Optimal dynamic hedging in commodity futures markets with a stochastic convenience yield

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Abstract **This version:December 2007**

The main objective of this paper is to fill the gap in the literature by addressing, in a continuous-time context, the issue of using commodity futures as vehicles for hedging purposes when, in particular, the convenience yield as well as the market prices of risk evolve randomly over time. Following the martingale route and by operating a suitable constant relative risk aversion utility function (CRRA) specific change of numéraire, we derive optimal demands for commodity futures contracts by an unconstrained investor, who can freely trade on the underlying spot asset and on a discount bond. Although the optimal demand exhibits a classical structure in that it is composed of a speculative part and of a hedging term, our model has four main distinctive features and goes beyond the existing studies. First, the speculative and hedging components may be decomposed in a convenient way underlining, in particular, the effect of the stochastic behavior of both the market prices of risk and the convenience yield on optimal demands. As a consequence, the investor is able to exactly asses their impact on optimal demands. Second, the interaction between the prices of risk associated especially with the spot commodity and the convenience yield combined with their mean-reverting character determine the sign and the magnitude of the speculative and the hedging proportions. Third, the futures contract turns out to be the appropriate instrument to hedge the idiosyncratic source of risk of the convenience yield. Furthermore, in contrast to Breeden's (1984) results, the primitive assets are effective in hedging the specific risk of the spot commodity and the interest rate. Finally, optimal demands can be computed in a recursive way, which greatly facilitates the use of our model for practical considerations.

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1. Introduction

Futures markets have experienced dramatic growth, worldwide, of both trading volume and contracts written on a wide range of underlying assets. These features make it easier to use futures contracts as hedging instruments against unfavorable changes in the opportunity set, i.e. changes in state variables or factors describing the economic/financial environment. The growing activity of these markets has been accompanied, since the original normal backwardation of Keynes (1930) and Hicks (1939), by a substantial body of literature devoted to pricing and hedging with futures contracts[2]. The main objective of this paper is to bridge the gap in the literature by addressing, in a continuous-time context, the issue of using commodity futures, by an unconstrained investor[3], as vehicles for hedging purposes.

In an intertemporal portfolio choice framework, Merton (1971, 1973) and Breeden (1979) derived optimal asset allocation for an unconstrained investor, who maximizes his (her) expected lifetime utility function of consumption under the budget constraint. This demand encompasses the commonly referred as Merton-Breeden hedging terms reflecting the investor's wish to hedge against the random fluctuations of the investment opportunity set. As is well-known, the utility maximization approach implies, however, that the optimal demand includes an additional speculative position which depends on the investor's risk aversion, as well as on the instantaneous expected excess return of the risky assets.

An abundant literature has been devoted to pricing commodity futures[4]. The models developed explain the evolution of the futures prices through the random evolution of several relevant state variables. The stochastic processes of these variables are specified exogenously. The convenience yield turns out to be the crucial variable, which constitutes one of the main differences between spot

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[2] Interested readers could refer to Lioui and Poncet (2005).

[3] The unconstrained investor is allowed to freely trade on the primitive assets, namely the underlying spot asset and, if need be, other risky assets.

[4] See, for instance, Gibson and Schwartz (1990), Schwartz (1997), Hilliard and Reis (1998), Miltersen and Schwartz (1998), Yan (2002), Nielsen and Schwartz (2004), Casassus and Collin-Dufresne (2005) and Sorensen (2002).

commodity prices and prices of financial assets[5]. The recent sharp increase in commodity prices has revived the interest in commodity risk management. Derivatives securities or contingent claims, futures contracts in particular, are major tools used by investors for hedging in order to mitigate their exposure to changes in commodity prices. Surprisingly, while there are a number of models dealing with futures hedging, to our best knowledge, the specific case of commodity futures contracts with a stochastic convenience yield has not yet been addressed in the literature. An exception is Hong (2001) whose economic environment and objective differ considerably from ours in that he examined, especially, the impact of a stochastic convenience yield on the term structure of open interest, i.e., the total number of contracts outstanding. Moreover, in the literature market prices of risk are usually assumed to be constant when studying optimal asset allocation. Few exceptions are, for example, Kim and Omberg (1996) and Wachter (2002). Lioui and Poncet (2001), in particular, examine the effect of stochastic prices of risk on futures hedging demands. In our environment to the extent that spot commodity prices, futures prices and inventory decisions are related (see, for instance, Brennan, 1958; Litzenberger, Rabinowitz 1995; Routledge et al., 2000), we would expect market prices of risk to be stochastic.

This paper provides a theoretical model of hedging that could better account for how both stochastic convenience yield and stochastic market prices of risk affect the optimal demand of an unconstrained investor[6]. In order to do so, in the same vein as Schwartz (1997) and Hilliard and Reis (1998) - the reference models in the literature - the economic framework retains the spot commodity price, the instantaneous interest rate and convenience yield as the relevant stochastic state variables associated with the dynamics of the futures price. Furthermore, it is assumed that the prices of risk are

[5] Brennan (1991) defines the convenience yield as “the flow of services accruing to the owner of the physical inventory, but not to the owner of a contract for future delivery”. Indeed, physical inventory provides some services such as the possibility of avoiding shortages of the spot commodity and thus to maintain the production process or even to benefit from a (anticipated) future price increase.

[6] Other theoretical models examining dynamic asset allocation with futures contracts (see, among others, Ho, 1984; Stulz, 1984; Adler and Detemple, 1988a, b; Duffie and Jackson, 1990; Briys et al., 1990; Duffie and Richardson, 1991; Lioui et al., 1996) deal with a constraint utility maximizer investor. In a similar economic environment, the investor’s optimal futures demand consists of three terms: a mean-variance speculative term, a Merton-Breeden hedging component and a pure hedge element related to the non-traded position.

affine functions of the state variables (see also Duffee, 2002; Casassus and Collin-Dufresne, 2005). The optimal demand for commodity futures contracts is derived for an investor who maximizes the expected constant relative risk aversion (CRRA) utility function of his (her) lifetime consumption and final wealth. By clarifying the work of Lioui and Poncet (2001), Rodriguez (2002) and Munk and Sorensen (2004), an appropriate change of probability measure, specific to the CRRA utility function, is shown to be of key importance not only because it makes easier the resolution to the maximization problem, but notably because it helps to gain an insight into the intuition behind both the allocation problem itself and the main results of this paper. The investor's consumption-wealth problem reduces to the computation, under this measure, of an investor's specific expectation involving the market prices of risk and the interest rates risk. In a complete market framework, this expectation is unique and reveals how essential the stochastic prices of risk are for the derivation of the investor's optimal demand. Also, consistent with prior studies, the role played by the logarithmic utility separating the investors' hedging position according to their risk aversion appears in a natural way.

Although the optimal unconstrained investor's demand exhibits a classical structure in the sense that it is composed of a speculative part and of a hedging term, a thorough study of these components reveals, however, some appealing and distinctive features of our model. This can be accomplished by introducing into the economic framework two synthetic assets replicating the idiosyncratic sources of risk of both the interest rate and the convenience yield. They allow to enrich the analysis of optimal demands by going beyond the existing studies. First, it is worth pointing out that stochastic prices of risk induce stochastic speculative components that can be decomposed, by using the synthetic assets, in three terms corresponding to the three assets. Thus, as the three factors vary randomly over time, the agent will consequently change his (her) speculative position. This is in sharp contrast to the majority of the models focusing on hedging with futures where the speculative element is only modified by the passage of time.

Second, the hedging term can be split into two parts. The first, due solely to the random fluctuation of the interest rate, involves the covariance of the discount bond with a bond with a maturity equal to the investor's horizon (see Lioui and Poncet, 2001; Munk and Sorensen, 2004). More importantly, the presence of the second term results from the stochastic character of the prices of risk and underscores that of the convenience yield. Making the most of the replicable assets, we show that random prices of risk result in three Merton-Breeden-like components for each and every state variable.

Third, the effect of the prices of risk on the investor's demand is more subtle than the one reported in the literature, which usually considers a unique mean-reverting stochastic price of risk. In this paper, by distinguishing the prices of risk related to the state variables, we explicitly account for their relations. These relations determine the investor's position, short or long, which may be reversed as the level of the state variables is modified. A numerical illustration also shows, for instance, that the interaction between the prices of risk allows to derive a critical value of the sate variables at which speculative demands vanish whatever the investor's risk preferences.

Fourth, our analysis clarifies the role played by the primitive assets and the futures contract. Breeden (1984) studied the allocational role of futures markets and derived the demand for futures contracts by an unconstrained investor when the futures contracts are written on the state variables and have instantaneous maturity. As a consequence, the primitive assets are ineffective in hedging the risk of the state variables. Our analysis calls into question this result by assigning these assets a specific task: hedging the idiosyncratic risk of the spot commodity and the short rate. Besides, the idiosyncratic risk associated with the convenience yield is uniquely hedged by the futures contract.

Fifth, despite their differences, the speculative and the hedging term related to the prices of risk have two common characteristics. On the one hand, they may be computed in a recursive way. The demand of futures contracts is first derived and then used to calculate that of bonds. In turn, both serve as ingredients to obtain the optimal proportion in spot commodities. On the other hand, given the additive structure of these components, the investor is able to precisely assess their influence on his (her) optimal demand. (S)he can therefore rule on the relevance of the investment opportunity set. Indeed, the formulas derived in this paper constitute a useful and alternative means in choosing the most important factors when the investor seeks to allocate his (her) wealth among traded assets including commodities.

The remainder of the paper is organized as follows. In section 2, the economic framework is described and the investor's optimization problem is formulated. Section 3 is devoted to the derivation of the optimal asset allocation for the unconstrained investor. An illustration of the behavior of this demand, via a numerical example, is given in section 4. Section 5 offers some concluding remarks and suggests some potential future extensions. All the proofs have been gathered in the Appendix.

2. The general economic framework

Consider a continuous-time frictionless economy. The uncertainty in the economy is represented by a complete probability space (Ω, *F*, *P*) with a standard filtration $F = \{F_t : t \in [0,T]\}$, a finite time period [0, *T*], the historical probability measure *P* and a 3-dimensional vector of correlated standard Wiener processes, $Z(t)^{'} = (z_S(t), z_f(t), z_\delta(t))$, the correlated basis, defined on (Ω, F), where ′ stands for the transpose. Since these processes are correlated, as will become clear later, it is useful to operate on an orthogonal change of basis and to define a 3-dimensional vector of independent standard Brownian motions, $z(t)^{'} = (z_S(t), z_u(t), z_v(t))$, the orthogonal basis. These two vectors of Brownian motions are related through the following expression: $dZ(t) = \rho dz(t)$, where ρ is a correlation matrix that will be defined below[7].

In this section, following Schwartz (1997), Hilliard and Reis (1998) and Casassus and Collin-Dufresne (2005), three imperfectly correlated factors are assumed to be associated with the dynamics of the futures prices: the logarithm of spot commodity price, *X(t) = Ln (S(t))*, the instantaneous riskless interest rate, *r(t)*, and the instantaneous convenience yield *δ(t)*. In the sequel of the paper, $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, with $i \neq j$, represents the covariance, while ρ_{ij} denotes the correlation coefficient. σ_i, is the strictly positive instantaneous volatility of the state variables for $i = X(t), r(t), \delta(t)$. $Y(t)^{'} = [X(t) \quad r(t) \quad \delta(t)]$ stands for the vector of the state variables that describes the economy.

Since the specification of the process followed by the state variables is now well-known, we will directly give the stochastic differential equation (SDE hereafter) satisfied by the vector *Y(t)*, all the more so as in what follows we will use equations in a matrix form. However, for readers' convenience and to shed light on the intuition underlying our main results, the stochastic evolution of *Y(t)* will be

[7] The standard Cholevsky decomposition establishes the link between the correlated and the orthogonal basis.

expressed in terms of the correlated as well as of the independent Brownian motions. *Y(t)* follows the SDE:

$$dY(t) = [\bar{\mu}(t) - \bar{\mu}_Y Y(t)]dt + \bar{\sigma}_Y dZ(t) \quad (1)$$

$$dY(t) = [\bar{\mu}(t) - \bar{\mu}_Y Y(t)]dt + \sigma_Y dz(t) \quad (1')$$

with the initial condition *Y(0) = Y*. $\bar{\sigma}_Y = \begin{pmatrix} \sigma_S & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_\delta \end{pmatrix}$ and $\sigma_Y = \begin{pmatrix} \sigma_S & 0 & 0 \\ \rho_{Sr}\sigma_r & \rho_{ur}\sigma_r & 0 \\ \rho_{S\delta}\sigma_\delta & \rho_{u\delta}\sigma_\delta & \rho_{v\delta}\sigma_\delta \end{pmatrix}$ are 3-dimensional diffusion matrices under the correlated and the orthogonal basis respectively. $\rho_{ur} = \sqrt{1-\rho_{Sr}^2}$, $\rho_{u\delta} = \frac{\rho_{r\delta} - \rho_{Sr}\rho_{S\delta}}{\sqrt{1-\rho_{Sr}^2}}$ and $\rho_{v\delta} = \frac{\sqrt{1-\rho_{Sr}^2 - \rho_{S\delta}^2 - \rho_{r\delta}^2 + 2\rho_{Sr}\rho_{S\delta}\rho_{r\delta}}}{\sqrt{1-\rho_{Sr}^2}}$. Moreover, let $\bar{\sigma}_X' = [\sigma_S \quad 0 \quad 0]$, $\bar{\sigma}_r' = [\rho_{sr}\sigma_r \quad \rho_{ur}\sigma_r \quad 0]$ and $\bar{\sigma}_\delta' = [\rho_{S\delta}\sigma_\delta \quad \rho_{u\delta}\sigma_\delta \quad \rho_{v\delta}\sigma_\delta]$ denote the 3-dimensional diffusion vectors of the state variables under the orthogonal basis. Since the prices of risk are stochastic, the drift parameters $\bar{\mu}(t)$ and $\bar{\mu}_Y$ are defined below.

