Estimating and Validating Long-Run Probability of Default with respect to Basel II Requirements

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Abstract

Basel II adopting banks estimate and validate Long-Run Probability of Default (LRPD) for each of their Internal Risk Ratings (IRRs). In this study, we examine alternative methodologies in estimating and validating LRPD. We propose the maximum likelihood estimators incorporating both cross-sectional and serial asset correlations while being consistent with the economic model underlying the Basel II capital requirement formulation. We first adopt Basel's infinitely granular portfolio assumption and propose a LRPD estimation methodology for regulatory capital estimation. We then relax this assumption to examine alternative estimation methodologies and their performances for finite number of borrowers. Simulation-based performance studies show that the proposed estimators outperform the alternatives in terms of their accuracies even under a number of small sample settings. Using the simple average of default rates as an estimator is found to be prone to underestimation of LRPD. For the purpose of validating the *assigned* LRPDs, we also examine alternative ways of establishing confidence intervals (CIs). For most of the cases, the use of the CIs constructed based on the proposed maximum likelihood estimators results in fewer errors in hypothesis tests. We show that the proposed method enables the use of external default rate data to supplement internal default rate data in attaining a more accurate and representative estimate of LRPD.

Keywords: Basel II, Long-Run Probability of Default, Asset Correlation, Stress Condition, Validation, Confidence Interval, Hypothesis Test.

Basel II requires that those banks adopting the Internal Ratings-Based (IRB) approach estimate the Long-Run Probability of Default (LRPD) for each of their Internal Risk Ratings (IRRs) to be used in the computation of their regulatory capital requirements. It is critical to correctly estimate the LRPD to be assigned to each IRR as the under-(over-) estimation of the LRPD results in the under-(over-) estimation of the capital requirement.

LRPD is a measure of the *long-term average* of the probability of default (PD) of the borrower over a one-year horizon and is typically estimated from historical default rates observed from portfolios of credit instruments. Estimation of LRPD poses significant challenges in terms of both methodology and data availability. The former requires the distinction of the *implicit* PD of individual borrowers from the *explicitly observable* portfolio default rates. Besides the level of the PD of individual borrowers, any pair-wise (cross-sectional) asset correlation may also affect the distribution of the default rates observed by tracking a uniform credit portfolio over time. Specifically, the higher the asset correlation across the borrowers, the higher is the chance that a large number of (joint) defaults occur. Moreover, PD is likely to be time-dependent and thus serially correlated through time. As a result, both cross-sectional and serial correlations need to be accounted for in order to accurately estimate the LRPD. A number of previous studies ignore these cross-sectional and time dependencies in the estimations of PD. For example, in constructing the confidence sets of the default (and transition) probabilities through bootstrap experiments, Christensen et al. (2004) assume credit risk is time-homogenous over the time windows. Nickell et al. (2000) also assume cross-sectional independence in computing the standard errors of their estimates of default (and transition) probabilities. Blochwitz et al. (2004) conduct a number of simulation exercises and conclude that the confidence intervals of the PD estimate established by ignoring the cross-sectional and time dependencies do not result in unacceptably high errors in the relevant hypothesis tests. The highest asset correlation scenario examined is however only 9%, which is believed to be less than those of most of the practical situations.

From the data availability standpoint, there are typically insufficient *internal* default rate data (especially for IRRs of low default risk) for banks to conduct their LRPD estimations. Not to mention the ability to collect sufficient default rate data that spans more than one credit cycle. On the other hand, default rate data collected and compiled by rating agencies (*external* default rate data) are richer and may dated back to the early 80s (e.g. S&P's CreditPro® database), thus covering multiple credit cycles. This makes a robust use of the external data in LRPD estimation invaluable.

In this paper, we examine alternative methodologies in estimating and validating LRPD. In Section 1, we first propose a maximum likelihood estimator (MLE) that is consistent with the economic model underlying the Basel II capital requirement formulation, which is essentially a single-factor structural model of infinitely-granular portfolio composition. The proposed estimator, while being consistent with Basel II's economic model, incorporates both cross-sectional and time dependent default event.[1]

[1] In conducting simulation exercises on portfolios of realistic characteristics, Tarashev and Zhu (2007) conclude that the violation of the infinitely-granular single-factor model assumption has virtually

Consistency with the underlying economic model of Basel II ensures that the LRPDs estimated would be appropriate inputs to the capital requirement formula of Pillar 1. The meaning of LRPD and thus its estimation methodologies can be very different according to the specifically assumed economic model used in the formulation of the capital requirement. As a result, a robust LRPD estimator obtained under an economic framework which is different from that of Basel II is not readily suitable to serve as an appropriate input to the capital formula, even though the dependent structure of the default events is correctly modeled.

In Section 2, by conducting a number of simulation exercises, we examine the performance of the proposed LRPD estimator and compare it with those of the alternative estimators. We judge the performance not only based on the accuracy of the point estimate, but also on the potential errors in the hypothesis tests of the assigned PD. The proposed MLE estimator is the best performer on both counts even under the typical small sample settings. We found that simple averaging of the defaults rates as an estimator of LRPD is prone to underestimation of LRPD under the assumed economic model of Basel II. Besides looking for an accurate and efficient estimator, banks might need to answer a different question in their validation process. Many banks have already assigned LRPDs to their IRRs using alternative methodologies and sometimes educated guesses. Validation of these assigned LRPDs with respect to the observed default rates becomes critical in complying with Basel II. For this validation task, we examine alternative ways of establishing the confidence intervals (CIs) of LRPD and compare their performances in conducting the relevant hypothesis tests.

In Section 3, we extend the methodology to enable multiple portfolio estimation so that we can use internal and external default rate data together. The ability to jointly estimate the LRPDs of the internal and external portfolio allows us to supplement the insufficient internal data with the much richer external data to improve the accuracy and efficiency of the estimation, while controlling for the difference in the characteristics of the two portfolios.

In Section 4, we relax the infinitely granular portfolio assumption and examine alternative MLE estimators and their performances for finite number of initial borrowers as seen in practice. The application of the resulting LRPD estimator would arguably be more appropriate (especially when the sample size is small) for pricing credit risk, computation of risk adjusted return on capital (RAROC) and estimation of economic (credit) capital, in which the advanced simulation techniques already incorporate the reality of finite number of borrowers.[2] We compare the performances of different estimators by conducting a number of simulation exercises. The proposed MLE estimator is again the best performer even under the small sample settings considered. Nevertheless, the estimator obtained from assuming infinitely granular portfolio performs almost equally well for portfolio of relatively high default risk.

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inconsequential implications on the measurement of portfolio credit risk, while any mis-specification in the asset correlation has a much larger impact.

[2] Most financial institutions prefer to use Point-in-Time (PIT) PDs for pricing and RAROC purposes. PIT PDs can be thought of as LRPDs conditioned on observable information.

In Section 4, we also ask the same question we ask in Section 2: *How to validate the LRPDs which have already been assigned to the IRRs using observed default rates?* For this validation task, we examine alternative ways of establishing the CIs in a number of *small sample settings* when making statistical inference in the hypothesis tests on the assigned LRPDs. These hypothesis tests serve as the building blocks of the validation exercise for Basel II compliance. Through a number of simulation exercises, we document which alternatives are best justified under different cases of portfolio sizes, lengths of historical time period and the degrees of the underlying default risk. Finally, we conclude with a couple of remarks in Section 5.

1. Basel II Capital Formula and Long-Run Probability of Default (LRPD)

In this section, we examine the Basel II capital formula in light of the underlying economic model. We want to identify the parameter to be estimated, which can serve as the appropriate LRPD input to be used in computing the capital requirement consistent with the underlying assumptions. To derive the Basel II capital formula, we start with a single-factor model of the variations of asset values under Merton's structural model. This is the model used by Vasicek (1987) in generating the loss distribution of a credit portfolio. Gordy (2003) derives the conditions that ensure the *portfolio-invariant* property of the resulting Value-at-Risk (VaR), which underpins the validity of the Basel II risk-weight function. We are not the first to utilize the Basel II framework. For example, using the same framework, Balthazar (2004) establishes the confidence intervals of default rate given a certain value of LRPD. In this study, we are interested in directly estimating LRPD and its statistical properties (e.g. confidence interval) given the observed default rates. We also go one step further by incorporating serially-correlated systematic factor.

Suppose borrowers are uniform in terms of their credit risks within a certain segment of the portfolio. Under a single-factor model, individual borrower's PD risk p_t at time t is driven by both the systematic PD risk P_t and the borrower-specific PD risk e_t. For example, for borrower i,

$$p_t^i = R \times P_t + \sqrt{1 - R^2} \times e_t^i \qquad (1)$$

We assume p_t, P_t and e_t follow the standard normal distribution, where P_t and e_t are independent. Under the Merton's framework, we can interpret p_t as a latent variable, which is a normalized function of the borrower's asset value. Borrower defaults when p_t becomes less than a certain (constant) default point (*DP*). The coefficient R is uniform across borrowers and measures the sensitivity of individual borrower's risk to the systematic PD risk. The parameter R^2 is therefore the pair-wise correlation in asset values among borrowers as a result of the systematic risk factor. It therefore governs the *cross-sectional dependency* of credit risk. Credit risk may also be *time dependent* if the systematic factor P_t is serially correlated. The time dependent property is explicitly modeled in the subsequent section.

Since p_t is assumed to be normally distributed with mean zero and unit variance, the *unconditional* probability of default of any borrower is simply $\Phi(DP)$, where $\Phi(\bullet)$ is the cumulative standard normal distribution function. As shown subsequently, it is exactly the LRPD we need to estimate for Basel II risk-weight function in order to be consistent with the underlying economic model.

It can be shown (e.g. in Vasicek (1987)) that the probability of default of borrower i *conditional on* observing the systematic PD risk P_t can be expressed as:

$$\Pr\left[p_t^i < DP \middle| P_t\right] = \Phi\left(z\left(P_t, R, DP\right)\right) \tag{2}$$

where

$$z\left(P_t, R, DP\right) = \frac{1}{\sqrt{1-R^2}}\left(DP - R \cdot P_t\right)$$

Even if the systematic factor is independent through time, default rate is not *time-homogenous*. Conditional default probability (of equation (2)) varies randomly with P_t through time. In this set up, it will only be time-homogenous if asset correlation is equal to zero.

