Adjusting Multi-Factor Models for Basel II-consistent Economic Capital

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Summary. Understanding and analytically measuring concentration risk in credit portfolios is one of the major challenges in recent research. The measurement is necessary for the determination of regulatory capital under pillar 2 of Basel II as well as for managing the portfolio and allocating economic capital. For this task, the Asymptotic Single Risk Factor (ASRF) framework has to be left to capture these risks. This particularly refers to the correlation structure and granularity. In the literature there exist some multi-factor models that in-depth deal with concentration risk such as name and sector concentrations in which are several shortcomings. To achieve meaningful results the approaches should be determined and applied consistent with the pillar 1 capital requirements which is often not satisfied. For this reason we adjust the multi-factor models to achieve Basel II-consistent results. Therefore, we determine an implied intra-sector concentration formula. Furthermore, we show that the mostly used risk measure "Value at Risk" is problematic when leaving the ASRF framework. Thus, we perform a matching procedure to make sure that the use of the coherent Expected Shortfall complies with conditions of Basel II. We apply these modifications to some multi-factor approaches and perform an extensive numerical study to get a closed form approximation formula. We test the impact of sector concentrations on several portfolios and compare the results within the different models. Finally, we carry out a simulation study to compare the accuracy of different models in a more general manner. Inter alia it is shown that the model of Pykhtin (2004) mostly provides an easier implementation with similar accuracy to the model of Cespedes et al. (2006) but the latter has advantages for ad-hoc and sensitivity analyses. In principle both approaches are suitable for measuring Basel II-consistent economic capital when applied in the proposed way.

Keywords: Concentration Risk; Basel II; Multi-Factor Models; Coherency; Intra-Sector Correlation; Inter-Sector Correlation

JEL classification: G21, G28

1 Introduction

In recent years there have been significant improvements in understanding and measuring concentration risk in credit portfolios such as undiversified idiosyncratic risk and industry or country risk. The measurement of these risks is important against the background of regulatory capital needs as well as for computing the economic capital. Unfortunately, the existing approaches are mostly not fully consistent with the new capital adequacy framework (Basel II) – sometimes within the derivation and sometimes within the implementation – so that the benefit of these approaches is restricted. Furthermore, comparative analyses on these models are scarce. Against this background we address the following questions:

- How can the existing approaches be modified and adjusted to be consistent with the Basel framework? How can we deal with the problems that arise when leaving the assumptions of the Basel framework?
- Which methods are capable to measure concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

For answering these questions, we firstly investigate the assumptions underlying the Basel framework. The Basel II formula for measuring the Value at Risk of credit portfolios is based on the so-called asymptotic single risk factor (ASRF) framework as explained in Gordy (2003). In this framework it is assumed that

- the portfolio is infinitely fine grained and thus it consists of a nearly infinite number of credits with small exposures, and
- only one systematic risk factor influences the default risk of all loans in the portfolio.

The first assumption implies that there are no name concentrations within the portfolio, thus all idiosyncratic risk is diversified completely. The second assumption implicates that there are no sector concentrations such as industry- or country-specific risk concentrations. These are idealizations that can be problematic for real world portfolios.

The Basel Committee on Banking Supervision (BCBS) already recognized the high importance of credit risk concentrations in the Basel framework: "Risk concentrations are arguably the single most important cause of major problems in banks."[1] Since it is difficult to incorporate credit risk concentrations in analytic approaches, in Basel II there is no quantitative approach mentioned how to deal with risk concentrations. Instead, it is only qualitatively demanded in pillar 2 of Basel II that "Banks should have in place effective internal policies, systems and controls to identify, measure, monitor, and control their credit risk concentra-

[1] See BCBS (2005) §770.

tions."[2] Thus, it is each bank's task how to meet these requirements concretely. But of course the measurement and management of risk concentrations are not only important for the determination of regulatory capital but also for the measurement of the "true" portfolio risk. The capital needs regarding this "true" risk will be denoted as economic capital in the following.

From the mentioned types of concentration risk, name concentrations are better understood than sector concentrations. The theoretical derivation of the so-called granularity adjustment that accounts for name concentrations was done by Wilde (2001) and improved by Pykhtin and Dev (2002) and Gordy (2003). This can be called "portfolio name concentration" because the approach refers to the finite number of credits in the portfolio. The adjustment formulas are derived in a more straightforward approach by Martin and Wilde (2002), Rau-Bredow (2002) and Gordy (2004). Furthermore, the adjustment is extended and numerically analyzed in detail by Gürtler et al. (2008). A related approach is the granularity adjustment from Gordy and Lütkebohmert (2007). In contrast, the semi-asymptotic approach from Emmer and Tasche (2005) refers to name concentrations due to a single name while the rest of the portfolio remains infinitely granular, so this can be called "single name concentration".

There also exist analytic and semi-analytic approaches that account for sector concentrations. One rigorous analytical approach is Pykhtin (2004) that is based on a similar principle as in Martin and Wilde (2002). An alternative is the semi-analytic model from Cespedes et al. (2006) that derives an approximation formula through a complex numerical mapping procedure. Another approach from Düllmann (2006) extends the binomial extension technique (BET) model from Moody's. Tasche (2006) suggests an ASRF-extension in an asymptotic multi-factor setting. Some numerical work on the performance of the Pykhtin model is done by Düllmann and Masschelein (2006). Furthermore, Düllmann (2008) presents a first comparison of different approaches on sector concentration risk. The problem is that the derivation and the application of the approaches is often inconsistent with the Basel II framework what is critical for the following reasons:

- Banks are demanded to measure concentration risks and "explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under

[2] See BCBS (2005) § 773. Furthermore, because of the importance of this topic for the stability of the banking system, the Basel Committee launched the "Research Task Force Concentration Risk" that presented its final report in BCBS (2006). The Task Force collected information about the state of the art in current practice and academic literature, analyzed the impact of departures from the ASRF model and reviewed some methodologies to measure name and sector concentrations. An additional workstream focused on stress testing against the background of risk concentrations.

Pillar 2" of Basel II. Even if a bank uses a high-sophisticated multi-factor model, the results are not comparable with the Pillar 1 capital requirement if the results are not consistent to the Basel framework. Thus, it remains unclear if or how much additional regulatory capital is needed regarding risk concentrations.

- Generally, it is not worthwhile to have a major gap between the regulatory and the "true" economic capital. A homogenization of these values is one goal of the new Capital Accord and would simplify the management of the credit portfolio.

For these reasons we demonstrate how multi-factor models can be used in a way that is consistent with the Basel II framework and thereby avoid the problems that arise when leaving the ASRF framework. Furthermore we compare the capability of different multi-factor approaches in approximating the "true" portfolio risk through a simulation study.

The rest of the paper is outlined as follows. In section 2 we briefly describe the ASRF framework and the Basel formula. Moreover, we discuss the problems of the non-coherent Value at Risk in the context of concentration risk and present how the coherent Expected Shortfall can be used consistent with Basel II. In section 3 we introduce multi-factor models in general, and the Pykhtin as well as the Cespedes model in particular. In this context we demonstrate how these approaches could be modified to achieve meaningful results. We compare the performance of the models with a simulation study in section 4. The paper concludes with section 5.

2 Coherent Concentration Risk Measurement in the Context of the Basel Framework

2.1 The ASRF Framework and the Basel II Formula

As mentioned before, the Basel II risk quantification formula is based upon the ASRF framework that assumes an infinitely granular portfolio and the existence of only one systematic risk factor $\tilde{x}$. If these two assumptions are fulfilled the relative portfolio loss $\tilde{L}$ in t = T almost surely equals the expected loss (EL) conditional on the realization of the systematic factor $\tilde{x}$

$$\tilde{L} - E\left(\tilde{L} \mid \tilde{x}\right) \rightarrow 0 \quad \text{a.s.}^{3} \tag{1}$$

[3] See Gordy (2003).

If the loss given default (LGD) is assumed to be deterministic, the conditional expectation can be written as

$$E\left(\tilde{L} \mid \tilde{x}\right) = \sum_{i=1}^{n} E\left(w_i \cdot LGD_i \cdot \tilde{I}_{Default,i} \mid \tilde{x}\right) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot E\left(\tilde{I}_{Default,i} \mid \tilde{x}\right), \tag{2}$$

where $\tilde{I}_{Default}$ represents the indicator function that is 1 in the event of default and 0 in case of survival of the obligor, n stands for the number of credits, and w_i denotes the weight of credit i in the credit portfolio ($i \in \{1, \ldots, n\}$). For the concrete application of formula (2), the conditional default expectation has to be determined. In the Basel II framework, the well known Vasicek model is used.[4] In this one-period one-factor model the return of each obligor is driven by two components that realize at a future point in time T: a systematic part $\tilde{x}$ that influences all firms and a firm-specific (idiosyncratic) part $\tilde{\varepsilon}_i$.[5] Thus, the "normalized" asset returns[6] $\tilde{a}_i$ of each obligor i in t = T can be represented by the following model

$$\tilde{a}_i = \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i, \tag{3}$$

in which $\tilde{x} \sim N(0,1)$ and $\tilde{\varepsilon}_i \sim N(0,1)$ are independently and identically normally distributed with mean zero and standard deviation one. In this model, the correlation structure of each firm i is represented by the firm-specific correlation $\sqrt{\rho_i}$ to the common factor. Hence, the correlation between two firms i, j can be expressed as $\sqrt{\rho_i} \cdot \sqrt{\rho_j}$ or simply as ρ for the case of a homogeneous correlation structure.

Further, the probability of default of each obligor is exogenously given as PD_i.[7] Corresponding to formula (3), an obligor i defaults at t = T when its "normalized" return falls below a default threshold b_i which can be characterized by

$$\tilde{a}_i < b_i \quad \Leftrightarrow \quad \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i < b_i. \tag{4}$$

Against this background the threshold b_i is determined by the exogenous specification of PD_i:[8]

$$PD_i = prob\left(\tilde{a}_i < b_i\right) = N(b_i) \quad \Leftrightarrow \quad b_i = N^{-1}(PD_i). \tag{5}$$

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[4] See e.g. Vasicek (1987, 1991, 2002) and Finger (1999, 2001).

[5] To keep track of the model, stochastic variables are marked with a tilde "~".

[6] The returns are normalized by subtracting the expected return and dividing the resulting term by the standard deviation in order to get standard normally distributed variables.

[7] The probability of default could either be determined by the institution itself or by a rating agency.

[8] The term prob(A) stands for the probability of the occurrence of an uncertain event A. N(.) characterizes the cumulative standard normal distribution and $N^{-1}(.)$ stands for the inverse of N(.).