In contrast to the majority of the models dealing with dynamic asset allocation and hedging, the market prices of risk associated with the state variables are not constant but stochastic and depend on the levels of the state variables. To allow for an analytical tractability of our model, we opt for an affine specification of these prices of risk. To characterise the dependence of the spot price on the level of inventories (see, for instance, Brennan, 1958; Dincerler et al. 2005), the price of risk associated with the (log) of the spot price process is an affine function of the level of both the (log) of the spot price and the convenience yield: $\lambda_X(S(t), \delta(t)) = \lambda_{X0} + \lambda_{XX}X(t) + \lambda_{X\delta}\delta(t)$. The prices of risk related to the interest rate and the convenience yield are also affine functions: $\lambda_r(r(t)) = \lambda_{r0} + \lambda_{rr}r(t)$ and $\lambda_\delta(\delta(t)) = \lambda_{\delta 0} + \lambda_{\delta\delta}\delta(t)$. $\lambda_{X0}, \lambda_{XX}, \lambda_{X\delta}, \lambda_{r0}, \lambda_{rr}, \lambda_{\delta 0}$ and $\lambda_{\delta\delta}$ are constants. *λ(t)* is a stochastic vector of the market prices of risk under the orthogonal basis whose expression can be couched in terms of the stochastic prices of risk, *Λ(t)*, associated with the correlated Wiener processes:

$$\lambda(t)=\rho^{-1}\Lambda(t)\Rightarrow\begin{pmatrix}\lambda_X(t)\\ \lambda_u(t)\\ \lambda_v(t)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ \rho_{Sr} & \rho_{ur} & 0\\ \rho_{S\delta} & \rho_{u\delta} & \rho_{v\delta}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_X(t)\\ \lambda_r(t)\\ \lambda_\delta(t)\end{pmatrix}=\begin{pmatrix}\lambda_X(t)\\ -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_X(t)+\frac{1}{\rho_{ur}}\lambda_r(t)\\ \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_X(t)-\frac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_r(t)+\frac{1}{\rho_{v\delta}}\lambda_\delta(t)\end{pmatrix}$$

$$\lambda(t)=\lambda_0+\lambda_Y Y(t) \tag{2}$$

where λ_0 and λ_Y are given in Appendix A.

We are now able to give the expressions of the drift parameters of the SDEs (1) and (1'). $\bar{\mu}(t)=\begin{bmatrix}\sigma_S\lambda_{X0}-\frac{1}{2}\sigma_S^2\\ \vartheta(t)\\ k\bar{\delta}\end{bmatrix}$ and $\bar{\mu}_Y=\begin{bmatrix}-\sigma_S\lambda_{XX} & -1 & 1-\sigma_S\lambda_{X\delta}\\ 0 & \alpha-\sigma_r\lambda_{rr} & 0\\ 0 & 0 & k\end{bmatrix}$. The short rate and the convenience yield behave stochastically over time following mean-reverting processes. In particular, the drift in the stochastic process of the short rate is a deterministic function, $\vartheta(t)$, such that the model incorporates all the information present in the current term structure (see Hull and White, 1990; Heath et al., 1992). $\vartheta(t)=\alpha f(0,t)+\frac{\partial f(0,t)}{\partial t}+\sigma_r^2 D_{2\alpha}(t)+\sigma_r\lambda_{r0}$, where *f(0,t)* describes the initial forward yield curve.

In addition to the spot commodity, there are in the economy a locally riskless asset, the savings account, such that: $\beta(t)=\exp\left\{\int_0^t r(s)ds\right\}$, with initial condition $\beta(0)=1$, and two risky traded assets. The first risky security is a discount bond with maturity T_B, whose price, at time *t*, $0\le t\le T_B$, is $B(r,t,T_B)\equiv B(t,T_B)$. The second additional risky asset is a futures contract written on a commodity with maturity T_H, whose price, at date *t*, $0\le t\le T_H\le T_B$, is denoted $H(Y(t),t,T_H)\equiv H(t,T_H)$. The Feynman-Kac representation allows us to find a closed form solution for the discount bond and the futures price: $B(t,T_B)=\exp\{-r(t)D_\alpha(t,T_B)+C(t,T_B)\}$ and $H(t,T_H)=\exp\{X(t)-\delta(t)D_\kappa(t,T_H)+r(t)D_\alpha(t,T_H)+K(t,T_H)\}$ respectively[8] with the terminal

[8] There is no need to specify the expression of *C(t,T_B)* and *K(t,T_H)*, since it will not be used in the rest of the paper; besides we are not interested in the pricing of bonds and of futures contracts.

condition $D_\alpha(T_B,T_B)=D_\alpha(T_H,T_H)=D_\kappa(T_H,T_H)=C(T_B,T_B)=K(T_H,T_H)=0$, where $D_x(t,y)=\frac{1-e^{-x(y-t)}}{x}$.

Assuming that the risky securities price functions are twice continuously differentiable in the state variables, their price dynamics can be written, in the orthogonal basis, as follows:

$$\begin{pmatrix} \frac{dS(t)}{S(t)} \\ \frac{dB(t,T_B)}{B(t,T_B)} \\ \frac{dH(t,T_H)}{H(t,T_H)} \end{pmatrix} = \begin{pmatrix} \mu_S(t) \\ \mu_B(t,T_B) \\ \mu_H(t,T_H) \end{pmatrix} dt + \begin{pmatrix} \sigma_S & 0 & 0 \\ \rho_{Sr}\sigma(t,T_B) & \rho_{ur}\sigma(t,T_B) & 0 \\ \sigma_{HS}(t,T_H) & \sigma_{Hu}(t,T_H) & \sigma_{Hv}(t,T_H) \end{pmatrix} \begin{pmatrix} dz_S(t) \\ dz_u(t) \\ dz_v(t) \end{pmatrix} \tag{3}$$

$$dV(t)=I_V(t)\left[\mu(t)dt+\sigma dz(t)\right]$$

with the initial condition *V(0) = V*. $V(t)=\left[S(t) \quad B(t) \quad H(t)\right]'$, $I_V(t)$ is a diagonal matrix, σ is the 3-dimensional volatility matrix, which is of full rank, hence the market is dynamically complete. $\sigma_{HS}(t,T_H)=\sigma_S+\rho_{Sr}\sigma(t,T_H)-\rho_{S\delta}\sigma_\delta D_k(t,T_H)$, $\sigma_{Hu}(t,T_H)=\rho_{ur}\sigma(t,T_H)-\rho_{u\delta}\sigma_\delta D_k(t,T_H)$ and $\sigma_{Hv}(t,T_H)=-\sigma_\delta\rho_{v\delta}D_k(t,T_H)$. $\sigma(t,T_B)$, the volatility of the discount bond is supposed to be deterministic and is restricted to the exponential case: $\sigma(t,T_B)=\sigma_r D_\alpha(t,T_B)$. $\sigma_B(t,T_B)'=\left[\rho_{sr}\,\sigma(t,T_B) \quad \rho_{ur}\sigma(t,T_B) \quad 0\right]$ and $\sigma_H(t,T_H)'=\left[\sigma_{HS}(t,T_H) \quad \sigma_{Hu}(t,T_H) \quad \sigma_{Hv}(t,T_H)\right]$ are the diffusion vectors of the discount bond and the futures price respectively.

Since we are interested in futures contracts, the futures price changes are credited to or debited from a margin account with interest at the continuously compounded interest rate *r(t)*. The futures contract is indeed assumed to be marked to market continuously rather than on a daily basis, and then to have always a zero current value. The current value of the margin account, *M(t)*, is then equal to:

$$M(t)=\int_0^t \exp\left\{\int_u^t r(v)dv\right\}\theta_H(u,T_H)dH(u,T_H)$$

Applying Itô's lemma to the above equation yields:

$$dM(t)=r(t)M(t)dt+\theta_H(t,T_H)dH(t,T_H) \tag{4}$$

where $\theta_H(t,T_H)$ represents the number of the futures contracts held at time *t*.

The unconstrained investor has an investment horizon T_I, $0\le t\le T_I\le T_H\le T_B$, and (s)he is supposed to have a utility function that exhibits constant relative risk aversion equal to γ, such that:

$$U(c,W(T_I))=\int_t^{T_I}\frac{c(s)^{1-\gamma}}{1-\gamma}ds+\frac{W(T_I)^{1-\gamma}}{1-\gamma} \tag{5}$$

where *U(.)* is a Von Neumann-Morgenstern utility function, $c(t) \geq 0$ and $W(T_I)$ represent consumption at time *t* and the agent's terminal wealth respectively. When $\gamma = 1$, the "reference" utility in the finance literature is obtained, that is, the logarithmic utility function characterizing a Bernoulli investor: $U(c,W(T_I))=\int_t^{T_I} Ln(c(s))ds + Ln(W(T_I))$. In this case, the investor behaves myopically in such a way that his (her) hedging demand will not include any component associated with a stochastic opportunity set.

To determine the optimal consumption and asset allocation (commodity, bond and futures contract), each investor maximizes the expected utility function of his (her) lifetime consumption and terminal wealth. The market described above is dynamically complete, since the number of sources of risk (Brownian motions) is equal to that of the traded risky securities. Karatzas et al. (1987) and Cox and Huang (1989; 1991) used the martingale approach to study the consumption-portfolio problem in a continuous-time setting. Their main idea is to transform this dynamic problem into the following static one (program Π):

$$\max_{\{c,W(T_I)\}} \mathrm{E}\left[\int_t^{T_I}\frac{c(s)^{1-\gamma}}{1-\gamma}ds+\frac{W(T_I)^{1-\gamma}}{1-\gamma}\Big|F_t\right] \tag{6}$$

$$\text{s.t. } \frac{W(t)}{G(t)}=\mathrm{E}\left[\int_t^{T}\frac{c(s)}{G(s)}ds+\frac{W(T_I)}{G(T_I)}\Big|F_t\right]$$

where $\mathrm{E}[\cdot|F_t]\equiv E_t[.]$ denotes the expectation, under *P*, conditional on the information, F_t, available at time *t* and $G(t)=\frac{\beta(t)}{\xi(t)}=\exp\left\{\int_0^t r(u)ds+\frac{1}{2}\int_0^t\|\lambda(u)\|^2+\int_0^t\lambda(u)'dz(u)\right\}$, with $G(0) = 1$, represents the numéraire or optimal growth portfolio such that the value of any admissible portfolio relative to this numéraire is a martingale under *P* (see Long, 1990; Merton, 1990; Bajeux-Besnainou and Portait, 1997). $\| \ \|$ stands for the norm in R^3 and $\xi(t)$ is the Radon-Nikodym derivative of the so-called, unique, risk-neutral probability measure Q equivalent to the historical probability P , such that the relative price (with respect to the savings account chosen as numéraire), of any risky security is a Q-martingale (see Harrison and Pliska, 1981).

3. Optimal dynamic strategies

Having described the economic framework, we will examine the optimal consumption and portfolio strategy problem for our unconstrained investor when the financial market is dynamically complete.

Given the CRRA utility function and the numéraire portfolio $G(t)$, the solution to the static problem (6), which is a standard Lagrangian optimization problem, determines the investor's optimal consumption and wealth at time t:

$$c(t)^* = \frac{W(t)^*}{\Phi(\gamma, t, T_I)} \tag{7}$$

$$W(t)^* = \zeta^{-\frac{1}{\gamma}} G(t)^{\frac{1}{\gamma}} \Phi(\gamma, t, T_I) \tag{8}$$

where ζ is the Lagrangian associated with the static program and $\Phi(\gamma, t, T_I) = \int_t^{T_I} E_t\left[\left(\frac{G(t)}{G(s)}\right)^{1-\frac{1}{\gamma}}\right] ds + E_t\left[\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}}\right]$ is the wealth-to-consumption ratio.

It follows from expressions (7) and (8) that the computation of $c(t)^*$ and $W(t)^*$ involves that of $\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}}$. To make this calculation easier and to underline the intuition as well as the financial interpretation of our results, this term may be rewritten:

$$\left(\frac{G(t)}{G(T_I)}\right)^{1-\frac{1}{\gamma}} = B(t,T_I)^{1-\frac{1}{\gamma}} \left(\frac{\frac{B(T_I,T_I)}{G(T_I)}}{\frac{B(t,T_I)}{G(t)}}\right)^{1-\frac{1}{\gamma}} = B(t,T_I)^{1-\frac{1}{\gamma}} \left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}$$

where $B(t,T_I) = B(0,T_I)\exp\left\{\int_0^t \left[r(u) + \lambda(u)'\sigma_B(u,T_I) - \frac{1}{2}\sigma_B(u,T_I)'\sigma_B(u,T_I)\right]du + \int_0^t \sigma_B(u,T_I)'dz(u)\right\}$

is the price of a discount bond of maturity T_I and $R(t,T_I)=B(t,T_I)/G(t)$.