Since e_t^i is assumed to be independent of e_t^j for $i \neq j$, the probability of observing k_t defaults out of n_t initial number of borrowers within a uniform portfolio given the realization of P_t is equal to:

$$\Omega\left(k_t, n_t; P_t, R, DP\right) = \binom{n_t}{k_t} \times \left(\Phi\left(z(P_t, R, DP\right)\right)^{k_t} \times \left(1 - \Phi\left(\left(z(P_t, R, DP\right)\right)\right)^{n_t - k_t} \tag{3}$$

and thus the unconditional cumulative probability that the number of defaults does not exceed k_t can be obtained by integrating over P_t,

$$F_{n_t}\left(\frac{k_t}{n_t}\right) = \sum_{j=0}^{k_t} \int_{-\infty}^{\infty} \Omega\left(j, n_t; P_t, R, DP\right) dP_t \tag{4}$$

Let's use θ_t to denote the observed default rate at time t, that is $\theta_t = k_t / n_t$. As n_t approaches infinity (i.e. in a portfolio of *infinite granularity*), it can be shown that $F_\infty\left(\theta_t\right)$ and its density function $f_\infty\left(\theta_t\right)$ can be respectively expressed as:[3]

$$F_\infty\left(\theta_t\right) = \Phi\left(\frac{1}{R_{PD}}\left(\sqrt{1-R^2} \times \Phi^{-1}\left(\theta_t\right) - DP\right)\right) \tag{5}$$

$$f_\infty\left(\theta_t\right) = \frac{\sqrt{1-R^2}}{R} \times \exp\left(-\frac{1}{2R^2}\left(\sqrt{1-R^2} \times \Phi^{-1}\left(\theta_t\right) - DP\right)^2 + \frac{1}{2}\left(\Phi^{-1}\left(\theta_t\right)\right)^2\right) \tag{6}$$

where $\Phi^{-1}(\bullet)$ is the inverse of the cumulative standard normal distribution function. Equation (5) shows that $\Phi^{-1}\left(\theta_t\right)$ is in fact normally distributed with mean and standard

[3] Please refer to Vasicek (1987) for details.

deviation equal to $DP\big/\sqrt{1-R^2}$ and $R\big/\sqrt{1-R^2}$ respectively. The observed default rate θ_t is actually a transformation of P_t.

$$\Phi^{-1}(\theta_t) = \frac{1}{\sqrt{1-R^2}}\left(DP - R \cdot P_t\right) \tag{7}$$

Equation (5) can also be used to derive the portfolio VaR. Let's use $\theta_{99.9\%}$ to denote the critical value of the default rate of which there will be 99.9% of not exceeding. We can then obtain an expression of $\theta_{99.9\%}$ by reverting equation (5).

$$\theta_{99.9\%} = \Phi\left(\frac{1}{\sqrt{1-R^2}}DP + \frac{R}{\sqrt{1-R^2}}\Phi^{-1}(0.999)\right) \tag{8}$$

The VaR (at 99.9%) per dollar amount of portfolio exposure is therefore equal to:

$$VaR_{99.9\%} = LGD \times \theta_{99.9\%} = LGD \times \Phi\left(\frac{1}{\sqrt{1-R^2}}DP + \frac{R}{\sqrt{1-R^2}}\Phi^{-1}(0.999)\right) \tag{9}$$

where *LGD* denotes the constant and uniform loss given default. Capital requirement can therefore be obtained by subtracting the expected loss (EL) from the above VaR.

$$Cap = VaR_{99.9\%} - EL$$

$$\Rightarrow Cap = LGD \times \Phi\left(\frac{1}{\sqrt{1-R^2}}DP + \frac{R}{\sqrt{1-R^2}}\Phi^{-1}(0.999)\right) - LGD \times \Phi(DP) \tag{10}$$

Equation (10) is exactly the Basel II Pillar I risk-weight function, in which R^2 is the asset correlation factor "*r*" that is determined based on asset class as specified in Basel II. Moreover, comparing equation (10) with the risk-weight function suggests the appropriate LRPD to be used is in fact $\Phi(DP)$, which is also the unconditional probability of default. In the next section, we derive *efficient estimators* of this Basel II parameter using the linear representation of $\Phi^{-1}(\theta_t)$ of equation (7).

2. Estimation of LRPD

In this section, we consider different approaches to estimate LRPD of a *sufficiently large* and *uniform* portfolio when we observe its annual default rates over a period of time. We derive an estimator that is asymptotically most efficient under the assumptions of the economic model underlying Basel II capital formula described in the previous section. Our purpose is therefore to estimate LRPDs especially as inputs to the Basel II regulatory (credit) capital formula. Through a simulation exercise, we also compare the performances of the alternative estimators in small sample settings.

Based on the single-factor model described in the previous section, the transformation of the observed default rate θ_t is a linear function of the random variable (systematic factor) P_t via equation (7) as the number of borrowers approaches infinity. We can therefore estimate the implicit DP (and thus LRPD) by estimating the intercept of the time-series regression equation (7) of $\Phi^{-1}(\theta_t)$. If the systematic factor P_t is i.i.d., the Ordinary Least Squares (OLS) estimator is asymptotically most efficient.[4] A number of studies (e.g. Carling et al. (2007) and Chava et al. (2006)) however suggest P_t covariates with macroeconomic variables. It is also likely to be positively serially correlated, where next period credit risk is more likely to be above average if the credit risk of this period is above average. We consider an autoregressive model of P_t with a single lag.

$$P_t = \beta \times P_{t-1} + \sqrt{1-\beta^2} \times \varepsilon_t \tag{11}$$

where ε_t is the i.i.d. innovations which follow the standard normal distribution and β is the lag-one serial correlation. If we assume both R^2 and β are known, the Generalized Least Squares (GLS) estimator is in fact the MLE estimator. It is also asymptotically most efficient.[5] The GLS estimator of DP is given by:[6]

$$DP_{MLE} = \frac{(1-\beta)\sqrt{1-R^2}}{(T-2)\beta^2 - 2(T-1)\beta + T}\left(\Phi^{-1}(\theta_1) + \sum_{t=2}^{T-1}\Phi^{-1}(\theta_t)\cdot(1-\beta) + \Phi^{-1}(\theta_T)\right) \tag{12}$$

The corresponding LRPD estimator is therefore equal to:

$$LRPD_{MLE} = \Phi(DP_{MLE}) \tag{13a}$$

A confidence interval for the LRPD may be constructed based on the sample variance of the GLS estimator.

$$\begin{aligned} CI_{MLE} &= [L_{MLE}, U_{MLE}] \\ U_{MLE} &= \Phi\left(DP_{MLE} + z_{CL} \times \frac{R \times \sqrt{1-\beta^2}}{\sqrt{(T-2)\beta^2 - 2(T-1)\beta + T}}\right) \\ L_{MLE} &= \Phi\left(DP_{MLE} - z_{CL} \times \frac{R \times \sqrt{1-\beta^2}}{\sqrt{(T-2)\beta^2 - 2(T-1)\beta + T}}\right) \end{aligned} \tag{14a}$$

where z_{CL} is the critical value corresponding to the pre-specified confidence level (CL) under the standard normal distribution (e.g. $z_{CL} = 1.96$ if CL is 95%).

[4] In this case, the OLS estimator is the same as the MLE estimator.

[5] We could have estimated both R^2 and β together with LRPD by MLE.

[6] Please refer to Appendix A for more details.

If $\beta = 0$ (i.e. time-independent), we have

$$LRPD_{MLE} = \Phi\left(\frac{\sqrt{1-R^2}}{T}\left(\sum_{t=1}^{T}\Phi^{-1}(\theta_t)\right)\right) \tag{13b}$$

$$\begin{aligned} U_{MLE} &= \Phi\left(\frac{\sqrt{1-R^2}}{T}\left(\sum_{t=1}^{T}\Phi^{-1}(\theta_t)\right) + z_{CL} \times \frac{R}{\sqrt{T}}\right) \\ L_{MLE} &= \Phi\left(\frac{\sqrt{1-R^2}}{T}\left(\sum_{t=1}^{T}\Phi^{-1}(\theta_t)\right) - z_{CL} \times \frac{R}{\sqrt{T}}\right) \end{aligned} \tag{14b}$$

Besides being consistent with the underlying model of Basel II capital formula, the MLE estimator of equation (13) is *asymptotically most efficient* in the sense that it attains the *Cramér-Rao Lower Bound* as *T* approaches infinity. There is however no guarantee that the performance is equally superior in a small sample setting, which is typically the case in practice.[7] Financial institutions rarely have internal historical default rates that go back to more than 10 years. Sometimes they may be able to supplement their internal data with external default rate data compiled by the rating agencies, which may then go back to the early 80s.

In the rest of this section, we would like to investigate the performance of the proposed MLE estimator of LRPD in a couple of small sample settings and compare with the performances of a number of alternative estimators commonly used in practice. By choosing parametric values that represent typical credit portfolios encountered by banks, we conduct a simulation exercise by generating time-series of default rates based on the infinitely granular single-factor model of the Basel II capital formula. We judge the performances based on the resulting distributions of the different estimators. Specifically, we are looking for an estimator which is unbiased and at the same time with small standard deviation. We also compare the sizes of the *Type I Error* (i.e. the probability of rejecting a true null) of alternative ways in establishing the confidence intervals around the point estimates. We consider four different estimators of LRPD:

1. Simple average of observed default rates (***$LRPD_{ave}$***)
2. The LRPD that results in the average default rates attaining the mode which matches the simple average of the observed default rates (***$LRPD_{mod}$***)
3. Proposed MLE estimator (***$LRPD_{MLE1}$*** based on equations (12) and (13a))
4. Proposed MLE estimator, but naively assuming that credit risk is time-series independent, i.e. β equals to zero (***$LRPD_{MLE2}$*** based on equation (13b))

All four are believed to be consistent estimators, while $LRPD_{ave}$ is also unbiased. $LRPD_{MLE1}$ is however asymptotically most efficient. $LRPD_{mod}$ is considered by the Canadian financial institutions regulator, the Office of the Superintendent of Financial Institutions (OSFI), to serve as a *conservative estimator* of LRPD which caters for the

[7] The confidence intervals of equations (14a) and (14b) tighten up with increasing *T*. The degree of accuracy therefore increases with the length of the default rate data.

difference between the mean and mode of the distribution of the average default rates. Detail information of computing this estimator is provided in Appendix B.

We consider the four sets of parametric values (Case 1 to 4) presented in Table 1. They therefore represent the "true" parametric values which govern the data generating process in the subsequent simulation exercises. Case 1 and 3 therefore represent portfolios of relatively good credit quality, with Case 3 having the luxury of observing annual default rates over 25 years. Case 2 and 4 are respectively the counterparts of Case 1 and 3, but now of lower credit quality. We choose the values of R^2 and β which are believed to be reasonable given our experience. For each of the four cases, we simulate 5,000 time-paths of default rates and compute the four estimators for each simulated time-path of default rates. Summary statistics of the sample distributions of the estimators across the 5,000 simulations are reported in Table 2.

