Heterogeneous Beliefs, Option Prices, and Volatility Smiles*

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Abstract

In an economy in which investors with different time preferences have heterogeneous beliefs about a dividend's mean growth rate, the volatility of the stock that claims the dividend is stochastic in equilibrium. The prices of the vanilla European options that are written on this stock admit closed-form solutions, hence their hedging deltas. The Black-Scholes implied volatility surface exhibits the observed patterns that are widely documented in various options markets and depends on the wealth distribution, investors' beliefs, and subjective discount rates. In addition, the prices of barrier options and hedging deltas can be approximated at any desired level of accuracy. In some cases, barrier and one-touch option prices and their hedging deltas can be closely bounded by closed-form formulae. In summary, the options pricing model that is developed in this paper not only offers a rationale for the observed implied volatility patterns in an equilibrium setting but also is easy to use in practice.

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1 Introduction

It is well documented that the celebrated Black and Scholes (1973) model does not fit real-world option prices, and in fact produces systematic biases. In general, the volatilities that back out from the Black-Scholes formulae are downward sloping against the moneyness and expirations. These biases are well known as implied volatility smiles or smirks. Since the discovery of the volatility smile, researchers and practitioners alike have been trying to build realistic option pricing models. One of the key reasons that the Black-Scholes model fails is that stock (or index) volatility is stochastic.

Unlike the main approach in the literature that either directly models the dynamics of volatility or makes volatility depend on price, this paper models an equilibrium stock price that exhibits stochastic volatility. Although there is only one source of uncertainty, the heterogeneities of investors make the volatility vary over time in a stochastic fashion. Because of the single source of risk, options that are written on the stock are redundant, and can be hedged by the underlying stock. The vanilla European options written on this stock are priced by simple formulae that match some of the key features of the implied volatility that is observed in options markets. Furthermore, the prices of certain exotic options, such as barrier options, can be approximated by simple integrals that can be solved numerically at any desired level of accuracy. This ability of the model seems to be unique in the current literature, in which even evaluations of vanilla European options are quite involved. In summary, the option pricing model that is developed in this paper not only explains the main features of the implied volatility, but is also simple to implement as in the spirit of the Black-Scholes model. The key assumption of the model is that investors are heterogeneous in their preferences and beliefs

The base model is a simple version of the general equilibrium asset pricing model with heterogeneous beliefs that is developed in Li (2007). In this economy, investors with a logarithmic utility[1] have different time preferences (discount rate) and heterogeneous beliefs about the only economic fundamental, which is a dividend or an endowment stream. Unlike the homogeneous or single representative agent setup of Lucas (1978), the trading of stock is motivated not only by risk sharing among investors, but also by speculation. Different trading strategies redistribute wealth among investors over time, and this redistribution effect adds another layer of variation to that which is caused by the variation of the fundamental, that is, the dividend with a constant volatility. The equilibrium price does not follow a Markovian process in terms of the price itself; detailed wealth distribution is needed to completely describe the equilibrium. The equilibrium stock volatility is stochastic, even though the volatility of the dividend is constant. As a result, the implied volatility surface is also changing over time due to the change in wealth distribution.

[1] This makes the model very tractable. See Li (2000) for the case of general power utility.

In a similar effort, Buraschi and Jiltsov (2006) employ a model, which is similar to Detemple and Murthy (1994),[2] to investigate option pricing with heterogeneous beliefs. Without any derivatives the original markets are incomplete, hence one of the options is not redundant.[3] While, in their model, heterogeneous beliefs affect stock volatility through agents' learning and hedging, in this model, such volatility effects are directly caused by agents' different time preferences. Thus, without resorting to the learning effects, the model yields simpler option pricing formulae for both vanilla and some exotic options and yet it is capable of generating various kinds of implied volatility surfaces that are observed in different options markets. In contrast to the case of options on stock indices in which negative smirks are more pronounced, the implied volatility for individual stocks and currencies exhibits positive, negative, and symmetric smiles. Therefore, the model has potential to be applied in different options markets.

There are several other studies that also try to explain implied volatility in equilibrium settings in which investors face uncertainty about the structure of economic fundamentals. However, in most of these models, prices can be only solved numerically, even for vanilla options. David and Veronesi (2002) propose an equilibrium, continuous-time model in which a dividend stream has two possible growth rates and investors have to make an inference about the current rate that the dividend follows. They show that the options that are written on the stock can generate an implied volatility smile. A similar idea is also investigated by Guidolin and Timmermann (2003) in a binomial tree setting. In Yan (2000), investors also continuously update the estimate of the mean dividend growth rate, which follows a mean reverting process. Liu, Pan, and Wang (2005) study an equilibrium model in which jumps in asset price are due to jumps in the underlying dividend, and hence implied volatility smiles due to the jumps in stock price. Garcia, Luger, and Renault (2003) directly assume some structures for the processes of pricing kernel (aggregate consumption) and dividend, the parameters of which follow a two-state Markov chain. Benzoni, Collin-Dufresne, and Goldstein (2005) also take a similar approach but they assume recursive preferences and that the expected growth rates of both aggregate consumption and dividend follow a jump-diffusion process.

There are many studies that aim to generalize the Black-Scholes model without an equilibrium setting. In the spirit of Black and Scholes (1973), such approaches are adopted to derive flexible pricing formulae in applications. Relaxing the constant volatility of a stock to a stochastic process seems to be an obvious step forward. Models that adopt this

[2]Zapatero (1998) and Basak (2000) further explore other implications of financial innovations and extraneous risk on asset pricing with heterogeneous beliefs in a similar setup, respectively.

[3]Although the model in this paper assumes frictionless and complete markets, the demand of options can easily motivated by short-sale constraints or stringent margin requirements on trading stocks. See, e.g., Li (2000).

approach are known as stochastic volatility models, and include those of Hull and White (1987), Heston (1993), and others. Another way to make stock volatility vary over time is to assume that volatility is a deterministic function of stock price, as Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) have done. Such models are also called local volatility models. One more approach is to add a jump component to the stock price dynamics as in the works of Merton (1976) and Bates (1991). There are also models that mix stochastic volatility and jumps together. Although these models enjoy some successes in terms of explaining the observed implied volatility surface, empirical studies, such as those of Bakshi, Cao, and Chen (1997), Das and Sundaram (1999), and Jones (2003), among others, show that such success is limited. In addition, these models also suffer some hedging problems in practice. For local volatility models, options can be hedged by the underlying stock only, but these models suffer from consistency problems and perform poorly empirically, as is shown by Dumas, Fleming, and Whaley (1998). Obviously, these models more or less aim to ascertain how to model stock dynamics such that the prices of the options that are written on the stock can explain the real-world option prices with little details on the economics.

From an economic viewpoint, equilibrium stock prices have stochastic volatility or jump components due to their fundamentals, that is, their dividends or earnings.[4] However, it is difficult to identify or verify the dynamics that the underlying fundamentals follow, because we do not have the necessary data to carry out the empirical analysis. Usually, information on economic fundamentals is difficult to quantify and subject to different interpretations.[5] An alternative is to assume that economic fundamentals follow a simple structure but that investors do not have perfect knowledge of the structure. Therefore, uncertainty about the structure of the fundamentals, through investors' learnings, may induce the stochastic volatility of stocks. This is the common approach that is adopted in several of the recent studies that have been mentioned. Such studies do provide additional economic insight, but, the pricing formulae in these models are quite complicated and inflexible, and hence are quite limited in practice. In contrast to the existing models, the options prices in the model that is proposed in this paper are not only derived from an equilibrium setting, but also easy to use in practice. In fact the resulting option price formulae are simpler and easier to use in practice than those in most of the reduced-form models.

[4]Indeed, this is the approach that is taken by Liu, Pan, and Wang (2005), but their main purpose is to show that event risk (jumps in dividend) may cause high equity premium when investors have ambiguity aversion. Presumably, we can also take the volatility of a stock's dividend to be stochastic. Given the fact that dividend or aggregate consumption is quite smooth, however, this approach may only produce what the reduced-form models have achieved, and hence offers little additional economic insight into options pricing.

[5]Knight (1964) classifies outcomes that are related to such information as risk with uncertainty, and contrasts it to the outcome of a poker game, which comprises risk without uncertainty. The main formalization of Knight's idea about risk with uncertainty is known as Knightian uncertainty in the literature.

The rest of the paper is organized as follows. An equilibrium asset pricing model is developed in the next section. Vanilla European options prices and the related hedging deltas are derived in Section 3. The Black-Scholes implied volatility surface is also studied in this section. Section 4 shows how the price of a barrier option can be approximated by an integral, and a simulation method to compute this integral is proposed. It is also shown that both barrier and one-touch options can be approximated by closed-form formulae in some cases. Section 5 contains the conclusion and discussions on possible extensions of the model. Appendix A provides all of the proofs that are omitted in the main text. Appendix B provides a lemma, which is used in pricing barrier options, for the boundary crossing probability of Brownian motion and some additional results on barrier and one-touch options.

2 A Model of Equilibrium Asset Prices

2.1 Setup

We consider a pure-exchange, continuous-time competitive economy over an infinite time horizon. Our model is similar to the Lucas (1978) model, except that here investors have different beliefs or models about the structure of a dividend process.

There is one risky security in the economy, which yields a nonnegative dividend process δ. Investors believes that the dividend process $\delta(t)$ admits the following decomposition form[6]

$$\delta(t) = \delta_0 + \int_0^t \mu_\delta(s)\delta(s)\,ds + \int_0^t \sigma_\delta\delta(s)\,dZ(s), \tag{1}$$

where $Z(s)$ is a one-dimensional Brownian motion and σ_δ is a constant. We assume that both $\mu_\delta(t)$, the mean growth rate, and Z are *unobservable*. Although investors agree on the *form* of the decomposition of the dividend process, they have different beliefs about the model that the process μ_δ follows due to the unobservability of $\mu_\delta(s)$ and $Z(s)$. Thus investors forecast the future mean growth rates differently. In general, μ_δ could follow a wide class of processes.[7] However, as the focus of this paper is on pricing derivatives in an economy with heterogeneous beliefs, the most convenient assumption about beliefs is

[6] The number of possible decompositions is infinite even when μ_δ is a constant if $Z(s)$ is not observable. This is an implication of the Girsanov theorem.

[7] Some of the possible setups are discussed in Li (2000) or Li (2007). Traditionally, heterogeneous beliefs are modeled as investors share the same underlying model or structure that governs some fundamentals but have different beliefs on the initial values or priors. Li (2007) extends the traditional view to include heterogeneous beliefs about the structure or underlying models. Hence investors might never agree with each other or their posterior beliefs might not converge as the traditional view predicts. Of course, another way to get this non-convergence is to introduce random periodic structure shocks.

the extreme case, in which investors believe the growth rate to be a constant and know it perfectly. As we will see later, this assumption enables closed-form solutions for vanilla European option prices and closed-form approximations for barrier options.[8] Specifically, we assume that there are three investors, or three kinds of investors, that are indexed by i, n, and p. Each believes that $\mu_\delta = \mu_k$, where μ_k is a constant for $k \in \{i,\, n,\, p\}$. Therefore, the dividend process (1) under investor k's belief effectively becomes

$$\delta(t) = \delta_0 + \int_0^t \mu_k \delta(s)\, ds + \int_0^t \sigma_\delta \delta(s)\, dZ_k(s). \tag{2}$$

where Z_k is a Brownian motion under investor k's beliefs, it is also called innovation process. An application of Itô Lemma shows

$$Z_k(t) = \frac{1}{\sigma_\delta}\left[\ln\delta(t) - \ln\delta(0) - \left(\mu_k - \frac{1}{2}\sigma_\delta^2\right)t\right]. \tag{3}$$

Hence, observing the dividend is equivalent to observing Z_k given belief μ_k. The price of the risky security satisfies the Itô process

$$S(t) + \int_0^t \delta(s)\, ds = S(0) + \int_0^t \mu_k^S(s) S(s)\, ds + \int_0^t \sigma_k^S(s) S(s)\, dZ_k(s) \tag{4}$$

from investor k's perspective.

There also is a risk-free asset (money market account), the price of which is

$$B(t) = \exp\left(\int_0^t r(s)\, ds\right),$$

where $r(s)$ is the instantaneous interest rate, which will be determined in equilibrium.

As has been mentioned, there are three classes of investors in the economy, and each investor has the utility function

$$U_k(t,c) = E_k\left[\int_t^T e^{-\rho_k s}\ln(c(s))\, ds \,\middle|\, \mathcal{F}^\delta(t)\right],$$

where $\mathcal{F}^\delta$ denotes the information structure that is generated by the dividend process, E_k is the expectation operator according to investor k's belief, and ρ_k is investor k's subjective discount rate. We assume that $0 < \rho_p \le \rho_n \le \rho_i$, which means that investor p is the most patient and investor i the most impatient in the economy. Investor n's patience lies somewhere in between. Furthermore, investor k is endowed with ω_k shares of the risky security, where the total number of shares is normalized to equal 1, that is $\omega_i + \omega_n + \omega_p = 1$.

[8] Most derivatives are short-term securities, thus ignoring the updating or learning effects of investors on the equilibrium might not have a major impact on the pricing of derivatives, though several studies mentioned in the introduction do show learning is important in pricing options.

2.2 Stock Price Dynamics and State Price Densities

The uncertainty about the drift of the dividend process and the heterogeneity of beliefs among investors lead to different stock price dynamics or forecasts. Namely, for investor k, we have

$$S(t) + \int_0^t \delta(s)\, ds = S(0) + \int_0^t \mu_k^S(s) S(s)\, ds + \int_0^t \sigma_k^S(s) S(s)\, dZ_k(s).$$

As the stock is a traded asset, in equilibrium for $k \in \{i,\, p\}$, we must have

$$\begin{aligned} & \int_0^t \mu_k^S(s) S(s)\, ds + \int_0^t \sigma_k^S(s) S(s)\, dZ_k(s) \\ = \; & \int_0^t \mu_n^S(s) S(s)\, ds + \int_0^t \sigma_n^S(s) S(s)\, dZ_n(s) \\ = \; & \int_0^t \mu_n^S(s) S(s)\, ds + \frac{1}{\sigma_\delta} \int_0^t \sigma_n^S(s) [\mu_k(s) - \mu_n(s)]\, ds + \int_0^t \sigma_n^S(s) S(s)\, dZ_k(s) \end{aligned}$$

for all t, where the last equation is obtained by using equation (3). This implies, by the fact that two identical stochastic processes must have the same finite variation and martingale parts, that

$$\sigma_k^S(s) = \sigma_n^S(s) = \sigma^S(s)$$

and

$$\frac{\mu_k^S(s) - \mu_n^S(s)}{\sigma^S(s)} = \frac{\mu_k - \mu_n}{\sigma_\delta} \tag{5}$$

for all s. Based on this observation, the stock price from now on satisfies the stochastic differential equation

$$S(t) + \int_0^t \delta(s)\, ds = \int_0^t \mu_k^S(s) S(s)\, ds + \int_0^t \sigma^S(s) S(s)\, dZ_k(s), \tag{6}$$

where $\mu_k^S(s)$ for $k \in \{i,\, n,\, p\}$ satisfies equation (5).

Heterogeneous beliefs also lead to different state price densities for each investor. Let

$$\theta_k(s) = \frac{\mu_k^S(s) - r(s)}{\sigma^S(s)}$$

denote investor k's price of risk. Investor k's state price density is then

$$\xi_k(t) = \exp\left[-\int_0^t r(s)\, ds - \frac{1}{2} \int_0^t \theta_k^2(s)\, ds - \int_0^t \theta_k(s)\, dZ_k(s) \right]. \tag{7}$$

By equation (5), we have

$$\theta_k(s) - \theta_n(s) = \frac{\mu_k - \mu_n}{\sigma_\delta} \equiv \beta_k \tag{8}$$

for $k \in \{p,\, i\}$, where β_k is the normalized difference of beliefs between investors k and n. This relation regarding investors' personal prices of risk must be satisfied in equilibrium.