Conditional on a realization of the systematic factor the probability of default of each obligor is[9]

$$\text{prob}\left(\tilde{a}_i < b_i \mid \tilde{x}\right) = E\left(\tilde{I}_{\tilde{a}_i < b_i} \mid \tilde{x}\right) = N\left(\frac{N^{-1}(PD_i) - \sqrt{\rho_i} \cdot \tilde{x}}{\sqrt{1-\rho_i}}\right) =: p_i(\tilde{x}). \tag{6}$$

Applying formula (6) from the Vasicek model to formula (2) from the ASRF framework, the portfolio loss distribution can be computed. For quantification of the credit risk, the Value at Risk (VaR) on confidence level z can be used, that is the z-quantile q_z of the loss variable, in which $z \in (0,1)$ is the target solvency probability. Precisely, like Gordy (2004), we define the VaR as the loss that is only exceeded with the probability of at most 1–z, i.e.

$$\text{VaR}_z\left(\tilde{L}\right) := q_z\left(\tilde{L}\right) := \inf\left(l:\ \text{prob}\left(\tilde{L} \le l\right) \ge z\right). \tag{7}$$

In the context of the ASRF framework, the VaR can be computed similarly to formula (1) as

$$\text{VaR}_z\left(\tilde{L}\right) - E\left(\tilde{L} \mid \tilde{x} = q_{1-z}(\tilde{x})\right) \to 0 \quad \text{a.s.}, \tag{8}$$

where $q_z(\tilde{x})$ stands for the z-quantile of the systematic factor. Recalling formula (2), (6), and the normality of the systematic factor, the VaR of the portfolio equals

$$\begin{aligned} \text{VaR}_z^{(\text{Basel})}\left(\tilde{L}\right) &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i\left(q_{1-z}(\tilde{x})\right) \\ &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left(\frac{N^{-1}(PD_i) + \sqrt{\rho_i} \cdot N^{-1}(0.999)}{\sqrt{1-\rho_i}}\right), \end{aligned} \tag{9}$$

if we insert the confidence level z = 0.999. This is the (well established) VaR formula used in Basel II. Obviously, the credit risk only relies on the systematic factor since due to the infinite number of exposures the idiosyncratic risks associated with each individual obligor cancel out each other and are diversified completely.

2.2 Concentration Risk and Coherency

In recent years there is an extensive discussion about reasonable risk measures. Artzner et al. (1999) formulated four axioms that a risk measure should satisfy to be a coherent risk measure: translation invariance, subadditivity, positive homogeneity, and monotonicity. Unfortunately, the commonly used VaR is not coherent because it is not necessarily subadditive. As long as we stay in the ASRF framework this characteristic is not problematic because in this context the VaR is exactly additive. This can be seen in formula (8) considering that the ex-

[9] In the following “E” denotes the expectation operator.

pectation operator is additive. But if we leave the ASRF framework, this behavior is not guaranteed anymore. This is true for non-asymptotic portfolios as well as for multi-factor models. However, many contributions that deal with concentration risk in the context of the Basel II framework use the VaR to quantify credit risk without questioning the risk measure (possibly to be consistent with the ASRF-framework) even if the subadditivity could get problematic if concentration risk is considered.[10] Thus, it could be beneficial to change the measure of risk, e.g. to use the coherent Expected Shortfall (ES), that is defined as[11]

$$ES_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left(E\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)} + q_z \cdot \left[(1-z) - \text{prob}\left(\tilde{L} \geq q_z\right)\right]\right) \tag{10}$$

with q_z for the VaR on confidence level z (see formula (7)), or simply as

$$ES_z\left(\tilde{L}\right) = (1-z)^{-1} \cdot \left[E\left(\tilde{L}\right) \cdot \tilde{I}_{(\tilde{L} \geq q_z)}\right] = E\left(\tilde{L} \mid \tilde{L} \geq q_z\right) \tag{11}$$

for continuous distributions. But before we change the risk measure we study the characteristics of the VaR for credit portfolios and analyze the need for using the ES.

For our analyses we start with removing the first assumption of the ASRF framework leading to a finite number of loans. Therefore we use the binomial model of Vasicek, assuming homogeneous credits. If we recall the conditional probabilities of default from formula (6) we identify the individual default events to be independent. Thus, the (conditional, still uncertain) number of defaults $\tilde{M} \mid x$ (and the gross loss rate) of the portfolio are binomially distributed with probability $p(x)$, i.e.

$$\tilde{M} \mid x \sim B\left(n; p(x)\right). \tag{12}$$

With reference to Vasicek (1987) (see also Gordy and Heitfield (2000)) we are able to calculate the unconditional probability of the occurrence of m defaults and we get

$$\text{prob}\left(\tilde{L} = \frac{m}{n}\right) = \int_{-\infty}^{+\infty} \binom{n}{m} \cdot p(x)^m \cdot \left(1 - p(x)\right)^{n-m} \cdot dN(x). \tag{13}$$

Finally, the VaR of the portfolio in the Vasicek model is

$$VaR_z^{(Vasicek)}\left(\tilde{L}\right) = \inf\left(l:\ \text{prob}\left(\tilde{L} \leq l\right) = \sum_{m=l}^{[l \cdot n]} \text{prob}\left(\tilde{L} = \frac{m}{n}\right) \geq z\right).^{12} \tag{14}$$

For an infinite number of credits the VaR of the Vasicek model converges towards the VaR of

[10] See e.g. Heitfield, Burton, and Chomsisengphet (2006), Cespedes et al. (2006), Düllmann (2006), as well as Düllmann and Masschelein (2006).

[11] See Acerbi and Tasche (2002).

[12] The symbolism $[l \cdot n]$ denotes the highest natural number that is smaller or equal to $(l \cdot n)$.

the ASRF framework.[13]

Now, we compute the VaR on a confidence level z = 99.9% for non-asymptotic portfolios with PD = 0.5% and ρ = 20%. In Figure 1 we plot the VaR for the ASRF framework and for the Vasicek binomial model in the cases of n = 1 to n = 300 homogeneous credits. The VaR for an infinite number of credits is 9.1%. For a finite number of credits the risk is higher because the unsystematic risk can not be diversified. The problem is that the risk should be monotonously decreasing with a higher number of credits, but this behaviour is not reflected by the VaR as a risk measure. Although the subadditivity axiom is not violated in the example, it is obvious that the risk should not increase with a higher number of credits and thus a better diversification. It is also possible to construct superadditive examples in the context of credit risk but this example gives a clear demonstration that it is problematic to use the VaR if there is concentration risk such as name concentration.

- Figure 1 about here -

For comparison, we compute the ES for the same portfolio setting. For calculation of the ES for the Vasicek model we have to apply formula (10) by using formula (14). The ES in the Basel II framework can be calculated with formula (11) and (9). With respect to Acerbi and Tasche (2002) and Pykhtin (2004) we get

$$ES_z^{(Basel)}\left(\tilde{L}\right) = \sum_{i=1}^{n} \frac{w_i \cdot LGD_i}{1-z} \cdot N_2\left(-N^{-1}(z), N^{-1}(PD_i), \sqrt{\rho_i}\right), \tag{15}$$

where $N_2(\cdot)$ stands for the bivariate cumulative normal distribution. As can be seen in Figure 2, the ES can satisfy the mentioned intuition of diversification. Further, our analyses show that the approximation formulas lead to better results if the ES is used instead of the VaR, particularly if there is a high degree of concentration risk.[14] Thus, it is advisable to use the ES instead of the VaR if the portfolio includes concentration risk. At this point the measured economic capital would be significantly higher by the use of ES, what is not the intended consequence of the change from VaR to ES. In our example even the ASRF solution rises from 9.1% to 11.81%. Instead, we would only like to use the appreciated properties for concentration risk without to be bound to increase the amount of economic capital. Therefore, we will adjust the confidence level as described in the next section.

[13] See Vasicek (2002) or Bluhm, Overbeck, and Wagner (2003).

[14] A numerical study can be requested from the authors.

- Figure 2 about here -

2.3 Adjusting for Coherency in Concentrated Portfolios

Against the background of the preceding section it seems to be reasonable to change the risk measure from VaR to ES if there are violations of the ASRF framework. But if we change the risk measure we have to ensure that the new risk measure (the ES) on the one hand is consistent with the Basel II framework (particular pillar 2) to get meaningful results for additional capital requirements stemming from concentration risk. On the other hand the new risk measure should match the capital requirements of Basel II if the portfolio under consideration fulfills the assumptions of the ASRF framework. I.e. in the context of the ASRF framework, the capital requirements should not differ whether the risk is measured by the VaR or by the ES. Therefore, we examine the $\mathrm{VaR}_{99.9\%}$ on the given confidence level z = 99.9% for several (infinitely granular) bank portfolios of different quality. As a next step we determine the confidence level of the ES that is necessary to match the results for both risk measures. We define this ES-confidence level z (= z(ES)) implicitly as

$$\mathrm{ES}_z^{(\mathrm{Basel})}(\tilde{\mathrm{L}}) = \mathrm{VaR}_{0.999}^{(\mathrm{Basel})}(\tilde{\mathrm{L}}), \tag{16}$$

with $\mathrm{VaR}_{0.999}^{(\mathrm{Basel})}$ given by formula (9) and $\mathrm{ES}_z^{(\mathrm{Basel})}$ presented in formula (15).

Firstly, we investigate the extreme cases that all creditors of a bank have a rating of (I) AAA or (VII) CCC.[15] As can be seen in Table 1, the ES-confidence level must be in a range between 99.67% and 99.74%. Using these confidence levels the economic capital is almost identical regardless of whether VaR or ES is used.

- Table 1 about here -

Additionally, we use five portfolios with different credit quality distributions (very high, high, average, low, and very low) that are visualized in Figure 3.[16] All resulting confidence levels are between 99.71% and 99.73% with mean 99.72%. Even if there is some interconnection

[15] We used the idealized default rates from Standard & Poors, see Brand and Bahar (2001), ranging from 0.01% to 18.27%, but the results do not differ widely for different values.

[16] The portfolios with high, average, low, and very low quality are taken from Gordy (2000). We added a portfolio with very high quality.

between the confidence level and the portfolio quality, an ES-confidence level of z = 99.72% seems to be accurate for most real world portfolios.