INSERT TABLE 1 AND 2 ABOUT HERE

The benefit of using the more efficient MLE estimators can be illustrated by comparing the resulting sample standard deviations of the different estimators in Table 2. Even in these small sample settings, the distributions of the MLE are much *tighter* than those of the other estimators. For example, in Case 3, the standard deviation of $\text{LRPD}_{\text{MLE1}}$ is only about 76% and 68% of that of LRPD_{ave} and LRPD_{mod} respectively. Moreover, judging from the smaller difference between the 97.5 and 2.5 percentiles, the chance of significantly over- or under-estimating the true LRPD is also much smaller when MLE is used.

In Figure 1, we plot the ratio of the sample standard deviation of $\text{LRPD}_{\text{MLE1}}$ to that of LRPD_{ave} against the length of the historical time-series (i.e. T) under the case where the true LRPD (i.e. $\Phi(DP)$), R^2 and β are equal to 2.0%, 25% and 10% respectively. As expected, the benefit of using MLE increases with T. Even in the case where T is as short as 5 years, we can still reduce about 7% of the variation of the estimator by using MLE.

In Figure 2, we plot the same ratio of the sample standard deviations but now against the true LRPD under the case where T is equal to 10 years, while R^2 and β are again equal to 25% and 10% respectively. The comparative advantage of MLE is higher in the estimation of the probability of default of borrowers of higher credit quality. The benefit becomes minimal when the true LRPD is 10% or higher.

INSERT FIGURE 1 AND 2 ABOUT HERE

It can be observed from Table 2 that there is a slight upward bias in terms of the sample means of the MLE. As expected, the bias is much smaller for T equals to 25 years than 10 years. In terms of the sample medians, there are always almost exactly equal chances that MLE over- or under-estimates the true LRPD. On the other hand, there is always a higher chance that LRPD_{ave} will underestimate rather than overestimate the true LRPD, which therefore suggests simple average may not produce a conservative estimator of LRPD. The opposite is the case for $LRPD_{mod}$. The chance of overestimation is always larger than that of underestimation. This is not surprising given that this estimator involves an upward adjustment from the mode to the mean of the positively-skewed distribution of average default rates. $LRPD_{mod}$ may therefore be too conservative. The chance of underestimation (overestimation) of $LRPD_{ave}$ ($LRPD_{mod}$) is the highest when the true LRPD is low and the historical data series is short. Comparing the sample statistics of $LRPD_{MLE1}$ and $LRPD_{MLE2}$, the benefit of knowing and modeling for the serial correlation of the systematic factor is found to be minimal when T is large. There is only marginal improvement of efficiency in using $LRPD_{MLE1}$ rather than $LRPD_{MLE2}$ when T is small.

So far we focus on the estimation of LRPDs from historical default rates. In the rest of this section, we would like to answer a different question. Many banks have already assigned LRPDs to their IRRs using alternative methodologies and sometimes educated guesses. Validation of these assigned LRPDs is critical and also required for Basel II compliance. *How can we validate these "pre-assigned" LRPDs given the observed default rates?* For this validation task, we examine alternative ways of establishing the CIs when making statistical inference in the relevant hypothesis tests on the assigned LRPD. Specifically, we would like to study the validity of the proposed CIs of equation (14) for hypothesis tests on LRPD in small sample settings. The CIs are bound to be too narrow when T is small and thus resulting in a Type 1 error (i.e. the probability of incorrectly rejecting the true null) which is larger than the specified confidence level. We conduct the same simulation exercises described above in obtaining Table 2. For each simulated time-path of default rates, we construct the 95% CI (based on equation (14)) of a *two-tail* test of LRPD and check if the true null (i.e. LRPD of either 0.5% or 2.0%) lies outside of the interval and thus being rejected. We then count the percentage of simulations which lead to a rejection.

Besides the above two-tail test, we also measure the Type 1 error of the following one-tail test. In each simulation, we ask ourselves whether we would like to reject the (true) null because we think it is (statistically) too high. This is the case where the true null turns out to be higher than the upper bound of the CI. The consequence of this type of rejection may lead to the imprudent action of the financial institutions to wrongfully reduce the assigned probability of default which is in fact correct in the first place. Specifically, we want to find out the Type 1 error of this one-tail test at the 97.5% confidence level.

We measure the Type 1 errors of four different CIs each corresponding to one of the estimators considered above. We construct a CI (CI_{MLE1}) around $LRPD_{MLE1}$ based on equation (14a). The CI (CI_{MLE2}) around $LRPD_{MLE2}$ is constructed based on equation (14b), that is, by setting β to zero. This represents the case where we naively assume the systematic factor is time-independent. The results of Table 2 suggest this assumption might not hinder the efficiency of the resulting estimator. We then construct an interval around $LRPD_{ave}$ by invoking the *central limit theorem* (CLT). If the systematic factor is time-independent, CLT ensures $LRPD_{ave}$ converges (as T approaches infinity) to a normal distribution with mean equals to the true LRPD and standard deviation equal to the sample standard deviation of the observed default rates over the sample period divided by the square root of T. In a finite sample setting, the resulting CI_{ave} is again bound to be too narrow, especially since the systematic factor is in fact not i.i.d. Finally, we construct a CI (CI_{mod}) around $LRPD_{mod}$ by assuming its standard deviation is also equal to that of $LRPD_{ave}$ described above.[8] We report the simulated Type 1 errors in Table 3. We again conduct the simulations for each of the four cases of Table 1.

INSERT TABLE 3 ABOUT HERE

From Table 3, using CI_{MLE1} results in Type I errors of the two- and one-tail tests that are very close to the theoretically correct values of 5% and 2.5% respectively even in these small sample settings. As expected, CI_{ave} is too narrow, especially when LRPD is small and T is short. More problematically, in comparing Panel A and B of Table 3, the majority of the rejection cases are those where we wrongfully reject the true LRPD because it is believed to be statistically too high. As a result, in using CI_{ave}, there might be a high probability that it may result in the financial institution taking the imprudent action of wrongfully reducing the assigned probability of default which is correct in the first place. The performance of CI_{mod} is found to be better than that of CI_{ave}, even though the resulting Type 1 errors are still significant. Unlike the findings in Table 2, there is now a payoff in correctly modeling for the serial correlation of the systematic factor. Ignoring the serial correlation (i.e. CI_{MLE2}) results in significantly larger errors, especially when T is short.

3. The Use of External Data together with Internal Data in Estimating LRPD

Rarely is the case that banks have historical default rate data under the new internal risk rating system (of Basel II) which covers more than a full credit cycle. How can the banks justify to the regulators that the sample period over which the above estimation is conducted is *sufficiently stressed* relative to the long-run? Considering that the last few years coincided with the favourable part of the credit cycle, resulting in low default rates, the regulators' concern that the LRPD estimated from those default rates may be understated is understandable.

Due to the lack of a long enough time-series of internal default rate, some banks fully rely on the use of external default rate data collected by rating agencies (e.g. S&P's and Moody's) which easily goes back to the early 80's. The longer external data series therefore easily spans more than a couple of credit cycles and should serve as a more credible dataset in the estimation of LRPD. The problem is it needs to be *pre-adjusted* to cater for the differences in the following characteristics between the bank's credit portfolio and those of the overall credit market from which the external data are collected.

- Regional (country) composition;
- Industry composition;
- General level of credit risk (i.e. LRPD); and
- Asset (cross-sectional) correlation (i.e. R^2), which is also a measure of the sensitivity to the systematic factor in the single-factor model described above.

[8] We are not aware of a closed-form asymptotic standard deviation for $LRPD_{mod}$.

In practice, the adjustments made are typically on an ad-hoc basis and may not be readily defendable scientifically. Moreover, it is a waste to completely ignore the internal default rate data collected even though it is not considered to be long enough. In this section, we extend the previous single-factor model in a bivariate setting to enable the joint estimation of the LRPDs of two different credit portfolios of which default rates are observed over time periods which are only partially overlapping. It therefore allows us to use both the *shorter but more relevant* internal data series together with the *longer but less relevant* external data in the estimation of the LRPDs of banks' internal default ratings, while at the same time control for the differences in the portfolio characteristics underlying the two datasets.

Suppose borrowers making up the *internal* portfolio are uniform in terms of their credit risks and are governed by the single-factor model of equation (1), specifically

$$p_t^i = R \times P_t + \sqrt{1 - R^2} \times e_t^i$$

where P_t is the systematic PD risk, while P_t and e_t are independent and follow the standard normal distribution. We again interpret p_t as the normalized asset value of the borrower. Borrower defaults when p_t becomes less than the default point (*DP*), which dictates the LRPD that we want to estimate. Parameter R^2 is the pair-wise correlation in asset values.

We then assume borrowers making up the *external* portfolio are again governed by a single-factor model.

$$p_{x,t}^i = R_x \times P_{x,t} + \sqrt{1 - R_x^2} \times e_{x,t}^i \tag{15}$$

where $P_{x,t}$ is the systematic PD risk specific to the external portfolio, while $P_{x,t}$ and $e_{x,t}$ are independent and follow the standard normal distribution. Borrower defaults when $p_{x,t}$ is less than the default point (DP_x) specific to this external portfolio. The external portfolio may also have a different pair-wise correlation (R_x^2). We expect P_t and $P_{x,t}$ vary over time in a correlated fashion with correlation coefficient equal to ρ which is straightly less than one.

In this bivariate model, we therefore assume the difference between the two portfolios can be fully captured by the differences in values of the respective default point and pair-wise correlation. For example, in the subsequent numerical example, we consider the external default rates of speculative grade instruments together with the internal default rates of a commercial bank. We expect DP_x to be higher than *DP*, given that the probability of default of the speculative grade instruments is believed to be in general higher than that of the bank's borrowers. We however expect R_x^2 to be lower than R^2 given that the bank's portfolio is believed to be less diversified in terms of its industrial compositions.