2.3 Optimality and Equilibrium

Equipped with the results in the last section, we now turn to investigate how the investors choose their consumption plans and portfolios to maximize their expected utilities.

A *feasible* consumption and trading strategy of investor k is a collection of $(c_k, \pi_k) = \{c_k(t), \pi_k(t)\}_0^\infty$ such that the following are satisfied.

1. $c_k(t)$ is nonnegative and $\mathcal{F}^\delta(t)$-adapted and satisfies $\int_0^T c_k(t)\,dt < \infty\ \forall T > 0$.

2. $\pi_k(t)$, which is the portion of investor k's wealth $W_k(t)$ invested in the stock, is $\mathcal{F}^\delta(t)$-adapted and satisfies $\forall T > 0$,

$$\int_0^T |W_k(t)r(t) + [\mu_k^S(t) - r(t)]\pi_k(t)W_k(t)|\,dt + \int_0^T [\pi_k(t)W_k(t)\sigma^S(t)]^2\,dt < \infty.$$

3. $c_k(t)$ and $\pi_k(t)$ satisfy the following intertemporal budget constraint

$$\begin{aligned} W_k(t) = W_k(0) + \int_0^t W_k(s)\{r(s) + \pi_k(s)[\mu_k^S(s) - r(s)] - c_k(s)\}\,ds \\ + \int_0^t W_k(s)\pi_k(s)\sigma^S(s)\,dZ_k(s), \end{aligned}$$

where $W_k(0) = \omega_k S(0)$ is the initial wealth.

The standard portfolio choice theory, or *the martingale approach* (Cox and Huang, 1989), applies to each investor's optimization problem in our model, which transforms a dynamic problem into a static problem. Equivalently, investor k seeks to maximize

$$E_k\left[\int_0^\infty e^{-\rho_k t}\ln(c_k(t))\,dt\right]$$

subject to

$$E_k\left[\int_0^\infty \xi_k(t)c_k(t)\,dt\right] \le W_k(0).$$

The following results are straightforward.

Lemma 1 *The optimal trading strategy for investor k is*

$$\pi_k^*(t) = \frac{\mu_k^S(t) - r(t)}{(\sigma^S(t))^2},$$

and the optimal consumption plan is

$$c_k^*(t) = \rho_k W_k(0)\xi_k^{-1}(t)e^{-\rho_k t}. \tag{9}$$

Furthermore, investor k's wealth at time t is given by

$$W_k(t) = e^{-\rho_k t}\xi_k^{-1}(t)W_k(0).$$

Due to the assumption that all investors have a logarithmic utility function, the individual investor's optimization problem has a very simple solution. In particular, the feature that the optimal consumption is deterministically proportional to an investor's wealth enables us to compute the equilibrium explicitly.

An *equilibrium* is a pair of interest rate and stock price processes $(r, S) = \{r(t), S(t)\}_0^\infty$ such that given (r, S), all of the investors maximize their expected utilities based on their own beliefs and information sets, and all of the markets—the perishable consumption good and the securities markets—are cleared. These market clearing conditions lead to the following.

Define

$$\eta_k(t) \equiv \frac{W_k(t)}{W_n(t)} = \frac{\omega_k}{\omega_n} \times \frac{\xi_k^{-1}(t)}{\xi_n^{-1}(t)} e^{-(\rho_k - \rho_n)t} \tag{10}$$

to be the ratio of wealth between investors $k \in \{p,\, i\}$ and n. Note that by (9), we have

$$\frac{c_k^*(t)}{c_n^*(t)} = \lambda_k \eta_k(t), \tag{11}$$

where $\lambda_k \equiv \frac{\rho_k}{\rho_n}$ is the ratio of propensity to consume between investors $k \in \{i,\, p\}$ and n.

Given the optimal policies, the market clearing condition for the perishable consumption good is

$$c_i^*(t) + c_n^*(t) + c_p^*(t) = \sum_{k \in \{i,n,p\}} \rho_k W_k(0) \xi_k^{-1}(t) e^{-\rho_k t} = \delta(t). \tag{12}$$

This implies that the state price for investor n

$$\xi_n(t) = [1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)] \frac{\rho_n W_n(0)}{\delta(t)} e^{-\rho_n t}, \tag{13}$$

and that for investor $k \in \{i,\, p\}$

$$\xi_k(t) = \frac{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)}{\lambda_k \eta_k(t)} \times \frac{\rho_k W_k(0)}{\delta(t)} e^{-\rho_k t}. \tag{14}$$

Proposition 1 *In equilibrium, the individual prices of risk of investors are*

$$\theta_n(t) = \sigma_\delta - \frac{\lambda_i \eta_i(t) \beta_i + \lambda_p \eta_p(t) \beta_p}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)},$$

$$\theta_i(t) = \sigma_\delta + \frac{\beta_i + \lambda_p \eta_p (\beta_i - \beta_p)}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)},$$

$$\theta_p(t) = \sigma_\delta + \frac{\beta_p + \lambda_i \eta_i (\beta_p - \beta_i)}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)},$$

respectively, where β_k is the normalized difference of beliefs between investors k and n as defined by equation (8). The equilibrium interest rate is

$$r(t) = \frac{\rho_n + \lambda_i \eta_i(t)\rho_i + \lambda_p \eta_p(t)\rho_p}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)} + \frac{\mu_n + \lambda_i \eta_i(t)\mu_i + \lambda_p \eta_p(t)\mu_p}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)} - \sigma_\delta^2.$$

Moreover, the price of the stock is[9]

$$S(t) = \frac{1}{\rho_n} \times \frac{1 + \eta_i(t) + \eta_p(t)}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)} \delta(t). \tag{15}$$

The equilibrium interest rate is a consumption weighted average of the subjective discount rate of investors plus their estimate of the instantaneous growth rate of the dividend minus the instantaneous variance of the dividend process. When investors share homogeneous beliefs or the same subjective discount rate, the stock price is independent of the wealth ratios η_k. Recall that in our pure exchange economy, all securities markets are automatically cleared when the consumption good market is cleared. This implies that the stock price is determined by the consumption behavior of investors. When investors with logarithmic preferences share the same subjective discount rate, the ratio of consumption to wealth is the same across all investors. Therefore, due to the fact that aggregate consumption is exogenously given to equal the current dividend, the redistributing wealth has no effect on the stock price.

One of the important observations from equation (15) is that the stock price itself does not follow a Markovian process. Therefore, observation on price only is not enough to describe the dynamics of stock price; one has to know the distribution of wealth among investors. Because investors with heterogeneous beliefs employ different trading strategies, wealth distribution varies stochastically over time, so as does stock volatility. This has many implications for the behaviors of the stock price, but, the focus of this paper is the implications for pricing options.

3 Vanilla Options and Implied Volatility

Bonds and options are redundant assets in this economy,[10] thus we can use the state prices that are derived in the previous section to price derivatives. The assumption that investors have constant beliefs enables the prices of many derivatives to have closed-form solutions.

[9]Although we assume investors have different fixed beliefs, the expression of stock price does not depend on any particular learning models due to the logarithmic preferences. See Li (2007) for more details.

[10]Introducing options might have impacts on the equilibrium of the economy if there are margins requirements to trade securities. Li (2005) studies the effects of margins on the stock volatility in a similar setup.

The definition of the wealth ratio (10), the relations among prices of risk (8), and the innovation processes (3) imply that

$$\eta_k(T) = \eta_k(t) \exp \left[- \left(\rho_k - \rho_n + \frac{1}{2}\beta_k^2 \right) (T-t) + \beta_k \left(Z_n(T) - Z_n(t) \right) \right]. \tag{16}$$

We also know that

$$\delta(T) = \delta(t) \exp \left[\left(\mu_n - \frac{1}{2}\sigma_\delta^2 \right) (T-t) + \sigma_\delta (Z_n(T) - Z_n(t)) \right]. \tag{17}$$

These two identities imply the following result.

Lemma 2 *Suppose that $\beta_i \leq 0$ and $\beta_p \geq 0$. Then, $S(T) \geq K$ if and only if $Z_n(T) - Z_n(t) \geq \bar{y}$, where $\bar{y}$ is the solution to*

$$\begin{aligned} & g(\tau) \frac{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)}{1 + \eta_i(t) + \eta_p(t)} \\ & \qquad \times \frac{1 + f_i(\tau)\eta_i(t)e^{\beta_i y} + f_p(\tau)\eta_p(t)e^{\beta_p y}}{1 + \lambda_i f_i(\tau)\eta_i(t)e^{\beta_i y} + \lambda_p f_p(\tau)\eta_p(t)e^{\beta_p y}} \times e^{\sigma_\delta y} = \frac{K}{S(t)}, \end{aligned} \tag{18}$$

where

$$f_k(\tau) = \exp \left[- \left(\rho_k - \rho_n + \frac{1}{2}\beta_k^2 \right) \tau \right]$$

for $k \in \{i, p\}$, and

$$g(\tau) = \exp \left[\left(\mu_n - \frac{1}{2}\sigma_\delta^2 \right) \tau \right],$$

where $\tau = T - t$.

Given the state prices, we can price any securities with payoff $G(t)$ using

$$\frac{1}{\xi_k(t)} E_k \left[\int_t^\infty \xi_k(s) G(s) \, ds \middle| \mathcal{F}^\delta(t) \right],$$

for $k \in \{n, i, p\}$. We will use $\xi_n(t)$ to price the options.[11] In the case of a European call option, we have

$$C(t) = \frac{1}{\xi_n(t)} E_n \left[\xi_n(T)[S(T) - K]^+ \middle| \mathcal{F}^\delta(t) \right],$$

where K is the strike price and T is the expiration time. Using the state price (13) and the stock price (15), we can rewrite the call option price as

$$C(t) \;\; = \;\; \frac{\delta(t)e^{-\rho_n(T-t)}}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)} E_n \left[\frac{1 + \lambda_i \eta_i(T) + \lambda_p \eta_p(T)}{\delta(T)} \right.$$

[11]All investors have the same risk-neutral probability.

$$
\begin{aligned}
&\times \left(\frac{1}{\rho_n} \frac{1+\eta_i(T)+\eta_p(T)}{1+\lambda_i\eta_i(T)+\lambda_p\eta_p(T)} \delta(T) - K \right)^+ \Bigg| \mathcal{F}^\delta(t) \Bigg] \\
= \quad & \frac{\delta(t) e^{-\rho_n(T-t)}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} \qquad\qquad (19) \\
&\times E_n \left[\left(\frac{1+\eta_i(T)+\eta_p(T)}{\rho_n} - \frac{1+\lambda_i\eta_i(T)+\lambda_p\eta_p(T)}{\delta(T)} K \right)^+ \Bigg| \mathcal{F}^\delta(t) \right].
\end{aligned}
$$

A direct calculation yields the European call option price.

Proposition 2 *Suppose that $\beta_i \leq 0$ and $\beta_p \geq 0$. Let $\tau = T - t$. Define*

$$
d_1^n = -\frac{\bar{y}}{\sqrt{\tau}}, \quad d_1^k = -\frac{\bar{y}}{\sqrt{\tau}} + \sqrt{\tau}\beta_k,
$$

and $d_2^n = d_1^n - \sqrt{\tau}\sigma_\delta$, $d_2^k = d_1^k - \sqrt{\tau}\sigma_\delta$ for $k \in \{i,\, p\}$, where $\bar{y}$ is the solution to (18).

The price of a European call option with strike price K and expiration time $T > t$ is then

$$
\begin{aligned}
&C(t, S(t), \eta_i(t), \eta_p(t); T, K) \\
&\quad = \quad S(t) \frac{e^{-\rho_n\tau} N(d_1^n) + \eta_i(t) e^{-\rho_i\tau} N(d_1^i) + \eta_p(t) e^{-\rho_p\tau} N(d_1^p)}{1+\eta_i(t)+\eta_p(t)} \\
&\qquad\quad - K \frac{e^{-r_n\tau} N(d_2^n) + \lambda_i\eta_i(t) e^{-r_i\tau} N(d_2^i) + \lambda_p\eta_p(t) e^{-r_p\tau} N(d_2^p)}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)},
\end{aligned}
$$

where $N(\cdot)$ is the cumulative distribution function of a normal random variable with a mean of 0 and a variance of 1 and

$$
r_k = \rho_k + \mu_k - \sigma_\delta^2
$$

for $k \in \{p,\, n,\, i\}$ is the interest rate of an economy in which only investor k prevails.

In addition, the bond price is given by

$$
B(t, \eta_i(t), \eta_p(t); T) = \frac{e^{-r_n\tau} + \lambda_i\eta_i(t) e^{-r_i\tau} + \lambda_p\eta_p(t) e^{-r_p\tau}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)}.
$$

The prices for the put options can be easily derived. Using the fact that $C(T) - P(T) = S(T) - K$ and the state price, we have

$$
\frac{1}{\xi_n(t)} E_n \left[\xi_n(T)[C(T) - P(T)] \middle| \mathcal{F}^\delta(t) \right] = \frac{1}{\xi_n(t)} E_n \left[\xi_n(T)[S(T) - K] \middle| \mathcal{F}^\delta(t) \right],
$$

which implies that

$$
C(t) - P(t) = \frac{e^{-\rho_n\tau} + \eta_i(t) e^{-\rho_i\tau} + \eta_p(t) e^{-\rho_p\tau}}{1+\eta_i(t)+\eta_p(t)} S(t) - B(t, \eta_i(t), \eta_p(t); T) K.
$$

Rearranging the terms of the equation yields the put option price

$$
\begin{aligned}
&P(t, S(t), \eta_i(t), \eta_p(t); T, K) \\
&= K\frac{e^{-r_n\tau}N(-d_2^n) + \lambda_i\eta_i(t)e^{-r_i\tau}N(-d_2^i) + \lambda_p\eta_p(t)e^{-r_p\tau}N(-d_2^p)}{1 + \lambda_i\eta_i(t) + \lambda_p\eta_p(t)} \\
&\quad - S(t)\frac{e^{-\rho_n\tau}N(-d_1^n) + \eta_i(t)e^{-\rho_i\tau}N(-d_1^i) + \eta_p(t)e^{-\rho_p\tau}N(-d_1^p)}{1 + \eta_i(t) + \eta_p(t)}.
\end{aligned}
$$

3.1 Hedging Deltas

As has been mentioned, the dynamics of the stock does not follow a Markovian process in terms of stock price itself, and is obvious that the options that are written on the stock do not follow a simple monotone relationship with the underlying stock. This result is consistent with the empirical findings of Bakshi, Cao, and Chen (2000) in the US options markets.

Nonetheless, the options written on the stock can be hedged by using the stock only, because the wealth ratios are locally perfectly correlated with the stock. Hence, the hedging delta has two parts: one for the movements of the stock caused by the changes in dividend and one for the stock price changes caused by the variations of the wealth ratios.