The optimal wealth may be rewritten in the following convenient way:

$$W(t)^* = \zeta^{-\frac{1}{\gamma}} G(t)^{\frac{1}{\gamma}} \left[\int_t^{T_I} B(t,s)^{1-\frac{1}{\gamma}} E_t\left[\left(\frac{R(s,s)}{R(t,s)} \right)^{1-\frac{1}{\gamma}} \right] ds + B(t,T_I)^{1-\frac{1}{\gamma}} E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)} \right)^{1-\frac{1}{\gamma}} \right] \right] \quad (9)$$

The resolution of the expectation in expression (9) may be simplified by making an appropriate change of numéraire. As shown by Lioui and Poncet (2001) and by Munk and Sorensen (2004), a zero-coupon bond, $B(t,T_I)$, whose maturity, T_I, coincides with that of the investor's horizon, is a useful numéraire. We take a step forward by operating a change of probability measure that is specific to CRRA utility functions. To obtain optimal demands, Rodriguez (2002) uses a change of probability measure related to a CRRA utility function, but in this paper, we attempt to clarify this change of measure. $R(t,T_I)$ is the relative price of this discount bond with respect to the numéraire *G(t)*. Note that $R(t,T_I)$ is a martingale under the probability measure *P*. In contrast, $R(t,T_I)^{1-\frac{1}{\gamma}}$, for $\gamma<\infty$, is neither a financial asset nor a martingale under *P*. To see this, applying Ito's lemma to $R(t,T_I)^{1-\frac{1}{\gamma}}$ gives:

$$\frac{dR(t,T_I)^{1-\frac{1}{\gamma}}}{R(t,T_I)^{1-\frac{1}{\gamma}}} = -\frac{\gamma-1}{2\gamma^2}\left[\lambda(t)-\sigma_B(t,T_I)\right]'\left[\lambda(t)-\sigma_B(t,T_I)\right]dt - \left(\frac{\gamma-1}{\gamma}\right)\left[\lambda(t)-\sigma_B(t,T_I)\right]' dz(t) \quad (10)$$

$$\begin{aligned} R(t,T_I)^{1-\frac{1}{\gamma}} &= B(0,T_I)^{1-\frac{1}{\gamma}} \exp\left\{ -\frac{\gamma-1}{2\gamma^2} \int_0^t \left\| \lambda(u)-\sigma_B(u,T_I) \right\|^2 du \right\} \\ &\quad \exp\left\{ -\frac{(1-\gamma)^2}{2\gamma^2} \int_0^t \left\| \lambda(u)-\sigma_B(u,T_I) \right\|^2 du - \frac{1-\gamma}{\gamma} \int_0^t \left[\lambda(u)-\sigma_B(u,T_I)\right]' dz(u) \right\} \end{aligned} \quad (10')$$

Remark that both the instantaneous expected changes and the variance of $R(t,T_I)^{1-\frac{1}{\gamma}}$ reflect the agent's risk aversion, as well as the risk associated with both the optimal growth portfolio and the discount bond $B(t,T_I)$. However, these two moments are not equal - a feature that turns out to be important for the change of the probability measure and the derivation of the agent's optimal demand.

Let us define:

$$\Delta_\gamma(t,T_I) = \exp\left\{\int_0^t \frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du\right\} \equiv \exp\left\{\int_0^t y_\gamma(u,T_I)du\right\}$$

with the initial condition $\Delta_\gamma(0,T_I)=1$. A simple inspection of equation (10) reveals that minus $y_\gamma(t,T_I)$ is nothing other than the instantaneous expected return of $R(t,T_I)^{1-\frac{1}{\gamma}}$. $\Delta_\gamma(t,T_I)$ is an adjustment factor arising from the fact that $R(t,T_I)^{1-\frac{1}{\gamma}}$ is not a martingale. When $\gamma\to\infty$, $\Delta_\gamma(t,T_I)=1$, which is a special case of the CRRA utility functions. Now $\overline{R}(t,T_I)=B(0,T_I)^{-\left(1-\frac{1}{\gamma}\right)}\Delta_\gamma(t,T_I)R(t,T_I)^{1-\frac{1}{\gamma}}$, which corresponds to the second exponential in the right hand side of equation (10'), is a martingale under *P*. Since $\overline{R}(t,T_I)>0$ and $E\left[\overline{R}(T_I,T_I)\right]=1$, $\overline{R}(t,T_I)$ is a potential candidate as the Radom-Nikodym derivative for a change of the probability measure in our specific case. The objective is to find a non-dividend-paying financial asset as numéraire associated with this probability measure. We suggest the following numéraire $N(t,T_I)=\overline{R}(t,T_I)G(t)$, such that any financial asset divided $N(t,T_I)$ is a martingale under this new probability. The dynamics of this numéraire are governed by:

$$\frac{dN(t,T_I)}{N(t,T_I)} = \left[r(t)+\lambda(t)'\left[\frac{1}{\gamma}\lambda(t)+\left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I)\right]'\right]dt + \left[\frac{1}{\gamma}\lambda(t)+\left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I)\right]' dz(t) \quad (11)$$

Following Geman et al. (1995), the Radon-Nikodym derivative, defining the probability measure $P^{(\gamma,T_I)}$ equivalent to *P*, is given by:

$$\xi_\gamma(t,T_I) \equiv \left.\frac{dP^{(\gamma,T_I)}}{dP}\right|_{F_t} = \overline{R}(t,T_I)$$

$$= \exp\left\{-\frac{(\gamma-1)^2}{2\gamma^2}\int_0^t \left\|\lambda(u)-\sigma_B(t,T_I)\right\|^2 du - \frac{\gamma-1}{\gamma}\int_0^t \left[\lambda(u)-\sigma_B(u,T_I)\right]' dz(u)\right\} \quad (11')$$

It follows that :

$$E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}\right] = E_t\left[\frac{\Delta_\gamma(t,T_I)}{\Delta_\gamma(T_I,T_I)}\frac{\xi_\gamma(T_I,T_I)}{\xi_\gamma(t,T_I)}\right] = E_t\left[\exp\left\{-\frac{\gamma-1}{2\gamma^2}\int_t^{T_I}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du\right\}\right.$$

$$\left.\exp\left\{-\frac{(1-\gamma)^2}{2\gamma^2}\int_t^{T_I}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2 du - \frac{1-\gamma}{\gamma}\int_t^{T_I}\left[\lambda(u)-\sigma_B(u,T_I)\right]' dz(u)\right\}\right]$$

We can use Bayes' rule to get:

$$E_t\left[\left(\frac{R(T_I,T_I)}{R(t,T_I)}\right)^{1-\frac{1}{\gamma}}\right]=E_t^{\left(P^{\gamma,T_I}\right)}\left[\frac{\Delta_\gamma(t,T_I)}{\Delta_\gamma(T_I,T_I)}\right]=E_t^{\left(P^{\gamma,T_I}\right)}\left[\exp\left\{-\int_t^{T_I}\frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,T_I)\right\|^2du\right\}\right]$$

$$=E_t^{\left(P^{\gamma,T_I}\right)}\left[\exp\left\{-\int_t^{T_I}y_\gamma(u)du\right\}\right]=B_\gamma(Y(t),t,T_I)\equiv B_\gamma(t,T_I) \tag{12}$$

The same procedure may be used to compute:

$$\int_t^{T_I}E_t\left[\left(\frac{R(s,s)}{R(t,s)}\right)^{1-\frac{1}{\gamma}}\right]ds=\int_t^{T_I}E_t\left[\frac{\Delta_\gamma(t,s)}{\Delta_\gamma(s,s)}\frac{\xi_\gamma(s,s)}{\xi_\gamma(t,s)}\right]ds=\int_t^{T_I}E_t^{\left(P^{\gamma,s}\right)}\left[\frac{\Delta_\gamma(t,s)}{\Delta_\gamma(s,s)}\right]ds$$

$$=\int_t^{T_I}E_t^{\left(P^{\gamma,s}\right)}\left[\exp\left\{-\int_t^{s}\frac{\gamma-1}{2\gamma^2}\left\|\lambda(u)-\sigma_B(u,s)\right\|^2du\right\}\right]ds$$
$$=\int_t^{T_I}B_\gamma(t,s)ds \tag{12'}$$

Inspired by the relevant literature of the term structure of interest rates (see Duffie and Kan, 1996; Dai and Singleton, 2000; Ahn et al., 2002), as shown in appendix A, $y_\gamma\left(t,T_I\right)$ is a quadratic function and $B_\gamma(t,T_I)$ may be viewed as an exponential quadratic function of the state variables:

$$y_\gamma(t,T_I)=A_0(\gamma,t,T_I)+A_1(\gamma,t,T_I)Y(t)+\frac{1}{2}Y(t)'A_2(\gamma,t,T_I)Y(t)$$

$$B_\gamma(t,T_I)=\exp\left\{B_0(\gamma,t,T_I)+B_1(\gamma,t,T_I)'Y(t)+\frac{1}{2}Y(t)'B_2(\gamma,t,T_I)Y(t)\right\}$$

with the terminal condition $B_0\left(\gamma,T_I,T_I\right)=B_1\left(\gamma,T_I,T_I\right)=B_2\left(\gamma,T_I,T_I\right)=0$. $A_0\left(t,T_I\right)$, $A_1\left(t,T_I\right)$ and $A_2\left(t,T_I\right)$ are given in Appendix A.

By substituting (12) and (12') in equation (9), the optimal wealth then becomes:

$$W\left(t\right)^*=\zeta^{-\frac{1}{\gamma}}G(t)^{\frac{1}{\gamma}}\Phi(\gamma,t,T_I) \tag{13}$$

$\Phi(\gamma,t,T_I)$ can be rewritten in the following way $\Phi(\gamma,t,T_I)=\left[\int_t^{T_I}\varphi(\gamma,t,s)ds+\varphi(\gamma,t,T_I)\right]$, where

$\varphi(\gamma,t,T_I)=B\left(t,T_I\right)^{1-\frac{1}{\gamma}}B_\gamma\left(t,T_I\right)$ is like an investor's specific zero-coupon bond. $\Phi(\gamma,t,T_I)$ is then analogous to a coupon bond that pays a coupon of one monetary unit and may be referred to as the investor's specific coupon bond[9].

The following important remarks are in order. First, $B_\gamma(t,T_I)$ is stochastic because of the stochastic character of the prices of risk. For a more risk-averse investor than the logarithmic utility agent (*γ > 1*), $y_\gamma(t,T_I)>0$, and $B_\gamma(t,T_I)$ is like a discounting factor. Conversely, when (s)he is less risk-averse than the Bernoulli investor (*γ <1*), $y_\gamma(t,T_I)<0$, and $B_\gamma(t,T_I)$ is comparable to a compounding factor. $y_\gamma(t,T_I)$ may be considered as a state variable incorporating the risk generated by the prices of risk and the yield curve. Also, $B_\gamma(t,T_I)$, which results from the agent's consumption-investment problem solution, is investor specific, since it is a function of his (her) risk aversion coefficient and horizon. Second, the assumption that markets are complete implies the uniqueness of the probability measure $P^{(\gamma,T_I)}$, hence that of the first and second expectations on the right hand side of equation (12). It follows that $B_\gamma(t,T_I)$, $c(t)^*$ and $W(t)^*$ are also uniquely obtained. Finally, under $P^{(\gamma,T_I)}$, the investor's optimization problem consists in calculating $B_\gamma(t,T_I)$. For both the Bernoulli ($\gamma=1$) and the infinitely risk-averse ($\gamma=\infty$) investors, $B_\gamma(t,T_I)=1$, which leads, for these two special cases, to a direct derivation of the optimal consumption and wealth. In general, the solution of $B_\gamma(t,T_I)$, as will be shown below, requires numerical methods.

At any date *t*, the wealth of the investor is composed of $\theta_S(t)$, $\theta_B(t)$ and $\theta_\beta(t)$ units of the spot commodity, the discount bonds and the riskless asset respectively, and the margin account:

$$W(t)=\theta_S(t)S(t)+\theta_B(t)B(t,T_B)+\theta_\beta(t)\beta(t)+M(t)$$

Applying Itô's lemma to the above expression, the dynamics of the unconstrained investor's wealth may be written:

$$\frac{dW(t)}{W(t)}=\left[r(t)+\pi(t)'\sigma\lambda(t)\right]dt+\pi(t)'\sigma dz(t) \tag{14}$$

[9] Obviously, $\Phi(\gamma,t,T_I)$ is not an asset, but as its expression is formally similar to that of a coupon bond, it will be qualified as the investor's coupon bond, although it is a misuse of language.

with the initial condition *W(0)* and $\pi(t)' = \left[\pi_S(t) \quad \pi_B(t) \quad \pi_H(t)\right]$. $\pi_S(t) \equiv \frac{\theta_S(t)S(t)}{W(t)}$, $\pi_B(t) \equiv \frac{\theta_B(t)B(t,T_B)}{W(t)}$ and $\pi_H(t,T_H) \equiv \frac{\theta_H(t)H(t,T_H)}{W(t)}$ denote the proportions of the total wealth invested in the commodity, the discount bond and the futures contract respectively. In order to optimally determine these proportions, the unconstrained investor solves the Program Π. The result obtained is presented in the following proposition.