- Figure 3 about here -

3 Basel II-consistent Credit Risk Modeling in a Multi-Factor Setting

3.1 Multi-Factor Models in Credit Risk Modeling

To obtain a more realistic modeling of correlated defaults in a credit portfolio, we will introduce a typical multi-factor model. In such a model the dependence structure between obligors is not driven by one global systematic risk factor but by sector specific risk factors. Additionally, the group of obligors is divided into S sectors. Hereby a suitable sector assignment is important,[17] i.e. asset correlations shall be high within a sector and low between different sectors. In contrast to the single factor model in which the correlation structure of each firm i is completely described by ρ, in a multi-factor model we differentiate between an inter-sector correlation ρ_{Inter} and an intra-sector correlation ρ_{Intra}. The inter-sector correlation describes the correlation between the sector factors and the intra-sector correlation characterizes the sensitivity of the asset return to the corresponding sector factor. Thus, the asset return of obligor i in sector s can be represented by

$$\tilde{a}_{s,i} = \sqrt{\rho_{Intra,i}} \cdot \tilde{x}_s + \sqrt{1-\rho_{Intra,i}} \cdot \tilde{\xi}_i, \tag{17}$$

where $\tilde{x}_s$ is the sector risk factor and $\tilde{\xi}_i$ stands for the idiosyncratic factor. $\tilde{x}_s$ and $\tilde{\xi}_i$ are normally distributed variables with mean zero and standard deviation one that are independent among each other. Since the sector risk factors $\tilde{x}_s$ are potentially dependent we make use of the possibility to present the sector risk factors as a combination of independently and standard normally distributed factors $\tilde{z}_k$ (k = 1, …, K)

$$\tilde{x}_s = \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k \quad \text{with} \sum_{k=1}^{K} \alpha_{s,k}^2 = 1, \tag{18}$$

where the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix, implying

[17] As shown by Morinaga and Shiina (2005) an assignment of borrowers to the wrong sectors leads to a higher estimation error than a non-optimal sector definition.

$$\rho_{s,t}^{\text{Inter}} = \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} \ . \tag{19}$$

From (17) and (18) the asset correlation between two obligors is given by

$$\operatorname{corr}(\tilde{a}_{s,i}, \tilde{a}_{t,j}) = \begin{cases} \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} & ,\text{if } s = t, \\ \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} & ,\text{if } s \neq t. \end{cases} \tag{20}$$

Obligors in the same sector will be highly correlated with one another when their intra-sector correlation is high. The correlation of obligors in different sectors also depends on the factor weights, which are derived from the inter-sector correlation. Hence the dependence structure in the multi-factor model is completely described by the intra- and inter-sector correlations.

Taking formula (5) into account, the portfolio loss distribution can be written as

$$\tilde{L} = \sum_{s=1}^{K} \sum_{i=1}^{n_s} w_{s,i} \cdot \tilde{I}_{\tilde{a}_{s,i} < N^{-1}(PD_i)}, \tag{21}$$

where n_s is the number of obligors in sector s.

In the next three sections we will present different approaches to determine the distribution and tail expectations. Furthermore, we will demonstrate how the models can be parameterized to be Basel II-consistent.

3.2 Monte-Carlo-Simulations and Parameterization through a Correlation Matching Procedure

A common approach to estimate the portfolio loss distribution is the use of Monte-Carlo-Simulations. In each simulation run the sector factors as well as the idiosyncratic factor of each obligor are randomly generated. Herewith the asset return is calculated according to (17). If $\tilde{a}_{s,i}$ is less then a threshold given by $N^{-1}(PD_i)$, the obligor i defaults. The portfolio loss is determined from formula (21) by summing up the exposure weights w_i multiplied by the LGD_i of each defaulted credit. To get a good approximation of the "true" loss distribution we choose 500,000 runs for our Monte-Carlo-Simulations. After running the simulation and sorting loss outcomes, we get the portfolio loss distribution. To obtain the ES for a given confidence level z, in principle the mean for all loss realizations equal or greater than q_z has to be calculated. q_z is given by the $z \cdot 500{,}000$th element of the simulated distribution.[18]

[18] The exact formulation is given in formula (10).

To calibrate the multi-factor model, most variables can be chosen identically to the single factor model. The only difference is the correlation structure that generally consists of inter- and intra-sector correlations as described above. The matrix of inter-sector correlations is usually derived from historical default rates or from equity correlations between industry sectors. The problem of a derivation based on historical default rates is that there are not always enough observations to get stable results. That is even more problematic if it is assumed (like in Basel II) that the correlation and the PD are interdependent. Furthermore, the results from the multi-factor model would normally not be consistent with Basel II because the correlation structure is completely different. Thus, it would not be possible to identify (consistently to pillar 1 of Basel II) if there is need for additional regulatory capital under pillar 2.

For both reasons the intra-sector correlations could be chosen analogous to the Basel II formula

$$\rho_{\text{Basel}} = 0,12 \cdot \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}} + 0,24 \cdot \left(1 - \frac{1-e^{-50 \cdot \text{PD}}}{1-e^{-50}}\right) \tag{22}$$

for corporates. This is what Cespedes et al. (2006) did in their analyses. But this approach is critical for the following reason: Using this formula for the intra-sector correlations is equivalent to the assumption that the Basel II framework is an upper barrier of the true risk and is only valid if there is only one sector or if all sectors are perfectly correlated. In all other cases there is an effect of sector diversification that leads to lower capital requirement compared to the Basel framework. In contrast to this assumption, the Basel II correlation formula is not intended by the Basel committee to reflect the intra-sector correlation only. Instead, the framework is calibrated on well-diversified portfolios implying the correlation formula in the single factor model to be a good approximation of the "true" risk that is based on the full correlation structure in a multi-factor model.[19] Cespedes et al. (2006) already recognized this criticism and mentioned that it should be possible to use some scaling up for the intra-sector correlations and the resulting capital, respectively, but their calculations are based on the formula above.

Alternatively, the intra-sector correlation could be chosen in a way that the regulatory capital can be matched with the economic capital that is simulated for a well-diversified portfolio within a multi-factor model. Therefore, we implicitly define the "implicit intra-sector correlation" $\rho_{\text{Intra}}^{(\text{Implied})}$ by

$$\text{EC}_{\text{Multi}}(\rho_{\text{Inter}}, \rho_{\text{Intra}}^{(\text{Implied})}) = \text{EC}_{\text{Single}}(\rho_{\text{Basel}}) \,. \tag{23}$$

[19] See Basel Committee on Banking Supervision (2006) and CEBS (2006).

Unfortunately, the portfolios for which the calibration was done by the Basel Committee the assumed inter-sector correlation are not publicly available. Thus, firstly we have to choose a concrete inter-sector correlation and determine the implicit intra-sector correlation for some hypothetical, well-diversified portfolios via Monte-Carlo-Simulations with several parameter trials. For the inter-sector correlation structure we use the matrix computed by Düllmann and Masschelein (2006) that is based on MSCI EMU industry indices (see Table 2).[20]

- Table 2 about here -

Our definition of a well-diversified portfolio is based on the overall sector concentration of the German banking system.[21] Even if it is theoretically possible to achieve lower capital requirements through different sector decomposition, this can only be done by a restricted number of banks since a deviation from the market structure of all banks immediately leads to a disequilibrium. The composition can be seen in Table 3. In addition, the total number of credits is assumed to be n = 5000 to guarantee low granularity.

- Table 3 about here -

If we assume a constant intra-sector correlation, the best match is achieved around $\rho_{\text{Intra}}^{(\text{Implied})} = 25\%$[22] but the concrete results vary with the portfolio quality (see Table 4).[23] Thus, using a constant intra-sector correlation can lead to a significant underestimation of economic capital for high-quality portfolios and to an overestimation for low-quality portfolios.

- Table 4 about here -

[20] The correlation structure based on the MSCI US is similar, see Düllmann and Masschelein (2006).

[21] Düllmann and Masschelein (2006) notice that the concentration is very similar to other countries like France, Belgium and Spain.

[22] This value results on the basis of both measures (VaR and ES) on the respective confidence level as described in section 2.3. The result is consistent with Düllmann and Masschelein (2006) who use a constant intra-sector correlation of 25% in their analysis.

[23] See Figure 3 for the portfolio characteristics.

To reduce the deviation, the intra-sector correlation should be decreasing in PD. We found that the following intra-sector correlation function leads to a good match for portfolios with different quality distributions:

$$\rho_{\text{Intra}}^{(\text{Implied})} = 0{,}185 \cdot \frac{1 - e^{-50 \cdot \text{PD}}}{1 - e^{-50}} + 0{,}34 \cdot \left(1 - \frac{1 - e^{-50 \cdot \text{PD}}}{1 - e^{-50}}\right) \tag{24}$$

Thus, in principle we use the correlation function from Basel but the correlation range is from 18.5% to 34% instead of 12% to 24%.

Hence, all additional input data needed for typical multi-factor models, e.g. using Monte-Carlo-Simulations, are given with Table 2 and formula (24). Using these values, the multi-factor models should be consistent with the Basel framework. Thus, the measured economic capital is only lower than the regulatory capital if the portfolio is less concentrated than a typical, well-diversified portfolio and the needed economic capital will be above the capital requirement of the regulatory framework if there is more concentration risk in the credit portfolio.

3.3 Implementation for the Pykhtin-Model

In this section we present the multi-factor adjustment of Pykhtin (2004). It is an extension of the granularity adjustment, introduced by Gordy (2003), Wilde (2001) and Martin and Wilde (2002), for multi-factor models and provides an analytical method for calculating the VaR and ES of a credit portfolio.