By observing the time-series of default rates θ_t and $\theta_{x,t}$ of the internal and external portfolios, we can simultaneously solve for the MLEs of the LRPDs of both portfolios, i.e. $\Phi(DP)$ and $\Phi(DP_x)$. Partial overlapping of the two samples is sufficient to allow for the joint estimation. The time series of θ_t is typically shorter than that of $\theta_{x,t}$. Suppose, we only observe $\theta_{x,t}$ from t = 1 to T_x-T, while we observe both θ_t and $\theta_{x,t}$ from $t = T_x$-$T+1$ to T_x.[9] The MLEs can therefore be obtained by maximizing the logarithmic of the joint likelihood.

$$[\Phi(DP),\Phi(DP_x)] = \arg\max\left(\log\left(\mathcal{F}_\infty\left(\theta_{T_x-T+1},\dots,\theta_{T_x},\theta_{x,1},\dots,\theta_{x,T_x}\right)\right)\right) \tag{16}$$

If the two systematic factors P_t and $P_{x,t}$ are i.i.d. through time, the log-likelihood can be expressed as the sum of a series of univariate unconditional and conditional log-density.[10] Please refer to the Appendix C for more details.

$$[\Phi(DP),\Phi(DP_x)] = \arg\max\left(\sum_{t=1}^{T_x}\log f_\infty\left(\theta_{x,t}\right) + \sum_{t=T_x-T+1}^{T_x}\log f_\infty\left(\theta_t \middle| \theta_{x,t}\right)\right) \tag{17}$$

where

$$f_\infty\left(\theta_{x,t}\right) = \frac{\sqrt{1-R_x^2}}{R_x}\times\exp\left(-\frac{z_{x,t}^2}{2R_x^2}+\frac{1}{2}\left(\Phi^{-1}\left(\theta_{x,t}\right)\right)^2\right) \tag{18}$$

$$f_\infty\left(\theta_t \middle| \theta_{x,t}\right) = \frac{\sqrt{1-R^2}}{R\sqrt{1-\rho^2}}\times\exp\left(-\frac{\left(z_t - R\cdot\rho\cdot z_{x,t}/R_x\right)^2}{2\left(1-\rho^2\right)R^2}+\frac{1}{2}\left(\Phi^{-1}\left(\theta_t\right)\right)^2\right) \tag{19}$$

$$z_{x,t} = \sqrt{1-R_x^2}\times\Phi^{-1}\left(\theta_{x,t}\right) - DP_x$$

$$z_t = \sqrt{1-R^2}\times\Phi^{-1}\left(\theta_t\right) - DP$$

If R^2, R_x^2 and ρ are known, we can then solve for the MLE of LRPD by maximizing the joint likelihood (i.e. equation (17)) of observing the two sets of default rates,[11]

$$LRPD_{MLE,x} = \Phi\left(\frac{\sqrt{1-R_x^2}}{T_x}\left(\sum_{t=1}^{T_x}\Phi^{-1}\left(\theta_{x,t}\right)\right)\right) \tag{20}$$

$$LRPD_{MLE} = \Phi\left(\frac{1}{T}\left(\sqrt{1-R^2}\sum_{t=1}^{T}\Phi^{-1}\left(\theta_t\right)+\frac{R\cdot\rho}{R_x}\left(T\cdot LRPD_{MLE,x}-\sqrt{1-R_x^2}\sum_{t=1}^{T}\Phi^{-1}\left(\theta_{x,t}\right)\right)\right)\right) \tag{21}$$

[9] The lengths of the internal and external time series are therefore T and T_x respectively.

[10] We could have modeled for the potential serial correlation of the systematic factors (e.g. similar to equation (11)).

[11] We could have estimated $\Phi(DP)$, $\Phi(DP_x)$, R^2, R_x^2 and ρ simultaneously. Obtaining the respective MLEs will then involving the solving of a set of simultaneous equations.

where $LRPD_{MLE}$ and $LRPD_{MLE,x}$ are the MLEs of the LRPD of the internal and external portfolio respectively. When comparing equation (21) with (13b), we can interpret the second term of equation (21) as the *correction* to be made to the *sub-sample LRPD estimate* of the internal portfolio given the benefit of observing the longer external dataset. If we could observe the internal default rates over the same period when we observe external default rates (i.e. $T = T_x$), the second term of equation (21) vanishes and we get back the same $LRPD_{MLE}$ of equation (13b) as if we have not observed the external default rates.

The corresponding confidence intervals of the MLEs are:

$$\begin{aligned} CI_{MLE,x} &= \left[L_{MLE,x}, U_{MLE,x}\right] \\ U_{MLE,x} &= \Phi\left(DP_{MLE,x} + z_{CL} \times \frac{R_x}{\sqrt{T_x + T\rho^2/\left(1-\rho^2\right)}}\right) \\ L_{MLE,x} &= \Phi\left(DP_{MLE,x} - z_{CL} \times \frac{R_x}{\sqrt{T_x + T\rho^2/\left(1-\rho^2\right)}}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} CI_{MLE} &= \left[L_{MLE}, U_{MLE}\right] \\ U_{MLE} &= \Phi\left(DP_{MLE} + z_{CL} \times \frac{R\sqrt{1-\rho^2}}{\sqrt{T}}\right) \\ L_{MLE} &= \Phi\left(DP_{MLE} - z_{CL} \times \frac{R\sqrt{1-\rho^2}}{\sqrt{T}}\right) \end{aligned} \tag{23}$$

where z_{CL} is the critical value corresponding to the pre-specified confidence level (CL) under the standard normal distribution (e.g. z_{CL} = 1.96 if CL is 95%). As expected, in comparing equation (23) and (14b), we notice that the MLE obtained here is more efficient and the CI tighter than when no (partially) overlapping external default rate data are observed.

In the rest of this section, we consider an example to illustrate the application of this joint estimation method. Suppose a bank has collected annual internal default rate data of a particular risk rating over a nine-year period from 1996 to 2004. Solely based on this information, it could estimate LRPD using equation (13). However, it might need to answer this question: Can it justify to the regulator that using data over this nine-year period will produce a prudent estimator of LRPD? Although this period covers the most recent credit downturn of 2001-02, it is still relatively short and barely covers a full business cycle. If it can also obtain a longer time-series of external default rate data, the bank can better answer the above question by adopting the joint-estimation method proposed earlier in this section.

Suppose the bank obtains S&P's speculative grade historical default rates from 1981 to 2004.[12] These external default rates together with its internal data are presented in Table 4. We expect DP_x to be higher than DP, given that the probability of default of speculative grades is believed to be in general higher than that of the bank's customers. We however expect R_x^2 to be lower than R^2 given that the bank's portfolio is believed to be less diversified in terms of its industrial composition.[13] By assuming $R^2 = 0.166$, $R_x^2 = 0.073$ and $\rho = 0.553$, we compute the MLEs of the LRPDs and the corresponding CIs based on the above joint-estimation approach (i.e. using equations (20) to (23)).[14] The results are reported in Table 5 together with those obtained when estimations are done *separately* (i.e. using equations (13b) and (14b)).

INSERT TABLE 4 AND 5 ABOUT HERE

From Table 5, the LRPD estimate of the bank's borrowers is reduced from 0.841% to 0.765% once we take into consideration of the longer external data. In other words, solely using the limited internal data over 1996 to 2004 actually results in a prudent estimation (0.841%) of the LRPD in the first place. Besides obtaining a more appropriate point estimate of LRPD, the joint-estimation also improves the efficiency and thus allowing for a more precise estimate (i.e. tighter CI). The power of any hypothesis test to be conducted on the LRPD is therefore enhanced.

In Figure 3, we plot the *most-likely* systematic factor P_t and $P_{x,t}$ over the sample period, which can be easily computed as follows.

$$P_t = \frac{\Phi^{-1}(LRPD_{MLE}) - \Phi^{-1}(\theta_t)\cdot\sqrt{1-R^2}}{R} \tag{24}$$

$$P_{x,t} = \frac{\Phi^{-1}(LRPD_{MLE,x}) - \Phi^{-1}(\theta_{x,t})\cdot\sqrt{1-R_x^2}}{R_x} \tag{25}$$

INSERT FIGURE 3 ABOUT HERE

It should be noted that default rate is negatively related to the systematic factor presented in Figure 3. The higher the systematic (asset) value, the lower is the probability of realizing a borrower-specific asset value that is below the default point DP. The timing and severity of the last two credit downturns (in early 90s and then 2001-02) are visualized by tracking the variation of the systematic factor implicit in S&P's speculative grades

[12] The bank uses speculative grade rather than investment grade default rates because it believes the former is more sensitive to credit cycle and thus is a better proxy for market-wide stress level than that of the latter.
[13] Readers will find out these beliefs are indeed confirmed in the subsequent estimations.
[14] The values of R^2, R_x^2 and ρ assumed here are the actual MLE estimates of these parameters if we estimate $\Phi(DP), \Phi(DP_x), R^2, R_x^2$ and ρ simultaneously based on the proposed joint-estimation method. To conserve space, we do not illustrate the details in this paper, which are however available from the authors upon request.

defaults. Not surprisingly, the two systematic factors are found to be moving in a (highly) correlated fashion over time (MLE of the correlation coefficient ρ is in fact found to be equal to 0.553). Moreover, the period from 1996 to 2004 may actually almost exactly span over one full credit cycle. Over the sub-sample of 1996 to 2004, the systematic factor of S&P's speculative grades ranges from +1.20 to -1.78, with a sample mean and standard deviation of -0.15 and 1.04 respectively.[15] From the slightly negative sample mean, we can conclude the average level of default risk over this nine-year period was marginally higher than the average over the full sample. That is why in Table 5 we should slightly reduce the estimated LRPD of the bank's borrowers from 0.841% to 0.765% to adjust for the difference in average default risk between this sub-sample and the full sample.

4. Finite number of initial borrowers

In Section 2, we derive the most efficient estimator of LRPD and its CI under the assumption of the single-factor infinitely-granular model underlying the Basel II capital formula. By conducting the simulation exercises, we examine their performances in various small sample settings of finite sample periods. In this section, we consider situations where the infinite-granularity assumption is violated. In practice, we never observe the default rate of a portfolio which is made up of infinite number of borrowers. Is the proposed estimator still appropriate in the estimation of LRPD using observed default rates from portfolios of finite number of borrowers? Moreover, banks might be interested in LRPD obtained from relaxing the *infinitely granular portfolio* assumption so that the resulting LRPD would arguably be more appropriate for the estimation of economic credit capital.[16]

We still maintain the previous assumption of a single systematic factor driving the asset values of borrowers of uniform credit risk. We therefore still conform to equations (1) to (4). However, the limiting representations of equations (5) to (7) become invalid. Nevertheless, we want to examine how the LRPD estimators (equations (13a) and (13b)) and the corresponding CIs (equations (14a) and (14b)) behave when the infinite-granularity assumption is violated.

