principal component loading matrix using equations (4) and (5).

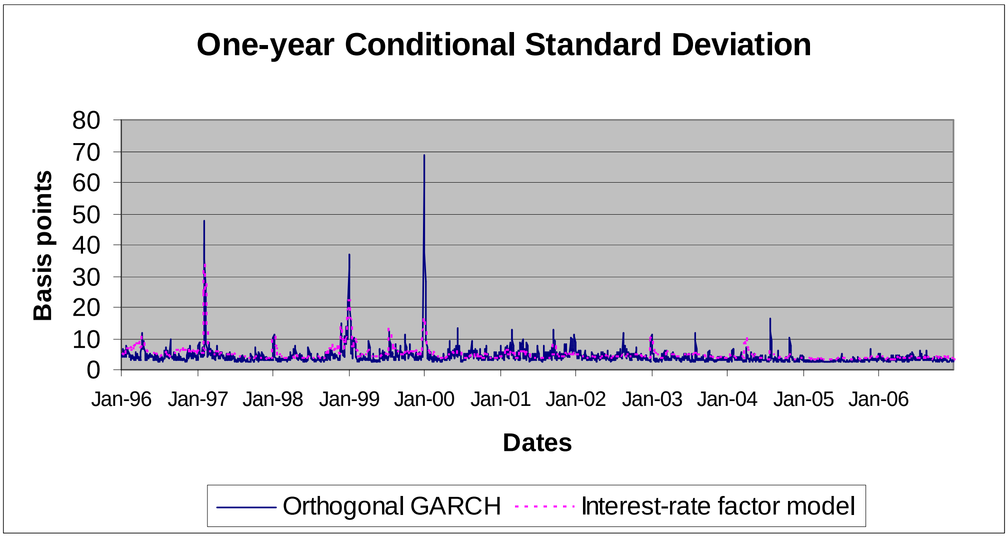
Specifically, conditional standard deviation for the interest rate changes for the whole sample period using this method has been calculated using equation (4). Graphs 1 to 3 show the comparison of this conditional standard deviation with the one obtained directly from the estimation of an univariate GARCH(1,1) model for the changes in interest rates with maturity of one-month, one-year, and ten-years.[6]

[6] Only three out of eleven graphs, corresponding to the key interest rate maturities, are plotted for space reasons.