The basic idea from Pykhtin is to approximate the portfolio loss $\tilde{L}$ in the multi-factor model with the respective portfolio loss $\tilde{\tilde{L}}$ in an accurately adjusted single factor model. This is done by mapping the correlation structure of the multi-factor model into a single correlation factor to approximate the distribution of L optimally. Concretely, Pykhtin defines a single risk factor $\tilde{\tilde{x}}$ in order to explain the original risk factors $\{\tilde{z}_k\}$ and to maximize the correlation between $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$.[24]

Via this approach it is possible, as shown in Wilde (2001), to approximate the z-quantile $q_z(\tilde{L})$ of the portfolio loss by a quadratic Taylor series as

$$q_z(\tilde{L}) \approx q_z(\tilde{\tilde{L}}) + \left.\frac{dq_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2} \cdot \left.\frac{d^2 q_z(\tilde{\tilde{L}} + \varepsilon \cdot \tilde{U})}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{25}$$

[24] The sets $\{\tilde{z}_k\}$ and $\{\tilde{x}_s\}$ are given as in section 3.1.

where ε is the scale of perturbation and U describes the approximation error between $\tilde{L}$ and $\tilde{\bar{L}}$, i.e. $\tilde{U} = \tilde{L} - \tilde{\bar{L}}$. The first summand on the right-hand side of (25) is the z-quantile of loss $\tilde{\bar{L}}$ for an infinitely fine grained portfolio in a single factor model. Since the conditions of the ASRF-model are satisfied, the loss distribution can be calculated by formula (9), so that

$$\tilde{\bar{L}} = l(\tilde{\bar{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\bar{x}}}{\sqrt{1-c_i}}\right], \tag{26}$$

where c_i is the correlation between the systematic risk factor $\tilde{\bar{x}}$ and the asset return $a_{s,i}$.[25] Instead of using ρ as it is done in the ASRF-model, the new correlation parameter c_i is used to match the correlation structure in the multi-factor model. Further, the loss quantile $q_z(\tilde{\bar{L}})$ is given by $l(N^{-1}(1-z))$. In addition, it can be shown that the first derivative in formula (25) is equal to zero, since $\tilde{\bar{L}} = E\left[\tilde{L} \mid \tilde{\bar{x}}\right]$.[26] Hence, the so-called multi-factor adjustment Δq_z is described completely by the second derivative. According to Pykhtin Δq_z can be written as

$$\Delta q_z = q_z(L) - q_z(\bar{L}) \approx -\frac{1}{2 \cdot l'(\bar{x})} \cdot \left[v'(\bar{x}) - v(\bar{x}) \cdot \left(\frac{l''(\bar{x})}{l'(\bar{x})} + \bar{x} \right) \right] \Bigg|_{\bar{x}=N^{-1}(1-z)}, \tag{27}$$

where $l'(\bar{x})$ and $l''(\bar{x})$ are the first and second derivative of formula (26) and $v(\bar{x})$ is the conditional variance of U.[27] Further, $v(\bar{x})$ can be decomposed into two terms, $v_\infty(\bar{x})$ and $v_{GA}(\bar{x})$.[28] The term $v_\infty(\bar{x})$ describes the systematic risk adjustment, which is given by the difference between the multi-factor and single-factor loss distribution. The other term $v_{GA}(\bar{x})$ is the granularity adjustment, which measures the influence of single-name concentration. Using these terms the multi-factor adjustment can be presented as

$$\Delta q_z = \Delta q_z^\infty + \Delta q_z^{GA}, \tag{28}$$

i.e. the multi-factor adjustment can be split into systematic risk adjustment component and a granularity adjustment component. Finally, the approximation of a loss quantile $q_z(\tilde{L})$ in (25) is given by (26) and the multifactor adjustment:

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[25] The derivation of c_i to obtain the maximum correlation between $\tilde{\bar{x}}$ and $\{\tilde{x}_s\}$ can be found in Appendix 1. From Appendix 1 we also know that both (the intra- and inter-sector) correlations are needed to determine c_i, which can be taken from section 3.2.

[26] Thus, the conditional expected obligor risk vanishes.

[27] The required formulas to determine can be found in Appendix 2.

[28] Again, see Appendix 2 for details.

$$q_z(\tilde{L}) = q_z(\tilde{\bar{L}}) + \Delta q_z^\infty + \Delta q_z^{GA}. \tag{29}$$

After dealing with the VaR we now determine the ES in a multi-factor model. In this context formula (11) can be rewritten as

$$ES_z(\tilde{L}) = ES_z(\tilde{\bar{L}}) + \frac{1}{1-z} \cdot \int_z^1 \Delta q_s(\tilde{L})\,ds =: ES_z(\tilde{\bar{L}}) + \Delta ES_z(\tilde{L}). \tag{30}$$

To get this result the quantile $q_z(\tilde{L})$ is substituted by the approximation (29). The first summand of the right-hand side describes the ES for the single factor portfolio and the second summand is the multi-factor adjustment.

As shown by Pykhtin (2004) $ES_z(\tilde{\bar{L}})$ and $\Delta ES_z(\tilde{L})$ can be calculated as

$$ES_z(\tilde{\bar{L}}) = \frac{1}{1-z} \sum_{i=1}^{n} w_i \cdot LGD_i \cdot N_2\left[N^{-1}(PD_i), N^{-1}(1-z), c_i\right], \tag{31}$$

and

$$\Delta ES_z(\tilde{L}) = -\frac{1}{2\cdot(1-z)} \cdot n\left[N^{-1}(1-z)\frac{v\left[N^{-1}(1-z)\right]}{l'\left[N^{-1}(1-z)\right]}\right], \tag{32}$$

with $n(\cdot)$ denoting the density function of the standard normal distribution. Again, the multi-factor adjustment can be decomposed into a systematic and an idiosyncratic part by decomposing the conditional variance. Hence the ES for a portfolio in a multi-factor model is given by

$$ES_z(\tilde{L}) = ES_z(\tilde{\bar{L}}) + \Delta ES_z^\infty(\tilde{L}) + \Delta ES_z^{GA}(\tilde{L}). \tag{33}$$

In principle it is straightforward to implement the Pykhtin model. For calculating the ES we have to compute formula (32).[29] If applied to large portfolios, its computation can be extremely time-consuming. The reason is that the calculation procedure inter alia requires n^2-times the computation of the conditional asset correlation,[30] with n being the number of credits. An alternative performed by Düllmann and Masschelein (2006) is to neglect the multi-factor adjustment and to use (26) only to aggregate all credits for each sector and thus using the formulas on sector and not on borrower level. In contrast, we propose to built PD-classes for each of the sectors and aggregate the credits to these buckets for the calculation of the multi-factor adjustment, so that the computation time is predominated by

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[29] To do so we need conditional variances presented in Appendix 2 in (A.11) and (A.14) by using (A.12), (A.6) and (A.8).

[30] The quadratic computation effort is due to the determination of a double sum (see Appendix 2, (A.11)).

$$\text{Loops} = (N_{\text{PD-classes}} \cdot S_{\text{Sectors}})^2\,, \tag{34}$$

where N_{PD} and S denote the number of PD-classes and sectors.[31] As the number of loops will not grow with bigger portfolios, it is possible to perform the adjustment on bucket level within reasonable time. Only the granularity adjustment should be calculated on borrower level but this is no computational burden.[32]

3.4 Implementation for the Cespedes-Model

Cespedes et al. (2006) present a method to relate the economic capital in the multi-factor model to the economic capital in a single-factor model via a diversification factor DF( ), which depends on two parameters:

- the average sector concentration CDI and
- the average weighted inter-sector correlation $\overline{\beta}$.

Herewith the economic capital of a portfolio can be approximated as:

$$EC^{mf} \approx DF \cdot EC^{sf}. \tag{35}$$

Thus, the economic capital in the multi-factor model EC^{mf} can be approximated by a well-defined diversification factor DF multiplied with the economic capital in the ASRF-model EC^{sf}. As mentioned before, Cespedes et al. assume that the Basel framework is an upper barrier of the true risk because no diversification effects between the sectors are considered, which implies that DF is always less than or equal to one. In contrast, if we use our definition of the intra-sector correlation ρ_{Intra} from section 3.2, it is possible to obtain $EC^{mf} > EC^{sf}$ as well as $EC^{mf} < EC^{sf}$, depending on the degree of diversification in comparison to the well-diversified portfolio defined in section 3.2. Hence, our later on calculated DF-function can be greater than one, i.e. the DF-function measures not only the benefit from sector diversification but also the risk resulting from high sector concentration. As the economic capital is additive in the ASRF-model (35) can be substituted by:

$$EC_z^{mf} = DF \cdot \sum_{k=1}^{K} EC_z^k\,, \tag{36}$$

[31] The results of the multi-factor adjustment do not differ whether different exposures with the same PD are aggregated or handled separately on borrower level. For details see Appendix 2.

[32] The computation time when calculating the multi-factor adjustment on bucket- instead on borrower-level can be reduced from 67 minutes to 2 seconds for a portfolio with 11 sectors and 5000 creditors on a PC with 3 GHz CPU.

where EC_z^{mf} is the economic capital in the multi-factor model and EC_z^k is the economic capital in the ASRF-model for sector k. In principle, the approach can be characterized as follows: Firstly, the EC_z^{mf} is calculated for a multitude of portfolios via Monte-Carlo-Simulations. For every simulated portfolio the diversification factor can be calculated according to formula (36). Finally, a regression is performed to get an approximation for DF as a function of the two parameters CDI and $\overline{\beta}$. If DF can capture the industry diversification effects, we are able to approximate EC_z^{mf} with formula (36) without additional Monte-Carlo-Simulations.

To derive the parameters which explain the effect of diversification and concentration in a multi-factor model, Cespedes et al. suggest to use the average inter-sector correlation $\overline{\beta}$. This can be interpreted as a scale of the dependence between the sectors. The formula for $\overline{\beta}$ is given as:

$$\overline{\beta} = \frac{\sum_{k=1}^{K}\sum_{j\neq k}\rho_{kj}^{inter}\cdot EC_z^k \cdot EC_z^j}{\sum_{k=1}^{K}\sum_{j\neq k} EC_z^k \cdot EC_z^j}. \tag{37}$$

The correlation is weighted by the expected shortfall in order to account for the contribution of each sector. The second suggested parameter is the capital diversification index denoted by CDI. It describes the sector concentration measured by the relative weight of each EC_z^k:[33]

$$CDI = \frac{\sum_{k=1}^{K}\left(EC_z^k\right)^2}{\left(\sum_{k=1}^{K} EC_z^k\right)^2}. \tag{38}$$

The parameter CDI lies between the two extreme values:

- $CDI = \frac{1}{n}$, i.e. perfect sector diversification,
- $CDI = 1$, i.e. perfect sector concentration.

To avoid a too complex model Cespedes et al. neglect further potential input parameters to determine the DF-function. To approximate the multi-factor model, formula (36) can be rewritten as:

$$EC_z^{mf}\left(CDI,\overline{\beta}\right) = DF\left(CDI,\overline{\beta}\right)\cdot\sum_{k=1}^{K} EC_z^k. \tag{39}$$

[33] This concentration measure is also known as the Herfindahl-Hirschmann-Index.

In the following, our procedure to estimate the DF-function is presented. To get a universally valid DF-factor as many portfolios as possible have to be generated and simulated. To reduce the necessary number of trials, the portfolios should be restricted to those with reasonable characteristics. Our portfolios are randomly generated using the following parameter setting. When we state several parameter values or a parameter range, the parameter is randomly drawn from this set.