Proposition 1. *Given the economic framework described above, the optimal demand for risky assets by the unconstrained investor is given by*:

$$\pi(t) = \frac{1}{\gamma}\Sigma^{-1}\sigma\lambda(t) + \left(1-\frac{1}{\gamma}\right)\Sigma^{-1}\sigma\left[\int_t^{T_I}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I)\right] \tag{15}$$

The optimal asset allocation may be decomposed in:

a) a traditional tangent component

$$\pi^{MV}(t) = \begin{pmatrix}\pi_H^{MV}(t)\\ \pi_B^{MV}(t)\\ \pi_S^{MV}(t)\end{pmatrix} = \frac{1}{\gamma}\Sigma^{-1}\sigma\lambda(t) = \frac{1}{\gamma}\Sigma^{-1}\begin{bmatrix}\mu_X(t)-r(t)\\ \mu_B(t)-r(t)\\ \mu_H(t)\end{bmatrix} \tag{16}$$

b) a hedging component related to the stochastic fluctuations of the interest rate

$$\pi^{HIR}(t) = \left(1-\frac{1}{\gamma}\right)\Sigma^{-1}\sigma\left[\int_t^{T_I}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_B(t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_B(t,T_I)\right] \tag{17}$$

c) a hedging component related to the random evolution of the (square) market prices of risk

$$\pi^{HMPR}(t) = \begin{pmatrix}\pi_H^{HMPR}(t)\\ \pi_B^{HMPR}(t)\\ \pi_S^{HMPR}(t)\end{pmatrix} = \Sigma^{-1}\sigma\left[\int_t^{T_I}\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma}(t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma}(t,T_I)\right] \tag{18}$$

where, $\sigma_\varphi(\gamma,t,T_I) = \left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I) + \sigma_{B_\gamma}(t,T_I)$, $\sigma_{B_\gamma}(t,T_I) = \sigma_Y'\left[B_1(\gamma,t,T_I) + B_2(\gamma,t,T_I)Y(t)\right]$, $\Sigma = \sigma\sigma'$ and $\Sigma_Y = \sigma_Y\sigma_Y'$. $B_1(\gamma,t,T_I)$ *and* $B_2(\gamma,t,T_I)$ *are solutions to the following ordinary differential equations (ODEs):*

$$B_{2t}(\gamma,t,T_I)-B_2(\gamma,t,T_I)\overline{\mu}_{Y\gamma}-\overline{\mu}_{Y\gamma}^{'}B_2(\gamma,t,T_I)+B_2(\gamma,t,T_I)\Sigma_Y B_2(\gamma,t,T_I)-A_2(\gamma,t,T_I)=0$$
$$B_{1t}(\gamma,t,T_I)-\overline{\mu}_{Y\gamma}^{'}B_1(\gamma,t,T_I)+\overline{\mu}_{\gamma}(t)^{'}B_2(\gamma,t,T_I)+B_2(\gamma,t,T_I)\Sigma_Y B_1(\gamma,t,T_I)-A_1(\gamma,t,T_I)=0$$

with the terminal condition $B_1(\gamma,T_I,T_I)=B_2(\gamma,T_I,T_I)=0$. $B_{1t}(\gamma,t,T_I)$ and $B_{2t}(\gamma,t,T_I)$ *are the first order derivatives with respect to t. The constant and deterministic functions* $\overline{\mu}_{Y\gamma}$ *and* $\overline{\mu}_{\gamma}(t)$ *are given in Appendix A.*

Proof. See Appendix A.

As shown in Proposition 1, the optimal demand for risky assets (equation 15) can be decomposed into two parts. The first one is the traditional mean-variance speculative portfolio proportional to the investor's risk tolerance (Proposition 2 below is dedicated to this term), whereas the second part is a hedge portfolio. The latter itself contains two components. The first one reveals how the investor should optimally hedge against unfavorable fluctuations of the interest rate. It depends on the covariances between the three traded assets and a discount bond. It is worth pointing out that the latter is not the traded bond with maturity T_B, but a discount bond with an expiration date, T_I, equal to that of the investor's horizon. This should not come as a surprise since the investor's objective is to hedge the fluctuations of his (her) opportunity set up to his (her) investment horizon (see also Lioui and Poncet, 2001; Munk and Sorensen, 2004).

More significantly, the second term arises because the prices of risk are stochastic and serves as a hedge against the risk generated by these prices of risk. It involves the investor's specific coupon bond as well as its standard error, and requires numerical methods to solve the ODEs[10]. The investor's coupon bond turns out to be the suitable instrument to hedge the risk stemming from the prices of risk, since it gathers all the sources of risk in the economy and reflects the agent's risk preferences. This component may be referred to as a Merton-Breeden hedging term in that the coupon bond depends on $B_{\gamma}(t,T_I)$, hence on $y_{\gamma}(t)$, which acts as a substitute for the state variables in the economy. As for the first hedging addend against shifts in interest rates, the risk is also measured by the standard error, but in this case, it is the standard error of the investor's coupon bond which, as a function of $\sigma_{B_{\gamma}}(t,T_I)$,

[10] Liu (2007) shown that the solution to the investor's problem reduces to that of a set of ODEs.

encompasses the volatility of the state variables. Notice that if market prices of risk were assumed to be constant or deterministic, then only interest rate risk would be likely to be hedged by the investor. Thus, in accordance with the results of Merton (1973), investors will not hedge at all those variables of the opportunity set that will not evolve randomly over time. Moreover, expression (18) separates the Merton-Breeden hedging term stemming from the maximization of the investor's utility function of consumption from that coming from the maximization of his (her) final wealth. However, a closer look reveals that these terms are related. Actually, the consumption part is the sum of the terminal wealth component over the agent's investment horizon. Thus, at each date, the investor makes the optimal hedging decision about his (her) consumption relative to his (her) optimal wealth at the same date.

Two special cases are worth mentioning. When the investor has a logarithmic utility function, (s)he behaves myopically, which leads to two standard results: the speculative term is independent of the investor's risk aversion and the hedging component disappears. Generally, the Bernoulli investor does not hedge stochastic variations in the investment opportunity set. In our case, (s)he is not concerned by the risk due to interest rate movements up to his (her) investment horizon, and nor is (s)he by that generated by the prices of risk as functions of the state variables. As expected, the demand of the infinitely risk-averse investor does not include any speculative element. However, the hedging part contains only the component related to interest rates changes, since the Merton-Breeden term vanishes. Then, in that sense, this last is not a "pure" hedging term.

The next propositions and corollary are devoted to a thorough study of the speculative and hedging terms. They try to elucidate the consequences on these terms of the stochastic opportunity set, especially the stochastic convenience yield, to highlight the role played by the traded primitive assets and the futures contract as hedging instruments, and, for practical considerations, to implement these terms in such a way that they depend on the measurable moments of the opportunity set.

To achieve these goals, two assets may be introduced into our analysis whose prices are denoted $B_u(t,T_B)$ and $H_v(t,T_H)$. These assets are assumed to be cash assets, i.e., they are not marked to market, and can be duplicated by a portfolio of four assets, namely the riskless asset, the discount bond with maturity T_B, the spot commodity and the futures contracts. They reflect idiosyncratic risks.

The first asset is associated with the idiosyncratic risk of the interest rate, while the second one is linked to that of the convenience yield. Note that the existing spot commodity spans the risk of $z_S(t)$.

$$\frac{dB_u(t,T_B)}{B_u(t,T_B)} = \mu_{Bu}(t,T_B)dt - \sigma_{Bu}(t,T_B)' dz(t)$$

$$\frac{dH_v(t,T_H)}{H_v(t,T_H)} = \mu_{Hv}(t,T_H)dt + \overline{\sigma}_{Hv}(t,T_H)' dz(t)$$

where $\sigma_{Bu}(t,T_B)' = \begin{bmatrix} 0 & \rho_{ur}\sigma(t,T_B) & 0 \end{bmatrix}$ and $\overline{\sigma}_{Hv}(t,T_H)' = \begin{bmatrix} 0 & 0 & \sigma_{Hv}(t,T_H) \end{bmatrix}$. Since the synthetic assets are cash assets, then $\mu_{Bu}(t,T_B) = r(t) - \sigma_{Bu}(t,T_B)'\lambda(t)$ and $\mu_{Hv}(t,T_H) = r(t) + \overline{\sigma}_{Hv}(t,T_H)'\lambda(t)$.

Equation (16) may further be manipulated to obtain more insightful expressions by introducing the two synthetic assets into our analysis. This leads to the following proposition.

Proposition 2. *The optimal mean-variance proportions can be couched in a recursive way:*

$$\pi_H^{MV}(t) = \frac{1}{\gamma}\frac{\mu_{Hv}(t,T_H) - r(t)}{\sigma_{Hv}^2(t,T_H)} \tag{19}$$

$$\pi_B^{MV}(t) = \frac{1}{\gamma}\left[\frac{\mu_{Bu}(t,T_B) - r(t)}{\sigma_{Bu}(t,T_B)'\sigma_{Bu}(t,T_B)} - \frac{\Sigma_{HB_u}(t,T_H,T_B)}{\sigma_{Bu}(t,T_B)'\sigma_{Bu}(t,T_B)}\pi_H^{MV}(t)\right] \tag{20}$$

$$\pi_S^{MV}(t) = \frac{1}{\gamma}\left[\frac{\mu_S(t) - r(t)}{\sigma_S^2} - \frac{\Sigma_{SB}(t,T_B)}{\sigma_S^2}\pi_B^{MV}(t) - \frac{\Sigma_{HS}(t,T_H)}{\sigma_S^2}\pi_H^{MV}(t)\right] \tag{21}$$

Proof. See Appendix B.

This formulation is useful for computational purposes since speculative demands are expressed in terms of excess returns, volatilities and covariances, and they are calculated in a recursive way: the speculative demand of futures contracts is first derived, which allows one then to determine that of the discount bond and finally the proportion of the spot commodity can be obtained as a function of the other two demands. It follows that the speculative proportions will not only be modified as time passes, but also as the state variables fluctuate stochastically over time. The investor will actively adjust his (her) speculative position as a function of the level of the state variables.

The investor's speculative demand consists of a fund including an element specific to the futures contract and a component proper to the two primitive risky assets. This decomposition sheds light on the crucial role played by the idiosyncratic risks captured by the two replicable assets and the spot commodity. The speculative demand for the futures contract depends on the excess return and the variance of the synthetic asset, $H_v(t,T_H)$. It reflects the investor's anticipations about the specific source of uncertainty of the convenience yield. The futures contract is thus the sole asset that will be used by the investor to form his (her) anticipations about the future evolution of the convenience yield. It follows that the mean-variance portfolio for futures contracts will be the only demand depending uniquely on the price risk of the convenience yield. The speculative demand for the discount bond is a function of the excess return and the variance of the synthetic asset, $B_u(t,T_B)$, which spans the idiosyncratic risk of the interest rate. Because of the correlation of the futures contract with the short rate, $\pi_B^{MV}(t)$ is, however, modified by a second term. This additional term involves the mean-variance portfolio for futures contracts weighted by the usual covariance/variance ratio $\frac{\Sigma_{HB_u}(t,T_H,T_B)}{\sigma_{Bu}(t,T_B)^{'}\sigma_{Bu}(t,T_B)}$. A similar argument applies to the speculative demand for commodities. The excess return of the spot commodity divided by its variance, spanning the idiosyncratic risk of the commodity, is now adjusted by two terms since the spot commodity is correlated with both the futures contract and the discount bond. If the convenience yield were non-stochastic, $\pi_H^{MV}(t)=0$, and then the speculative demand contains only the proportions of the two primitive assets.

The interaction between the three components of the investor's speculative demand can be examined through the covariances between the assets. On the one hand, since $\Sigma_{HB_u}(t,T_H,T_B)$ is supposed to take low real values, it has a weak impact on the investor's speculative position on the spot commodity. As expected, unlike $\pi_S(t)$, the proportion invested in the discount bond is strongly influenced by $\Sigma_{HB_u}(t,T_H,T_B)$, and therefore by the speculative demand of the futures contracts. On the other hand, as the spot commodity and the futures contract are highly positively correlated, the speculative proportion of the commodity will be largely driven by that of the futures contract.

Proposition 3. a) *The optimal hedging proportions spawned by the interest rate are only carried by the discount bond and write:*

$$\pi_H^{HIR}(t) = \pi_S^{HIR}(t) = 0$$

$$\pi_B^{HIR}(t) = \left(1 - \frac{1}{\gamma}\right)\left[\int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \frac{\sigma_B(t,s)}{\sigma_B(t,T_B)} ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \frac{\sigma_B(t,T_I)}{\sigma_B(t,T_B)}\right] \tag{22}$$

b) The optimal hedging proportions generated by the (square) market prices of risk can be expressed in a recursive way:

$$\pi_H^{HMPR}(t) = \frac{1}{\sigma_{Hv}(t)}\left[\int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma v}(t,s) ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma v}(t,T_I)\right] \tag{23'}$$

$$\pi_B^{HMPR}(t) = \frac{1}{\sigma_{Bu}(t)}\left[\int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma u}(t,s) ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma u}(t,T_I)\right] - \frac{\Sigma_{HB_u}(t)}{\sigma_{Bu}(t)^2} \pi_H^{HMPR}(t) \tag{23''}$$

$$\pi_S^{HMPR}(t) = \frac{1}{\sigma_S}\left[\int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma S}(t,s) ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \sigma_{B_\gamma S}(t,T_I)\right] - \frac{\Sigma_{SB}(t)}{\sigma_S{}^2} \pi_B^{HMPR}(t) - \frac{\Sigma_{HS}(t)}{\sigma_S{}^2} \pi_H^{HMPR}(t) \tag{23'''}$$

c) The optimal hedging proportions generated by the (square) market prices of risk can be decomposed in the following manner:

$$\pi^{HMPR}(t) = \pi^{HMPR\_X}(t) + \pi^{HMPR\_r}(t) + \pi^{HMPR\_\delta}(t) \tag{24}$$

$$\pi^{HMPR\_i}(t) = \Sigma^{-1}\sigma\overline{\sigma}_i\left[\int_t^{T_I} \left[I_l^{'}B_1(t,s) + I_l^{'}B_2(t,s)Y(t)\right]\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \left[I_l^{'}B_1(t,T_I) + I_l^{'}B_2(t,T_I)Y(t)\right]\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\right] \tag{25}$$

$$\pi^{HMPR\_i}(t) = \Sigma^{-1}\sigma\overline{\sigma}_i\left[\int_t^{T_I} \Psi_{\gamma_i}(t,s)\frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_i}(t,T_I)\frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\right] \tag{25'}$$

where $i \in \{X(t), r(t), \delta(t)\}$, *I is a 3-dimensional identity matrix and* I_l, *l = 1, 2, 3, represent its columns.* $\sigma_{B_\gamma}(t,T_I) = \lfloor \sigma_{B_\gamma S}(t,T_I) \quad \sigma_{B_\gamma u}(t,T_I) \quad \sigma_{B_\gamma v}(t,T_I) \rfloor$, $\Psi_{\gamma_i}(t,T_I) = \frac{B_{\gamma_i}(t,T_I)}{B_\gamma(t,T_I)}$ *and* $B_{\gamma_i}(t,T_I)$ *stands for the first order derivative of* $B_\gamma(t,T_I)$ *with respect to each state variable.*

Proof. See Appendix C.