Corollary 1 *The delta of the European call option that is specified in Proposition 2 is*

$$
\Delta(t, S(t), \eta_i(t), \eta_p(t); T, K) = \frac{\Sigma_C(t, S(t), \eta_i(t), \eta_p(t); T, K)}{S(t)\sigma^S(t, \eta_i(t), \eta_p(t))},
$$

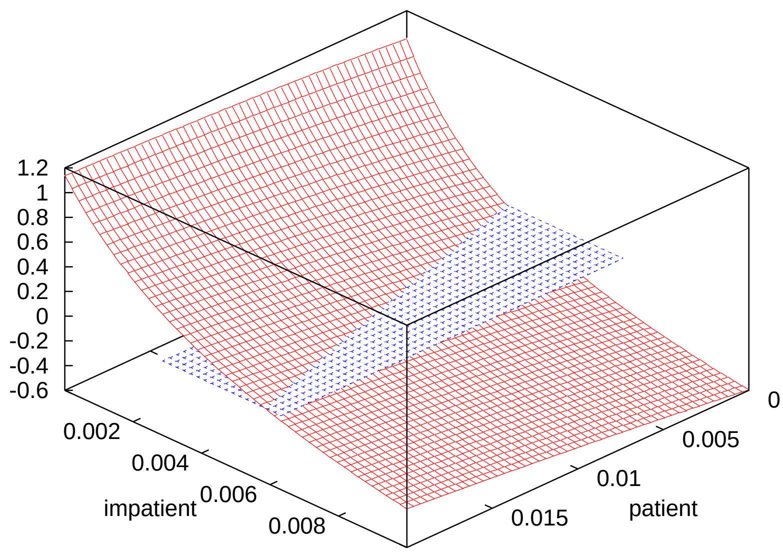
where

$$
\begin{aligned}
&\Sigma_C(t, S(t), \eta_i(t), \eta_p(t); T, K) \\
&= S(t)\frac{e^{-\rho_n\tau}\theta_n(t)N(d_1^n) + \eta_i(t)e^{-\rho_i\tau}\theta_i(t)N(d_1^i) + \eta_p(t)e^{-\rho_p\tau}\theta_p(t)N(d_1^p)}{1 + \eta_i(t) + \eta_p(t)} \\
&\quad - \frac{K}{[1 + \lambda_i\eta_i(t) + \lambda_p\eta_p(t)]^2}\Big[\lambda_i\eta_i(t)\beta_i\left(e^{-r_i\tau}N(d_2^i) - e^{-r_n\tau}N(d_2^n)\right) \\
&\qquad + \lambda_p\eta_p(t)\beta_p\left(e^{-r_p\tau}N(d_2^p) - e^{-r_n\tau}N(d_2^n)\right) \\
&\qquad + \lambda_i\eta_i(t)\lambda_p\eta_p(t)(\beta_p - \beta_i)\left(e^{-r_p\tau}N(d_2^p) - e^{-r_i\tau}N(d_2^i)\right)\Big]
\end{aligned}
$$

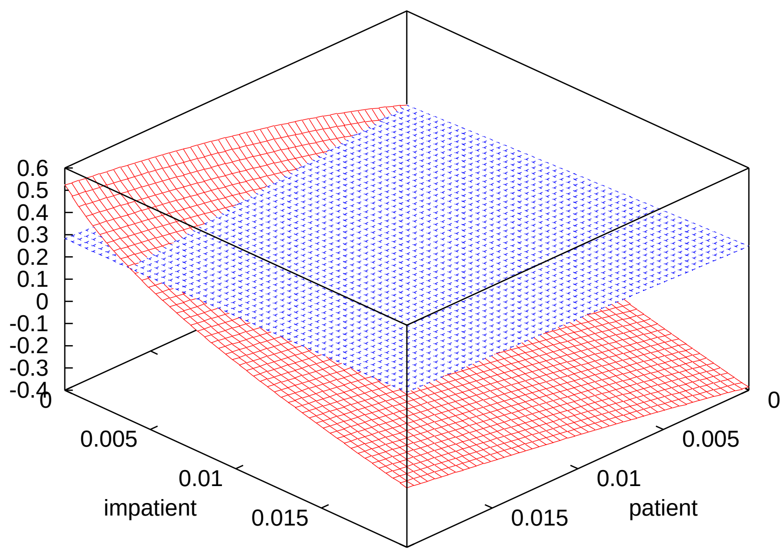
and

$$
\begin{aligned}
&\sigma^S(t, \eta_i(t), \eta_p(t)) \\
&= \sigma_\delta + \frac{(1-\lambda_i)\eta_i(t)\beta_i + (1-\lambda_p)\eta_p(t)\beta_p + (\lambda_i - \lambda_p)\eta_i(t)\eta_p(t)(\beta_p - \beta_i)}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i\eta_i(t) + \lambda_p\eta_p(t)]}
\end{aligned}
\tag{20}
$$

is the stock volatility, where θ_k is the state price of risk for investor k as defined in Proposition 1.



(a)



(b)

Figure 1: The Δs for a European call option against wealth ratios. The dotted planes are (a) the zero level; and (b) the Δs of Black-Scholes using the implied volatility. The strike price and time to expiration are (a) $K = 0.8S$ and $\tau = 0.1$; and (b) $K = 1.1S$ and $\tau = 1$. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 7.5\%$.

Although options can be hedged by the underlying stock, the deltas for call options can be negative or greater than 1, depending on the level of the wealth ratios. Figure 1(a) is a numerical example that illustrates this point. The key difference to the Black-Scholes model is the randomness of the wealth ratios. Although the wealth ratios are locally perfectly correlated with the stock return, neither the stock nor the options that are written on it can be a Markov system of the stock price itself. Thus the wealth ratios are needed to describe the dynamics of the stock and options.

It is not surprising that the hedging deltas as calculated by the Black-Scholes formula using the implied volatility are different from those that are given in Corollary 1, as shown in Figure 1(b). The poor approximation of the hedging deltas indicates that the Black-Scholes implied volatility cannot account for the full dynamics of the underlying stock in the Black-Scholes model. In general, matching price levels does not automatically entail the matching of the derivatives of the different pricing formulae.

3.2 Implied Volatility Surface

We now turn to investigate the properties of the implied volatility surface in this model. Let ρ be the current dividend yield, which is defined by

$$\rho(t) = \frac{\delta(t)}{S(t)} = \rho_n \frac{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)}{1 + \eta_i(t) + \eta_p(t)} = \frac{\rho_n + \eta_i(t)\rho_i + \eta_p(t)\rho_p}{1 + \eta_i(t) + \eta_p(t)}, \tag{21}$$

where we obtain the last equality by the definition of λ. Given the current interest rate in Proposition 1 and the volatility σ, the call option under the Black-Scholes model is priced as

$$C_{BS}(t, S(t); T, K; \sigma) = e^{-\rho(t)\tau} S(t) N(d_1) - e^{-r(t)\tau} K N(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r(t) - \rho(t) + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

The implied volatility for vanilla call options, $\sigma_{\text{imv}}(t; \tau, K)$, is the solution to[12]

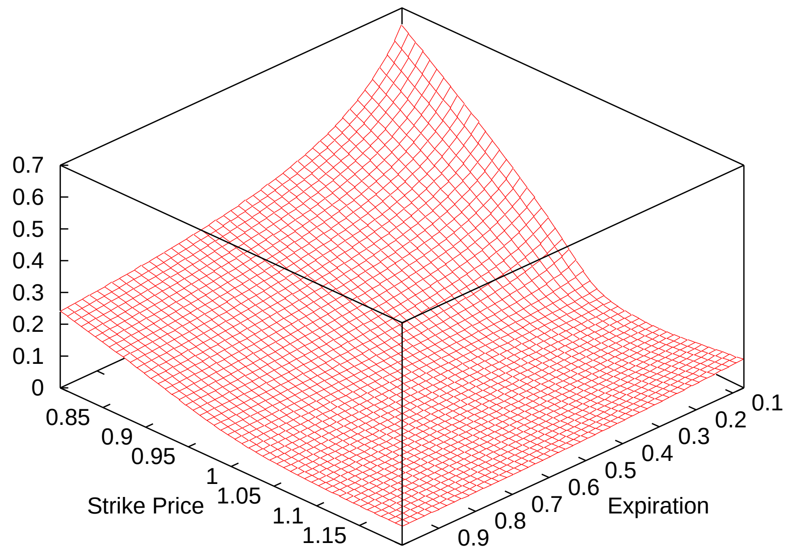
$$C_{BS}(t, S(t); T, K; \sigma_{\text{imv}}(t; \tau, K)) = C(t, S(t), \eta_i(t), \eta_p(t); T, K). \tag{22}$$

The implied volatility for put options can be defined in a similar fashion.

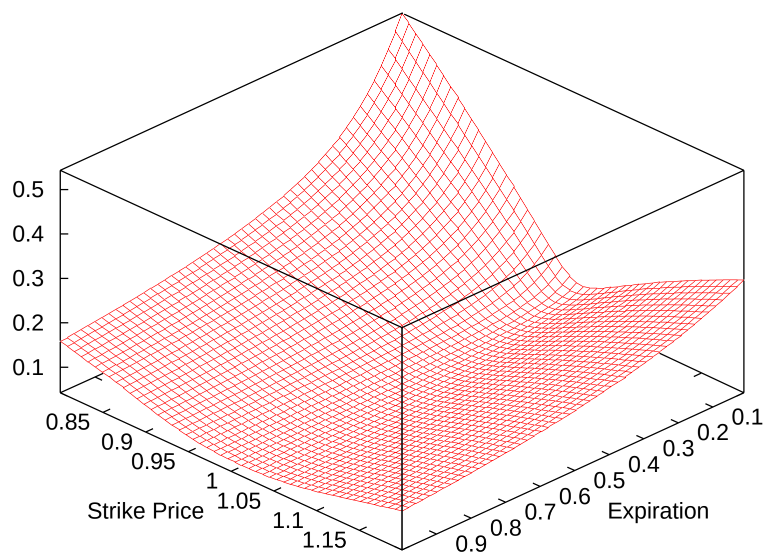
[12] In practice or empirical studies, implied volatility is sometimes calculated using the slightly different formula

$$C_{BS}(t, S(t); T, K; \sigma) = e^{-\rho(t)\tau} S(t) N(d_1) - B(t, T) K N(d_2),$$

where $B(t, T)$ is the price of the bond mature at T. Of course, this is not a strict interpretation of the Black-Scholes formula when the interest rate is stochastic.

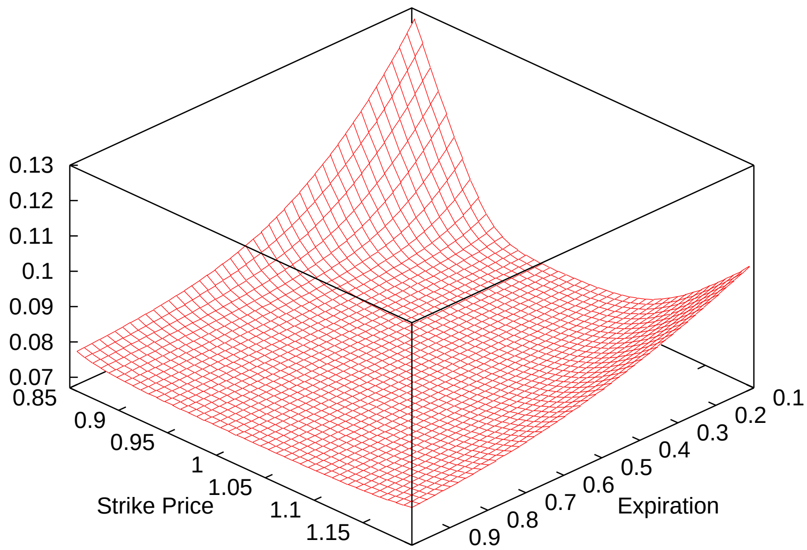


(a)

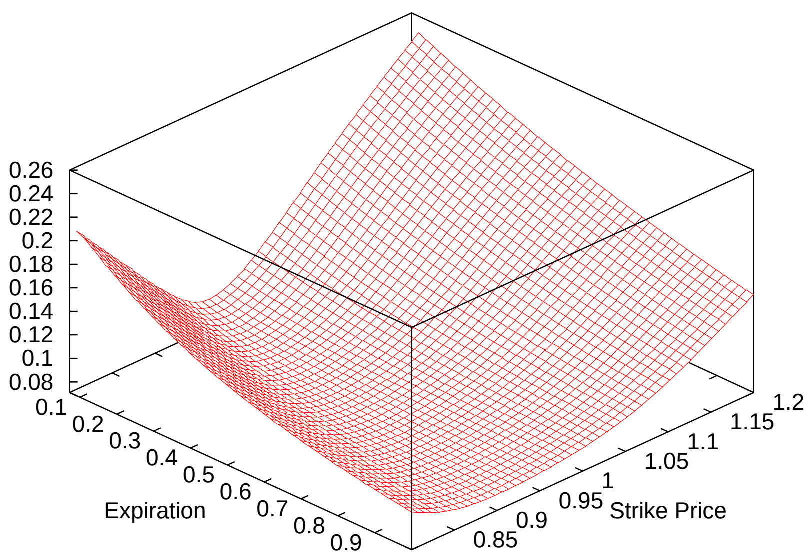


(b)

Figure 2: Implied volatility surface against moneyness and expiration. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 5\%$; initial wealth ratios (a) $\eta_i = 0.008$, $\eta_p = 0.000002$; (b) $\eta_i = 0.002$, $\eta_p = 0.008$.

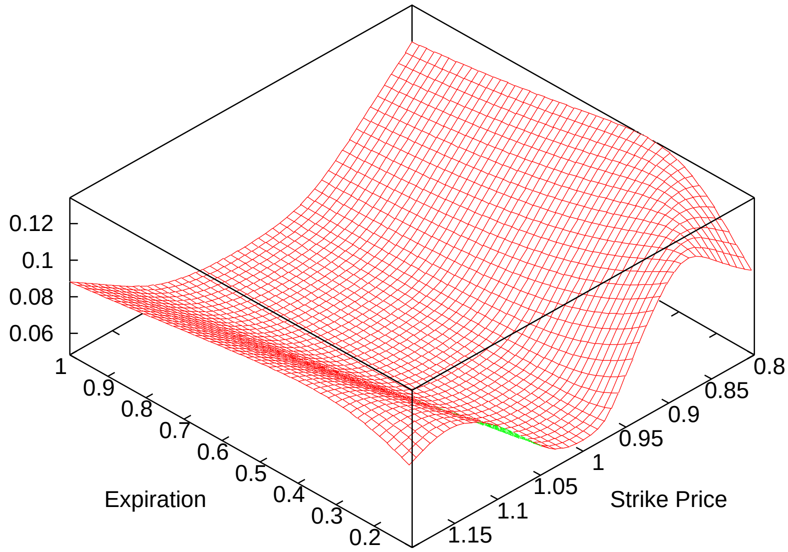


(a)

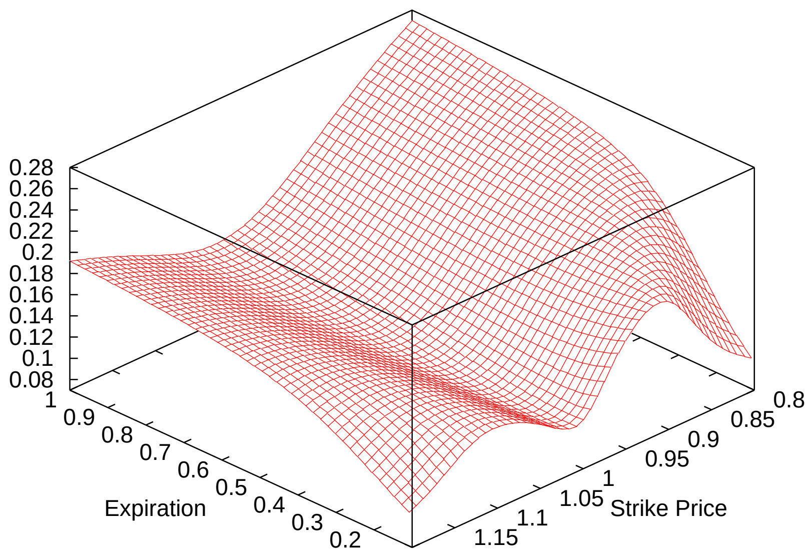


(b)

Figure 3: Implied volatility surface against moneyness and expiration. The parameters are as follows: beliefs $\mu_n = 3\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 7.5\%$; initial wealth ratios (a) $\eta_i = 0.00002$, $\eta_p = 0.0002$, and (b) $\eta_i = 0.0001$, $\eta_p = 0.02$.