For the intra-sector correlations we use the functional form of formula (24). The inter-sector correlation structure is taken from Table 2, so that all simulated portfolios are stemming from this sector definition. Each portfolio consists of {2, …, 11} sectors that are randomly drawn from the different industries. The sector weights are in [0, 1]. The total number of credits is 5000, equally divided for each sector. Each sector in turn consists of credits from the PD classes {AAA, AA, A, BBB, BB, B, CCC}. Instead of using equally distributed PD classes we draw the quality distribution from our predefined credit portfolio qualities {very high, high, average, low, very low} for every sector.[34] We draw 25,000 and 50,000 portfolios, respectively, and compute the economic capital in the multi-factor model for each portfolio.

To determine the economic capital we tried both Monte-Carlo-Simulations with 100,000 trials[35] for every portfolio and the Pykhtin formula from section 3.3. Because the computation time for Monte-Carlo-Simulations is materially longer, the corresponding results are based on 25,000 random portfolios whereas we computed the economic capital for 50,000 portfolios when using the Pykhtin formula instead.[36] Furthermore we used a different definition of economic capital as Cespedes et al. (2006), who define the economic capital as $\mathrm{EC} = \mathrm{VaR} - \mathrm{EL}$. We redefine the economic capital of the multi-factor model with respect to ES instead of VaR, $\mathrm{EC}^{\mathrm{mf}} = \mathrm{ES}^{\mathrm{mf}} - \mathrm{EL}$, as argued in section 2.3.[37] In contrast, for the ASRF-model we use the VaR, $\mathrm{EC}^{\mathrm{sf}} = \mathrm{VaR}^{\mathrm{sf}} - \mathrm{EL}$. The result could also be related to the Expected Shortfall in the ASRF-model but we detected that the results differ only marginally and the Value at Risk is

[34] The setting is similar to Cespedes et al. Until this point, the main difference is the definition of the intra- and inter-sector correlations.

[35] For determination of the economic capital for one specific portfolio the number of trials is slightly low but as we perform 25,000 simulations and the simulation noise of each simulation is unsystematic the error terms should cancel out each other to a large extent.

[36] In a Matlab environment the computation of the Monte-Carlo-Simulations took about one month on 3 PCs each with 3 GHz CPUs. In contrast, the computation time when using the Pykhtin formula in the proposed way is less than 7 hours on one of these PCs.

[37] We also tested the results when using the ES instead of the unexpected loss but the coefficient of determination is higher when subtracting the EL in the corresponding formulas when performing the simulations.

easier to implement in typical spreadsheet applications.[38] The results for the diversification factor DF are very similar whether they are based on Monte-Carlo-Simulations or on the Pykhtin formula. In Figure 4 the diversification factor when using the Pykhtin formula can be seen.

- Figure 4 about here -

For determination of the functional form of DF we use a regression of the type[39]

$$DF = a_0 + a_1 \cdot (1 - CDI) \cdot (1 - \overline{\beta}) + a_2 \cdot (1 - CDI)^2 \cdot (1 - \overline{\beta}) + a_3 \cdot (1 - CDI) \cdot (1 - \overline{\beta})^2 \tag{40}$$

in both cases. The resulting function when using Monte-Carlo-Simulations is

$$\begin{aligned} DF_{MC} &= 1.4626 - 1.4475 \times (1 - CDI) \times (1 - \overline{\beta}) \\ &\quad - 0.0382 \times (1 - CDI)^2 \times (1 - \overline{\beta}) + 0.3289 \times (1 - CDI) \times (1 - \overline{\beta})^2 \end{aligned} \tag{41}$$

with $R^2 = 95.5\%$. Analogously, we determined the DF-function when using the Pykhtin formula

$$\begin{aligned} DF_{Pykhtin} &= 1.4598 - 1.4168 \times (1 - CDI) \times (1 - \overline{\beta}) \\ &\quad - 0.0213 \times (1 - CDI)^2 \times (1 - \overline{\beta}) + 0.2421 \times (1 - CDI) \times (1 - \overline{\beta})^2 \end{aligned} \tag{42}$$

with an coefficient of determinination of $R^2 = 97.9\%$. The latter function is plotted in Figure 5.[40] To finally get the approximation for the multi-factor model, formula (39) has to be computed using either function (41) or (42).

- Figure 5 about here -

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[38] To determine the Expected Shortfall with formula (15) a bivariate cumulative normal distribution has to be computed whereas the Value at Risk only makes use of univariate distributions.

[39] We tried several different regressions but similar to Cespedes et al. this function worked best. In contrast to Cespedes et al. we did not set the first parameter a_0 to one because our DF-factor is not bound by the single-factor-model.

[40] The shape of the function is similar to Cespedes et al. but their range is from 0.1 to 1.0 whereas our function ranges from 0.2 to 1.5. In addition they received a little higher R^2 (99.4% instead of 95.5% and 97.9%, respectively) but this is mainly due to the different simulation setting. Cespedes et al. directly draw the parameter $\overline{\beta}$ as an input parameter for each simulation, implying $\overline{\beta}$ to fully define their correlation structure. We use a heterogeneous correlation structure instead and compute $\overline{\beta}$ for the portfolios. Thus, in our setting $\overline{\beta}$ does not reflect the complete correlation structure which results in a lower R^2 but does not imply a worse approximation.

It can be seen that the maximum diversification factor is about 1.46. Thus, in the case of (almost) no diversification effects the measured capital requirement is 46% above the regulatory capital under Pillar 1. This will appear in the case of being concentrated to a single sector, leading to $\mathrm{CDI} = 1$, as well as in the theoretical case of perfect correlations between the relevant sectors, leading to $\overline{\beta} = 1$. Furthermore, the diversification factor is strongly increasing in CDI and in $\overline{\beta}$ which is consistent with the intuition.

4 Performance of the Concentration Risk Models

4.1 Analysis for Deterministic Portfolios

To determine the quality of the presented models we start our analysis with calculating the expected shortfall for five deterministic portfolios of different quality.[41] We generate well-diversified portfolios consisting of 5,000 credits. Consequently, we have neither high name nor high sector concentration risk. For this we choose the sectors and their weights as given in Table 3. The inter-sector correlation is given in Table 2 whereas the intra-sector correlation is calculated on the basis of formula (24). The five portfolios differ in their PD distribution which is presented in Figure 3. Portfolio 1 is the portfolio with the highest and Portfolio 5 is the one with the lowest credit quality distribution.

In Table 5 we compare the results from Monte-Carlo-Simulation (MC-Sim.), the Basel II formula (Basel II), the Pykhtin model (Pykhtin), the Cespedes model with Monte-Carlo-Simulations (Cespedes I) and the Cespedes model with the Pykhtin formula (Cespedes II). As can be seen in the table, the benchmark portfolio is constructed in a way that the Basel II formula represents a very good approximation[42] of the "real" ES in a multi-factor model given by Monte Carlo Simulations.[43] The calculated values of the Pykhtin model are very good approximations of the ES in almost all cases, too. The outcomes of the Cespedes model are somewhat more imprecise in both cases. With better credit quality the estimation error is increasing, which leads to an underestimation of risk in high quality portfolios.

[41] The results refer to the total gross loss of a portfolio in terms of ES. To relate this to the unexpected net loss, the results have to be multiplied by the LGD and the EL has to be subtracted.

[42] The small mismatch is mainly due to keeping the ES-confidence level constant and not a result of the chosen intra-correlation function. If we directly compare the results from Monte-Carlo-Simulations with the ES in the ASRF-framework from formula (15), the relative root mean squared error is reduced from 0.97% to 0.28%.

[43] In our analyses the number of simulation runs is 500,000.

- Table 5 about here -

As a next step, we change the portfolio structure towards high sector concentration. Therefore, we increase the sector weights of two sectors. We assume that 45% of the creditors – in terms of their exposure – belong to the Information Technology sector and an equal amount belongs to the Telecommunication Services sector. The remaining 10% of exposure are equally assigned to the miscellaneous sectors. As shown in Table 6 the risk materially increases for all types of portfolio quality. Especially, the Basel formula underestimates the risk by 14 % to 20 % depending on the portfolio quality. This is the (relative) amount that should be considered in the assessment of capital adequacy under pillar 2. The approximation formula of Pykhtin can capture this concentration risk with a negligible error in all cases. Cespedes I leads to an underestimation of risk in high quality portfolios and to an overestimation of risk in low quality portfolios with a maximum deviation of nearly 4%. Contrary, Cespedes II underestimates the risk in most cases with up to 6%. Thus, the sector concentration risk is not fully captured for high quality portfolios.

- Table 6 about here -

Furthermore, we built credit portfolios with low sector concentration. For this purpose, we use the concept of naïve diversification so that every sector has an equal weight of 1/11. As can be seen in Table 7, the economic capital is significantly lower than the regulatory capital.[44] Moreover, this shows that it is easy to construct portfolios, which are better diversified than the overall credit market.[45] Again, the Pykhtin model leads to good approximations for all types of credit qualitiy. The Cespedes model I understimates the risk for high quality portfolios with up to 3%. The Cespedes model II underestimates the risk, too, but the approximation error is negligible.

- Table 7 about here -

[44] However, in the case of these negative deviations in comparison to the regulatory capital it is not allowed – at least at present – to reduce the regulatory capital.

[45] If we consider all 25,000 simulated portfolios from section 3.4, the lowest measured economic capital requirement was even 26% lower than the regulatory capital. This underlines the prospects of actively managing credit portfolios, e.g. with credit derivatives, but this is not in the scope of this paper.

4.2 Simulation Study for Homogeneous and Heterogeneous Portfolios

To achieve more general results we test the models for different, randomly generated portfolios. For this we implement four simulation studies. In these studies we analyze the accuracy for homogeneous as well as for heterogeneous portfolios with respect to PD and EAD. In each simulation run we generate a portfolio and determine its ES by the three models. After 100 runs we calculate the root mean squared error for the outcomes of the Pykhtin model and of the Cespedes model I and II[46] in absolute and relative terms to quantify its performance in comparison to Monte-Carlo-Simulations using 500,000 trials. In the following we describe the four simulation settings.

Simulation I: In this scenario we generate portfolios with homogenous exposure sizes and homogenous PDs, that is, $w_i = 1/5000$ and $PD_i = PD = \text{const}$ for each credit. To test the accuracy for different portfolio qualities a PD is drawn from a uniformly distribution between 0 % and 10 % before each new run. The sector structure and correlation is the same as in section 4.1.

Simulation II: We generate portfolios with homogenous exposure sizes but heterogeneous PDs. For each sector we determine randomly one of the quality distributions from section 2.3. After that we draw the PD for each credit of the sector according to this quality distribution. The exposure size remains as in Simulation I. Again, the sector structure and correlation is taken from section 4.1.