There is however a practical problem in using the MLE estimators of equation (13a) and (13b) when the number of borrowers is finite. Specifically, they are undefined when any of the observed default rate θ_t is zero.[17] It can be resolved by replacing the zero default rates with a *sufficiently small but finite* default rate $\underline{\theta}$. Rather than using the observed time series of default rates $\theta_1, \theta_2, ..., \theta_T$, we therefore operate on the *transformed* time-series of default rates $\psi_1, \psi_2, ..., \psi_T$, which are defined as

[15] The population mean and standard deviation of the systematic factor is zero and one respectively. A zero value of the systematic factor therefore represents a *cycle-neutral* level of default risk.

[16] Typically, the advance simulation techniques used in the estimation of economic credit capital are capable of modeling credit portfolios which are made up of finite numbers of borrowers.

[17] In practice, it is not uncommon to observe zero default rates for portfolios of relatively high credit ratings. On the other hand, regardless of the credit rating, there is zero probability of realizing $\theta_t = 0$ in a portfolio of infinite number of borrowers.

$$
\begin{aligned}
\psi_t &= \theta_t \ \text{ if } \theta_t \neq 0 \\
\psi_t &= \underline{\theta} \ \text{ if } \theta_t = 0
\end{aligned} \tag{26}
$$

The default rate $\underline{\theta}$ is computing by preserving the expected time-series variation of default rates. Given the fact that: (i) $\Phi^{-1}(\theta_t)$ is a linear transformation of P_t according to equation (7); and (ii) P_t follows the autoregressive model of equation (11), it can be shown (in Appendix D) that:

$$
E\left[\frac{1}{T-1}\sum_{t=1}^{T}\left(\Phi^{-1}(\theta_t)-\bar{\mu}_\theta\right)^2\right]=\frac{R^2}{\left(1-R^2\right)}\times\frac{\left(T-\dfrac{1_T'\Omega_\beta 1_T}{T}\right)}{(T-1)}, \tag{27}
$$

where 1_T is the unit column vector of length T, $\bar{\mu}_\theta=\frac{1}{T}\sum_{t=1}^{T}\Phi^{-1}(\theta_t)$, and

$$
\Omega_\beta=\begin{bmatrix} 1 & \beta & \beta^2 & . & \beta^{T-1} \\ \beta & 1 & \beta & . & \beta^{T-2} \\ \beta^2 & \beta & 1 & . & \beta^{T-3} \\ . & . & . & . & . \\ \beta^{T-1} & \beta^{T-2} & \beta^{T-3} & . & 1 \end{bmatrix}.
$$

We can therefore estimate $\underline{\theta}$ by minimizing the difference between the expected time-series variance given by equation (27) and its sample counterpart. That is,

$$
\hat{\underline{\theta}}=\arg\min\left\{\left[\frac{1}{T-1}\sum_{t=1}^{T}\left(\Phi^{-1}(\psi_t)-\bar{\mu}_\psi\right)^2\right]^{0.5}-\left[\frac{R^2}{\left(1-R^2\right)}\times\frac{\left(T-\dfrac{1_T'\Omega_\beta 1_T}{T}\right)}{(T-1)}\right]^{0.5}\right\}^2, \tag{28}
$$

where $\bar{\mu}_\psi=\frac{1}{T}\sum_{t=1}^{T}\Phi^{-1}(\psi_t)$.

We can of course relax the assumption of infinite number of borrowers and still obtain a MLE estimator for LRPD accordingly. We therefore consider the case where n_t is finite and it becomes essentially the *probit-normal Bernoulli mixture model* of Frey and McNeil (2003). They conduct simulations and the results suggest that the simple average of default rates as a LRPD estimator performs almost as good as an estimator developed under the correctly-specified model. Pluto and Tasche (2005) and Benjamin et al. (2006) also utilize the same approach, but they are more interested in using it to establish the upper confidence bounds of low default portfolios rather than estimating their LRPDs. Moreover, they have not addressed the performances of their proposed confidence bounds. Gordy and Heitfield (2002) also adopt a similar factor model but their interest is in the performances of the estimators of default correlations (i.e. factor loading). Their Monte Carlo study suggests that the failing to impose reasonable restrictions on the default correlation structure may lead to significant biases in the estimations given the limited length of the historical default data series. In their study, they assume time-independent systematic factor and they do not concern themselves with establishing confidence intervals for their estimators.

The cost of relaxing the assumption of infinite number of borrowers in practice is the fact that it becomes computationally intensive since analytical formulas of the estimator and its standard error are not available. It therefore also makes comparative static analysis difficult. For example, with the assumption of infinite number of borrowers, we can easily measure the sensitivity of the LRPD estimator on the serial correlation parameter β by evaluating the partial derivative of equation (13a) analytically. When the assumption is relaxed, we can only find out the sensitivity by conducting a numerical analysis. Although the estimator obtained from relaxing the assumption is more efficient asymptotically than that obtained with the assumption, we want to document the conditions under which the improvement in efficiency is so insignificant that it is not worthwhile to incur the additional computational costs.

In relaxing the assumption of infinite number of borrowers, the joint probability $\mathcal{G}$ of observing the time-series of default rates $k_1, k_2, ..., k_T$ out of the initial number of borrowers $n_1, n_2, ..., n_T$ in time 1, 2, …, T respectively can be expressed as:

$$\mathcal{G}(k_1, k_2, ..., k_T) = \sum_{t=1}^{T} \int_{-\infty}^{\infty} \Omega(k_t, n_t; P_t, R, DP)\, dP_t \tag{29}$$

where $\Omega(\bullet)$ is given in equation (3), which is the conditional probability of observing k_t defaults out of n_t borrowers given the realization of P_t. The systematic factor P_t is assumed to be serial correlated according to the autoregressive model of equation (11). We can therefore solve for the MLE estimate of LRPD by maximizing $\mathcal{G}$.

$$LRPD_{MLEf} = \arg\max\left(\log\left(\mathcal{G}(k_1, k_2, ..., k_T)\right)\right) \tag{30}$$

Solving for $LRPD_{MLEf}$ involves the numerical integrations of equation (29) over the multivariate normal distribution of the vector [P_1 P_2 … P_T]. Analytical asymptotic standard deviation of $LRPD_{MLEf}$ is unavailable, but it can be evaluated numerically based on the second derivative of the log-likelihood function.

$$\sigma_{MLEf} = \left[-\frac{\partial^2 \log(\mathcal{G}(\bullet))}{\partial(\Phi(DP))^2} \right]^{-0.5}_{\Phi(DP) = LRPD_{MLEf}} \tag{31}$$

In the rest of this section, similar simulation exercises as in Section 2 are conducted to examine the performances of various LRPD estimators in a number of small sample settings. Unlike in Section 2, we now generate defaults based on pre-specified finite initial numbers of borrowers. We consider six different estimators of LRPD.

1. Simple average of observed default rates (***LRPD_{ave}***)

2. The LRPD that results in the average default rates attaining the mode which matches the simple average of the observed default rates (***$LRPD_{mod}$***)
3. Proposed MLE estimator under *infinite* number of borrowers (***$LRPD_{MLE1}$*** based on equations (12) and (13a))
4. Proposed MLE estimator under *infinite* number of borrowers, but naively assuming that credit risk is time-series independent, i.e. β equals to zero (***$LRPD_{MLE2}$*** based on equation (13b))
5. Proposed MLE estimator under (exact) *finite* number of borrowers (***$LRPD_{MLEf1}$*** based on equation (30))
6. Proposed MLE estimator under (exact) *finite* number of borrowers, but naively assuming that credit risk is time-series independent (***$LRPD_{MLEf2}$*** based on equation (30), but setting β to zero)

All six are believed to be consistent estimators, while $LRPD_{MLEf1}$ is asymptotically most efficient. Again, $LRPD_{mod}$ is the estimator considered by OSFI as a conservative estimator of LRPD, which is also examined in Section 2. We again consider the four different combinations of parametric values presented in Table 1. They therefore represent the "true" parametric values, which govern the data generating process in the simulation exercises. For each set of parameters, we generate time-series of the number of defaults based on the serially-correlated single-factor model and under three different scenarios of constant initial number of borrowers (n_t = 50, 100 and 500) over time. In each case, we simulate 1,000 time-paths of the number of defaults and compute the six estimators for each simulated time-path. Summary statistics of the sample distributions of the estimators across the 1,000 simulations are reported in Table 6. We also report the percentage of simulations in which the true LRPD is underestimated by the respective estimator.

INSERT TABLE 6 ABOUT HERE

It can be observed from Table 6 that, even though simple averaging ($LRPD_{ave}$) is unbiased on average, there is always a higher chance of underestimating rather than overestimating the true LRPD. The understatement is most significant when both T and the true LRPD are small. For example, there can be as much as a 66% chance of underestimating the true LRPD of 0.5% when T is equal to 10. Judging from the standard deviation of $LRPD_{ave}$, its performance improves with the initial number of borrowers n.

As expected, the MLE estimator of $LRPD_{MLE1}$ performs the best when both n and T are large. Judging from the standard deviations, $LRPD_{MLE1}$ is always a more efficient estimator than $LRPD_{ave}$ when n is equal to 500. Besides, regardless of the values of n and T, the underestimation of the true LRPD by $LRPD_{MLE1}$ is always less than that of $LRPD_{ave}$ when the true LRPD is 2.0%. The use of $LRPD_{MLE1}$ becomes inappropriate when both n and the true LRPD are small (e.g. when n = 50 and LRPD = 0.5%).

Not surprisingly, based on the relative efficiency (i.e. the variability of the estimator), the performance of $LRPD_{MLEf1}$ is the best among all the estimators over all the cases considered here. Since it is an *exact* estimator in terms of finite values of n, its performance is essentially unaffected by the number of borrowers. $LRPD_{MLE1}$ however performs almost equally well when the true LRPD is equal to 2.0% and when n is not *too* small (e.g. n is at least equal to 100).

Similar to the findings in Section 2, $\mathrm{LRPD_{mod}}$ is always a conservative but inefficient estimator. Finally, judging from the similarity of the performances of $\mathrm{LRPD_{MLE1}}$ ($\mathrm{LRPD_{MLEf1}}$) and $\mathrm{LRPD_{MLE2}}$ ($\mathrm{LRPD_{MLEf2}}$), the modeling of the serial-correlation of the systematic factor is found to be of secondary importance in the estimation of LRPD. The barely noticeable differences are documented for the cases of T and n equal to 10 and 50 respectively. Given the results reported in Table 6, we may conclude that $\mathrm{LRPD_{MLEf1}}$ (or $\mathrm{LRPD_{MLEf2}}$) should be used in the estimation of LRPD. However, in the case of relatively high LRPD (e.g. 2.0%) and large number of borrowers (e.g. equal to or more than 100), the marginal benefit over the much more analytically tractable estimator of $\mathrm{LRPD_{MLE1}}$ (or $\mathrm{LRPD_{MLE2}}$) might not be worthwhile for the effort of conducting the required computationally intensive analysis.