(a)



(b)

Figure 4: Implied volatility surface against moneyness and expiration. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 3.2\%$, $\rho_p = 2.8\%$; volatility of dividend $\sigma_\delta = 5\%$; initial wealth ratios (a) $\eta_i = 0.002$, $\eta_p = 0.005$, and (b) $\eta_i = 0.1$, $\eta_p = 0.1$.

Figure 2 is a plot of the implied volatility surface, $\sigma_{\text{imv}}(t;\tau,K)$, for call options. It shows that the model that is presented here can generate some of the key features that are well documented in the empirical literature on real-world options markets. When the patient investor, who is also optimistic, holds a relatively small portion of wealth, the implied volatility smile is pronouncedly downward sloping and becomes a smirk. When the patient investor holds a relatively large portion of wealth, the implied volatility smile becomes more symmetric. Figure 3 shows a similar plot, but with a low relative wealth holding by the impatient (or pessimistic) investor. In this case, we have a flat volatility smile near the money, and a strong positively skewed volatility smile when the patient (or optimistic) investor has a higher relative wealth. These plots indicate that the shape of the volatility surface is determined by the relative distribution of wealth among investors.

These properties of the volatility smile seem to fit some casual empirical observations well. In the stock market, the fear of a crash seems to be a dominant factor, and thus the pessimistic (impatient) investor dominates the optimistic investor. This means that the negative skewness of the volatility smile occurs more often. However, in a currency market, both optimistic and pessimistic investors are present, and thus the U- or V-shaped volatility smile dominates.

The term structures of the implied volatility in Figures 2 and 3 are downward sloping in general. However, as shown in Figure 4, the term structures that are generated by the model can also be upward sloping or hump shaped. In this case, the time preferences of investors are quite close or the same, and the slope of the term structures can be quite high when the wealth ratios of the pessimistic and optimistic are relatively high.

In summary, the shape of the implied volatility surface depends on the time preferences, beliefs, and relative wealth of investors. The numerical examples show that the model can generate quite a variety of implied volatility surfaces that are consistent with empirical observations, such as those shown by Das and Sundaram (1999) and Derman (1999).

Another interesting observation from these plots is that the beliefs of both the pessimistic and optimistic investors are quite extreme,[13] but the wealth ratios are quite small. This means that it is not necessary for the extremely pessimistic and optimistic investors to hold significant portions of the aggregate wealth to replicate the observed patterns of the implied volatility surface. Indeed, the wealth of the extreme investors is very small in the numerical example. This makes the model more plausible.

The instantaneous stock volatility, σ^S that is given by equation (20) varies with wealth ratios but lacks of any dramatic characteristics, and almost represents a plane when both wealth ratios are in the range of $(0, 0.1)$. Options significantly "amplify" the future varia-

[13]In Garcia, Luger, and Renault (2003) (page 69), the estimated mean growth for the "crash" state is extremely negative (-32%) with a relatively high probability (11%) in the empirical estimation of their model using S&P 500 option price data.

tions of stock volatility through changes in the wealth ratios.[14]

In a comparative static sense, the implied volatility surface is dependent on the difference in beliefs and time preferences of investors. Dynamically, the wealth ratios are stochastic processes, and change over time, and so does the implied volatility surface. Hence, any shocks to the fundamental will result in changes to the implied volatility surface.

4 Barrier Options

It is usually difficult to obtain a closed-form solution to barrier options except with the Black-Scholes model. This is also true for the model in this paper, but as shown in the following, our model yields tight closed-form bounds for barrier options under certain conditions, and in general, the pricing bounds can be made to converge to the price using simple numerical methods. The reason for the ease of pricing barrier options in this model is the occurrence of deterministic barriers of Brownian motion, which are equivalent to constant stock barriers.

Let $b(s)$ be the solution to equation (18) for $T = t + s$ and $K = S_b$, where S_b is the barrier that is based on the stock price. Then $b(s)$ is the equivalent barrier that is based on the innovation process $Z_n(t+s) - Z_n(t)$ according to investor n's belief. The slope of $b(s)$ has constant upper and lower bounds that are shown by the following lemma.

Lemma 3 *For a constant barrier that is based on the stock price, the equivalent barrier that is based on the innovation processes Z_n, $b(s)$, is a deterministic function of time and satisfies*

$$\max\{h, l_i, l_p\} \geq \frac{\partial b(s)}{\partial s} = \frac{\sigma_\delta h + \Sigma_i(s, b(s))l_i + \Sigma_p(s, b(s))l_p}{\sigma_\delta + \Sigma_i(s, b(s)) + \Sigma_p(s, b(s))} \geq \min\{h, l_i, l_p\},$$

where

$$\Sigma_i(s, b(s)) = \frac{\beta_i \left(1 - \lambda_i + (\lambda_p - \lambda_i) f_p(s)\eta_p(t)e^{\beta_p b(s)}\right) f_i(s)\eta_i(t)e^{\beta_i b(s)}}{(1 + \sum_{k\in\{i,p\}} \lambda_k f_k(s)\eta_k(t)e^{\beta_k b(s)})(1 + \sum_{k\in\{i,p\}} f_k(s)\eta_k(t)e^{\beta_k b(s)})},$$

$$\Sigma_p(s, b(s)) = \frac{\beta_p \left(1 - \lambda_p + (\lambda_i - \lambda_p) f_i(s)\eta_i(t)e^{\beta_i b(s)}\right) f_p(s)\eta_p(t)e^{\beta_p b(s)}}{(1 + \sum_{k\in\{i,p\}} \lambda_k f_k(s)\eta_k(t)e^{\beta_k b(s)})(1 + \sum_{k\in\{i,p\}} f_k(s)\eta_k(t)e^{\beta_k b(s)})},$$

and

$$h = -\frac{\mu_n - \frac{1}{2}\sigma_\delta^2}{\sigma_\delta}, \quad l_k = \frac{\rho_k - \rho_n + \frac{1}{2}\beta_k^2}{\beta_k},$$

for $k \in \{i, p\}$, where $b(s)$ is the solution to (18) by setting $T = t + s$ and $K = S_b$.

[14] Future volatility does not feed back to the current volatility due to the assumption of logarithmic preferences. For general preferences, we should expect some feedback effects through investors hedging demands.

The fact that $b(s)$ is a deterministic function of time makes the pricing of barrier options much easier. Several numerical techniques are available to approximate the prices of barrier options, such as the hazard rate approximation that is proposed by Roberts and Shortland (1997) and the piece-wise linear approximations of Wang and Potzelberger (1997).[15] We use the latter method to approximate the prices of barrier options in this paper.

Let $0 = s_0 < s_1 <, ... < s_m = T - t$ and both $b_m^u(s)$ and $b_m^l(s)$ be linear functions on each of the intervals $[s_{j-1}, s_j]$ such that $b_m^l(s) \leq b(s) \leq b_m^u(s)$ for all $s \in [0, T-t]$. The pricing bounds are accessed by piece-wise linear bounds that either have closed-form solutions or can be easily calculated by numerical methods. Consider a special case of "barriers" that depend directly on the Brownian motion Z_n. For example, the payoff of the European "barrier" call option is $(S(T) - K)^+$ or 0, depending on whether $Z(s) \equiv Z_n(t+s) - Z_n(t)$ hits a piece-wise linear barrier $b_m(s)$. The pricing equation (19) implies that the price for an up-and-out call option is given by

$$C_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; b_m) = \frac{\delta(t)e^{-\rho_n \tau}}{1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)} E_n \left[\mathbf{1}_{\{Z(s) < b_m(s), s \leq \tau\}} \right.$$
$$\left. \times \left(\frac{1 + \eta_i(T) + \eta_p(T)}{\rho_n} - \frac{1 + \lambda_i \eta_i(T) + \lambda_p \eta_p(T)}{\delta(T)} K \right)^+ \middle| \mathcal{F}^\delta(t) \right], \tag{23}$$

where $\tau = T - t$. As shown by Lemma B.1, the probability density function of $\{Z(s) = Z_n(t+s) - Z_n(t) < b_m(s), s \leq \tau = T - t\}$ depends only on the joint distribution of the Brownian motion Z at m turning points $(s_1, ..., s_m)$. This observation yields the following proposition.

Proposition 3 *Given a piece-wise linear barrier $b_m(s)$ as previously defined, then, if $b_m(0)$ is positive, the price of a European up-and-out call option with a strike price K and a barrier $b_m(s)$ such that $Z_n(t+s) - Z_n(t) < b_m(s)$ is*

$$\begin{aligned} & C_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; b_m) \\ = \quad & e^{-\rho_n \tau} \int p(x, b_m) \mathbf{1}_{\{x_m \geq \bar{y}\}} \left(\frac{1 + \eta_i(t) f_i(\tau) e^{\beta_i x_m} + \eta_p(t) f_p(\tau) e^{\beta_p x_m}}{1 + \eta_i(t) + \eta_p(t)} S(t) \right. \\ & \left. - \frac{1 + \lambda_i \eta_i(t) f_i(\tau) e^{\beta_i x_m} + \lambda_p \eta_p(t) f_p(\tau) e^{\beta_p x_m}}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)] g(\tau) e^{\sigma_\delta x_m}} K \right) dx, \end{aligned} \tag{24}$$

where $p(x, b_m)$ is as defined in Lemma B.1, x and $b_m = (b_m(s_1), ..., b_m(s_m))^\top$ are m-dimensional vectors, $\bar{y}$ is determined by equation (18), and f_k for $k \in \{i, p\}$ and g are as defined in Lemma 2.

[15] Potzelberger and Wang (2001) also develop a similar linear approximation for double deterministic Brownian barriers, which can be used to price double barrier options.

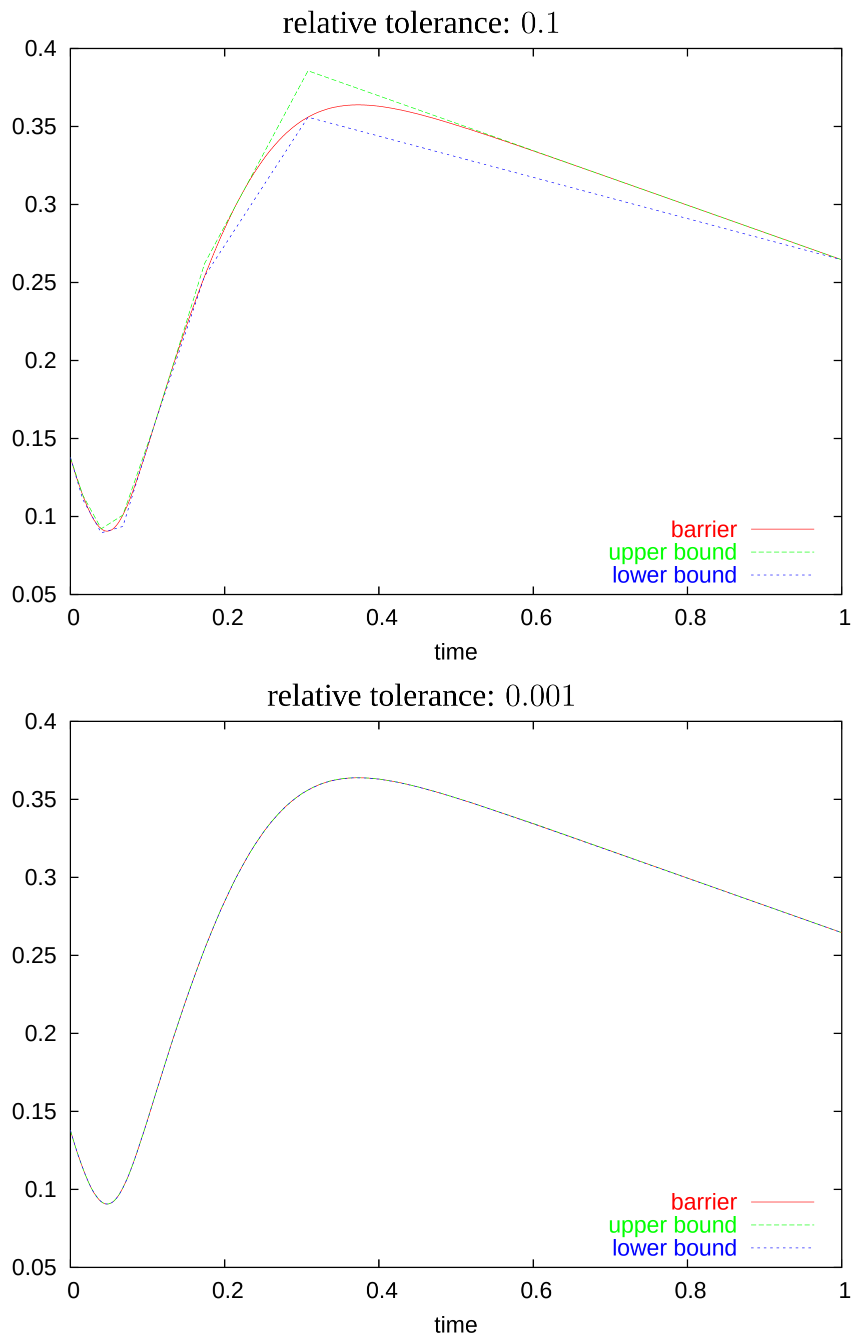


Figure 5: Linear approximations of stock barriers. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 5\%$; initial wealth ratios $\eta_i = 0.1$, $\eta_p = 0.05$.

The formula given by equation (24) is quite easy to calculate either by numerical integration when m is small or by simulation. It turns out that the simulation is easier to implement by using a slightly different density function $p(x, b_m)$. That is, by changing variable $x = b_m - \tilde{x}$, $p(x, b_m) = \tilde{p}(\tilde{x}, b_m)$, where $\tilde{p}$ is as defined in Lemma B.1 and $\tilde{x}$ is an m-dimensional normal random variable with a mean of b_m and a variance matrix of $\Sigma = M\,\mathrm{diag}(s_1 - s_0, ..., s_m - s_{m-1})M^\top$, and M is lower triangular matrix with nonzero elements that is equal to 1. The price of a European call barrier option can then be approximated by averaging

$$\begin{aligned} A(x) &= \mathbf{1}_{\{\tilde{x}_m \le b_m(\tau) - \bar{y}\}} \prod_{j=1}^{m} \mathbf{1}_{\{\tilde{x}_j > 0\}} \left(1 - \exp\left[-\frac{2\tilde{x}_{j-1}\tilde{x}_j}{s_j - s_{j-1}}\right]\right) \\ &\quad \times e^{-\rho_n \tau} \left(\frac{1 + \eta_i(t) f_i(\tau) e^{\beta_i(b_m(\tau) - \tilde{x}_m)} + \eta_p(t) f_p(\tau) e^{\beta_p(b_m(\tau) - \tilde{x}_m)}}{1 + \eta_i(t) + \eta_p(t)} S(t) \right. \\ &\quad \left. - \frac{1 + \lambda_i \eta_i(t) f_i(\tau) e^{\beta_i(b_m(\tau) - \tilde{x}_m)} + \lambda_p \eta_p(t) f_p(\tau) e^{\beta_p(b_m(\tau) - \tilde{x}_m)}}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)] g(\tau) e^{\sigma_\delta (b_m(\tau) - \tilde{x}_m)}} K \right) \end{aligned} \tag{25}$$

over a sample of draws, where $\tilde{x}_0 = b_m(0)$ and all other relevant variables and functions are as defined in Proposition 3. When using simulation to estimate integral (24) by $\bar{A} = \sum_j A(x^j)/N$, the standard error of this estimator is given by (see Wang and Potzelberger (1997))

$$\sqrt{\frac{\sum_j [A(x^j) - \bar{A}]^2}{(N-1)N}}, \tag{26}$$

where x^j is the jth sample of x and N is the sample size. This enables the assessment of accuracy when using the simulation to approximate barrier options prices.