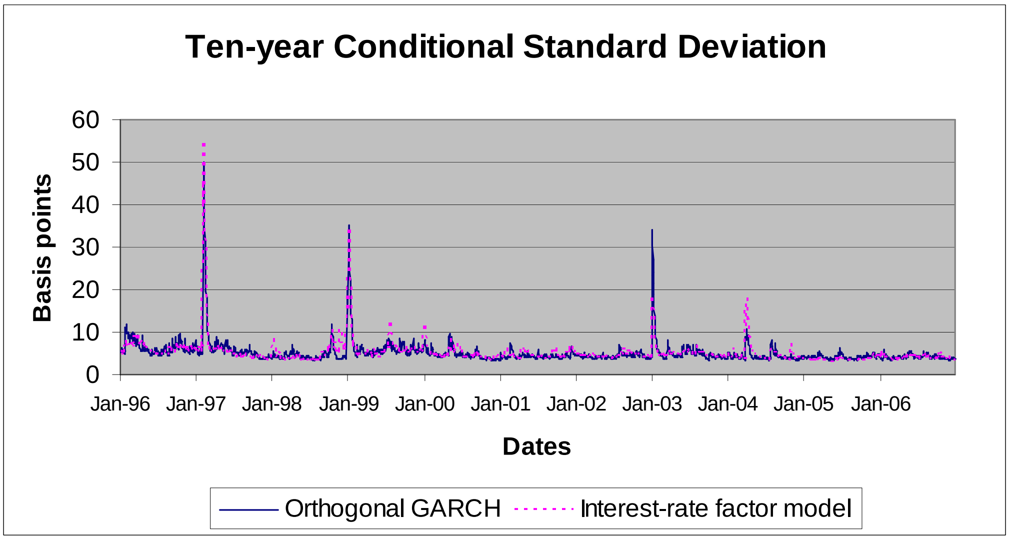
Graph 1



Graph 2



Graph 3

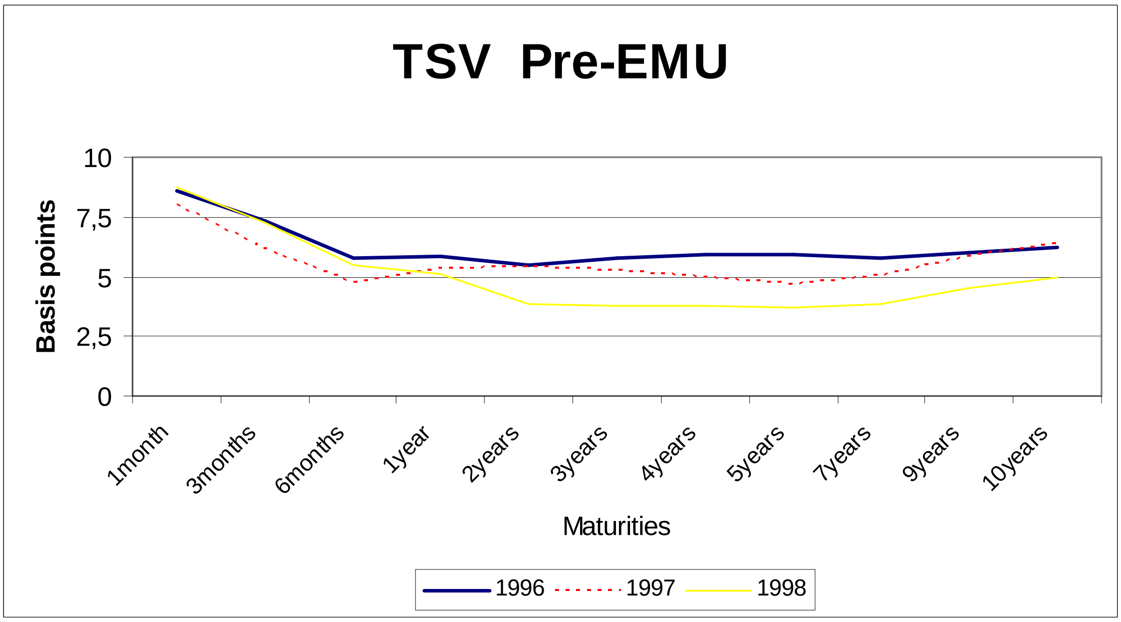


As it can be seen, in all cases the behaviour of the conditional standard deviation is very similar in both models but the results obtained working with the orthogonal GARCH model are smoother for short and long maturities.

Alternatively, and more interestingly for our purposes, 2781 daily TSV can be obtained drawing for each day the conditional standard deviation of spot rates changes as a function of their maturity. As the graphical representation of 2781 curves is not at all practical to extract conclusions, to summarize we have obtained the annual averages of these TSV. Graphs 4 to 6 show these 11 curves obtained as annual averages of TSV corresponding to the periods 1996-1998 (Pre-European Monetary Union period), 1999-2001 (European Monetary Union period), and 2002-2006 (Post-European Monetary Union period), respectively.