According to Proposition 3, the hedging demand for the discount bond (equation 23) is the only one including a term that hedges the risk due to the stochastic nature of the interest rate. This component is proportional to the ratio of the volatilities of the bonds with maturities respectively equal to T_I and T_B. When the two maturities coincide, this ratio is equal to one, and the hedging demand is merely a function of the investor's risk aversion. This is also the sole ingredient in the agent's optimal demand evolving deterministically over time. This feature is quite general, in the sense that it is not related to the Gaussian character of the short rate. Insofar as the variance of the interest rate is proportional to its level this characteristic remains valid. This would be the case, for instance, if the short-rate followed a square-root process.

Parts b) and c) of Proposition 1 indicate that the hedging term that stems from the stochastic character of the market prices of risk may admit two different decompositions pursuing two different objectives. The first, inspired by the speculative components (Proposition 2), expresses the hedging terms in a recursive way and establishes a relation between them. The second decomposition, given in expression (24), separates the hedging addend into three Merton-Breeden-like components; one for each and every state variable. In particular, introducing a stochastic convenience yield into the economy results in the presence of a hedging demand, $\pi^{HMPR\_\delta}(t)$, specific to this yield studied in the next corollary. This equation makes it possible to disentangle the hedging element related to each state variable from those associated with the other variables. As a consequence, our model has the ability to exactly measure the impact of these hedging terms on the investor's optimal demand.

In the light of expression (25'), this decomposition appears in a natural way and admits an economic interpretation. The investor wishes to hedge the random shifts in the prices of risk. As discussed above, $B_\gamma(t,T_I)$ incorporates these prices through $y_\gamma(t)$, which involves the state variables. The hedging demands $\pi^{HMPR\_i}(t)$ depend on the ratios $\Psi_{\gamma_i}(t,T_I)=\dfrac{B_{\gamma_i}(t,T_I)}{B_\gamma(t,T_I)}$, which determine the sensitivity of $B_\gamma(t,T_I)$ on the three state variables (see also Wachter, 2002). In other words, each $\Psi_{\gamma_i}(t,T_I)$ assesses the sensitivity of the hedging demands to changes in $B_\gamma(t,T_I)$ resulting from a change in the state variables.

Corollary 1. *Each of the Merton-Breeden-like component can also be decomposed in a recursive way:*

$$\pi^{HMPR\_X}(t) = \pi_S^{HMPR\_X}(t) I_1 = \left[\int_t^{T_I} \Psi_{\gamma_X}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} I_1 ds + \Psi_{\gamma_X}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} I_1 \right] \tag{26}$$

$$\pi^{HMPR\_r}(t) = \pi_S^{HMPR\_r}(t) I_1 + \pi_B^{HMPR\_r}(t) I_2 \tag{27}$$

$$\pi_S^{HMPR\_r}(t) = \frac{\Sigma_{rS}}{\sigma_S^2} \left[\int_t^{T_I} \Psi_{\gamma_r}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_r}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \right] - \frac{\Sigma_{BS}(t,T_B)}{\sigma_S^2} \pi_B^{HMPR\_r}(t) \tag{28}$$

$$\pi_B^{HMPR\_r}(t) = \frac{\Sigma_{rBu}(t,T_B)}{\sigma_{Bu}(t,T_B)' \sigma_{Bu}(t,T_B)} \left[\int_t^{T_I} \Psi_{\gamma_r}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_r}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \right] \tag{29}$$

$$\pi^{HMPR\_\delta}(t) = \pi_S^{HMPR\_\delta}(t) I_1 + \pi_B^{HMPR\_\delta}(t) I_2 + \pi_H^{HMPR\_\delta}(t) I_3 \tag{30}$$

$$\pi_H^{HMPR\_\delta}(t) = \frac{\Sigma_{\delta Hv}(t,T_H)}{\sigma_{Hv}^2(t,T_H)} \left[\int_t^{T_I} \Psi_{\gamma_\delta}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_\delta}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \right] \tag{31}$$

$$\begin{aligned} \pi_B^{HMPR\_\delta}(t) = \frac{\Sigma_{\delta Bu}(t,T_H)}{\sigma_{Hv}^2(t,T_H)} & \left[\int_t^{T_I} \Psi_{\gamma_\delta}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_\delta}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \right] \\ & - \frac{\Sigma_{HB_u}(t,T_H,T_B)}{\sigma_{Bu}(t,T_B)' \sigma_{Bu}(t,T_B)} \pi_H^{HMPR\_\delta}(t) \end{aligned} \tag{32}$$

$$\begin{aligned} \pi_S^{HMPR\_\delta}(t) = \frac{\Sigma_{\delta S}}{\sigma_S^2} & \left[\int_t^{T_I} \Psi_{\gamma_\delta}(t,s) \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)} ds + \Psi_{\gamma_\delta}(t,T_I) \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} \right] \\ & - \frac{\Sigma_{BS}(t,T_B)}{\sigma_S^2} \pi_B^{HMPR\_\delta}(t) - \frac{\Sigma_{HS}(t,T_H)}{\sigma_S^2} \pi_H^{HMPR\_\delta}(t) \end{aligned} \tag{33}$$

Proof. See Appendix D.

A parallel can be drawn between expressions (26) to (33) and equations (24) to (25’). Indeed, the Merton-Breeden hedging components, like the mean-variance proportions, may be computed in a recursive way requiring only the calculation of variances, covariances and of $B_\gamma(t,T_I)$. Furthermore, the covariances between the state variables and the assets which, in conjunction with the partial derivatives of $B_\gamma(t,T_I)$, determine the sign of the Merton-Breeden hedging demands, appear in a simple way facilitating the use of the above expressions.

By virtue of Corollary 1, the stochastic prices of risk will be hedged by both the primitive assets and the futures contract. It states that when the state variables are correlated with the risky securities, a portfolio of the latter will be manufactured to hedge the risk generated from the former. The proportions obtained in Corollary 1 differ markedly from those of Breeden (1984), who, in his study, considers futures contracts with an instantaneous maturity perfectly correlated with the state variables. This particular definition of futures contracts implies that the demand for the primitive assets that serve to hedge state variables vanishes. In contrast, our investor elaborates his (her) strategy by including the primitive assets in order to hedge against the risk of the state variables.

The price risk associated with the spot commodity price will be hedged exclusively by this asset, because it is not correlated with the synthetic assets ($\pi_H^{HMPR\_X}(t) = \pi_B^{HMPR\_X}(t) = 0$). In sharp contrast, from (27) and (30), it is obvious that two and three risky securities are needed to hedge the risk price of the interest rate and the convenience yield respectively. These two state variables are indeed imperfectly correlated with the spot commodity and $B_u(t, T_B)$. Moreover, the convenience yield is imperfectly correlated with $H_v(t, T_H)$. The risk of the convenience is entirely hedged by the futures contract. The discount bond is employed to hedge the idiosyncratic risk of the short rate, while the spot commodity is used to hedge its own risk. The proportions given by (29), (32) and (33) are modified by adjustments terms in the same manner as for the speculative demands.

The refinement achieved by the expressions provided in Corollary 1 presents another major advantage. It allows one to assess the weight of each state variable in the Merton-Breeden hedging terms and therefore to assess the relevance as well as the importance of the state variables included in the investment opportunity set when the investor’s objective is to implement hedging strategies. Actually, given the nature of the underlying commodity, some factors may have a strong or a negligible effect on these hedging elements implying that these factors may have or have not to be included in the opportunity set. Thus, this last may varied according to the nature of the spot commodity to be hedged. Since the speculative and the Merton-Breeden components are both affine functions, the investor has, in addition, the possibility to separate the impact of the state variables on these components and to better understand the overall behavior of his (her) optimal demand.

4. An illustrative example

To get more insights on the impact of the parameters on the model, various simulations are represented in figures 1 to 18. We simulate the reaction of the speculative and the hedging demands to the investor's horizon as well as to the state variables evolution. Table 1 summarizes the values of the parameters used in our simulations.

[INSERT TABLE 1 ABOUT HERE]

The parameter values are partly inspired from Schwartz's (1997) and Casassus and Collin-Dufresne's (2005) models. They are chosen in order to account for some features characterizing commodities. Commodity futures prices are frequently below the current spot price exhibiting backwardation (see Litzenberger and Rabinowitz, 1995), which is equivalent to a positive risk premium and implies a positive convenience yield. Commodity spot prices and convenience yields follow mean-reverting processes (e.g. Bessembinder et al., 1995), as well as the short rate, so that $\alpha > 0$ and $k > 0$. The constant components of the prices of risk are supposed to be positive, while $\lambda_{XX} < 0, \lambda_{X\delta} < 0,$ and $\lambda_{rr} < 0$ inducing also mean-reversion in prices of risk and strengthening that of the state variables (see Cassasus and Collin-Dufresne, 2005). The convenience yield and the spot price are related through inventory decisions (e.g. Routledge et al., 2000). During periods of low inventories, the probability that shortages will occur is greater, and hence the spot price as well as the convenience yield should be high. Conversely, when inventories are abundant, the spot price and the convenience yield tend to be low. It follows that a positive correlation between the convenience yield and the spot price may be predicted. Frankel and Hardouvelis (1985) and Frankel (1986) argued that high real interest rates reduce commodity prices, and vice-versa. This should imply a negative correlation between, on the one hand, interest rates and, on the other hand, spot prices and convenience yields.

To analyze the impact of risk aversion, optimal demands are depicted for four degrees of relative risk aversion (RRA). The first one is that of the investor who is less risk-averse than the Bernoulli one. As pointed out by Kim and Omberg (1996), the indirect utility function may explode for too low values when $\gamma < 1$. To avoid a such a problem, we put 0.7. The second risk aversion parameter is the traditional logarithmic function, $\gamma = 1$, that separates bounded from unbounded utility functions. For a more risk-averse investor than the log-utility investor, we retain a value of $\gamma = 3$ for our simulations. Finally, according to Mehra and Prescott (1985) risk aversion should be much higher than one. To take into account this feature, we choose $\gamma = 6$.

When studying the optimal proportions as a function of the investment horizon, we let this horizon vary in the interval $T_I \in [0, 2]$, and we set the maturity of the futures contract and the bond such as $T_H = T_I + 1/12$ and $T_B = T_I + 5$ respectively. That is the futures contract and the discount bond expire one month and five years respectively after the end of the investor's horizon.

The simulations show that the components of the optimal demand have, despite their differences, some common characteristics confirming the financial intuition and our main theoretical results. First, as a function of the investor's horizon, the mean variance and hedging demands of the spot commodity behave in an opposite way than that of the futures contract. A short (long) position in the spot commodity is partially offset by a long (short) position in the futures contract. Also, opposite patterns are followed by the speculative and hedging terms. Second, the lower the investor's degree of risk-aversion, the higher, in absolute values, the speculative proportions. Third, the Bernoulli investor appears as the dividing line between hedging and "reverse hedging" positions. Moreover, the Merton-Breeden hedging terms vanishe for an investor behaving myopically.

[INSERT FIGURES 1, 2, 3, 4, 5, 6 ABOUT HERE]

Figures 1 trough 6 picture the reaction to the investor's horizon of the optimal demands. To avoid any confusion, a preliminary remark is in order. The speculative demands are independent from the investor's horizon. However, as we let T_H and T_B vary with the investor's horizon, these proportions changes as T_I is modified. A clear distinction can be operated between the mean-variance elements related to the interest rate from those associated with the spot commodity and the futures contract. The latter are non-linear and sharply increase or decrease for a short horizon but they rapidly reach an asymptote for a longer term. This is due to the pattern of the synthetic assets price volatility,

$\sigma_{Hv}(t,T_H)$: it flattens for a long horizon but is highly non-linear when the horizon shrinks. In contrast, the terms relative to the short rate are almost linear. Indeed, as the correlation between the interest rate and both the convenience yield and the commodity is low, these terms are essentially driven by the volatility of the bond. This last slowly varies with the horizon, and, as a consequence, the demand for the bond. The evolution of the Merton-Breeden terms as time passes is more intriguing. In fact, the hedging components stemming from the prices of risk associated with the futures contract and the spot commodity, in particular, are not monotonic in T_I. They attain an optimum for a short horizon. For a longer horizon, they may evolve in a counter-intuitive way. For instance, for a less (more) risk-averse investor, $\pi_H^{HMPR}(t)$ slightly increases (decreases).