As for the case of vanilla options, this pricing formula also enables the implementation of a simple formula for the hedging delta of barrier options.

Corollary 2 *The hedging delta for the special barrier options that are described in Proposition 3 is given by*

$$\frac{\Sigma_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; b_m)}{S(t)\sigma^S(t, \eta_i(t), \eta_p(t))},$$

where

$$\begin{aligned} &\Sigma_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; b_m) = \\ &\quad e^{-\rho_n \tau} \int \mathbf{1}_{\{\tilde{x}_m \le b_m(\tau) - \bar{y}\}} \tilde{p}(\tilde{x}, b_m) \left[\left(\frac{[1 + \sum_{k \in \{i,p\}} \eta_k(t) f_k(\tau) e^{\beta_k(b_m(\tau) - \tilde{x}_m)}]\sigma_\delta}{1 + \eta_i(t) + \eta_p(t)} \right.\right. \\ &\qquad\qquad + \frac{\sum_{k \in \{i,p\}} [f_k(\tau) e^{\beta_k(b_m(\tau) - \tilde{x}_m)} - \lambda_k] \eta_k(t) \beta_k}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{[\lambda_i f_p(\tau) e^{\beta_p(b_m(\tau)-\tilde{x}_m)} - \lambda_p f_i(\tau) e^{\beta_i(b_m(\tau)-\tilde{x}_m)}]\eta_i(t)\eta_p(t)(\beta_p - \beta_i)}{[1+\eta_i(t)+\eta_p(t)][1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)]}\Bigg) S(t) \\
&- \Bigg(\frac{\sum_{k\in\{i,p\}} \lambda_k [f_k(\tau) e^{\beta_k(b_m(\tau)-\tilde{x}_m)} - 1]\eta_k(t)\beta_k}{[1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)]^2 g(\tau) e^{\sigma_\delta(b_m(\tau)-\tilde{x}_m)}} \\
&+ \ \frac{\lambda_i\lambda_p [f_p(\tau) e^{\beta_p(b_m(\tau)-\tilde{x}_m)} - f_i(\tau) e^{\beta_i(b_m(\tau)-\tilde{x}_m)}]\eta_i(t)\eta_p(t)(\beta_p - \beta_i)}{[1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)]^2 g(\tau) e^{\sigma_\delta(b_m(\tau)-\tilde{x}_m)}}\Bigg) K \Bigg] d\tilde{x}
\end{aligned}
$$

and $\sigma^S(t, \eta_i(t), \eta_p(t))$ *is as defined in Corollary 1.*

Figure 6 plots the upper bounds of the prices and hedging deltas of a barrier option under different wealth distributions. The numerical calculations are carried out by setting a relative tolerance of 0.1% between the true barrier and the approximated piece-wise linear barrier. The sample size of the simulation is half a million. The lower bounds are very similar. The relative differences between the upper and lower bounds are very small, with a mean of 0.1%, but the standard errors that are caused by the simulation are relatively large, with a mean of 0.8%. This indicates, roughly, that the simulation size has to be increased by 64 times, that is, to 32 million, to make the standard error that is caused by the simulation match the error that is caused by the piece-wise linear approximation of the nonlinear barrier. Keep in mind that the relative tolerance for the piece-wise linear approximation is very easy to achieve, and that the main obstacle to gaining price accuracy is the simulation size. However, this will not cause a serious problem, because the simulation is for a multivariate normal random variable.

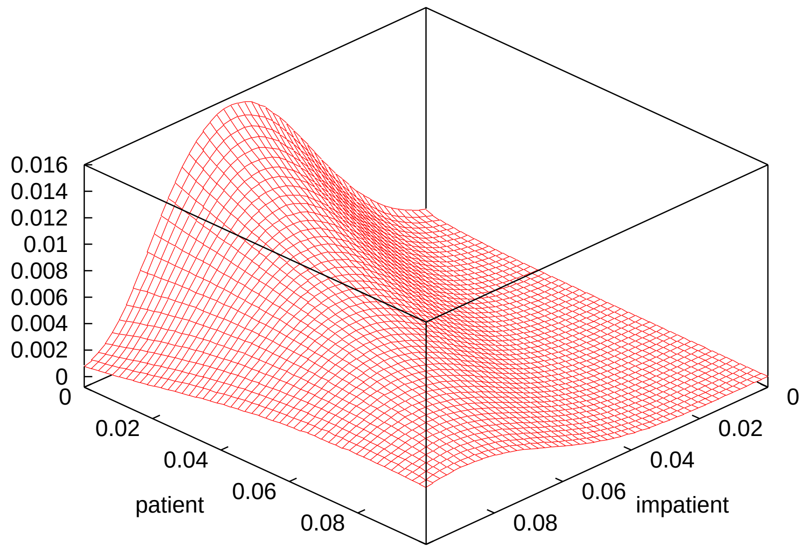
For the case of $m = 1$, the pricing formula in Proposition 3 admits a closed-form expression, which is able to provide tight price bounds for barrier options under certain conditions. This closed-form formula can also be used to calculate the hedging bounds.

Proposition 4 *The price of an up-and-out European call option with a strike* K *and a special barrier* $b_1(s) = \alpha + \mu s$ *such that* $b_1(\tau) > \bar{y}$ *is given by*

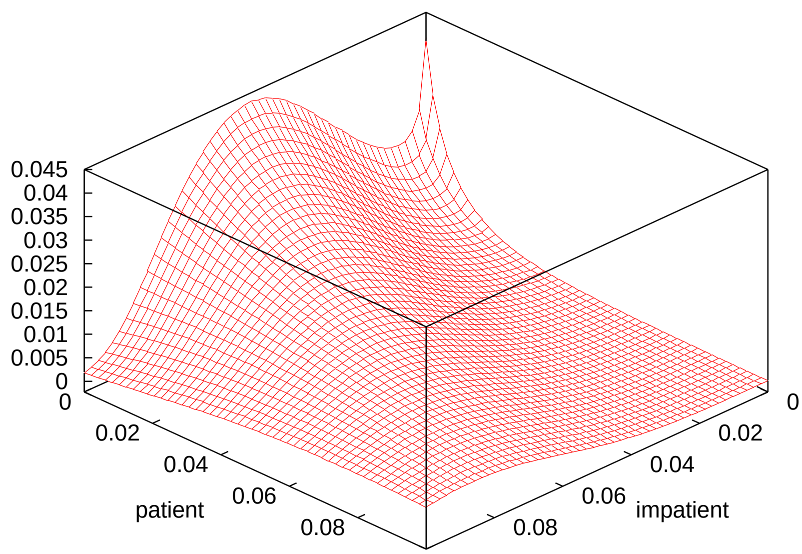
$$
\begin{aligned}
&C_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha, \mu) \\
&\quad = \ C(t, S(t), \eta_i(t), \eta_p(t); T, K) - C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha + \mu\tau, 0) \\
&\qquad\qquad - e^{-2\alpha\mu} \left[C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, \alpha) \right. \\
&\qquad\qquad\qquad \left. - \ C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha + \mu\tau, \alpha) \right],
\end{aligned}
$$

where

$$
\begin{aligned}
C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; y, \alpha) &= \frac{S(t)}{1+\eta_i(t)+\eta_p(t)} \left[e^{-\rho_n \tau} N\left(\frac{2\alpha - y}{\sqrt{\tau}} \right) \right. \\
&\quad \left. + \sum_{k\in\{i,p\}} \eta_k(t) e^{-\rho_k\tau + 2\alpha\beta_k} N\left(\frac{2\alpha + \beta_k\tau - y}{\sqrt{\tau}} \right) \right]
\end{aligned}
$$



(a)



(b)

Figure 6: Up-and-out European call option price and delta upper bounds against wealth ratios. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 5\%$. Strike price: $K = 1.1S$; expiration: $\tau = 1$; barrier $S_b = 1.2S$. Plot (a): price, (b) delta.

$$
- \frac{K e^{-2\alpha\sigma_\delta}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} \Bigg[e^{-r_n\tau} N\left(\frac{2\alpha-\sigma_\delta\tau-y}{\sqrt{\tau}}\right) + \sum_{k\in\{i,p\}} \lambda_k\eta_k(t) e^{-r_k\tau+2\alpha\beta_k} N\left(\frac{2\alpha+(\beta_k-\sigma_\delta)\tau-y}{\sqrt{\tau}}\right)\Bigg],
$$

where $\bar{y}$ and r_k for $k=i,p$ is as defined in Proposition 2.

This special barrier option provides lower and upper bounds for the barrier option prices with a constant barrier on stock price.

Similar to the barrier options, we can also find the bounds for an American type one-touch option price.

Proposition 5 *For a linear touch barrier $b_1(s)=\alpha+\mu s$, the price of a call option that pays \$1 at the moment when the barrier is reached by Z_n is*

$$
\begin{aligned}
& C_Z^{ot}(t,\eta_i(t),\eta_p(t);T;\alpha,\mu) \\
= \quad & \frac{e^{-(\mu+\sigma_\delta)\alpha-|\alpha|\sqrt{(\mu+\sigma_\delta)^2+2r_n}}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} N\left(\frac{\sqrt{(\mu+\sigma_\delta)^2+2r_n}\tau-|\alpha|}{\sqrt{\tau}}\right) \\
& + \frac{e^{-(\mu+\sigma_\delta)\alpha+|\alpha|\sqrt{(\mu+\sigma_\delta)^2+2r_n}}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} N\left(-\frac{\sqrt{(\mu+\sigma_\delta)^2+2r_n}\tau+|\alpha|}{\sqrt{\tau}}\right) \\
& + \sum_{k\in\{i,p\}} \frac{\lambda_k\eta_k(t) e^{-(\mu+\sigma_\delta-\beta_k)\alpha-|\alpha|\sqrt{\Lambda_k}}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} N\left(\frac{\sqrt{\Lambda_k}\tau-|\alpha|}{\sqrt{\tau}}\right) \\
& + \sum_{k\in\{i,p\}} \frac{\lambda_k\eta_k(t) e^{-(\mu+\sigma_\delta-\beta_k)\alpha+|\alpha|\sqrt{\Lambda_k}}}{1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)} N\left(-\frac{\sqrt{\Lambda_k}\tau+|\alpha|}{\sqrt{\tau}}\right)
\end{aligned}
$$

if both $(\mu+\sigma_\delta)^2+2r_n$ and $\Lambda_k=(\mu+\sigma_\delta-\beta_k)^2+2r_k$, for $k\in\{i,p\}$, are positive.

This closed-form formula can be used to approximate the prices for one-touch options if the touch barrier $b(s)$ that is based on a constant strike has tight linear bounds.

Figure 7 plots the linear barrier bounds that are calculated by the formula in Proposition 4 for several examples in which the stock barrier $b(s)$ is either concave or convex. In light of Proposition 5, one-touch options prices can also be bounded by the prices under the linear barriers.

We also use these price bounds to evaluate a common practice in pricing barrier options. Because the price for an up-and-out barrier option does not have a monotone relation to its underlying stock volatility in the Black-Scholes model, some practitioners use two implied volatilities to price up-and-out options: the implied volatility for a vanilla call with the same strike price and the implied volatility for a liquid one-touch option with the same

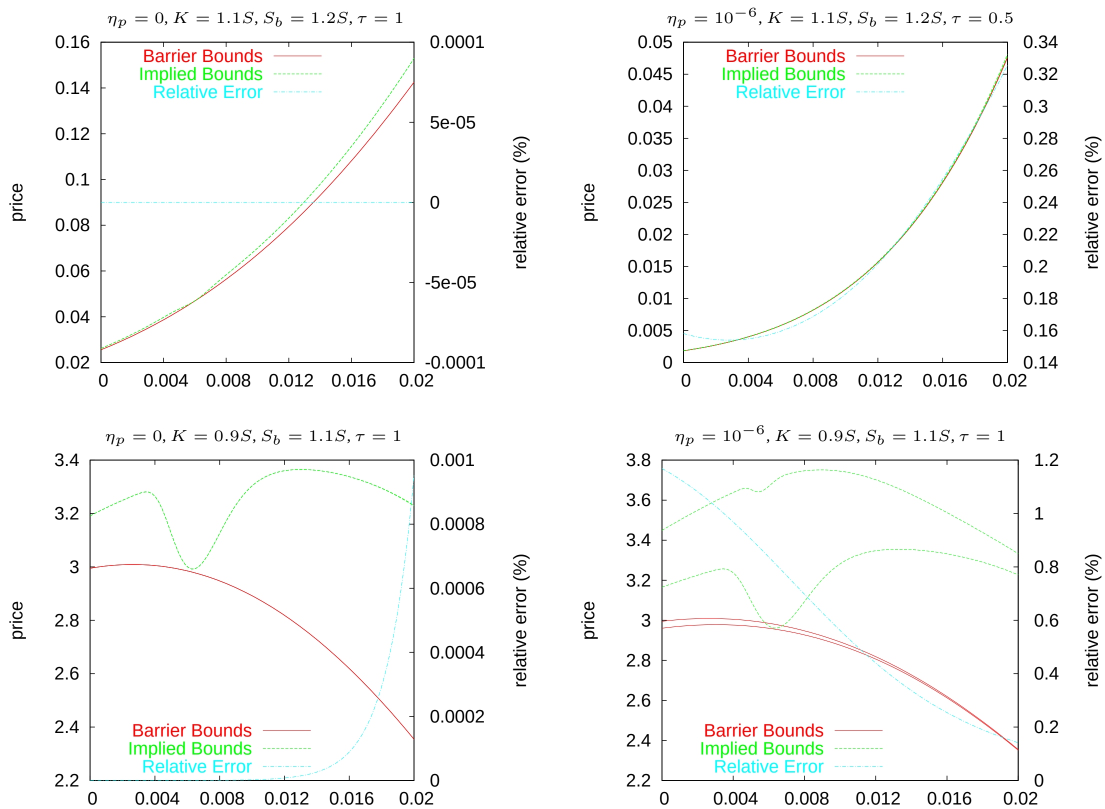


Figure 7: Up-and-out European call option price bounds against wealth ratio η_i. The parameters are as follows: beliefs $\mu_n = 1\%$, $\mu_i = -50\%$, $\mu_p = 30\%$; discount rates $\rho_n = 3\%$, $\rho_i = 10\%$, $\rho_p = 1\%$; volatility of dividend $\sigma_\delta = 5\%$. "Barrier bounds" are the linear barriers; "relative errors" are the relative differences of the linear bounds; "implied bounds" are calculated by the implied volatility bounds of one-touch options.

stock barrier. As the price for an up-and-out option equals a vanilla call price minus the price for an up-and-in barrier option, the call price is valuated by using the first implied volatility, and the "in" option is valuated by the one-touch implied volatility in the Black-Scholes model. For our numerical examples, there are two implied volatility bounds that are based on the two price bounds for the one-touch options. Such bounds for up-and-out prices are labeled "implied bounds" in Figure 7.