Simulation III: We generate portfolios with homogenous PDs as in Simulation I but with heterogeneous exposure sizes. Firstly, we choose the number of sectors randomly between 4 and 11. Then we apply a uniform distribution between 0 and 1 for the weight of every sector and scale this so that the weights sum up to one. The weights for the credits in each sector are determined in the same manner. The correlations remain unchanged.

Simulation IV: In this setting the PDs as well as the exposure sizes of the generated portfolios are heterogeneous. The PDs are determined as in Simulation II and the exposure sizes as in Simulation III.

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[46] Again, Cespedes I corresponds to the DF-function based on Monte-Carlo-Simulation and Cespedes II on the Pykhtin formula.

In each simulation we calculate the intra-sector correlations with formula (24) and choose 5,000 credits. These portfolios contain a relatively low amount of name concentration. Instead we focus on sector concentration. The reason is that the identical methodology for measuring name concentrations, the granularity adjustment can be used within both approaches. Thus, we prefer to avoid name concentrations to be able to separately analyze the effect of sector concentrations. The results of our analyses can be found in Table 8.

- Table 8 about here -

Again, the outcomes of the Pykhtin model are a good approximation of the "true" result from the Monte-Carlo Simulations. Especially, when EADs are homogeneous the results are very good. Both types of the Cespedes model lead to very stable results in all simulation settings. As the approximation accuracy of the alternative implementations Cespedes I and II is almost identically but the computation time for determination of the DF-function is significantly lower for Cespedes II, we strongly propose to use the Pykhtin model to calibrate the model. Interestingly, the Cespedes model performs even better when PDs are heterogeneous, probably because the portfolios used for calculation of the functional form have heterogeneous PDs, too, and thus the resulting portfolios are more similar. Somewhat surprising, the overall performance of the Cespedes model is better than the Pykhtin model even if the Pykhtin formula is used for determination of the diversification factor. Probably the approximation errors of the Pykhtin model are partially smoothed by the regression from formula (40).

5 Conclusion

In this paper we proposed a methodology to perform multi-factor models that are able to measure concentration risk in credit portfolios in terms of economic capital and still deliver results that are consistent with Basel II. Furthermore, we applied this to different multi-factor approaches and compared their performance. It could be shown that it is possible to achieve good approximations in reasonable time when the approaches are adjusted in the proposed way.

We showed that it is problematic to use the Value at Risk if there is concentration risk in the portfolio what is often disregarded in the literature. Instead, it is advisable to use a coherent risk measure like the Expected Shortfall. As the ES is by definition higher than the VaR, we perform a mapping procedure that determines the confidence level ($z = 99.72$ %) that should be used for the ES to get reasonable values. This assures to get results for the economic capital that are consistent to Basel II and simultaneously avoids the problems of the VaR when leaving the ASRF framework.

Furthermore, we chose input parameters, especially the inter- and intra-sector correlations, in a way that the results are comparable with the regulatory pillar 1 capital. Thus, we do not follow some approaches that assume a pure diversification effect compared with the Basel II formula. Instead, we relate the results to a well-diversified portfolio as assumed when calibrating the Basel II formula and determine a function for the implied intra-sector correlation. Hence, it is possible to directly consider the extent of credit risk concentrations in the assessment of capital adequacy under Pillar 2. Using these modifications, we performed an extensive numerical study similar to Cespedes et al. (2006) to get a closed form approximation formula. In addition, we suggest computing the multi-factor adjustment on bucket instead of borrower level. This allows to compute the Pykhtin formula much faster than Monte-Carlo-Simulations even for a high number of credits.

Having assured a Basel II consistent capital requirement, we analyzed the impact of credit concentration risk and carried out a simulation study to compare the performance of the (modified) models from Cespedes et al. (2006) and Pykhtin (2004). We found that the Pykhtin model leads to very good results for homogeneous as well as heterogeneous PDs when EADs are homogeneous. The performance is slightly lower for heterogeneous EADs. The results of the Cespedes model have a throughout high accuracy. Interestingly, the approach works better for heterogeneous portfolios. In general, both models can be used for approximating the economic capital in a multi-factor setting when adjusted in the proposed way. The main advantage of the Pykhtin model is that it can be directly applied to an arbitrary portfolio type, whereas the approach of Cespedes et al. (2006) should not be used without initially performing the demonstrated extensive numerical work when the portfolio structure is very different. On the contrary, the results of the Cespedes model were slightly better for heterogeneous portfolios and it allows for ad-hoc analyses including sensitivity analyses when the non-recurring extensive numerical work is progressed.

In further analyses it would be interesting to analyze the approach of Cespedes et al. (2006) when adjusted to a specific bank portfolio. Under the (plausible) assumption that a bank's portfolio will only be faced to minor changes for a finite period, it should be possible to get a higher accuracy for this bandwidth of scenarios. Moreover, it would be helpful to know how much numerical work is necessary when the parameters are highly restricted to these realistic cases to achieve stable results because the extensive computation time is still a challenge.

Appendix 1

To relate $\tilde{\tilde{L}}$ to L the systematic factor $\tilde{\tilde{x}}$ is defined as

$$\tilde{\tilde{x}} = \sum_{k=1}^{K} b_k \cdot \tilde{z}_k, \qquad \text{where } \sum_{k=1}^{K} b_k^2 = 1. \tag{A.1}$$

On condition that $\tilde{\tilde{L}} = E\left[\tilde{L} \mid \tilde{\tilde{x}}\right]$ Pykhtin (2004) shows that for obligor i in sector s the c_i can be calculated as

$$c_i = \rho_{Intra,i} \cdot \overline{\rho}_i^2 = \rho_{Intra,i} \cdot \left(\sum_{k=1}^{K} \alpha_{s,k} \cdot b_k \right)^2, \tag{A.2}$$

where $\overline{\rho}_i$ is the correlation between $\tilde{\tilde{x}}$ and $\tilde{x}_s$.

Since there is no unique method to determine the coefficients $\{b_k\}$, we use the approach presented by Pykhtin (2004). Thus, the coefficients are chosen in a way that the correlation between $\tilde{\tilde{x}}$ and $\{\tilde{x}\}$ will be maximized, in order to minimize the difference given by (27) between the quantiles $q_z(\tilde{L})$ and $q_z(\tilde{L})$. This leads to the following maximization problem:

$$\max_{\{b_k\}} \left(\sum_{i=1}^{n} d_i \cdot \overline{\rho}_i \right) \quad \text{such that } \sum_{k=1}^{K} b_k^2 = 1. \tag{A.3}$$

The solutions of $\{b_k\}$ are given as

$$b_k = \sum_{i=1}^{n} \frac{d_i \cdot \alpha_{ik}}{\lambda}, \tag{A.4}$$

where the Lagrange multiplier λ is chosen so that $\{b_k\}$ satisfy the constraint. There is no obvious choice of the weighting factors d_i but

$$d_i = w_i \cdot LGD_i \cdot N\left[\frac{N^{-1}(PD_i) + \sqrt{\rho_{Intra,i}} \cdot N^{-1}(q)}{\sqrt{1-\rho_{Intra,i}}} \right], \tag{A.5}$$

leads to good results, which is the VaR formula in a single factor model. The intuition behind this is that obligors with a high exposure in terms of VaR should get a high weight in the maximization problem.

Appendix 2

The derivatives of (26) are calculated by

$$l'(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i'(\tilde{\tilde{x}}) \tag{A.6}$$

and

$$l''(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i''(\tilde{\tilde{x}}). \tag{A.7}$$

The derivatives $p_i'(\tilde{\tilde{x}})$ and $p_i''(\tilde{\tilde{x}})$ of the conditional default probability are calculated by differentiation of equation (6) as

$$p_i'(\tilde{\tilde{x}}) = -\frac{\sqrt{c_i}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}}\right] \tag{A.8}$$

and

$$p_i''(\tilde{\tilde{x}}) = -\frac{c_i}{1-c_i} \cdot \frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}} \cdot n\left[\frac{N^{-1}(PD_i) - \sqrt{c_i} \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i}}\right]. \tag{A.9}$$

Since $\tilde{\tilde{L}}$ is deterministic for given $\tilde{\tilde{x}}$, $v(\tilde{\tilde{x}})$ equals the conditional variance of $\tilde{L}$, this means $v(\tilde{\tilde{x}}) = var(\tilde{L} - \tilde{\tilde{L}} \mid \tilde{\tilde{x}}) = var(\tilde{L} \mid \tilde{\tilde{x}})$. To calculate $v(\tilde{\tilde{x}})$ the conditional variance can be decomposed as the sum of systematic and idiosyncratic parts:

$$v(\tilde{\tilde{x}}) = \underbrace{var[E(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_\infty(\tilde{\tilde{x}})} + \underbrace{E[var(L \mid \{\tilde{z}_k\}) \mid \tilde{\tilde{x}}]}_{v_{GA}(\tilde{\tilde{x}})}. \tag{A.10}$$

The first summand $v_\infty(\tilde{\tilde{x}})$ of (A.10) can be calculated as

$$\begin{aligned} v_\infty(\tilde{\tilde{x}}) = \sum_{i=1}^{n}\sum_{j=1}^{n} & w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot \\ & \left[N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \rho_{ij}^{\tilde{\tilde{x}}}\right) - p_i(\tilde{\tilde{x}}) \cdot p_j(\tilde{\tilde{x}})\right], \end{aligned} \tag{A.11}$$

where $\rho_{ij}^{\tilde{\tilde{x}}}$ describes the conditional asset correlation

$$\sqrt{\rho_{ij}^{\tilde{\tilde{x}}}} = \frac{\sqrt{\rho_{Intra,i} \cdot \rho_{Intra,j}} \cdot \sum_{k=1}^{K} \alpha_{ik} \cdot \alpha_{jk} - \sqrt{c_i \cdot c_j}}{\sqrt{(1-c_i) \cdot (1-c_j)}}. \tag{A.12}$$

The first derivative of $v_\infty(\tilde{\tilde{x}})$ is given by:

$$\begin{aligned} v_\infty'(\tilde{\tilde{x}}) = 2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} & w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot p_i'(\tilde{\tilde{x}}) \\ & \cdot \left[N\left(\frac{N^{-1}\left[p_j(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ij}^{\tilde{\tilde{x}}}} \cdot N^{-1} \cdot \left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ij}^{\tilde{\tilde{x}}}}}\right) - p_j(\tilde{\tilde{x}})\right]. \end{aligned} \tag{A.13}$$