[INSERT FIGURES 7, 8, 9, 10, 11, 12 ABOUT HERE]

We turn now to the study of the impact of the changes of the state variables on the speculative demand and of the γ parameter, which are displayed in figures 7 to 13. We set the investor's horizon $T_I = 1$, while the other parameters values remain unchanged. As expected the mean variance components are inversely related to γ and tend to zero as γ goes to infinity. Our numerical simulations show that the interest rate has a weak impact on the investor's demand, except for the proportions specific to this variable. We will then omit the majority of the figures related to the short rate and will mainly focus our analysis on the influence of the spot commodity price and the convenience yield, first, on the speculative demands, and, second, on the hedging terms. To examine the role of the spot commodity, its price ranges from 80 dollars to 120 dollars. For low values of the spot price, the speculative component of the futures contract is positive and decreasing, while for high values it is negative and decreasing. This result may essentially be explained by the prices of risk of both the spot commodity and the convenience yield. On the one hand, as indicated below Proposition 2, this component captures the idiosyncratic risk of the convenience yield and is a function of the price of risk associated with this source of uncertainty. Formally, we have $\lambda_v(t) = -\sigma_\delta D_k(t,T_H)\left[\frac{\rho_{Sr}\rho_{u\delta} - \rho_{S\delta}\rho_{ur}}{\rho_{ur}}\lambda_X(t) - \frac{\rho_{u\delta}}{\rho_{ur}}\lambda_r(t) + \lambda_\delta(t)\right]$. This price in turn depends, for the values used in these simulations, negatively on the price of risk related to the (log)spot price

$(\rho_{Sr}\rho_{u\delta} - \rho_{S\delta}\rho_{ur} < 0)$ and positively to that of the convenience yield $(\rho_{v\delta} > 0)$. On the other hand, mean-reversion in the price of risk of the (log)spot price implies that as the latter raises the former declines. Moreover, since the convenience yield and the futures price are negatively correlated, this component is multiplied by minus. In overall, for low values of the spot price, its price of risk outweighs that of the convenience yield so that the mean-variance term is positive, while for high values of the spot price the inverse holds. An inspection of Figure 7 shows that there exists a critical value of the spot price at which $\lambda_v(t) = 0$, separating positive from negative speculative demands for futures contracts[11]. In other words, the interaction between the prices of risk in conjunction with their mean-reverting behavior determines whether the speculator goes short or long. The speculative proportion of the discount bond evolves contrary to that of the futures contracts (see Figure 9). A critical value of the spot price can be derived at which the expected return of the synthetic bond exactly compensates the speculative position in the futures contracts. Figure 13 performs a similar analysis when the speculative element of the discount bond varies as a function of the interest rate. A critical value of the short rate can also be determined distinguishing long from short positions.

Contrary to the other two mean-variance terms, that of the spot commodity is positive and monotonic increasing in the spot price (see Figure 8). When this last is low, a high, due to mean-reversion, instantaneous expected return of the spot commodity is negatively adjusted by both the speculative position in the futures contract $(\Sigma_{HS}(t,T_H) > 0)$ and in the bond $(\Sigma_{SB}(t,T_B) < 0)$. This results in a low speculative demand of the spot commodity. Conversely, although a high spot price reduces the expected return, a positive mean-variance component of the futures contract and of the bond offset this fall inducing a high speculative demand of the spot commodity.

To underscore the importance of the convenience yield we let it vary between –5% and +15%. The speculative demand of the futures contract is an increasing function of the convenience yield and takes negative values (see Figure 10). Following the same reasoning as for the spot price, notice that

[11] Kim and Omberg (1996) determine a value of the risk premium at which the position in the risky assets is zero (i.e. the speculative term perfectly counterbalances the hedging one), but in a different context: there are not futures contracts in their economy and the risk premium does not explicitly depend on the asset price.

the effect of the price of risk associated with the convenience yield dominates that of the spot commodity and the speculative demand is negative. However, as the convenience yield approaches to zero and becomes positive the difference between the two prices of risk lessens and tends to zero. For sufficiently high values of a positive convenience yield the speculative demand may be positive. The speculative proportion of the spot commodity is positive and decreasing in the convenience yield. Indeed, the joint effect of mean-reversion in the expected return of the spot commodity and the behavior of the speculative demand of the futures contract lead to this result.

Figures 14 to 18 depict the reaction of the Merton-Breeden hedging addends to the changes of the state variables for different values of the γ parameter. As can be seen in Figure 15, a less risk-averse investor (solid line) than the logarithmic one (dashed-dotted line) holds a negative increasing proportion of the spot commodity, while a more risk-averse agent (dotted and dashed lines) holds a positive decreasing proportion. The intuition behind this result may be explained as follows. More risk-averse agents wish to hedge against uncertainty and prefer a higher risk premium than a less risk-averse individual. Thanks to mean-reversion, low spot prices imply a high positive risk premium associated with the spot price and vice versa. Thus, a more risk-averse investor will hold more of the spot commodity than the myopic proportion (an increasing proportion), while a less risk-averse investor will hold less. This is in essence the explanation given by Kim and Omberg (1996), Wachter (2002) and Chacko and Viceira (2005)[12]. In our case, however, this analysis must be qualified. Indeed, unlike these papers, the investor has the possibility to use the discount bond and the futures contract as hedging instruments. $\pi_S^{HMPR}(t)$ is thus affected by the hedging terms of the futures contracts and of the bond. This causes $\pi_S^{HMPR}(t)$ to decrease when $\gamma > 1$ and to increase otherwise. The convenience yield affects $\pi_S^{HMPR}(t)$ in a similar but more moderate way than the spot commodity, since it acts on both the prices of risk associated with the spot commodity and the convenience yield. For those agents who are

[12] Wachter (2002) and Campbell and Viceira (2005) invoke the income and the substitution effects to explain the behavior of the hedging terms. Due to the income effect, an improvement of the investment opportunities (a higher risk premium) results in a higher consumption. It is compensated by the substitution effect which implies that the greater the investment opportunities the higher the savings. For investors with a $\gamma > 1$, the income effect dominates.

committed to a long (short) hedging position in the spot commodity, the position in the futures contract is short (long).

4. Concluding remarks

In this paper, optimal hedging decisions involving commodity futures contracts have been studied in a continuous-time environment (i) for an unconstrained investor with a constant relative risk aversion utility function (CRRA), (ii) when spot prices, interest rates and, especially, the convenience yield evolve randomly over time, and (iii) market prices of risk are stochastic and affine functions of the state variables. In this setting, by using a suitable CRRA specific change of probability measure, we derive the investor's optimal demand, which consists of a speculative component affine in the state variables and two so called hedging terms. The first hedging component is associated with interest rates uncertainty. This term, which vanishes in the case of constant interest rates, involves a discount bond with a maturity equal to the investor's investment horizon. The second one deserves a great attention because it has some interesting properties, partly shared with the speculative element, distinguishing our results from those of other papers. It is composed of affine in the state variables Merton-Breeden hedging terms resulting from the (square) market prices of risk. They underline the role played by the primitive assets and the futures contracts as hedging instruments against the idiosyncratic risk of the state variable, the convenience yield in particular. Both the speculative component and the Merton-Breeden hedging terms can be couched in a recursive way depending on the first two moments of the state variables and on correlation coefficients. The main implication of these properties is that the investor may measure the effect of each state variable on his (her) optimal demand and decide on which of those variables are effectively important when s(he) pursues a hedging objective.

The economic framework of this paper can be extended in several directions. First, the general setting may usefully be adapted to the investor's allocation problem in the case of stocks paying a dividend. Second, a natural extension of this paper is to derive optimal demands for a constrained investor. Third, commodities markets are highly volatile and spot assets exhibit jumps (see, for instance, Hilliard and Reis, 1998; Yan, 2002). The effect of jumps on the optimal asset allocation with commodities remains an open question. Fourth, another observed characteristic distinguishing commodities from financial assets is that commodity prices exhibit seasonal patterns (see Richter and Sorensen, 2006). It would be of great interest to examine how seasonality modifies the investor's hedging behavior. Finally, it is now acknowledged in the relevant literature that the convenience yield is not observable: indeed, in a partially observable economy (see, for instance, Dothan and Feldman 1986; Detemple 1986; Gennotte 1986; Xia 2001) an agent can estimate one or more unobserved state variable(s) given information conveyed by past observations spawned by observable state variables via the continuous-time Kalman-Bucy filter. One important extension would therefore be to study how the incomplete information affects optimal asset allocation.

Appendix. Proofs

Appendix A. Proof of Proposition 1.

By using expression (2) and by operating the appropriate calculations $y_\gamma(t)$ can be expressed as a quadratic function of the state variables.

$$
\begin{aligned}
y_\gamma(t,T_I) &= \frac{\gamma-1}{2\gamma^2}\left[\lambda(t)-\sigma_B(t,T_I)\right]\left[\lambda(t)-\sigma_B(t,T_I)\right] = \frac{\gamma-1}{2\gamma^2}\left[\lambda_0-\sigma_B(t,T_I)+\lambda_Y Y(t)\right]\left[\lambda_0-\sigma_B(t,T_I)+\lambda_Y Y(t)\right] \\
&= \frac{\gamma-1}{2\gamma^2}\left\{\left\|\lambda_0-\sigma_B(t,T_I)\right\|^2+2\left[\lambda_0-\sigma_B(t,T_I)\right]^{'}\lambda_Y Y(t)+Y(t)^{'}\lambda_Y^{'}\lambda_Y Y(t)\right\} \\
&= A_0(\gamma,t,T_I)+A_1(\gamma,t,T_I)Y(t)+\frac{1}{2}Y(t)^{'}A_2(\gamma,t,T_I)Y(t)
\end{aligned}
$$

where $\lambda_0 = \rho^{-1}\begin{pmatrix}\lambda_{X0}\\ \lambda_{r0}\\ \lambda_{\delta 0}\end{pmatrix} = \begin{pmatrix}\lambda_{X0}\\ -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{X0}+\frac{1}{\rho_{ur}}\lambda_{r0}\\ \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{X0}-\frac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_{r0}+\frac{1}{\rho_{v\delta}}\lambda_{\delta 0}\end{pmatrix}$ and

$$
\lambda_Y = \rho^{-1}\begin{pmatrix}\lambda_{XX} & 0 & \lambda_{X\delta}\\ 0 & \lambda_{rr} & 0\\ 0 & 0 & \lambda_{\delta\delta}\end{pmatrix} = \begin{pmatrix}\lambda_{XX} & 0 & \lambda_{X\delta}\\ -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{XX} & \frac{1}{\rho_{ur}}\lambda_{rr} & -\frac{\rho_{Sr}}{\rho_{ur}}\lambda_{X\delta}\\ \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{XX} & -\frac{\rho_{u\delta}}{\rho_{ur}\rho_{v\delta}}\lambda_{rr} & \frac{\rho_{Sr}\rho_{u\delta}-\rho_{S\delta}\rho_{ur}}{\rho_{ur}\rho_{v\delta}}\lambda_{X\delta}+\frac{1}{\rho_{v\delta}}\lambda_{\delta\delta}\end{pmatrix}
$$

The dynamics of the state variables under the probability $P^{(\gamma,T_I)}$ are given by:

$$
dY(t) = \left[\overline{\mu}_\gamma(t)-\overline{\mu}_{Y,\gamma}Y(t)\right]dt+\sigma_Y^{'}dz^{(\gamma,T_I)}(t)
$$

where $\bar{\mu}_{Y\gamma}(t) = \bar{\mu}(t) - \left(1 - \frac{1}{\gamma}\right)\sigma_Y^{'}\left[\lambda_0 - \sigma_B(t, T_I)\right]$, $\bar{\mu}_{Y\gamma} = \bar{\mu}_Y + \left(1 - \frac{1}{\gamma}\right)\sigma_Y^{'}\lambda_Y$ and, by Girsanov's theorem, $dz^{(\gamma,T_I)}(t) = dz(t) + \left(1 - \frac{1}{\gamma}\right)\left[\lambda(t) - \sigma_B(t, T_I)\right]dt$ is a Brownian motion under $P^{(\gamma,T_I)}$.

The Feynman-Kac formula for quadratic processes under the probability $P^{(\gamma,T_I)}$ implies that equation (12) is solved by the partial differential equation (PDE):

$$\frac{1}{2}tr\left[\Sigma_Y B_{\gamma_{YY}}(t, T_I)\right] + \left(\bar{\mu}_\gamma(t) - \bar{\mu}_{Y,\gamma} Y(t)\right)^{'} B_{\gamma_Y}(t, T_I) + B_{\gamma_t}(t, T_I) - y_\gamma(t) B_\gamma(t, T_I) = 0 \tag{A.1}$$

with the terminal condition $B_\gamma(T_I, T_I) = 1$. *tr[.]* denotes the trace of a matrix, and $B_{\gamma_t}(t, T_I)$, $B_{\gamma_Y}(t, T_I)$ and $B_{\gamma_{YY}}(t, T_I)$ represent the first and second order partial derivatives with respect to *t* and *Y(t)* respectively.