Figure 7 shows that the price bounds for barrier options are quite tight. Any other barrier option with an expiration that is shorter than those in the examples has even tighter bounds. Therefore, the closed-form formulae given in Propositions 4 and 5 can be used to price such barrier and one-touch options. In addition, the bounds for hedging deltas can also be bounded by closed-form formulae, which are given in Appendix B.

The practice of using two implied volatilities to price up-and-out options does not seem to work well, especially for in-the-money options. This is another indication that the implied volatility does not fully account the details of stock dynamics.

5 Conclusion

This paper studies option pricing by deriving the underlying asset price in an equilibrium model. Although options are redundant assets and can be hedged by the underlying asset only, the equilibrium asset price dynamics do not admit any existing reduced-form option pricing models. However, option pricing for both vanilla and certain exotic options in this model is straightforward, and simpler than it is in most reduced-from models, such as the stochastic volatility model. The hedging strategies are also straightforward in our model. However, the option prices in this model are not trivial, and share the major characteristics that are observed in various options markets.

Although option pricing is presented in the context of the stock market, the model can also be applied in other options markets, such as the currency options market, because of the ability of the model to generate various smiles of implied volatility — both symmetric and asymmetric — and different term structures of implied volatility.

There are several ways to extend the current model. Adding more classes of investors appears to be straightforward if richer price dynamics are needed. The pricing of other exotic options is also plausible and interesting. A more challenging vein of future research might be to study the implications of margin requirements for both the underlying stock and options. Of course, the ultimate test of an options pricing model is to examine its empirical performance. We leave this task to future research.

Appendix A: Proofs

Proof of Lemma 1

These are standard results. □

Proof of Proposition 1

The definition of state price (7) and optimal consumption plan (9) imply that

$$\begin{aligned} dc_k^*(t) &= c_k^*(t)\left(-\rho_k + r(t) + \theta_k^2(t)\right)\, dt + c_k^*(t)\theta_k(t)\, dZ_k(t) \\ &= c_k^*(t)\left(-\rho_k + r(t) + \theta_k^2(t) - \beta_k\theta_k(t)\right)\, dt + c_k^*(t)\theta_k(t)\, dZ_n(t). \end{aligned}$$

Using this, an application of the Itô Lemma to (12) shows that

$$\frac{\sum_{k\in\{n,i,k\}} dc_k^*(t)}{\sum_{k\in\{n,i,k\}} c_k^*(t)} = \frac{d\delta(t)}{\delta(t)}.$$

Notice that the expression for consumption ratios (11) and $\lambda_n \equiv 1$, $\eta_n \equiv 1$ leads to

$$\frac{\sum_{k\in\{n,i,k\}} \lambda_k\eta_k(t)\left(-\rho_k + r(t) + \theta_k^2(t) - \beta_k\theta_k(t)\right)}{\sum_{k\in\{n,i,k\}} \lambda_k\eta_k(t)} = \mu_n$$

and

$$\frac{\sum_{k\in\{n,i,k\}} \lambda_k\eta_k(t)\theta_k(t)}{\sum_{k\in\{n,i,k\}} \lambda_k\eta_k(t)} = \sigma_\delta.$$

Using (8), we can first solve $\theta_k(t)$, then $r(t)$.

For the stock price, it is trivial by the recognition that

$$S(t) = \sum_{k\in\{n,i,p\}} W_k(t)$$

by the clearing conditions in the securities markets. Using the clearing condition for the good market (12), we have

$$\frac{S(t)}{\delta(t)} = \frac{\sum_{k\in\{n,i,p\}} W_k(t)}{\sum_{k\in\{n,i,p\}} c_k^*(t)}.$$

Finally, using the definitions of wealth ratio (10) and consumption ratio (11), we have

$$\frac{S(t)}{\delta(t)} = \frac{W_n(t)}{c_n^*(t)} \frac{\sum_{k\in\{n,i,p\}} \eta_k(t)}{\sum_{k\in\{n,i,p\}} \lambda_k\eta_k(t)} = \frac{1}{\rho_n} \frac{\sum_{k\in\{n,i,p\}} \eta_k(t)}{\sum_{k\in\{n,i,p\}} \lambda_k\eta_k(t)}.$$

This is the stock price that is stated in the proposition. □

Proof of Lemma 2

Let $y = Z_n(T) - Z_n(t)$ and $\tau = T - t$. Substituting equations (16) and (17) into stock price expression (15) then yields

$$S(T) = \frac{1}{\rho_n} \times \frac{1 + f_i(\tau)\eta_i(t)e^{\beta_i y} + f_p(\tau)\eta_p(t)e^{\beta_p y}}{1 + \lambda_i f_i(\tau)\eta_i(t)e^{\beta_i y} + \lambda_p f_p(\tau)\eta_p(t)e^{\beta_p y}} g(\tau)\delta(t)e^{\sigma_\delta y}.$$

From this, we have

$$\begin{aligned} \frac{\partial S(T)}{\partial y} &= \frac{1}{\rho_n} \times \frac{1 + f_i(\tau)\eta_i(t)e^{\beta_i y} + f_p(\tau)\eta_p(t)e^{\beta_p y}}{1 + \lambda_i f_i(\tau)\eta_i(t)e^{\beta_i y} + \lambda_p f_p(\tau)\eta_p(t)e^{\beta_p y}} g(\tau)\delta(t)e^{\sigma_\delta y} \qquad (27) \\ &\times \left(\sigma_\delta + \frac{\sum_{k\in\{i,p\}}(1-\lambda_k)f_k(\tau)\eta_k(t)e^{\beta_k y}\beta_k + (\lambda_i - \lambda_p)\eta_i(t)\eta_p(t)(\beta_p - \beta_i)}{\left[1 + \sum_{k\in\{i,p\}}\lambda_k f_k(\tau)\eta_k(t)e^{\beta_k y}\right]\left[1 + \sum_{k\in\{i,p\}} f_k(\tau)\eta_k(t)e^{\beta_k y}\right]}\right), \end{aligned}$$

which is positive for all $y \in R$ when $\beta_i \leq 0$ and $\beta_p \geq 0$, because $1 - \lambda_i < 0$ and $1 - \lambda_p > 0$. That is, $S(T) \geq K$ if and only if $y = Z_i(T) - Z_i(t) \geq \bar{y}$, which solves

$$\frac{1}{\rho_n} \times \frac{1 + f_i(\tau)\eta_i(t)e^{\beta_i y} + f_p(\tau)\eta_p(t)e^{\beta_p y}}{1 + \lambda_i f_i(\tau)\eta_i(t)e^{\beta_i y} + \lambda_p f_p(\tau)\eta_p(t)e^{\beta_p y}} g(\tau)\delta(t)e^{\sigma_\delta y} = K.$$

Using stock price expression (15) again shows that the foregoing equation is equivalent to equation (18) in the lemma. □

Proof of Proposition 2

Rewrite equations (16) and (17) as

$$\eta_k(T) = \eta_k(t)f_k(\tau)e^{\beta_k(Z_n(T)-Z_n(t))}, \quad \delta(T) = \delta(t)g(\tau)e^{\sigma_\delta(Z_n(T)-Z_n(t))}.$$

Substituting these identities into pricing equation (19) and using the fact that $Z_n(T) - Z_n(t)$ follows a normal distribution with a mean of 0 and a variance of $\tau = T - t$ means that Lemma 2 implies that

$$\begin{aligned} C(t) &= \frac{\delta(t)e^{-\rho_n(T-t)}}{1 + \lambda_i\eta_i(t) + \lambda_p\eta_p(t)} \times \frac{1}{\sqrt{2\pi\tau}} \int_{\bar{y}}^{\infty} \left(\frac{1 + \sum_{k\in\{i,p\}} \eta_k(t)f_k(\tau)e^{\beta_k x}}{\rho_n} \right. \\ &\quad - \left. \frac{1 + \sum_{k\in\{i,p\}} \lambda_k\eta_k(t)f_k(\tau)e^{\beta_k x}}{\delta(t)g(\tau)e^{\sigma_\delta x}} K \right) e^{-\frac{x^2}{2\tau}} dx. \qquad (28) \end{aligned}$$

Using the identity

$$\frac{1}{\sqrt{2\pi\tau}} \int_{\bar{y}}^{\infty} e^{\gamma x} e^{-\frac{x^2}{2\tau}} dx = \exp\left(\frac{\gamma^2\tau}{2}\right) N\left(-\frac{\bar{y} - \gamma\tau}{\sqrt{\tau}}\right), \qquad (29)$$

we have

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi\tau}}\int_{\bar{y}}^{\infty}\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(t)f_k(\tau)e^{\beta_k x}}{\rho_n}\right.\\
&\qquad\qquad\left.-\ \frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)f_k(\tau)e^{\beta_k x}}{\delta(t)g(\tau)e^{\sigma_\delta x}}K\right)e^{-\frac{x^2}{2\tau}}dx\\
&= \frac{1}{\rho_n}\left[N\left(-\frac{\bar{y}}{\sqrt{\tau}}\right)+\sum_{k\in\{i,p\}}\eta_k(t)f_k(\tau)\exp\left(\frac{\beta_k^2\tau}{2}\right)N\left(-\frac{\bar{y}-\beta_k\tau}{\sqrt{\tau}}\right)\right]\\
&\quad -\frac{K}{\delta(t)g(\tau)}\left[\exp\left(\frac{\sigma_\delta^2\tau}{2}\right)N\left(-\frac{\bar{y}+\sigma_\delta\tau}{\sqrt{\tau}}\right)\right.\\
&\qquad\left.+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)f_k(\tau)\exp\left(\frac{(\beta_k-\sigma_\delta)^2\tau}{2}\right)N\left(-\frac{\bar{y}-(\beta_k-\sigma_\delta)\tau}{\sqrt{\tau}}\right)\right]\\
&= \frac{1}{\rho_n}\left[N\left(-\frac{\bar{y}}{\sqrt{\tau}}\right)+\sum_{k\in\{i,p\}}\eta_k(t)e^{(\rho_n-\rho_k)\tau}N\left(-\frac{\bar{y}-\beta_k\tau}{\sqrt{\tau}}\right)\right]\\
&\qquad -\frac{K}{\delta(t)}\left[e^{-(\mu_n-\sigma_\delta^2)\tau}N\left(-\frac{\bar{y}+\sigma_\delta\tau}{\sqrt{\tau}}\right)\right.\\
&\qquad\qquad\left.+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)e^{(\rho_n-\rho_k-\mu_k+\sigma_\delta^2)\tau}N\left(-\frac{\bar{y}-(\beta_k-\sigma_\delta)\tau}{\sqrt{\tau}}\right)\right],
\end{aligned}
$$

where the equality is obtained by the definitions of f_k, g, and β_k. This shows that

$$
\begin{aligned}
C(t) &= \frac{1}{\rho_n}\frac{\delta(t)}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)}\left[e^{-\rho_n\tau}N(d_1^n)+\sum_{k\in\{i,p\}}\eta_k(t)e^{-\rho_k\tau}N(d_1^k)\right]\\
&\quad -\frac{K}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)}\left[e^{-r_n\tau}N(d_2^n)+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)e^{-r_k\tau}N(d_2^k)\right].
\end{aligned}
$$

Using the expression of the stock price then yields the call option price that is stated in the proposition.

Finally, the bond price is a direct calculation of $\frac{1}{\xi_n(t)}E_n[\xi_n(T)|\mathcal{F}^\delta(t)]$. □

Proof of Corollary 1

It seems to be messy to work directly with the option price. Instead, it is quite simple to work with the formula that is given in the proof of Proposition 2. Rearrange equation (28) as

$$
C(t) = \frac{e^{-\rho_n\tau}}{\sqrt{2\pi\tau}}\int_{\bar{y}}^{\infty}\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(t)f_k(\tau)e^{\beta_k x}}{\rho_n[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]}\delta(t)\right.
$$

$$
\left. - \frac{1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t) f_k(\tau) e^{\beta_k x}}{[1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)] g(\tau) e^{\sigma_\delta x}} K \right) e^{-\frac{x^2}{2\tau}} dx.
$$

Note that the partial derivative of $C(t)$ with respect to $\bar{y}$ equals 0, though $\bar{y}$ depends on $\eta_i(t)$, $\eta_p(t)$, and $\delta(t)$. Using Itô's Lemma and ignoring the drift term, the diffusion term of the option is then

$$
\begin{aligned}
&\Sigma_C(t, S(t), \eta_i(t), \eta_p(t); T, K) \\
&= \frac{e^{-\rho_n \tau}}{\sqrt{2\pi\tau}} \left[\int_{\bar{y}}^{\infty} \left(\frac{1+\sum_{k\in\{i,p\}} \eta_k(t) f_k(\tau) e^{\beta_k x}}{\rho_n [1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]} \sigma_\delta \delta(t) \right.\right. \\
&\qquad + \frac{\sum_{k\in\{i,p\}} [f_k(\tau) e^{\beta_k x} - \lambda_k] \eta_k(t) \beta_k}{\rho_n [1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]^2} \delta(t) \\
&\qquad + \frac{[\lambda_i f_p(\tau) e^{\beta_p x} - \lambda_p f_i(\tau) e^{\beta_i x}] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{\rho_n [1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]^2} \delta(t) \\
&\qquad - \frac{\sum_{k\in\{i,p\}} \lambda_k [f_k(\tau) e^{\beta_k x} - 1] \eta_k(t) \beta_k}{[1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]^2 g(\tau) e^{\sigma_\delta x}} K \\
&\qquad \left.\left. - \frac{\lambda_i \lambda_p [f_p(\tau) e^{\beta_p x} - f_i(\tau) e^{\beta_i x}] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{[1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]^2 g(\tau) e^{\sigma_\delta x}} K \right) e^{-\frac{x^2}{2\tau}} dx \right] \\
&= e^{-\rho_n \tau} \left\{ \frac{\delta(t)}{\rho_n [1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]} \left[\sigma_\delta \left(N(d_1^n) + \sum_{k\in\{i,p\}} \eta_k(t) f_k(\tau) e^{\frac{\beta_k^2}{2}\tau} N(d_1^k) \right) \right.\right. \\
&\qquad + \frac{1}{1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)} \sum_{k\in\{i,p\}} \beta_k \eta_k(t) \left(f_k(\tau) e^{\frac{\beta_k^2}{2}\tau} N(d_1^k) - \lambda_k N(d_1^n) \right) \\
&\qquad \left. + \frac{\eta_i(t)\eta_p(t)(\beta_p - \beta_i)}{1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)} \left(\lambda_i f_p(\tau) e^{\frac{\beta_p^2}{2}\tau} N(d_1^p) - \lambda_p f_i(\tau) e^{\frac{\beta_i^2}{2}\tau} N(d_1^i) \right) \right] \\
&\quad - \frac{K}{[1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)]^2 g(\tau)} \left[\sum_{k\in\{i,p\}} \lambda_k \eta_k(t) \beta_k \left(f_k(\tau) e^{\frac{(\beta_k - \sigma_\delta)^2}{2}\tau} N(d_2^k) \right.\right. \\
&\qquad\qquad \left. - e^{\frac{\sigma_\delta^2}{2}\tau} N(d_2^n) \right) \\
&\quad \left.\left. - \lambda_i \eta_i(t) \lambda_p \eta_p(t) (\beta_p - \beta_i) \left(f_p(\tau) e^{\frac{(\beta_p - \sigma_\delta)^2}{2}\tau} N(d_2^p) - f_i(\tau) e^{\frac{(\beta_i - \sigma_\delta)^2}{2}\tau} N(d_2^i) \right) \right] \right\} \\
&= \frac{S(t)}{1+\sum_{k\in\{i,p\}} \eta_k(t)} \left[e^{-\rho_n \tau} \left(\sigma_\delta - \frac{\sum_{k\in\{i,p\}} \lambda_k \eta_k(t) \beta_k}{1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)} \right) N(d_1^n) \right. \\
&\qquad + e^{-\rho_i \tau} \eta_i(t) \left(\sigma_\delta + \frac{\beta_i + \lambda_p \eta_p(t)(\beta_i - \beta_p)}{1+\sum_{k\in\{i,p\}} \lambda_k \eta_k(t)} \right) N(d_1^i) \\
&\qquad \left. + e^{-\rho_p \tau} \eta_p(t) \left(\sigma_\delta + \frac{\beta_p + \lambda_i \eta_i (\beta_p - \beta_i)}{1+\sum \lambda_k \eta_k(t)} \right) N(d_1^p) \right]
\end{aligned}
$$

$$
- \frac{K}{[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]^2}\left[\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)\beta_k\left(e^{-r_k\tau}N(d_2^k)-e^{-r_n\tau}N(d_2^n)\right)\right.
$$
$$
\left. +\ \lambda_i\eta_i(t)\lambda_p\eta_p(t)(\beta_p-\beta_i)\left(e^{-r_p\tau}N(d_2^p)-e^{-r_i\tau}N(d_2^i)\right)\right],
$$

where we use identity (29) in the proof of Proposition 2 to obtain the second equality and use the definitions of f_k, g, and the stock price to obtain the third equality. Then, by the definition of the delta of the option (the ratio of diffusions between the option and its underlying stock), the result follows. □