The second summand $v_{GA}(\tilde{\tilde{x}})$ of (A.10) and its derivative $v_{GA}'(\tilde{\tilde{x}})$ are

$$v_{GA}(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot \left(LGD_i^2\left[p_i(\tilde{\tilde{x}}) - N_2\left(N^{-1}\left[p_i(\tilde{\tilde{x}})\right], N^{-1}\left[p_j(\tilde{\tilde{x}})\right], \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}}\right)\right]\right) \tag{A.14}$$

and

$$v'_{GA}(\tilde{\tilde{x}}) = \sum_{i=1}^{n} w_i^2 \cdot p'_i\ (\tilde{\tilde{x}}) \cdot \left(LGD_i^2 \left[1 - 2 \cdot N\left(\frac{N^{-1}\left[p_i(\tilde{\tilde{x}})\right] - \sqrt{\rho_{ii}^{\tilde{\tilde{x}}}} \cdot N^{-1}\left[p_i(\tilde{\tilde{x}})\right]}{\sqrt{1-\rho_{ii}^{\tilde{\tilde{x}}}}} \right) \right] \right). \quad (A.15)$$

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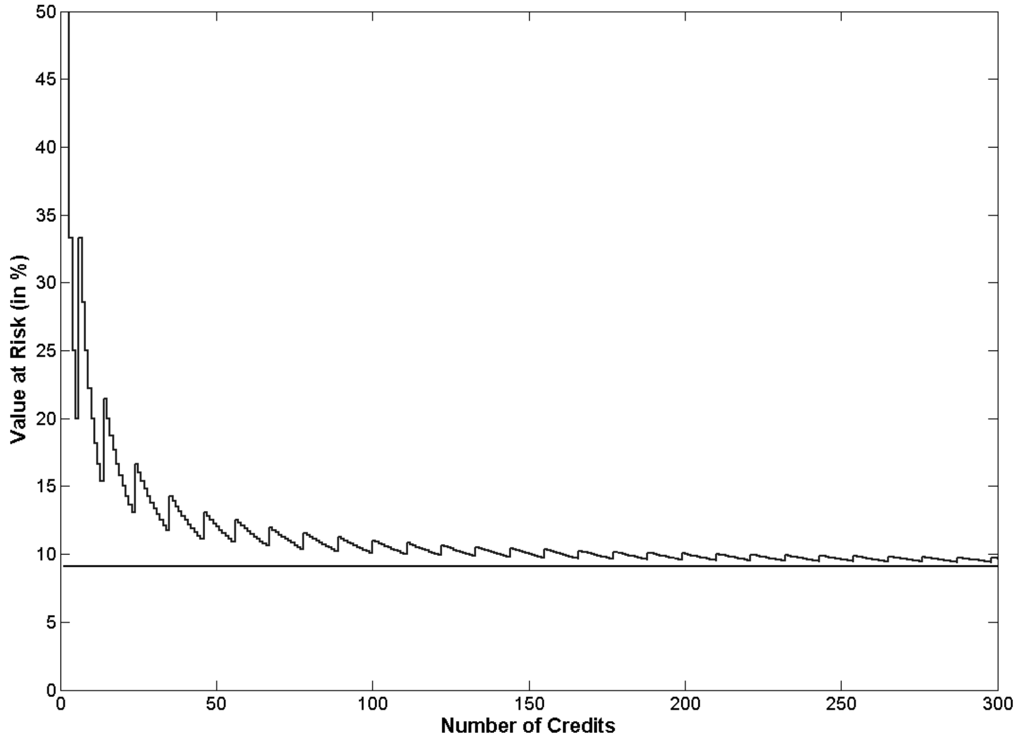