We use the standard separation of variables method and consider a discount bond price function of the form:

$$B_\gamma(t, T_I) = \exp\left\{B_0(\gamma, t, T_I) + B_1(\gamma, t, T_I)^{'} Y(t) + \frac{1}{2} Y(t)^{'} B_2(\gamma, t, T_I) Y(t)\right\} \tag{A.2}$$

where $B_0(\gamma, t, T_I)$, $B_1(\gamma, t, T_I)$ and $B_2(\gamma, t, T_I)$ are functions of time to maturity that are assumed to be continuously differentiable. These functions satisfy the terminal condition: $B_0(\gamma, T_I, T_I) = B_1(\gamma, T_I, T_I) = B_2(\gamma, T_I, T_I) = 0$. Differentiating (A.2) with respect to *t* and *Y(t)* yields:

$$B_{\gamma_t}(t, T_I) = B_\gamma(t, T_I)\left[B_{0_t}(\gamma, t, T_I) + B_{1_t}(\gamma, t, T_I)^{'} Y(t) + \frac{1}{2} Y(t)^{'} B_{2_t}(\gamma, t, T_I) Y(t)\right]$$
$$B_{\gamma_Y}(t, T_I) = B_\gamma(t, T_I)\left[B_1(\gamma, t, T_I) + B_2(\gamma, t, T_I) Y(t)\right]$$

$$B_{\gamma_{YY}}(t, T_I) = B_\gamma(t, T_I)\left[B_1(\gamma, t, T_I) B_1(\gamma, t, T_I)^{'} + B_2(\gamma, t, T_I) + B_1(\gamma, t, T_I) Y(t)^{'} B_2(\gamma, t, T_I) + \right.$$
$$\left. B_2(\gamma, t, T_I) Y(t) B_1(\gamma, t, T_I)^{'} + B_2(\gamma, t, T_I) Y(t) Y(t)^{'} B_2(\gamma, t, T_I)\right]$$

Substituting these partial derivatives into the PDE (A.1), collecting terms in *Y(t)*, $Y(t)^2$, and terms independent of the state variables gives the following ordinary differential equations (ODEs) subject to the terminal conditions:

$$B_{2t}(\gamma, t, T_I) - B_2(\gamma, t, T_I)\bar{\mu}_{Y,\gamma} - \bar{\mu}_{Y,\gamma}^{'} B_2(\gamma, t, T_I) + B_2(\gamma, t, T_I)\Sigma_Y B_2(\gamma, t, T_I) - A_2(\gamma, t, T_I) = 0$$

$$B_{1t}(\gamma, t, T_I) - \bar{\mu}_{Y,\gamma}^{'} B_1(\gamma, t, T_I) + \bar{\mu}_\gamma(t)^{'} B_2(\gamma, t, T_I) + B_2(\gamma, t, T_I)\Sigma_Y B_1(\gamma, t, T_I) - A_1(\gamma, t, T_I) = 0$$
$$B_{0t}(\gamma, t, T_I) + \bar{\mu}_\gamma(t)^{'} B_1(\gamma, t, T_I) + \frac{1}{2} B_1(\gamma, t, T_I)^{'} \Sigma_Y B_1(\gamma, t, T_I) + \frac{1}{2} tr\left[\Sigma_Y B_2(\gamma, t, T_I)\right] - A_0(\gamma, t, T_I) = 0$$

By using Itô lemma, $B_\gamma(t, T_I)$ and $\varphi(\gamma, t, T_I)$ follow respectively the SDEs:

$$\frac{dB_\gamma(t,T_I)}{B_\gamma(t,T_I)} = \mu_\gamma(t,T_I)dt + \sigma_{B_\gamma}(t,T_I)^{'} dz(t)$$

$$\frac{d\varphi(\gamma,t,T_I)}{\varphi(\gamma,t,T_I)} = \mu_\varphi(t,T_I)dt + \sigma_\varphi(\gamma,t,T_I)^{'} dz(t)$$

where $\sigma_{B_\gamma}(t,T_I) = \sigma_Y^{'} \frac{B_{\gamma_Y}(t,T_I)}{B_\gamma(t,T_I)} = \sigma_Y^{'}\left[B_1(\gamma,t,T_I) + B_2(\gamma,t,T_I)Y(t)\right]$, $\quad \sigma_\phi(\gamma,t,T_I) = \left(1-\frac{1}{\gamma}\right)\sigma_B(t,T_I) + \sigma_{B_\gamma}(t,T_I)$.

Note that $\mu_\gamma(t,T_I)$ and $\mu_\phi(t,T_I)$ are irrelevant for our allocation problem and will not be specified. By using Leibniz type rule for stochastic integrals (see Munk and Sorensen, 2004), $\Phi(\gamma,t,T_I)$ obeys the following equation:

$$\frac{d\Phi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)} = \mu_\Phi(\gamma,t,T_I)dt + \left[\int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I)\right]^{'} dz(t) \qquad \text{(A.3)}$$

By using Itô's lemma, the instantaneous return of the optimal wealth (13) may be written:

$$\frac{dW(t)^*}{W(t)^*} = \mu_W(t)dt + \left[\frac{\lambda(t)}{\gamma} + \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I)\right]^{'} dz(t) \qquad \text{(A.4)}$$

Identifying the diffusion terms of the admissible wealth (14) and the optimum wealth (A.4) yields:

$$\sigma^{'}\pi = \frac{\lambda(t)}{\gamma} + \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I) \qquad \text{(A.5)}$$

$$\sigma\sigma^{'}\pi = \frac{1}{\gamma}\sigma\left[\frac{\lambda(t)}{\gamma} + \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_\varphi(\gamma,t,T_I)\right] \qquad \text{(A.6)}$$

which leads to equation (15).

Parts a), b) and c) of proposition 1 can directly be obtained from this equation.

Appendix B. Proof of Proposition 2.

The following matrix products give:

$$\Sigma^{-1}\sigma = \begin{pmatrix} \frac{1}{\sigma_S} & -\frac{\rho_{Sr}}{\rho_{ur}\sigma_S} & -\frac{\rho_{ur}\sigma_{HS}(t,T_H)-\rho_{Sr}\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma_S\sigma_{Hv}(t,T_H)} \\ 0 & \frac{1}{\rho_{ur}\sigma(t,T_B)} & -\frac{\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma(t,T_B)\sigma_{Hv}(t,T_H)} \\ 0 & 0 & \frac{1}{\sigma_{Hv}(t,T_H)} \end{pmatrix} \tag{B.1}$$

$$\Sigma^{-1}\sigma\lambda(t) = \begin{pmatrix} \frac{\lambda_s(t)}{\sigma_S} - \frac{\lambda_u(t)\rho_{Sr}}{\rho_{ur}\sigma_S} - \frac{\lambda_v(t)}{\sigma_S}\frac{\rho_{ur}\sigma_{HS}(t,T_H)-\rho_{Sr}\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma_{Hv}(t,T_H)} \\ \frac{\lambda_u(t)}{\rho_{ur}\sigma(t,T_B)} - \frac{\lambda_v(t)\sigma_{Hu}(t,T_H)}{\rho_{ur}\sigma(t,T_B)\sigma_{Hv}(t,T_H)} \\ \frac{\lambda_v(t)}{\sigma_{Hv}(t,T_H)} \end{pmatrix}$$

By using equation (16) and by rearranging terms, we obtain:

$$\pi_{\mathrm{H}}(\mathrm{t}) = \frac{\lambda_v(t)\sigma_{Hv}(t,T_H)}{\sigma^2_{Hv}(t,T_H)} = \frac{\mu_{Hv}(t,T_H)-r(t)}{\sigma^2_{Hv}(t,T_H)} \tag{B.2}$$

$$\pi_{\mathrm{B}}(\mathrm{t}) = \frac{\rho_{ur}\sigma(t,T_B)\lambda_u(t)}{\rho^2_{ur}\sigma(t,T_B)\sigma(t,T_B)'} - \frac{\rho_{ur}\sigma(t,T_B)\sigma_{Hu}(t,T_H)}{\rho^2_{ur}\sigma(t,T_B)\sigma(t,T_B)'}\frac{\lambda_v(t)\sigma_{Hv}(t,T_H)}{\sigma^2_{Hv}(t,T_H)} \tag{B.3}$$

$$\pi_{\mathrm{S}}(\mathrm{t}) = \frac{\lambda_S(t)}{\sigma_S} - \frac{\rho_{uf}\sigma_S\sigma(t,T_B)}{\sigma^2_S}\pi_{\mathrm{B}}(\mathrm{t}) - \frac{\sigma_S\sigma_{HS}(t,T_H)}{\sigma^2_S}\pi_{\mathrm{H}}(\mathrm{t}) \tag{B.4}$$

By replacing the first two moments and the appropriate covariances of the synthetic assets into (B.3) and (B.4), equations (19), (20) and (21), in the main text, are obtained.

Appendix C. Proof of Proposition 3.

a) Expression (22) can easily be derived by operating the computation of $\Sigma^{-1}\sigma\sigma_B(t,T_B)$.

b) $\sigma_{B_\gamma}(t,T_I)$ may be written, in the orthogonal basis, in the following manner:

$$\sigma_{B_\gamma}(t,T_I) = \left[\sigma_{B_\gamma S}(t,T_I) \quad \sigma_{B_\gamma u}(t,T_I) \quad \sigma_{B_\gamma v}(t,T_I)\right]'$$

Equation (18) may thus be expressed as:

$$\pi^{HMPR}(t) = \Sigma^{-1}\sigma \begin{bmatrix} \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma S}(t,T_I) \\ \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma u}(t,T_I) \\ \int_t^{T_I} \frac{\varphi(\gamma,t,s)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,s)ds + \frac{\varphi(\gamma,t,T_I)}{\Phi(\gamma,t,T_I)}\sigma_{B_\gamma v}(t,T_I) \end{bmatrix}$$

Using (B.1) leads to equations (23).

c) Let *I* be a 3-dimensional identity matrix and I_1, I_2, I_3 be its columns. Then, we have

$$\begin{aligned}\sigma_{B_\gamma}(t,T_I) = \sigma_Y^{'}\left[B_1(\gamma,t,T_I) + B_2(\gamma,t,T_I)Y(t)\right] = \overline{\sigma}_X\left[I_1^{'}B_1(\gamma,t,T_I) + I_1^{'}B_2(\gamma,t,T_I)Y(t)\right] + \\ \overline{\sigma}_r\left[I_2^{'}B_1(\gamma,t,T_I) + I_2^{'}B_2(\gamma,t,T_I)Y(t)\right] + \overline{\sigma}_\delta\left[I_3^{'}B_1(\gamma,t,T_I) + I_3^{'}B_2(\gamma,t,T_I)Y(t)\right]\end{aligned} \quad \text{(C.1)}$$

Plugging the above expression into equation (18) and rearranging terms leads to equations (24) and (25).

The first order derivative of $B_\gamma(t,T_I)$ with respect to the state variables can be written: $B_{\gamma_i}(t,T_I) = B_\gamma(t,T_I)\left[I_l^{'}B_1(\gamma,t,T_I) + I_l^{'}B_2(\gamma,t,T_I)Y(t)\right],\ i \in \{X(t), r(t), \delta(t)\}$ and l = *1, 2, 3*. Replacing these derivatives into (C.1) gives:

$$\sigma_{B_\gamma}(t,T_I) = \sigma_Y^{'}\left[B_1(\gamma,t,T_I) + B_2(\gamma,t,T_I)Y(t)\right] = \overline{\sigma}_X \frac{B_{\gamma_X}(t,T_I)}{B_\gamma(t,T_I)} + \overline{\sigma}_r \frac{B_{\gamma_r}(t,T_I)}{B_\gamma(t,T_I)} + \overline{\sigma}_\delta \frac{B_{\gamma_\delta}(t,T_I)}{B_\gamma(t,T_I)} \quad \text{(C.2)}$$

Expressions (C.2) and (25) allow to establish equation (25').

Appendix D. Proof of Corollary 1.

By using (B.1), computing the matrix product $\Sigma^{-1}\sigma\overline{\sigma}_i$, $i \in \{X(t), r(t), \delta(t)\}$ and substituting the result into equation (25) for each state variable. By calculating the appropriate covariances between the state variables and, on the one hand, the traded assets and, on the other hand, the synthetic assets, and by rearranging terms allow one to derive expressions (26) to (31).

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Table 1

Numerical values of the parameters used in the model

r	δ	X	α	κ	$\overline{\delta}$	σ_S	σ_r	σ_δ	*f(0,t)*
0.04	0.07	4.6	0.25	1.5	0.05	0.35	0.01	0.25	0.04
λ_{X0}	λ_{r0}	$\lambda_{\delta 0}$	λ_{XX}	$\lambda_{X\delta}$	λ_{rr}	$\lambda_{\delta\delta}$	ρ_{Sr}	$\rho_{S\delta}$	$\rho_{r\delta}$
12	0.15	0.5	-2.5	-1.5	-6	-2	-0.15	0.7	-0.1

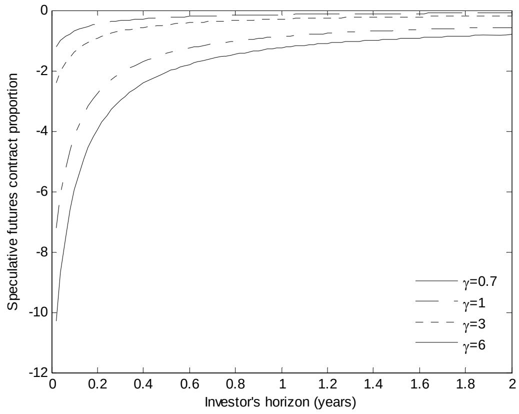

Fig. 1. *Speculative futures proportion varying with the investor's horizon.* This figure plots $\pi_H^{MV}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

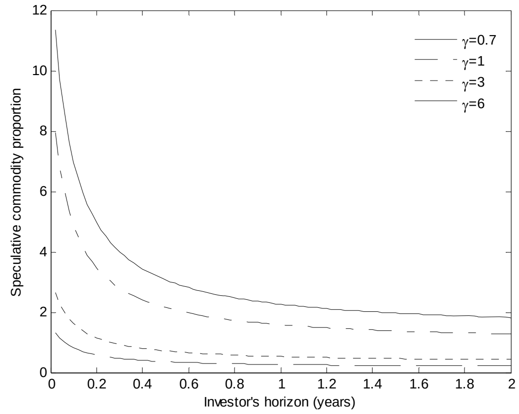

Fig. 2. *Speculative commodity proportion varying with the investor's horizon.* This figure plots $\pi_S^{MV}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