Proof of Lemma 3

Taking the derivatives of both sides of equation (18) with respect to s yields

$$
\begin{aligned}
& \sigma_\delta\left(\frac{\partial b(s)}{\partial s}-h\right) \\
+ \quad & \beta_i\left(\frac{\partial b(s)}{\partial s}-l_i\right)\frac{\left(1-\lambda_i+(\lambda_p-\lambda_i)f_p\eta_p e^{\beta_p b}\right)f_i\eta_i e^{\beta_i b}}{(1+\sum_{k\in\{i,p\}}\lambda_k f_k\eta_k e^{\beta_k b})(1+\sum_{k\in\{i,p\}}f_k\eta_k e^{\beta_k b})} \\
+ \quad & \beta_p\left(\frac{\partial b(s)}{\partial s}-l_p\right)\frac{\left(1-\lambda_p+(\lambda_i-\lambda_p)f_i\eta_i e^{\beta_i b}\right)f_p\eta_p e^{\beta_p b}}{(1+\sum_{k\in\{i,p\}}\lambda_k f_k\eta_k e^{\beta_k b})(1+\sum_{k\in\{i,p\}}f_k\eta_k e^{\beta_k b})}=0.
\end{aligned}
$$

Rearranging terms gives

$$
\frac{\partial b(s)}{\partial s}=\frac{\sigma_\delta h+\Sigma_i(s,b(s))l_i+\Sigma_p(s,b(s))l_p}{\sigma_\delta+\Sigma_i(s,b(s))+\Sigma_p(s,b(s))}.
$$

As both σ_δ and Σ_k for $k\in\{i,\,p\}$ are positive, the lemma follows. □

Proof of Proposition 3

After noting that $\delta(T)$, $\eta_k(T)$ for $k\in\{i,p\}$ can be rewritten as time to expiration τ and x_m, the result is straightforward by applying Lemma B.1 to the expectation in pricing equation (23).

Proof of Corollary 2

Following a similar method as for the case of vanilla call options, we have

$$
\Sigma_Z^{uo}(t,S(t),\eta_i(t),\eta_p(t);T,K;b_m)=
$$
$$
e^{-\rho_n\tau}\int \mathbf{1}_{\{x_m\le b_m(\tau)-\bar{y}\}}\tilde{p}(\tilde{x},b_m)\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(t)f_k(\tau)e^{\beta_k(b_m(\tau)-\tilde{x}_m)}}{\rho_n[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]}\sigma_\delta\delta(t)\right.
$$

$$
\begin{aligned}
&+\frac{\sum_{k\in\{i,p\}}[f_k(\tau)e^{\beta_k(b_m(\tau)-\tilde{x}_m)}-\lambda_k]\eta_k(t)\beta_k}{\rho_n[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]^2}\delta(t)\\
&+\frac{[\lambda_i f_p(\tau)e^{\beta_p(b_m(\tau)-\tilde{x}_m)}-\lambda_p f_i(\tau)e^{\beta_i(b_m(\tau)-\tilde{x}_m)}]\eta_i(t)\eta_p(t)(\beta_p-\beta_i)}{\rho_n[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]^2}\delta(t)\\
&-\frac{\sum_{k\in\{i,p\}}\lambda_k[f_k(\tau)e^{\beta_k(b_m(\tau)-\tilde{x}_m)}-1]\eta_k(t)\beta_k}{[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]^2 g(\tau)e^{\sigma_\delta(b_m(\tau)-\tilde{x}_m)}}K\\
&-\left.\frac{\lambda_i\lambda_p[f_p(\tau)e^{\beta_p(b_m(\tau)-\tilde{x}_m)}-f_i(\tau)e^{\beta_i(b_m(\tau)-\tilde{x}_m)}]\eta_i(t)\eta_p(t)(\beta_p-\beta_i)}{[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)]^2 g(\tau)e^{\sigma_\delta(b_m(\tau)-\tilde{x}_m)}}K\right)d\tilde{x}.
\end{aligned}
$$

Rearranging the terms and using the expression for the stock price in Proposition 1 yields the result. □

Proof of Proposition 4

From equation (23), the pricing of the option means the calculation of

$$
\begin{aligned}
I \;=\; E_n\Bigg[\mathbf{1}_{\{Z(s)<b_1(s),0\le s\le\tau\}}\Bigg(&\frac{1+\sum_{k\in\{i,p\}}\eta_k(T)}{\rho_n}\\
&-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(T)}{\delta(T)}K\Bigg)^+\Bigg|\mathcal{F}^\delta(t)\Bigg].
\end{aligned}
$$

Since both $\eta_k(T)$ and $\delta(T)$ are functions of time and $Z_n(T)-Z_n(t)$, the calculation of the foregoing expectation is straightforward if the distribution density function of $Z_n(T)-Z_n(t)$ is known. By Lemma B.1 or equation (36) in Appendix B, the probability density function of $Z(s)=Z_n(t+s)-Z_n(t)$ is $p(x,b_1)$, and we then have

$$
\begin{aligned}
I &= \int\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(T)}{\rho_n}-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(T)}{\delta(T)}K\right)^+p(x,b_1)dx\\
&=\frac{1}{\sqrt{2\pi\tau}}\int_{\bar{y}}^{\alpha+\mu\tau}\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(T)}{\rho_n}-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(T)}{\delta(T)}K\right)e^{-\frac{x^2}{2\tau}}dx\\
&-\frac{e^{-2\alpha\mu}}{\sqrt{2\pi\tau}}\int_{\bar{y}}^{\alpha+\mu\tau}\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(T)}{\rho_n}-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(T)}{\delta(T)}K\right)e^{-\frac{(x-2\alpha)^2}{2\tau}}dx\\
&= I(\bar{y},0)-I(\alpha+\mu\tau,0)-e^{-2\alpha\mu}[I(\bar{y},\alpha)-I(\alpha+\mu\tau,\alpha)],
\end{aligned}
$$

where

$$
\begin{aligned}
&I(y,\alpha) \qquad (30)\\
&=\frac{1}{\sqrt{2\pi\tau}}\int_y^\infty\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k(T)}{\rho_n}-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(T)}{\delta(T)}K\right)e^{-\frac{(x-2\alpha)^2}{2\tau}}dx\\
&=\frac{1}{\sqrt{2\pi\tau}}\int_y^\infty\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k f_k e^{\beta_k x}}{\rho_n}-\frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k f_k e^{\beta_k x}}{\delta(t)g(\tau)e^{\sigma_\delta x}}K\right)e^{-\frac{(x-2\alpha)^2}{2\tau}}dx.
\end{aligned}
$$

Integral $I(\bar{y}, 0)$ is calculated in the proof of Lemma 2, and the other integrals can be calculated by using the following identity for any constant γ and y.

$$\frac{1}{\sqrt{2\pi\tau}}\int_y^\infty e^{\gamma x}e^{-\frac{(x-2\alpha)^2}{2\tau}}dx = \exp\left(\frac{\gamma^2\tau+4\alpha\gamma}{2}\right)N\left(\frac{2\alpha+\gamma\tau-y}{\sqrt{\tau}}\right).$$

Then, for any constant y and α

$$\begin{aligned}
I(y,\alpha) &= \frac{1}{\sqrt{2\pi\tau}} \\
&\times\int_y^\infty\left(\frac{1+\sum_{k\in\{i,p\}}\eta_k f_k(\tau)e^{\beta_k x}}{\rho_n} - \frac{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k f_k(\tau)e^{\beta_k x}}{\delta(t)g(\tau)e^{\sigma_\delta x}}K\right)e^{-\frac{(x-2\alpha)^2}{2\tau}}dx \\
&= \frac{1}{\rho_n}\left[N\left(\frac{2\alpha-y}{\sqrt{\tau}}\right)+\sum_{k\in\{i,p\}}\eta_k(t)f_k(\tau)e^{\frac{\beta_k^2\tau+4\alpha\beta_k}{2}}N\left(\frac{2\alpha+\beta_k\tau-y}{\sqrt{\tau}}\right)\right] \\
&\quad -\frac{K}{\delta(t)g(\tau)}\left[e^{\frac{\sigma_\delta^2\tau-4\alpha\sigma_\delta}{2}}N\left(\frac{2\alpha-\sigma_\delta\tau-y}{\sqrt{\tau}}\right)+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)f_k(\tau)\right. \\
&\qquad \left.\times e^{\frac{(\beta_k-\sigma_\delta)^2\tau+4\alpha(\beta_k-\sigma_\delta)}{2}}N\left(\frac{2\alpha+(\beta_k-\sigma_\delta)\tau-y}{\sqrt{\tau}}\right)\right] \\
&= \frac{1}{\rho_n}\left[N\left(\frac{2\alpha-y}{\sqrt{\tau}}\right)+\sum_{k\in\{i,p\}}\eta_k(t)e^{(\rho_n-\rho_k)\tau+2\alpha\beta_k}N\left(\frac{2\alpha+\beta_k\tau-y}{\sqrt{\tau}}\right)\right] \\
&\quad -\frac{K}{\delta(t)}\left[e^{-(\mu_n-\sigma_\delta^2)\tau-2\alpha\sigma_\delta}N\left(\frac{2\alpha-\sigma_\delta\tau-y}{\sqrt{\tau}}\right)\right. \\
&\quad \left.+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)e^{(\rho_n-\rho_k-\mu_k+\sigma_\delta^2)\tau+2\alpha(\beta_k-\sigma_\delta)}N\left(\frac{2\alpha+(\beta_k-\sigma_\delta)\tau-y}{\sqrt{\tau}}\right)\right],
\end{aligned}$$

where the last equality is the direct implication of the definitions of f_k, g, and β_k.

Let

$$\begin{aligned}
&C_Z^u(t,S(t),\eta_i(t),\eta_p(t);T,K;y,\alpha) \\
&= \frac{\delta(t)e^{-\rho_i\tau}}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)}I(y,\alpha) \qquad (31)\\
&= \frac{S(t)}{1+\sum_{k\in\{i,p\}}\eta_k(t)}\left[e^{-\rho_n\tau}N\left(\frac{2\alpha-y}{\sqrt{\tau}}\right)\right. \\
&\qquad \left.+\sum_{k\in\{i,p\}}\eta_k(t)e^{-\rho_k\tau+2\alpha\beta_k}N\left(\frac{2\alpha+\beta_k\tau-y}{\sqrt{\tau}}\right)\right] \\
&\quad -\frac{Ke^{-2\alpha\sigma_\delta}}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)}\left[e^{-r_n\tau}N\left(\frac{2\alpha-\sigma_\delta\tau-y}{\sqrt{\tau}}\right)\right.
\end{aligned}$$

$$+ \sum_{k\in\{i,p\}} \lambda_k \eta_k(t) e^{-r_k\tau+2\alpha\beta_k} N\left(\frac{2\alpha+(\beta_k-\sigma_\delta)\tau-y}{\sqrt{\tau}}\right)\Bigg],$$

where r_k for $k = n, i, p$ is defined as in Proposition 2.

Given the results, we have

$$\begin{aligned}
&C_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha, \mu) \\
&= C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, 0) \\
&\qquad - C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha+\mu\tau, 0) \\
&\quad -e^{-2\alpha\mu}\left[C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, \alpha)\right. \\
&\qquad \left. - C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha+\mu\tau, \alpha)\right].
\end{aligned}$$

Combining this with the fact that

$$C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, 0) = C(t, S(t), \eta_i(t), \eta_p(t); T, K)$$

leads to the conclusion. □

Proof of Proposition 5

Let $Z(s) = Z_n(t+s) - Z_n(t)$. Then,

$$\inf\{s \geq 0 | Z(s) = \alpha + \mu s\} = \inf\{s \geq 0 | Z(s) - \mu s = \alpha\}.$$

From Karatzas and Shreve (1991) (see page 196-197), the probability density function of the first touch time s is

$$\psi(t) = \frac{|\alpha|}{\sqrt{2\pi s^3}} \exp\left[-\frac{(\alpha+\mu s)^2}{2s}\right].$$

The one-touch digital price is then

$$\begin{aligned}
&C_Z^{ot}(t, \eta_i(t), \eta_p(t); T; \alpha, \mu) \\
&= \frac{1}{\xi_n(t)} \int_0^\tau \xi_n(t+s)\psi(s)\,ds \\
&= \frac{\delta(t)}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)} \int_0^\tau \frac{[1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t+s)]e^{-\rho_n s}}{\delta(t+s)}\psi(s)\,ds \\
&= \frac{e^{-\sigma_\delta\alpha}}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)} \int_0^\tau e^{-\left(r_n+\frac{1}{2}\sigma_\delta^2+\sigma_\delta\mu\right)s}\psi(s)\,ds \\
&\quad + \sum_{k\in\{i,p\}} \frac{\lambda_k\eta_k(t)e^{(\beta_k-\sigma_\delta)\alpha}}{1+\sum_{k\in\{i,p\}}\lambda_k\eta_k(t)} \int_0^\tau e^{-\left(r_k+\frac{1}{2}(\sigma_\delta-\beta_k)^2+(\sigma_\delta-\beta_k)\mu\right)s}\psi(s)\,ds,
\end{aligned} \tag{32}$$

where we use $Z_n(t+s) - Z_n(t) = \alpha + \mu s$ when $Z(s)$ hits the barrier.