FIGURE 1: Value at Risk in the ASRF and the Vasicek model

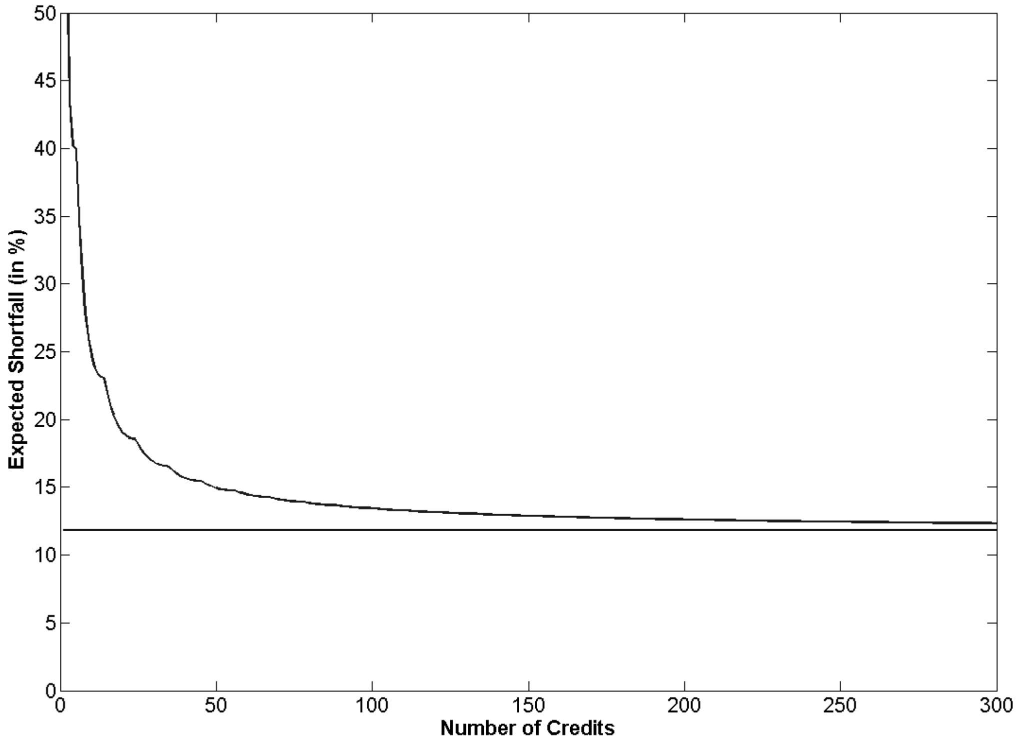

FIGURE 2: Expected Shortfall in the ASRF and the Vasicek model

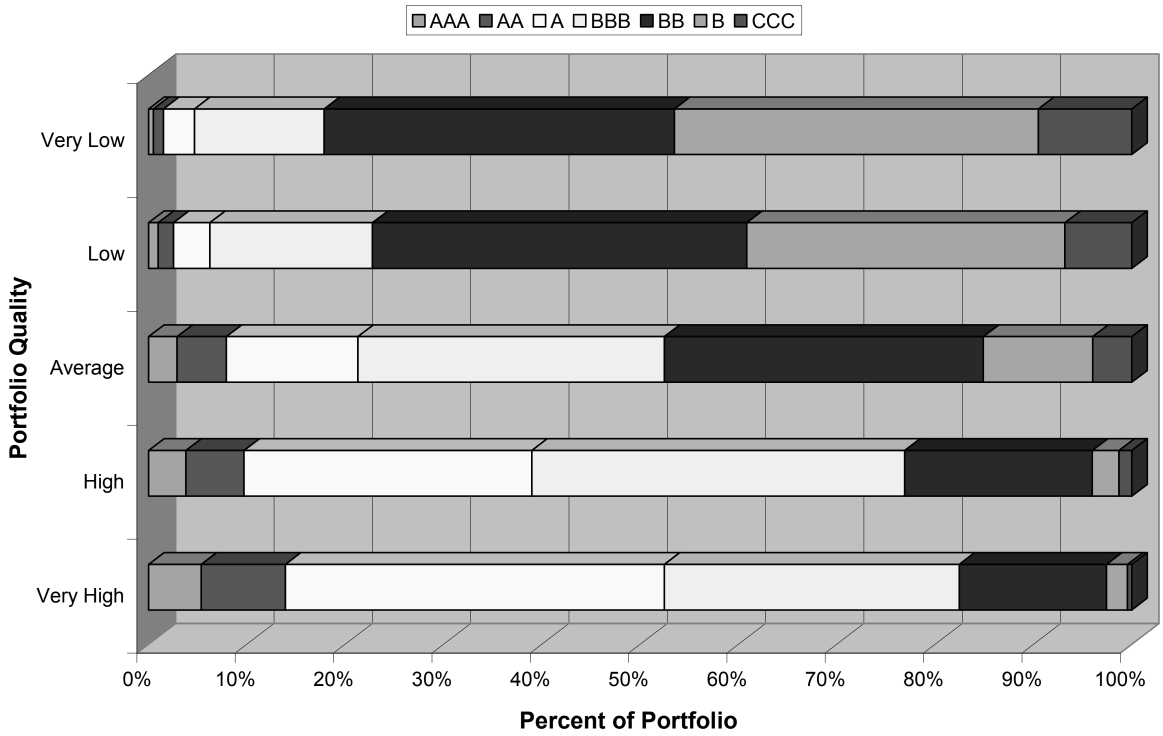

FIGURE 3: Portfolio quality distributions

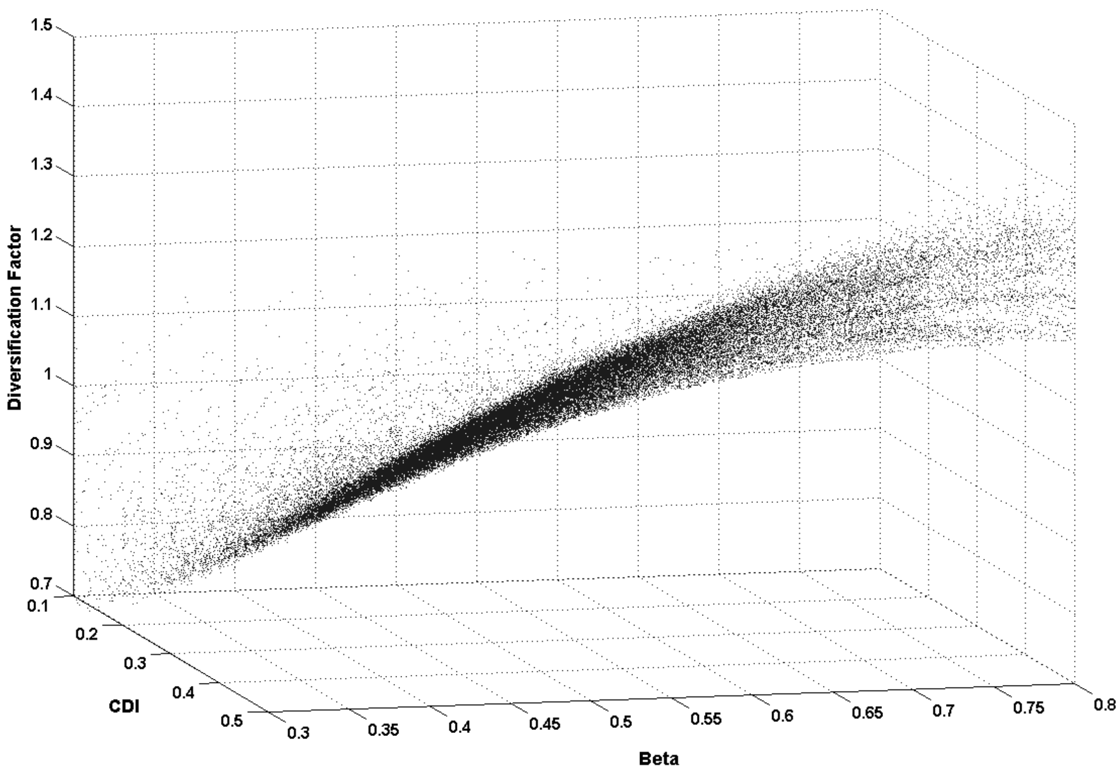

FIGURE 4: Diversification Factor of 50.000 simulations

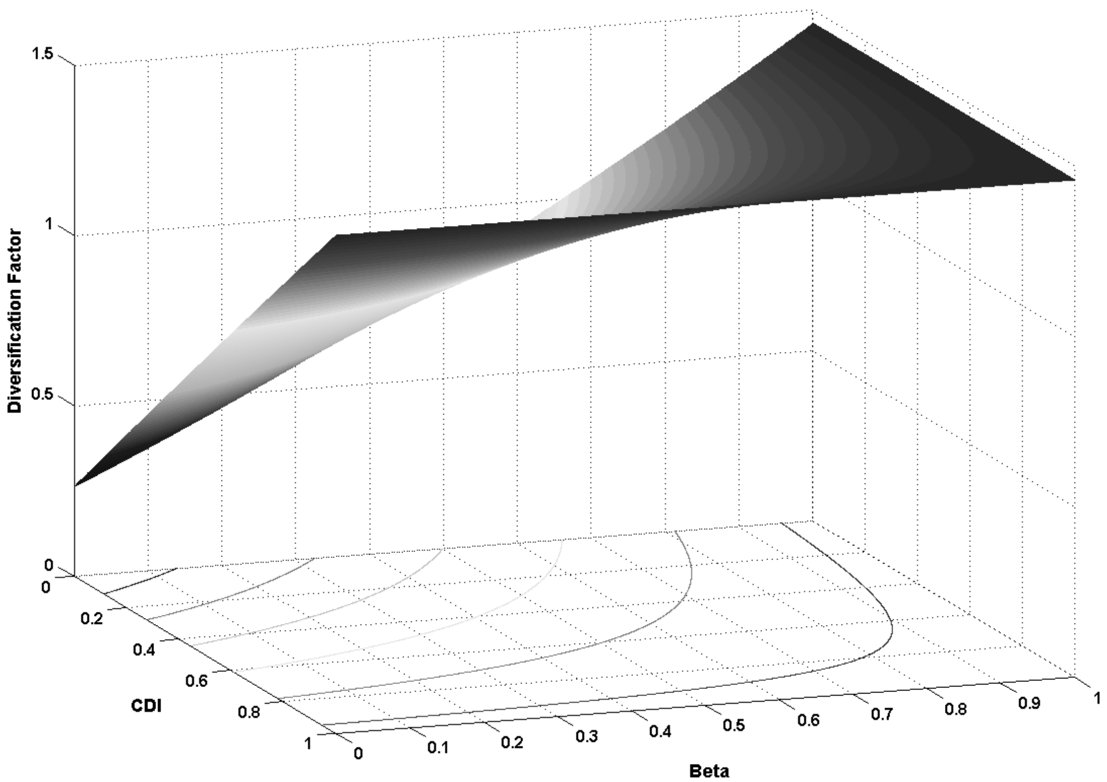

FIGURE 5: Surface plot of the DF-function

TABLE 1: Confidence level for the ES so that the ES is matched with the $VaR_{99.9\%}$ for portfolios of different quality

Portfolio Type / Quality	$VaR_{99,9\%}$ & ES_z	Confidence Level z (ES)
(I) AAA only	0.57%	99.672%
(II) Very High	6,12%	99.709%
(III) High	7.59%	99.711%
(IV) Average	12.94%	99.719%
(V) Low	20.89%	99.726%
(VI) Very Low	23.30%	99.727%
(VII) CCC only	57.00%	99.741%

TABLE 2: Inter-sector correlation structure based on MSCI industry indices (in %)

Sector	A	B	C1	C2	C3	D	E	F	H	I	J
A: Energy	100	50	42	34	45	46	57	34	10	31	69
B: Meterials		100	87	61	75	84	62	30	56	73	66
C1: Capital Goods			100	67	83	92	65	32	69	82	66
C2: Comm. svs & Supplies				100	58	68	40	8	50	60	37
C3: Transportation					100	83	68	27	58	77	67
D: Consumer Discretionary						100	76	21	69	81	66
E: Consumer Staples							100	33	46	56	66
F: Health Care								100	15	24	46
H: Information Technology									100	75	42
I: Telecommunication Services										100	62
J: Utilities											100

TABLE 3: Overall sector composition of the German banking system

Sector	Exposure Weight
A: Energy	0.18%
B: Meterials	6.01%
C1: Capital Goods	11.53%
C2: Comm. svs & Supplies	33.69%
C3: Transportation	7.14%
D: Consumer Discretionary	14.97%
E: Consumer Staples	6.48%
F: Health Care	9.09%
H: Information Technology	3.20%
I: Telecommunication Services	1.04%
J: Utilities	6.67%

TABLE 4: Implicit intra-sector correlations for different portfolio quality

Portfolio Type / Quality	Implicit Intra-Sector Correlation
(I) Very High	30%
(II) High	28%
(III) Average	25%
(IV) Low	23%
(V) Very Low	21%

TABLE 5: Comparison of the models for the 5 benchmark portfolios
with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	6.23%	7.68%	12.95%	20.88%	23.15%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	-1.1 bp	-0.9 bp	0.0 bp	0.1 bp	1.1 bp
	Relative Error	-1.77%	-1.17%	0.00%	0.05%	0.48%
Pykhtin	ES	6.21%	7.66%	12.91%	20.80%	23.20%
	Absolute Error	-0.2 bp	-0.2 bp	-0.4 bp	-0.8 bp	0.5 bp
	Relative Error	-0.32%	-0.26%	-0.31%	-0.38%	0.22%
Cespedes I	ES	6.07%	7.51%	12.70%	20.43%	22.79%
	Absolute Error	-1.6 bp	-1.7 bp	-2.5 bp	-4.5 bp	-3.6 bp
	Relative Error	-2.57%	-2.21%	-1.93%	-2.16%	-1.56%
Cespedes II	ES	6.00%	7.45%	12.68%	20.48%	22.87%
	Absolute Error	-2.3 bp	-2.3 bp	-2.7 bp	-4.0 bp	-2.8 bp
	Relative Error	-3.69%	-2.99%	-2.08%	-1.92%	-1.21%

TABLE 6: Comparison of the models for 5 high concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	7.69%	9.22%	15.41%	24.41%	27.10%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	-15.7 bp	-16.3 bp	-24.6 bp	-35.2 bp	-38.4 bp
	Relative Error	-20.42%	-17.68%	-15.96%	-14.42%	-14.17%
Pykhtin	ES	7.66%	9.29%	15.46%	24.39%	27.03%
	Absolute Error	-0.3 bp	0.7 bp	0.5 bp	-0.2 bp	-0.7 bp
	Relative Error	-0.35%	0.76%	0.31%	-0.08%	-0.24%
Cespedes I	ES	7.40%	9.08%	15.59%	25.07%	27.95%
	Absolute Error	-2.9 bp	-1.4 bp	1.8 bp	6.6 bp	8.5 bp
	Relative Error	-3.77%	1.52%	1.17%	2.70%	3.14%
Cespedes II	ES	7.22%	8.86%	15.19%	24.38%	27.14%
	Absolute Error	-4.7 bp	-3.6 bp	-2.2 bp	-0.3 bp	0.4 bp
	Relative Error	-6.11%	-3.90%	-1.43%	-0.12%	0.15%

TABLE 7: Comparison of the models for 5 low concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES	5.66%	6.98%	12.16%	19.78%	22.06%
Basel II	VaR	6.12%	7.59%	12.95%	20.89%	23.26%
	Absolute Error	4.6 bp	6.1 bp	7.9 bp	11.1 bp	12.0 bp
	Relative Error	8.13%	8.74%	6.50%	5.61%	5.44%
Pykhtin	ES	5.67%	6.98%	12.14%	19.74%	22.08%
	Absolute Error	0.1 bp	0.0 bp	-0.2 bp	-0.4 bp	0.2 bp
	Relative Error	0.26%	-0.07%	-0.16%	-0.21%	0.09%
Cespedes I	ES	5.66%	6.94%	11.92%	19.17%	21.38%
	Absolute Error	0.0 bp	-0.4 bp	-2.4 bp	-0.61 bp	-0.68 bp
	Relative Error	0.0%	-0.57%	-1.97%	-3.08%	-3.08%
Cespedes II	ES	5.64%	6.94%	12.06%	19.52%	21.81%
	Absolute Error	-0.2 bp	-0.4 bp	-1.0 bp	-2.6 bp	-2.5 bp
	Relative Error	-0.35%	-0.57%	-0.82%	-1.31%	-1.13%

TABLE 8: Comparison of the models resulting from simulation studies with different parameter settings

	Pykhtin-Model		**Cespedes I**		**Cespedes II**	
	Ø Relative Error	Ø Absolute Error	Ø Relative Error	Ø Absolute Error	Ø Relative Error	Ø Absolute Error
Simulation I	0.64%	1.4 bp	1.73%	5.4 bp	1.72%	5.4 bp
Simulation II	0.81%	1.1 bp	0.79%	1.1 bp	0.84%	1.2 bp
Simulation III	3.64%	7.2 bp	1.66%	4.4 bp	1.69%	4.6 bp
Simulation IV	2.89%	3.7 bp	1.07%	1.5 bp	1.07%	1.5 bp