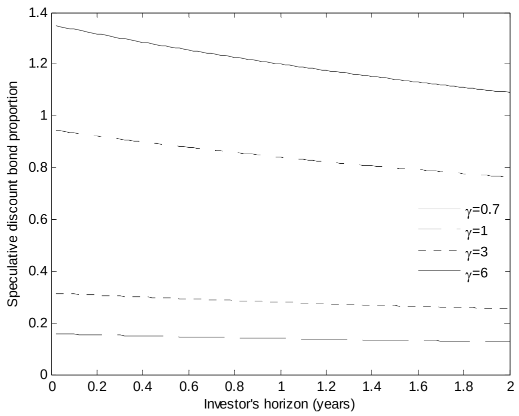

Fig. 3. *Speculative bond proportion varying with the investor's horizon.* This figure plots $\pi_B^{MV}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

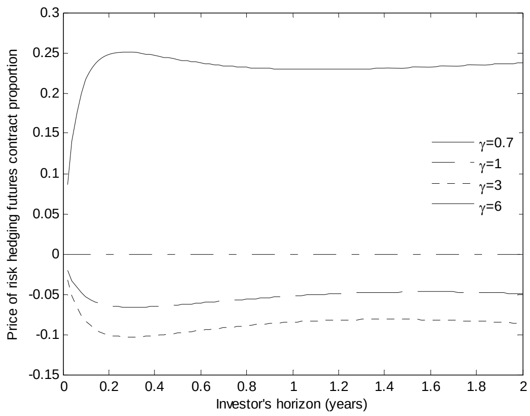

Fig. 4. *Price of risk hedging futures proportion varying with the investor's horizon.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

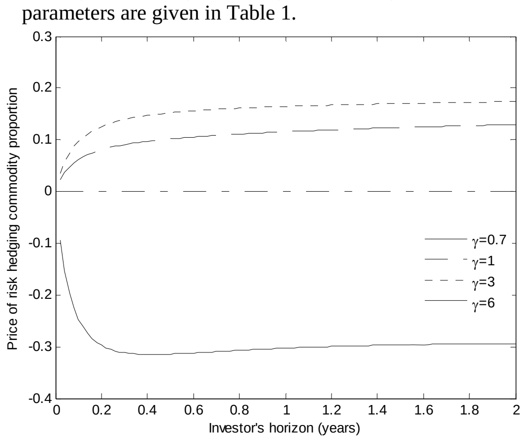

Fig. 5. *Price of risk hedging commodity proportion varying with the investor's horizon.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

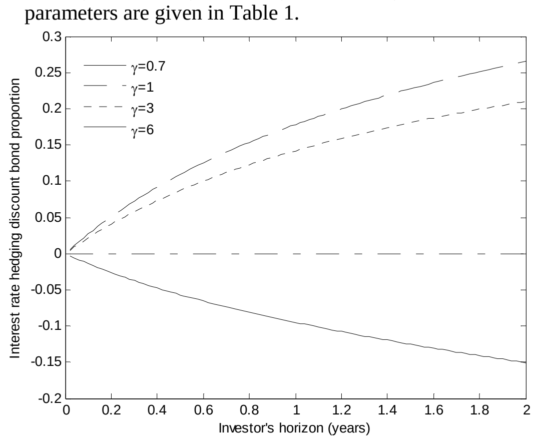

Fig. 6. *Interest rate hedging bond proportion varying with the investor's horizon.* This figure plots $\pi_B^{HMPR}(t)$ as a function of the investor horizon ranging from 0 to 2 years for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

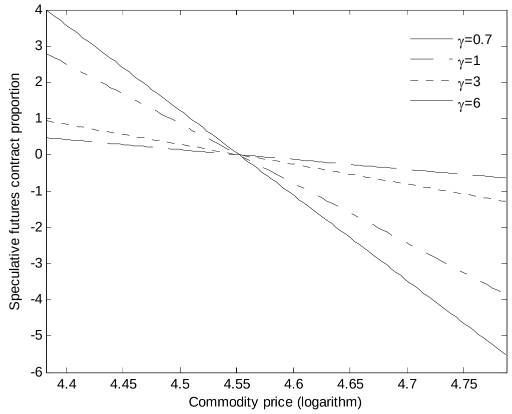

Fig. 7. *Speculative futures proportion varying with the commodity price.* This figure plots $\pi_H^{MV}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

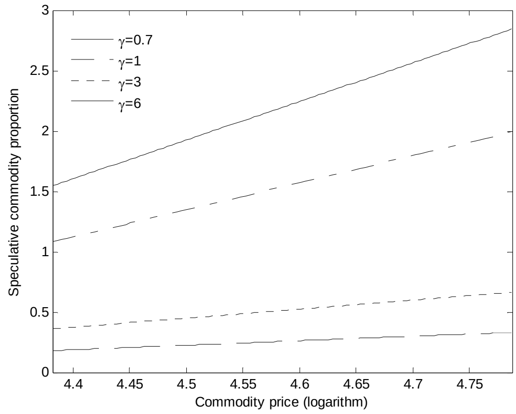

Fig. 8. *Speculative commodity proportion varying with the commodity price.* This figure plots $\pi_S^{MV}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

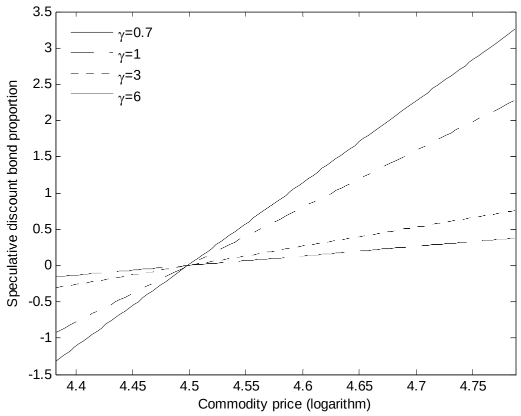

Fig. 9. *Speculative bond proportion varying with the commodity price.* This figure plots $\pi_B^{MV}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

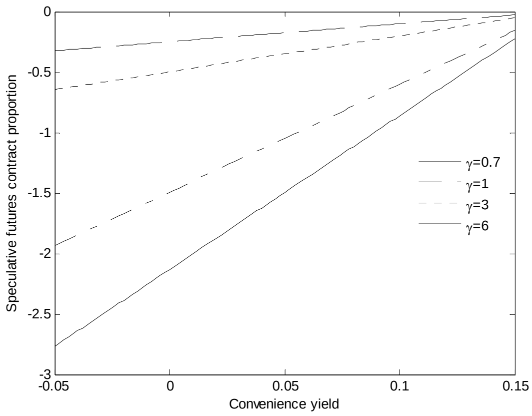

Fig. 10. *Speculative futures proportion varying with the convenience yield.* This figure plots $\pi_H^{MV}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

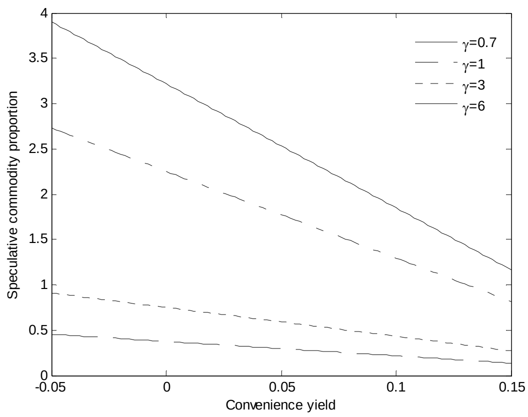

Fig. 11. *Speculative commodity proportion varying with the convenience yield.* This figure plots $\pi_S^{MV}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

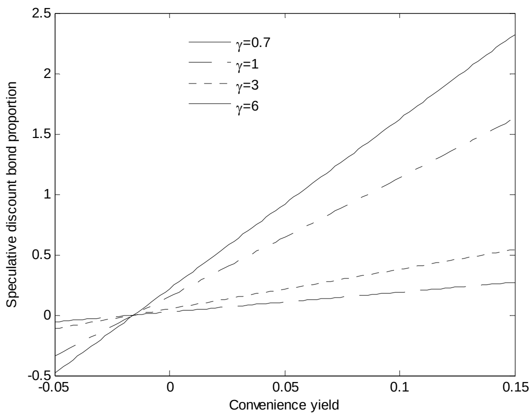

Fig. 12. *Speculative bond proportion varying with the convenience yield.* This figure plots $\pi_B^{MV}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

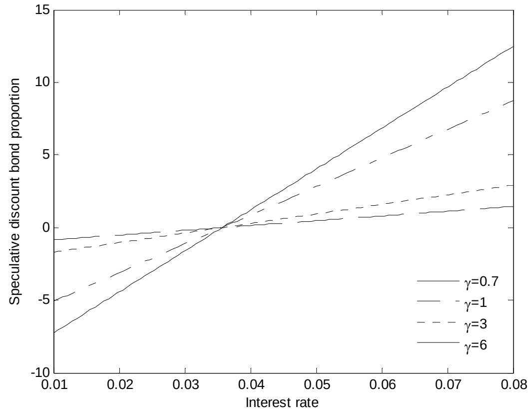

Fig. 13. *Speculative bond proportion varying with the interest rate.* This figure plots $\pi_B^{MV}(t)$ as a function of the interest rate ranging from 1% to 8% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

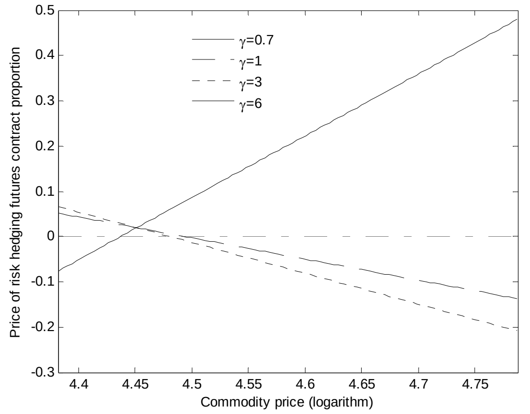

Fig. 14. *Price of risk hedging futures proportion varying with the commodity price.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

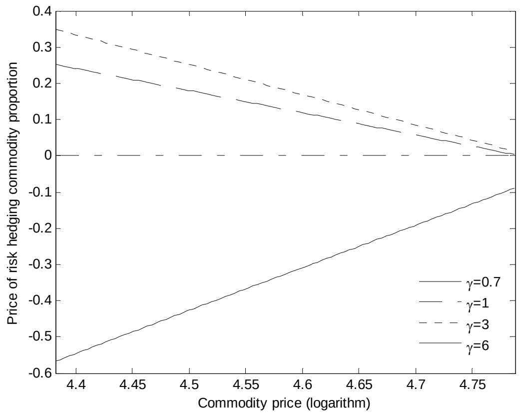

Fig. 15. *Price of risk hedging commodity proportion varying with the commodity price.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the (logarithm) commodity price ranging from \$80 to \$120 for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

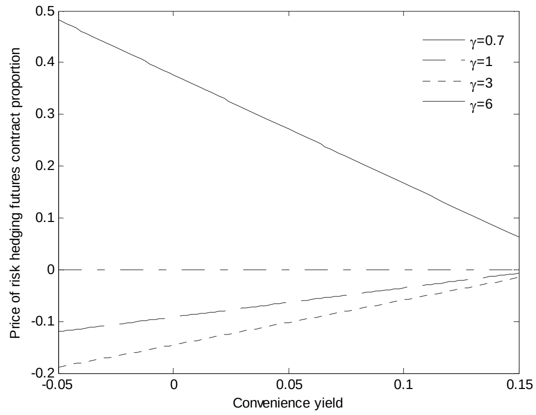

Fig. 16. *Price of risk hedging futures proportion varying with the convenience yield.* This figure plots $\pi_H^{HMPR}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

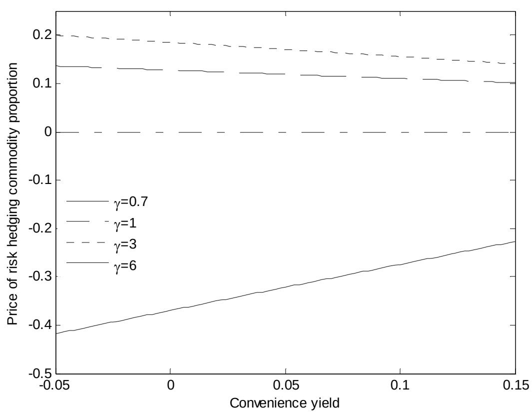

Fig. 17. *Price of risk hedging commodity proportion varying with the convenience yield.* This figure plots $\pi_S^{HMPR}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.

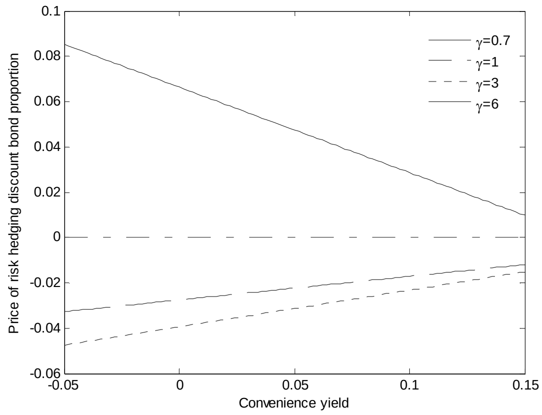

Fig. 18. *Price of risk hedging bond proportion varying with the convenience yield.* This figure plots $\pi_B^{HMPR}(t)$ as a function of the convenience yield ranging from -5% to 15% for $\gamma = 0.7$ (solid line), $\gamma = 1$ (dashed-dotted line), $\gamma = 3$ (dotted line), $\gamma = 6$ (dashed line). The other parameters are given in Table 1.