Since, for any $\gamma \geq -\mu^2/2$, we have

$$\begin{aligned}
&\int_0^\tau e^{-\gamma s} \frac{|\alpha|}{\sqrt{2\pi s^3}} \exp\left[-\frac{(\alpha+\mu s)^2}{2s}\right] ds \\
= \;& e^{-\alpha\mu} \int_0^\tau \frac{|\alpha|}{\sqrt{2\pi s^3}} \exp\left[-\frac{\alpha^2 + (\mu^2+2\gamma)s^2}{2s}\right] ds \\
= \;& e^{-\alpha\mu - |\alpha|\sqrt{\mu^2+2\gamma}} \int_0^\tau \frac{1}{2} \frac{|\alpha| + \sqrt{\mu^2+2\gamma}s}{\sqrt{2\pi s^3}} \exp\left[-\frac{(|\alpha| - \sqrt{\mu^2+2\gamma}s)^2}{2s}\right] ds \\
& + e^{-\alpha\mu + |\alpha|\sqrt{\mu^2+2\gamma}} \int_0^\tau \frac{1}{2} \frac{|\alpha| - \sqrt{\mu^2+2\gamma}s}{\sqrt{2\pi s^3}} \exp\left[-\frac{(|\alpha| + \sqrt{\mu^2+2\gamma}s)^2}{2s}\right] ds \\
= \;& e^{-\alpha\mu - |\alpha|\sqrt{\mu^2+2\gamma}} N\left(\frac{\sqrt{\mu^2+2\gamma}\tau - |\alpha|}{\sqrt{\tau}}\right) \\
& \qquad + e^{-\alpha\mu + |\alpha|\sqrt{\mu^2+2\gamma}} N\left(-\frac{\sqrt{\mu^2+2\gamma}\tau + |\alpha|}{\sqrt{\tau}}\right).
\end{aligned} \tag{33}$$

Substituting

$$\gamma = r_n + \frac{1}{2}\sigma_\delta^2 + \sigma_\delta \mu$$

and

$$\gamma = r_k + \frac{1}{2}(\sigma_\delta - \beta_k)^2 + (\sigma_\delta - \beta_k)\mu$$

into (33) yields the result. □

Appendix B: Auxiliary Lemmas and Further Results for the Pricing of Barrier Options

Appendix B.1 Crossing Probability of a Brownian Motion for a Piece-Wise Linear Barrier

Lemma B.1 *Let $0 = s_0 < s_1 ... s_{m-1} < s_m = \tau$ and $b_m(s)$ be linear functions of s on each of the intervals $[s_{j-1}, s_j]$ for $j = 1, ..., m$. The probability density function that a Brownian motion $Z(s)$ does not cross barrier $b_m(s)$ for all $0 \leq s \leq \tau$ is then*

$$p(x, b_m) = \psi(x, b_m) \prod_{j=1}^m \left(\mathbf{1}_{\{b_m(0)>0\}} \mathbf{1}_{\{x_j < b_m(s_j)\}} + \mathbf{1}_{\{b_m(0)<0\}} \mathbf{1}_{\{x_j > b_m(s_j)\}}\right),$$

where

$$\psi(x, b_m) = \prod_{j=1}^m \left(1 - \exp\left[-\frac{2[b_m(s_{j-1}) - x_{j-1}][b_m(s_j) - x_j]}{s_j - s_{j-1}}\right]\right)$$

$$\times \frac{1}{\sqrt{2\pi(s_j - s_{j-1})}} \exp\left[\frac{(x_j - x_{j-1})^2}{2(s_j - s_{j-1})}\right], \tag{34}$$

where $b_0 = b_m(0)$*,* $b_m = (b_m(s_1), ..., b_m(s_m))^\top$*,* $x_0 = 0$*,* $x = (x_1, ..., x_m)^\top$*.*

Let $\tilde{x} = b_m - x$*. The probability density function that* $Z(s)$ *does not cross the piece-wise linear barrier* $b_m(s)$ *for* $0 \le s \le \tau$ *is then*

$$\tilde{p}(\tilde{x}, b_m) = \tilde{\psi}(\tilde{x}, b_m) \prod_{j=1}^{m} \left(\mathbf{1}_{\{b_m(0)>0\}}\mathbf{1}_{\{\tilde{x}_j>0\}} + \mathbf{1}_{\{b_m(0)<0\}}\mathbf{1}_{\{\tilde{x}_j<0\}}\right),$$

where

$$\begin{aligned} \tilde{\psi}(\tilde{x}, b_m) &= \frac{1}{\sqrt{\det(2\pi\Sigma)}} \prod_{j=1}^{m} \mathbf{1}_{\{\tilde{x}_j>0\}} \left(1 - \exp\left[-\frac{2\tilde{x}_{j-1}\tilde{x}_j}{s_j - s_{j-1}}\right]\right) \\ &\quad \times \exp\left[-\frac{(\tilde{x} - b_m)^\top \Sigma^{-1} (\tilde{x} - b_m)}{2}\right], \end{aligned} \tag{35}$$

where $\tilde{x}_0 = b_m(0)$*,* $\Sigma = M\,\mathrm{diag}(s_1 - s_0, ..., s_m - s_{m-1}) M^\top$*, and* M *is the lower triangular matrix with all nonzero elements equal to 1.*

Proof: See Wang and Potzelberger (1997). □

In the case of $m = 1$, let $b_1(s) = \alpha + \mu s$. Then, $b_1(0) = \alpha$ and $b_1 = \alpha + \mu\tau$. Substituting these into equation (34) yields

$$p(x, b_1) = \left(\mathbf{1}_{\{\alpha>0\}}\mathbf{1}_{\{x<\alpha+\mu\tau\}} + \mathbf{1}_{\{\alpha<0\}}\mathbf{1}_{\{x>\alpha+\mu\tau\}}\right) \psi(x, b_1),$$

where

$$\psi(x, b_1) = \frac{1}{\sqrt{2\pi\tau}} \left[\exp\left(\frac{x^2}{2\tau}\right) - e^{-2\alpha\mu} \exp\left(\frac{(x - 2\alpha)^2}{2\tau}\right)\right], \tag{36}$$

where we have used the fact that $x = x_1$ for the case of $m = 1$.

Appendix B.2 Pricing Other Barrier Options

There are basically four kinds of barrier options with a single barrier: down-and-in, down-and-out, up-and-in, and up-and-out. The payoff for an “out” option is

$$G_Z^{\text{out}}(S(T), K) = \begin{cases} \mathbf{1}_{\{H>\tau\}} G(S(T), K) & \text{if the barrier is NOT crossed} \\ 0 & \text{if the barrier is crossed,} \end{cases}$$

where $G(S(T), K)$ is a vanilla European option with a strike price K. The “in” option is defined in the opposite way. The immediate implication of these definitions is that

$$G_Z^{\text{in}}(S(T), K) + G_Z^{\text{out}}(S(T), K) = G(S(T), K).$$

This means that

$$V_Z^{\text{in}}(t) + V_Z^{\text{out}}(t) = V(t). \tag{37}$$

This shows that the "in" options can be priced by the relevant "out" options and vanilla European options. The prices for up-and-out options are given in the text, and those of down-and-out options are given by the following lemma.

Lemma B.2 *The price of a down-and-out call option with a constant barrier based on the innovation process* $b_1(s) = \alpha + \mu s$ *is given by*

$$\begin{aligned} &C_Z^{do}(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha, \mu) \\ &\quad = \begin{cases} C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \alpha + \mu\tau, 0) & \\ \quad -e^{-2\alpha\mu} C_Z^u(t, S(t), \eta(t), \eta_p(t); T, K; \alpha + \mu\tau, \alpha) & \bar{y} \le \alpha + \mu\tau \\ C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, 0) & \\ \quad -e^{-2\alpha\mu} C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; \bar{y}, \alpha) & \bar{y} > \alpha + \mu\tau, \end{cases} \end{aligned}$$

where $C_Z^u(t, S(t), \eta_i(t), \eta_p(t); T, K; y, \alpha)$ *is defined as in Proposition 4.*

Proof. Since the distribution of $Z(s) = Z_n(T) - Z_n(t)$ conditional on the barrier not having been hit up to time T is $p(x, b_1) = \mathbf{1}_{\{x > \alpha + \mu\tau\}} \psi(x, b_1)$, where ψ is as defined by equation (36), we have

$$\begin{aligned} C_Z^{do}(t) &= \frac{1}{\xi_n(t)} E_i\left[\xi_n(T) G_{\text{out}}^{\text{down}}(S(T), K) \mathbf{1}_{\{Z(s) > b_1(s), s \le \tau\}} \middle| \mathcal{F}^\delta(t) \right] \\ &= \frac{\delta(t) e^{-\rho_n \tau}}{1 + \sum \lambda_k \eta_k} \int_{\alpha+\mu\tau}^{\infty} \left(\frac{1 + \sum \eta_k(T)}{\rho_n} - \frac{1 + \sum \lambda_k \eta_k(T)}{\delta(T)} K \right)^+ \psi(x, b_1) dx \\ &= \frac{\delta(t) e^{-\rho_n \tau}}{1 + \sum \lambda_k \eta_k} \begin{cases} \int_{\alpha+\mu\tau}^{\infty} \left(\frac{1+\sum \eta_k(T)}{\rho_n} - \frac{1+\sum \lambda_k \eta_k(T)}{\delta(T)} K \right) \psi(x, b_1) dx & \bar{y} \le \alpha + \mu\tau \\ \int_{\bar{y}}^{\infty} \left(\frac{1+\sum \eta_k(T)}{\rho_n} - \frac{1+\sum \lambda_k \eta_k(T)}{\delta(T)} K \right) \psi(x, b_1) dx & \bar{y} > \alpha + \mu\tau \end{cases} \\ &= \frac{\delta(t) e^{-\rho_n \tau}}{1 + \sum \lambda_k \eta_k} \begin{cases} I(\alpha + \mu\tau, 0) - e^{-2\alpha\mu} I(\alpha + \mu\tau, \alpha) & \bar{y} \le \alpha + \mu\tau \\ I(\bar{y}, 0) - e^{-2\alpha\mu} I(\bar{y}, \alpha) & \bar{y} > \alpha + \mu\tau, \end{cases} \end{aligned}$$

where the last equality is obtained by using the identity (30) in the proof of Proposition 4. The application of (31) as defined in the proof of Proposition 4 yields the prices that are stated in the lemma. □

Appendix B.3 Hedging Delta for the Special Linear Barrier

Similar to the case of vanilla European options, we calculate the diffusion terms for the barrier and one-touch options. The hedging deltas then follow by taking the ratios between the diffusion terms and the diffusion term of the stock.

Appendix B.3.1 Delta for Barrier Options

Substituting $\tilde{x} = b_1(\tau) - x$ into the expression for Σ_Z^{uo} in Corollary 2 yields

$$
\begin{aligned}
&\Sigma_Z^{uo}(t, S(t), \eta_i(t), \eta_p(t); T, K; b_1) \\
&\quad = e^{-\rho_n \tau} \int \mathbf{1}_{\{x \geq \bar{y}\}} p(x, b_1) \Bigg[\Bigg(\frac{[1 + \sum_{k \in \{i,p\}} \eta_k(t) f_k(\tau) e^{\beta_k x}] \sigma_\delta}{1 + \eta_i(t) + \eta_p(t)} \\
&\qquad\qquad + \frac{\sum_{k \in \{i,p\}} [f_k(\tau) e^{\beta_k x} - \lambda_k] \eta_k(t) \beta_k}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]} \\
&\qquad + \frac{[\lambda_i f_p(\tau) e^{\beta_p x} - \lambda_p f_i(\tau) e^{\beta_i x}] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]} \Bigg) S(t) \\
&\qquad - \Bigg(\frac{\sum_{k \in \{i,p\}} \lambda_k [f_k(\tau) e^{\beta_k x} - 1] \eta_k(t) \beta_k}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]^2 g(\tau) e^{\sigma_\delta x}} \\
&\qquad + \frac{\lambda_i \lambda_p [f_p(\tau) e^{\beta_p x} - f_i(\tau) e^{\beta_i x}] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]^2 g(\tau) e^{\sigma_\delta x}} \Bigg) K \Bigg] dx \\
&\quad = \Bigg(\frac{e^{-\rho_n \tau} J(0) + e^{-\rho_i \tau} J(\beta_i) + e^{-\rho_p \tau} J(\beta_p)}{1 + \eta_i(t) + \eta_p(t)} \sigma_\delta \\
&\qquad\qquad + \frac{\sum_{k \in \{i,p\}} [e^{-\rho_k \tau} J(\beta_k) - \lambda_k e^{-\rho_n \tau} J(0)] \eta_k(t) \beta_k}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]} \\
&\qquad + \frac{[\lambda_i e^{-\rho_p \tau} J(\beta_p) - \lambda_p e^{-\rho_i \tau} J(\beta_i)] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{[1 + \eta_i(t) + \eta_p(t)][1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]} \Bigg) S(t) \\
&\qquad - \Bigg(\frac{\sum_{k \in \{i,p\}} \lambda_k [e^{-r_k \tau} J(\beta_k - \sigma_\delta) - e^{-r_n \tau} J(-\sigma_\delta)] \eta_k(t) \beta_k}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]^2} \\
&\qquad + \frac{\lambda_i \lambda_p [e^{-r_p \tau} J(\beta_p - \sigma_\delta) - e^{-r_i \tau} J(\beta_i - \sigma_\delta)] \eta_i(t) \eta_p(t) (\beta_p - \beta_i)}{[1 + \lambda_i \eta_i(t) + \lambda_p \eta_p(t)]^2} \Bigg) K,
\end{aligned}
$$

where we use the following identity

$$
\int_{\bar{y}}^{\alpha + \mu \tau} e^{\gamma x} p(x, b_1)\, dx = e^{\frac{\gamma^2 \tau}{2}} J(\gamma), \tag{38}
$$

where

$$
\begin{aligned}
J(\gamma) \;=\; & \left[N\left(\frac{\alpha + \mu \tau - \gamma \tau}{\sqrt{\tau}} \right) - N\left(\frac{\bar{y} - \gamma \tau}{\sqrt{\tau}} \right) \right] \\
& - e^{-2\alpha\mu + \alpha\gamma} \left[N\left(\frac{-\alpha + \mu \tau - \gamma \tau}{\sqrt{\tau}} \right) - N\left(\frac{\bar{y} - \gamma \tau - 2\alpha}{\sqrt{\tau}} \right) \right].
\end{aligned}
$$

Appendix B.3.2 Delta for One-Touch Options

Let

$$H(\gamma) = e^{-\alpha\mu-|\alpha|\sqrt{\mu^2+2\gamma}} N\left(\frac{\sqrt{\mu^2+2\gamma}\tau-|\alpha|}{\sqrt{\tau}}\right) + e^{-\alpha\mu+|\alpha|\sqrt{\mu^2+2\gamma}} N\left(-\frac{\sqrt{\mu^2+2\gamma}\tau+|\alpha|}{\sqrt{\tau}}\right).$$

Using this and equation (32) then yields

$$\begin{aligned}
&\Sigma_Z^{ot}(t,\eta_i(t),\eta_p(t),b_1) = \\
&\quad -\frac{e^{-\alpha\sigma_\delta}}{[1+\lambda_i\eta_i(t)+\lambda_p\eta_p(t)]^2}\left[\left(\lambda_i\eta_i(t)\beta_i+\lambda_p\eta_p(t)\beta_p\right) H\left(r_n+\frac{1}{2}\sigma_\delta^2+\sigma_\delta\mu\right)\right. \\
&\quad -e^{\alpha\beta_i}\lambda_i\eta_i(t)\left(\beta_i+\lambda_p\eta_p(t)(\beta_i-\beta_p)\right) H\left(r_i+\frac{1}{2}(\sigma_\delta-\beta_i)^2+(\sigma_\delta-\beta_i)\mu\right) \\
&\quad \left.-e^{\alpha\beta_p}\lambda_p\eta_p(t)\left(\beta_p+\lambda_i\eta_i(t)(\beta_p-\beta_i)\right) H\left(r_p+\frac{1}{2}(\sigma_\delta-\beta_p)^2+(\sigma_\delta-\beta_p)\mu\right)\right],
\end{aligned}$$

where $b_1(s) = \alpha + \mu s$.

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