For the remainder of this section, we would like to ask the same validation question we ask in Section 2: If the LRPDs have already been assigned, how can we validate them given the observed default rates? The choice of an appropriate CI is important in making statistical inferences during the validation process of Basel II. Through the simulation exercises, we judge their performances based on their Type I errors under both the one- and two-tail hypothesis tests similar to those examined in Section 2. Besides, we also examine the resulting Type II errors (i.e. the probability of wrongfully accepting an alternative hypothesis) in a couple of cases. Together with the CIs established from the proposed MLE estimators, we consider a total of eight different ways in constructing the interval.[18]

1. Wald CI (CI_W)
2. Agresti-Coull CI (CI_{AC})
3. Clopper-Pearson CI (CI_{CP})
4. CI from non-parametric bootstrap (CI_B)
5. CI_{MLE1}
6. CI_{MLE2}
7. CI_{MLEf1}
8. CI_{MLEf2}

Wald CI ($\mathbf{CI_W}$) is one of the most popular CI used for binomial distribution. It is based on the normal distribution approximation under the central limit theorem when the occurrences of individual default events are i.i.d. With the assumption of equal initial number of borrowers (i.e. $n_1 = n_2 = ... = n_T \equiv n$), it can be expressed as:

$$CI_W = \overline{\theta} \pm z_{CL} \times \sqrt{\frac{\overline{\theta}\left(1-\overline{\theta}\right)}{n \times T}} \tag{32}$$

where $\overline{\theta} = \sum_{t=1}^{T} \theta_t \Big/ T$ and z_{CL} is the critical value corresponding to the pre-specified confidence level (CL) under the standard normal distribution (e.g. z_{CL} = 1.96 if CL is 95%).

[18] The first four CIs are proposed and examined by Hanson and Schuermann (2005).

The coverage of CI_W is always too narrow under a finite value of $n \times T$. Moreover, since default incidences are correlated both cross-sectionally (because of non-zero R^2) and over time (because of non-zero β), the understatement of the coverage will even be more significant under the serially-correlated single-factor model considered here. It may therefore lead to an unacceptable level of Type I error, especially when both n and T are small. Furthermore, the erratic behaviours of CI_W (e.g. those documented by Brown et al., 2001) may further cast doubts on its validity as an appropriate CI.

Brown et al. (2001) show that the **Agresti-Coull CI (CI_{AC})** proposed by Agresti and Coull (1998) produces a coverage which is more accurate than that of CI_W in various small sample simulation exercises. In terms of our notations, CI_{AC} can be expressed as:

$$CI_{AC} = \bar{\theta}^* \pm z_{CL} \times \sqrt{\frac{\bar{\theta}^*\left(1-\bar{\theta}^*\right)}{n \times T + z_{CL}^2}} \tag{33}$$

where

$$\bar{\theta}^* = \frac{\sum_{t=1}^{T} k_t + 0.5 z_{CL}^2}{n \times T + z_{CL}^2}$$

Comparing with CI_W, CI_{AC} is essentially obtained from an upward adjustment of the average observed default rate. To make up for the deficiency of CI_W mentioned above, the adjustment becomes more significant when both n and T are small. CI_{AC} should therefore perform better than CI_W in small sample settings. However, it is not sure if the adjustment is sufficient to cope with the lack of modeling of the default correlation. Furthermore, the adjustment may increase the Type II error, especially when LRPD is small. For example, when n = 50 and T = 10, the minimum of the upper bound of CI_{AC} is about 92 bps at a CL of 95%.[19] The probability of correctly rejecting a false null of "LRPD being higher than a value which is less than 92 bps" (i.e. in a one-tail test) is therefore zero. In other words, the resulting Type II error is 100%. It may therefore result in an *overly-conservative* conclusion in the validation process. Specifically, regardless of the realized number of defaults, there is no way that the assigned PD can be further reduced if it is already set at a value lower than 92 bps.

The upper and lower bound of the **Clopper-Pearson CI (CI_{CP})** of Clopper and Pearson (1934) are solutions to the following equations.[20]

$$\sum_{i=0}^{\bar{\theta} \times n \times T} \binom{n \times T}{i} U_{CP}^i \left(1-U_{CP}\right)^{n \times T - i} = \frac{(1-CL)}{2} \tag{34a}$$

$$\sum_{i=\bar{\theta} \times n \times T}^{n \times T} \binom{n \times T}{i} L_{CP}^i \left(1-L_{CP}\right)^{n \times T - i} = \frac{(1-CL)}{2} \tag{34b}$$

[19] The minimum is attained when we observe no default incidence.

[20] It can be shown that the upper bound is the 1-(1-CL)/2 quantile of the beta distribution $Beta\left[\bar{\theta} \times n \times T + 1, n \times T \times \left(1-\bar{\theta}\right)\right]$, while the lower bound is the (1-CL)/2 quantile of the beta distribution $Beta\left[\bar{\theta} \times n \times T, n \times T \times \left(1-\bar{\theta}\right) + 1\right]$. The beta distribution representations are easier to be implemented in practice.

Unlike CI_W, the construction of CI_{CP} does not rely on the limiting condition of infinite $n \times T$ and thus it should be robust in finite sample settings. Moreover, it can be shown that the coverage of CI_{CP} is always equal to or above the corresponding CL under the assumption that individual default events are i.i.d. However, it is not sure if the interval is wide enough to make up for the lack of modeling of default correlation. Again, the cost of a wide interval might be a high Type II error, especially when LRPD is small. For example, when n = 50 and T = 10, the minimum of the upper bound of CI_{CP} is about 74 bps.[21] The Type II error will therefore be 100% in a one-tail hypothesis test of "LRPD being higher than a value which is less than 74 bps", potentially resulting in overly-conservative conclusions in the validation process.

We also examine the performance of $\mathbf{CI_B}$ constructed by **non-parametric bootstrap**, which is similar to the one considered by Hanson and Schuermann (2005). Out of the default rates observed over time T, we randomly draw *with replacement* 5,000 sets of T default rates and construct the empirical distribution of the time-average default rate. At a CL of 95%, the upper and lower bounds of CI_B therefore correspond respectively to the 97.5 and 2.5 percentiles of the distribution. Similar to CI_{CP}, it is robust in finite sample applications given that it does not rely on attaining any limiting condition. However, although the cross-sectional default correlation structure can be preserved, the serial-correlation of the systematic factor is destroyed in the sampling exercises. The potential understatement of coverage due to the lack of modeling of the serial-correlation is believed to be more significant when the sample period is short (i.e. small T).

We also construct four different CIs based on the proposed MLE estimators. $\mathbf{CI_{MLE1}}$ and $\mathbf{CI_{MLE2}}$ are the CIs considered in Section 2 based on infinite number of borrowers. CI_{MLE1} is constructed based on equation (14a), while CI_{MLE2} is construct in a similar fashion but naively assuming that defaults are time independent (i.e. according to equation (14b) when β = 0). The validity of these CIs relies on the asymptotic behaviour being attained when both n and T are *sufficiently* large. They are likely to be too narrow when both n and T are small. The question we would like to answer is: "Under what conditions of the values of n and T will the performances of these CIs become *unacceptable* and will require us to resort to more accurate but computationally more demanding methods in constructing the appropriate CI (e.g. CI_{MLEf1} and CI_{MLEf2} introduced subsequently)?"

Finally, $\mathbf{CI_{MLEf1}}$ and $\mathbf{CI_{MLEf2}}$ are obtained from the MLE estimator of equation (30) and its asymptotic standard deviation of equation (31). They are therefore the counterparts of CI_{MLE1} and CI_{MLE2} but relaxing the assumption of infinite number of borrowers. They are therefore *exact* with respect to finite values of n. Again, CI_{MLEf2} is obtained by assuming time-independent default rates. CI_{MLEf1} and CI_{MLEf2} are however still approximations when T is not *sufficiently* large. They however should be the best among the eight CIs considered here. The only draw-back is it is computationally intensive.

We conduct the same simulation exercise in obtaining Table 6. We consider the four different combinations of parametric values presented in Table 1. For each set of

[21] This is again at a CL of 95% and the minimum is attained when we observe no default incidence.

parameters, we generate time-series of the number of defaults based on the serially-correlated single-factor model and under three different scenarios of constant initial number of borrowers (n_t = 50, 100 and 500) over time. For each case, we simulate 1,000 time-paths of the number of defaults. For each simulation, we construct the eight CIs based on the generated time-series of numbers of defaults and check if the true LRPD (i.e. either 0.5% or 2.0%) is rejected according to each CI in a two-tail hypothesis test at a CL of 95%. The proportion of rejections of each CI across the 1,000 simulations therefore represents the Type I error of that CI under the specific combination of parametric values. Similar to the analysis performed in Section 2, we also measure the Type 1 error of a one-tail test. For each simulation, we check if we would like to reject the (true) null because we think it is (statistically) too high. This is the case where the true null turns out to be higher than the upper bound of the CI. The consequence of this type of rejection may lead to the imprudent action of the financial institutions to wrongfully reduce the assigned probability of default which is in fact correct in the first place. Specifically, we want to find out the Type 1 error of this one-tail test at the 97.5% CL. The results are reported in Table 7.

INSERT TABLE 7 ABOUT HERE

Our simulation results reported in Table 7 suggest that using CI_W in hypothesis tests can easily result in unacceptable level of Type I error. For example, when the true LRPD is 2.0%, T is 10 and n is 500, the Type I errors of the two-tail and one-tail tests are 72.0% and 44.5% respectively. The errors being increasing functions of n may be due to the fact that the existence of default correlation hinders the rate of convergence of the distribution.

Using CI_{AC} always results in a smaller Type I error than by using CI_W. The benefit is particularly significant in low LRPD, and when both T and n are small. It produces the smallest errors among all the eight CIs when the true LRPD is 0.5%, while T and n equal to 10 and 50 respectively. However, its errors are almost as bad as those of CI_W for high LRPD, and when both T and n are relatively large. Again, it might be attributed to the lack of modeling of the default correlation. Furthermore, the use of such a wide CI is likely to come with the price of large Type II errors. In the above case where the true LRPD is 0.5%, n = 50 and T = 10, further analysis (not reported here) suggests that using CI_{AC} results in the probabilities of wrongfully accepting the false alternatives of LRPD equals to 25bps and 75bps being 82.8% and 95.9% respectively in a two-tail test. The corresponding Type II errors are 73.7% and 55.4% when CI_W is used.