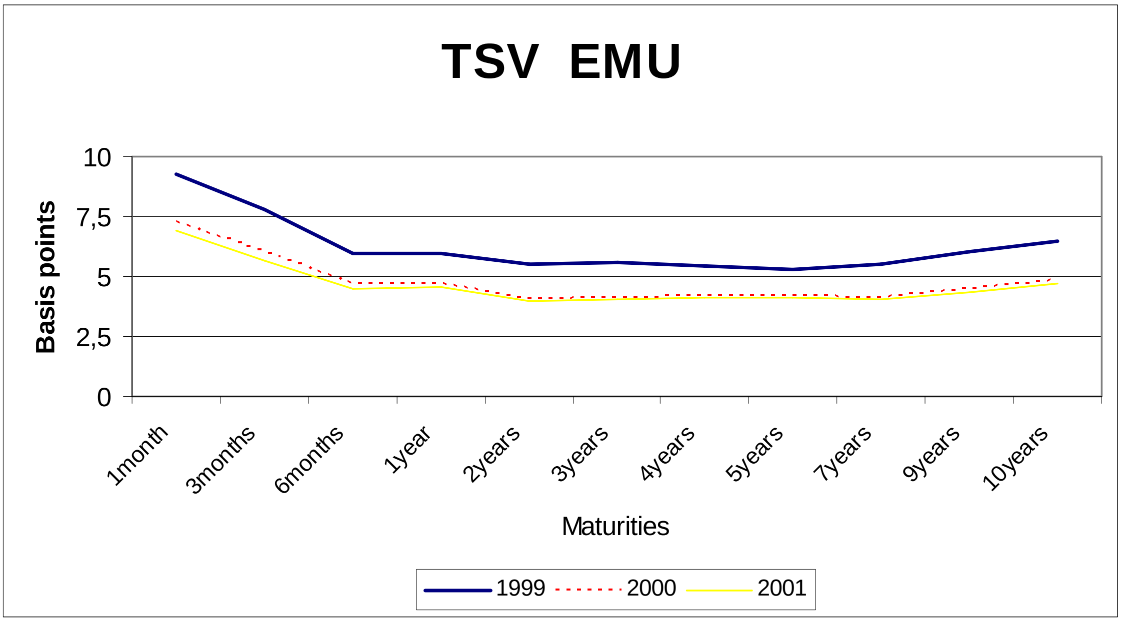
Graph 4

Annual averages of TSV for the period 1996-1998



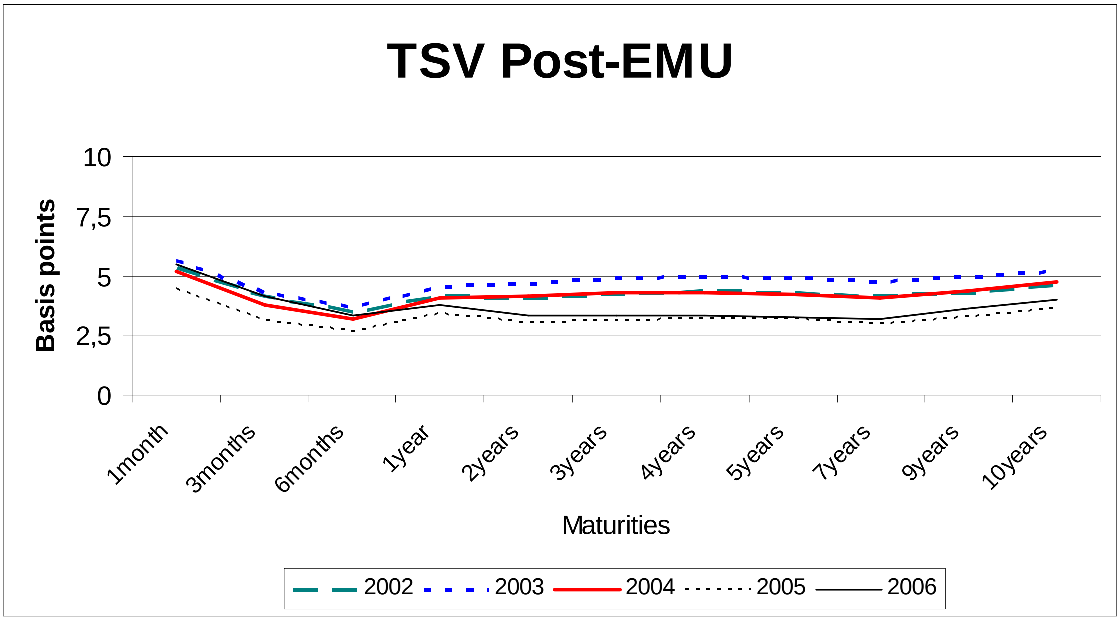
Graph 5

Annual averages of TSV for the period 1999-2001



Graph 6

Annual averages of TSV for the period 2002-2006



Several notable features emerge from these graphs. First, it is apparent that the estimated term structures of volatility of interest rates of the Spanish Public Debt market exhibit a quite similar pattern over the period of study, since most of the curves of volatility are roughly parallel. Specifically, as expected the volatility of changes in interest rates with shorter maturities is larger than the volatility of interest rates with longer maturities, although the changes in medium-term rates are the ones which show in general lower volatility.

Second, it can also be observed that the term structures of volatility of interest rates vary over time, showing a decreasing trend along the sample period. This decline in volatility of interest rates over the last few years may be closely related with some important changes and improvements in the conduct of monetary policy. In particular, the increasing gradualism in policy action (more frequent policy moves of smaller size), the greater transparency of central banks through improved communication about policy intentions, together with the historically low levels of interest rates, given the positive correlation between level and variability of interest rates usually observed, may have played a key role in this context.

5. CONCLUSIONS

Accurate information about the term structure of volatility (TSV) of interest rates is particularly valuable in many practical applications, for instance in risk management for calculations of Value at Risk measures, formulation and implementation of hedging strategies, interest rate derivatives pricing, asset allocation, or in the framework of monetary policy. In this paper, the TSV of interest rates of the Spanish Public Debt market has been estimated using the orthogonal GARCH model introduced by Alexander and Chibumba (1997) and Alexander (2000). The orthogonal GARCH model is a method especially appropriate to be applied on interest rate term structures and other highly correlated systems. It allows for large covariance matrices to be generated from just univariate GARCH models applied on the orthogonal principal components of the system under consideration and consequently avoids the usual estimation problems arising in large dimension systems.

The results of the empirical analysis show some interesting features of the TSV of interest rates of the Spanish market. First, interest rate volatility decreases with time to maturity, as observed in most analysis of the term structure of interest rates. Second, it can be seen that interest rate volatility has declined over the period of study to such an extent of reaching historically low levels. Recent improvements in monetary policymaking –greater gradualism and transparency of central banks– together with the very low level of interest rates during last years may have played a major role in the reduction of volatility in the Spanish Public Debt market.

Since the TSV of interest rates is analogous to the term structure of interest rates, it may provide information about expected future short-term volatilities. In this context, an interesting line of future research will be focused on using the TSV of interest rates for volatility forecasting purposes.

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