The performance of CI_{CP} is almost identical to that of CI_{AC} with marginally smaller errors in the two-tail tests. Although its coverage is always larger than the CL when default events are i.i.d., apparently it is not sufficient to cope with the widening of the CI due to default correlation, especially when the true LRPD is relatively high.

In terms of the resulting Type I errors, the performance of CI_B is quite uniform across all the cases considered in this simulation exercise. Its performance is much better than the three CIs discussed previously when n is equal to or larger than 100. It seems to be robust to the existence of default correlation given that the correlation structure is preserved in the bootstrapping. The errors are slightly larger when T is equal to 10 rather than 25. It is likely due to the fact that the modeling of the serial correlation, which is ignored in the bootstrapping, is more important in short sample.

Regardless of the number of borrowers, using CI_{MLE1} always results in a smaller error than CI_W when LRPD is equal to 2.0%. Its appropriateness however is questionable when LRPD is equal to 0.5%. Unless we are working with a large number of borrowers (e.g. n = 500), we should not be using CI_{MLE1} for low default rate portfolio. As expected, the asymptotically-exact but computationally-intensive CI_{MLEf1} produces consistently smaller errors than CI_W and CI_B across all the cases considered here. Nevertheless, the errors are still higher than the theoretical values of 5.0% and 2.5% for the two- and one-tail tests respectively, especially when T is small and LRPD is low. Moreover, the errors are almost as large as those produced by CI_B when LRPD is equal to 0.5%, T is equal to 10 and n is equal to or smaller than 100. On the other hand, the advantage over the analytically-tractable CI_{MLE1} disappears when LRPD is equal to 2.0% and n is equal to 500. Finally, in comparing CI_{MLEf1} (CI_{MLE1}) and CI_{MLEf2} (CI_{MLE2}), the marginal benefit of modeling for the serial correlation is hardly noticeable in terms of the resulting Type I errors.

To further examine the impact of the default correlation on the performances of the different CIs, we repeat the simulation analysis under the case of LRPD = 2.0%, T = 10 and n = 100, but with R^2 taking on values of 10%, 20%, 25%, 30% and 40% respectively. In Figure 4 and 5, we plot the resulting Type I errors of CI_W, CI_{AC}, CI_{CP} and CI_B in excess of those of CI_{MLE1} in the two- and one-tail tests respectively. As expected, the excess errors of CI_W, CI_{AC} and CI_{CP} increase with R^2. Even at relatively low level of default correlation (e.g. 10%), the excess errors can be quite significant if the default correlation is ignored. On the contrary, the performance of CI_B is quite robust to different values of default correlations.

From the results reported in Table 7, we may conclude that:

(i) When LRPD is relatively low, and T and n are both small, we might as well resort to bootstrapping rather than relying on the asymptotic CIs of CI_{MLE1} and CI_{MLEf1};

(ii) When LRPD is relatively high and n is large, the use of the analytically-tractable CI_{MLE1} is recommended over CI_{MLEf1}, given the ease of computation of the former and the almost identical performances; and

(iii) In other situations, it might be worthwhile to invest in the extra efforts in using CI_{MLEf1}.

INSERT FIGURE 4 AND 5 ABOUT HERE

5. Conclusions

In this paper we examine alternative methodologies in estimating LRPD from time series of historical default rates, using efficient estimators that incorporate both the cross-sectional and serial asset correlation exhibited in actual credit portfolio. We first, adopting Basel's infinitely granular portfolio assumption, propose a LRPD estimation methodology. Next we relax this assumption to examine alternative estimation methodologies and their performances for finite number of borrowers. The former would be appropriate for the estimation of regulatory (credit) capital under Basel II given the consistency of the underlying economic model. The latter is considered to be more appropriate for the estimation of economic (credit) capital, where the advance simulation techniques are capable of modeling credit portfolios which are made up of a specific number of borrowers.

We examine the performances of the proposed estimators relative to alternative estimators via a number of simulation exercises. In most of cases and for portfolios of both finite and infinite number of borrowers, the proposed estimators outperform the alternatives even in a number of small sample settings. The simulation results also suggest that simple averaging of the default rates as an estimator of LRPD is prone to underestimate the true LRPD. The proposed methodology also enables us to supplement internal data with external data which improves the accuracy and representation of the LRPD estimate. We also consider alternative ways of establishing CIs for the validation of pre-assigned LRPDs.

Appendices

Appendix A: Derivation of the GLS Estimator of *DP* and its CI presented in Equations (12) to (14)

Rewriting equation (7), we have

$$\Phi^{-1}(\theta_t) = \frac{DP}{\sqrt{1-R^2}} - \frac{R}{\sqrt{1-R^2}} P_t \tag{A1}$$

The GLS estimator of *DP* is asymptotically most efficient when P_t follows an AR(1) process of equation (11) with known serial-correlation coefficient β. It is given in Greene (1997) pg. 507 that:

$$DP_{GLS} = \left(X'\Lambda^{-1}X\right)^{-1} X'\Lambda^{-1} y \tag{A2}$$

where

$$X = \frac{1}{\sqrt{1-R^2}} 1_T \quad \text{of which } 1_T \text{ is the unit vector;}$$

$$y = \left[\Phi^{-1}(\theta_1) \quad \Phi^{-1}(\theta_2) \quad \ldots \quad \Phi^{-1}(\theta_T)\right]' \text{; and}$$

$$\Lambda = \frac{R^2}{1-R^2} \times \begin{bmatrix} 1 & \beta & \beta^2 & . & \beta^{T-1} \\ \beta & 1 & \beta & . & \beta^{T-2} \\ \beta^2 & \beta & 1 & . & \beta^{T-3} \\ . & . & . & . & . \\ \beta^{T-1} & \beta^{T-2} & \beta^{T-3} & . & 1 \end{bmatrix}.$$

It can be shown (e.g. by Greene (1997) on pg. 589) that:

$$\Lambda^{-1} = \frac{1-R^2}{R^2\left(1-\beta^2\right)} \times \begin{bmatrix} 1 & -\beta & 0 & . & 0 \\ -\beta & 1+\beta^2 & -\beta & . & 0 \\ 0 & -\beta & 1+\beta^2 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \end{bmatrix} \tag{A3}$$

Therefore,

$$\left(X'\Lambda^{-1}X\right)^{-1} = \frac{R^2\left(1-\beta^2\right)}{(T-2)\beta^2 - 2(T-1)\beta + T} \tag{A4}$$

$$X'\Lambda^{-1}y = \frac{(1-\beta)\sqrt{1-R^2}}{\left(1-\beta^2\right)R^2}\left(\Phi^{-1}(\theta_1) + \sum_{t=2}^{T-1}\Phi^{-1}(\theta_t)\cdot(1-\beta) + \Phi^{-1}(\theta_T)\right) \tag{A5}$$

Finally, substituting equations (A4) and (A5) into (A2), we have

$$DP_{GLS} = \frac{(1-\beta)\sqrt{1-R^2}}{(T-2)\beta^2 - 2(T-1)\beta + T}\left(\Phi^{-1}(\theta_1) + \sum_{t=2}^{T-1}\Phi^{-1}(\theta_t)\cdot(1-\beta) + \Phi^{-1}(\theta_T)\right). \quad \text{(A6)}$$

The variance of the GLS estimator is:

$$Var(DP_{GLS}) = \left(X'\Lambda^{-1}X\right)^{-1} = \frac{R^2\left(1-\beta^2\right)}{(T-2)\beta^2 - 2(T-1)\beta + T} \quad \text{(A7)}$$

Based on this variance, a confidence interval for the LRPD may therefore be constructed:

$$\begin{aligned} CI_{MLE} &= \left[L_{MLE}, U_{MLE}\right] \\ U_{MLE} &= \Phi\left(DP_{MLE} + z_{CL} \times \frac{R \times \sqrt{1-\beta^2}}{\sqrt{(T-2)\beta^2 - 2(T-1)\beta + T}}\right) \\ L_{MLE} &= \Phi\left(DP_{MLE} - z_{CL} \times \frac{R \times \sqrt{1-\beta^2}}{\sqrt{(T-2)\beta^2 - 2(T-1)\beta + T}}\right) \end{aligned} \quad \text{(A8)}$$

Appendix B: Description of the LRPD estimator: *$LPRD_{mod}$*

In this Appendix, we summarize the formulation of the estimator $LPRD_{mod}$, which was examined by the Office of the Superintendent of Financial Institutions (OSFI) of Canada. For detail illustrations, please refer to OSFI document: "Risk Quantification of IRB Systems at IRB Banks: Appendix - A Conservative Estimate of a Long-Term Average PD by a Hypothetical Bank," December 2004.

The OSFI document outlines a method to estimate LRPD by observing the time-series average default rate of a portfolio of uniform credit risk. Essentially, it proposes to estimate the LRPD which results in the highest chance to generate the observed average default rate. To achieve this objective, it suggests conducting simulations to obtain the distribution of average default rate. The distribution of default rates implicit in Basel II capital formula is utilized in the simulation exercise. Specifically, under the infinite-granular single-factor model of Basel II, $\Phi^{-1}(\theta_t)$ attains a normal distribution with mean and standard deviation equal to $DP\big/\sqrt{1-R^2}$ and $R\big/\sqrt{1-R^2}$ respectively (see equation (5)). Given a certain level of DP (i.e. LRPD) and R^2, we can therefore construct the time-series of default rates over a certain period of time (from t = 1 to T) by simulating i.i.d. normally distributed random numbers. The systematic factor is therefore assumed to be time-independent. By repeating the simulations 10,000 times, we can therefore approximate the distribution of the time-series average default rate θ_{ave}, which is contingent on the values of the assumed LRPD and R^2. The OSFI document suggests we can solve for the LRPD by matching the mode (i.e. the realization with the highest chance of occurrence) of the distribution of the simulated average default rate θ_{ave} with the observed average default rate $\hat{\theta}_{ave}$. That is, we solve for $LRPD_{mod}$ which satisfies equation (B1).[22]

$$\hat{\theta}_{ave} = \mathrm{mod}\left(LRPD_{\mathrm{mod}}, R^2\right) \qquad \text{(B1)}$$

where $\mathrm{mod}(\bullet)$ is the mode of the distribution of the average default rate given the arguments of the function. Since there is no explicit analytical equation to relate the mode of the distribution with the underlying LRPD, this approach involves changing the assumed value of LRPD until we obtain a mode (through simulations), which is identical (or close enough) to the observed average default rate $\hat{\theta}_{ave}$.

It should be noted that $LRPD_{mod}$ is not a maximum likelihood estimator. According to the proposed methodology, it is most likely to observe the realized average default rate (among all the possible values of average default rates) given the solved long-run PD (i.e. $LRPD_{mod}$). There is however no guarantee that, among all the possible values of LRPD, $LRPD_{mod}$ results in the highest probability of realizing the observed average default rate. Another value of LRPD might as well result in an even higher probability of realizing the observed average default rate, while the mode of the resulting distribution happens to be different from the observed value.

Appendix C: Derivations of Equations (18)-(23)

The limiting unconditional distribution of $\theta_{x,t}$ of equation (18) is identical to equation (6) of Section 1. Under the infinitely-granular single-factor model, $\Phi^{-1}\left(\theta_{x,t}\right)$ and $\Phi^{-1}\left(\theta_t\right)$ follow a bivariate normal distribution where the means are $DP_x \big/ \sqrt{1-R_x^2}$ and $DP \big/ \sqrt{1-R^2}$ respectively, and covariance matrix equals to:

$$\begin{bmatrix} \dfrac{R_x^2}{1-R_x^2} & \rho \dfrac{R_x R}{\sqrt{\left(1-R_x^2\right)\cdot\left(1-R^2\right)}} \\ \rho \dfrac{R_x R}{\sqrt{\left(1-R_x^2\right)\cdot\left(1-R^2\right)}} & \dfrac{R^2}{1-R^2} \end{bmatrix} \qquad \text{(C1)}$$

The distribution of $\Phi^{-1}\left(\theta_t\right)$ conditional on $\Phi^{-1}\left(\theta_{x,t}\right)$ is therefore also normal with mean and standard deviation equal to $\dfrac{DP}{\sqrt{1-R^2}} + \rho \dfrac{R\sqrt{1-R_x^2}}{R_x\sqrt{1-R^2}}\left(\Phi^{-1}\left(\theta_{x,t}\right) - \dfrac{DP_x}{\sqrt{1-R_x^2}}\right)$ and $\dfrac{R^2\left(1-\rho^2\right)}{1-R^2}$ respectively. That is, the conditional cumulative probability function is:

$$F_\infty\left(\theta_t \middle| \theta_{x,t}\right) = \Phi\left(\frac{1}{R\sqrt{1-\rho^2}}\left(z_t - \rho\frac{R}{R_x} z_{x,t}\right)\right) \qquad \text{(C2)}$$

---

[22] Here, we assume R is known.

where

$$z_t = \sqrt{1-R^2} \times \Phi^{-1}(\theta_t) - DP$$
$$z_{x,t} = \sqrt{1-R_x^2} \times \Phi^{-1}(\theta_{x,t}) - DP_x$$

The conditional probability density function of θ_t is therefore equal to the derivative of equation (C2) with respect to θ_t.

$$f_\infty(\theta_t | \theta_{x,t}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2R^2(1-\rho^2)} \left(z_t - \rho \frac{R}{R_x} z_{x,t} \right)^2 \right) \times \frac{\sqrt{1-R^2}}{R\sqrt{1-\rho^2}} \times \frac{d\Phi^{-1}}{d\theta_t}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2R^2(1-\rho^2)} \left(z_t - \rho \frac{R}{R_x} z_{x,t} \right)^2 \right) \times \frac{\sqrt{1-R^2}}{R\sqrt{1-\rho^2}} \times \sqrt{2\pi} \times \exp\left(\frac{\Phi^{-1}(\theta_t)^2}{2} \right)$$

$$= \frac{\sqrt{1-R^2}}{R\sqrt{1-\rho^2}} \times \exp\left(-\frac{(z_t - R \cdot \rho \cdot z_{x,t} / R_x)^2}{2(1-\rho^2)R^2} + \frac{1}{2}(\Phi^{-1}(\theta_t))^2 \right)$$

If R^2, R_x^2 and ρ are known, we can then solve for the MLE of LRPD by taking the first derivatives of the joint likelihood function of equation (17) with respect to $\Phi(DP_x)$ and $\Phi(DP)$, and setting them to zero. The MLE estimators are respectively:

$$LRPD_{MLE,x} = \Phi\left(\frac{\sqrt{1-R_x^2}}{T_x} \left(\sum_{t=1}^{T_x} \Phi^{-1}(\theta_{x,t}) \right) \right)$$

$$LRPD_{MLE} = \Phi\left(\frac{1}{T} \left(\sqrt{1-R^2} \sum_{t=1}^{T} \Phi^{-1}(\theta_t) + \frac{R \cdot \rho}{R_x} \left(T \cdot LRPD_{MLE,x} - \sqrt{1-R_x^2} \sum_{t=1}^{T} \Phi^{-1}(\theta_{x,t}) \right) \right) \right)$$

Finally, the asymptotic standard deviations of the MLEs are obtained by evaluating the second derivatives of the joint likelihood function at the MLE solutions.

$$\sigma_{MLE,x} = \left(\frac{R_x^2}{T_x + T\,\rho^2/(1-\rho^2)} \right)^{0.5} \cdot \left(\sqrt{2\pi} \exp\left(\frac{(\Phi^{-1}(\theta))^2}{2} \right) \right)^{-1}_{\theta = LRPD_{MLE,x}}$$

$$\sigma_{MLE} = \left(\frac{(1-\rho^2)R^2}{T} \right)^{0.5} \cdot \left(\sqrt{2\pi} \exp\left(\frac{(\Phi^{-1}(\theta))^2}{2} \right) \right)^{-1}_{\theta = LRPD_{MLE}}$$

Appendix D: Derivation of Equation (27)

$$E\left[\frac{1}{T-1} \sum_{t=1}^{T} \left(\Phi^{-1}(\theta_t) - \overline{\mu}_\theta \right)^2 \right]$$

$$
\begin{aligned}
&= \frac{1}{T-1} E\left[\sum_{t=1}^{T}\left(\Phi^{-1}(\theta_t)^2 - 2\Phi^{-1}(\theta_t)\overline{\mu}_\theta + \overline{\mu}_\theta^2\right)\right] \\
&= \frac{1}{T-1}\left[T\,\mathrm{var}\left(\Phi^{-1}(\theta_t)\right) + T\,\mathrm{var}\left(\overline{\mu}_\theta\right) - 2\sum_{t=1}^{T}\mathrm{cov}(\Phi^{-1}(\theta_t),\overline{\mu}_\theta)\right] \\
&= \frac{R^2}{\left(1-R^2\right)\left(T-1\right)}\left[T + \frac{1_T'\,\Omega_\beta 1_T}{T} - 2\frac{1_T'\,\Omega_\beta 1_T}{T}\right] \\
&= \frac{R^2}{\left(1-R^2\right)} \times \frac{\left(T - \frac{1_T'\,\Omega_\beta 1_T}{T}\right)}{\left(T-1\right)}
\end{aligned}
$$

References

Agresti, A., and B.A. Coull, 1998, "Approximate is Better than "Exact" for Interval Estimation of Binomial Proportions," *The American Statistician* 52, pp. 119-126

Balthazar, L., 2004, "PD Estimates for Basel II," *RISK*, pp. 84-85

Benjamin, N., A. Cathcart, and K. Ryan, 2006, "Low Default Portfolios: A Proposal for Conservative Estimation of Default Probabilities," working paper, Financial Services Authority

Blochwitz, S., S. Hohl, D. Tasche, and C.S. When, 2004, "Validating Default Probabilities on Short Time Series," working paper, Deutsche Bundesbank

Brown, L.D., T.T. Cai, and A. DasGupta, 2001, "Interval Estimation for a Binomial Proportion," *Statistical Science* 16, pp. 101-117

Carling, K., T. Jacobson, J. Lindé, and K. Roszbach, 2007, "Corporate Credit Risk Modeling and the Macroeconomy," *Journal of Banking & Finance* 31, pp. 845-868

Chava, S., C. Stefanescu, and S.M. Turnbull, 2006, "Modeling Expected Loss," working paper

Christensen, J.H.E., E. Hansen, and D. Lando, 2004, "Confidence Sets for Continuous-Time Rating Transition Probabilities," *Journal of Banking & Finance* 28, pp. 2575-2602

Clopper, C.J., and E.S. Pearson, 1934, "The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial," *Biometrika* 26, pp. 404-413

Frey, R., and A.J. McNeil, 2003, "Dependent Defaults in Models of Portfolio Credit Risk," *Journal of Risk* 6, pp. 59-92

Gordy, M.B., 2003, "A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules," *Journal of Financial Intermediation* 12, pp. 199-232

Gordy, M.B., and E. Heitfield, 2002, "Estimating Default Correlations from Short Panels of Credit Rating Performance Data," working paper, Federal Reserve Board

Greene, W.H., 1997, "Econometric Analysis, 3rd edition," Prentice Hall, Upper Saddle River, New Jersey

Hanson, S., and T. Schuermann, 2005, "Confidence Intervals for Probabilities of Default," working paper, Federal Reserve Bank of New York

Nickell, P., W. Perraudin, and S. Varotto, 2000, "Stability of Rating Transitions," *Journal of Banking & Finance* 24, pp. 203-227

Pluto, K., and D. Tasche, 2005, "Estimating Probabilities of Default for Low Default Portfolios," working paper, Deutsche Bundesbank

Tarashev, N., and H. Zhu, 2007, "Modelling and Calibration Errors in Measures of Portfolio Credit Risk," working paper, Bank for International Settlements

Vasicek, O., 1987, "Probability of Loss on Loan Portfolio," working paper, Moody's KMV Corporation

Table 1: Parametric Values of Simulation Exercises

	Case 1	**Case 2**	**Case 3**	**Case 4**
True LRPD	0.5%	2.0%	0.5%	2.0%
R^2	25%	25%	25%	25%
β	10%	10%	10%	10%
T	10 yrs.	10 yrs.	25 yrs.	25 yrs.

Table 2: Summary Statistics of the Distribution of the Estimators of LRPD (in percents)

Case 1: True LRPD =0.5%; *T* = 10

	$LRPD_{ave}$	$LRPD_{mod}$	$LRPD_{MLE1}$	$LRPD_{MLE2}$
Mean	0.510	0.771	0.566	0.567
Median	0.414	0.651	0.509	0.510
% of times LRPD is underestimated	62%	32%	49%	49%
Std. deviation	0.370	0.475	0.292	0.293
97.5 prctile	1.503	2.031	1.285	1.287
2.5 prctile	0.106	0.225	0.178	0.180
97.5-2.5 prctile	1.397	1.806	1.106	1.107

Case 2: True LRPD =2.0%; *T* = 10

	$LRPD_{ave}$	$LRPD_{mod}$	$LRPD_{MLE1}$	$LRPD_{MLE2}$
Mean	2.033	2.528	2.178	2.179
Median	1.808	2.302	2.031	2.034
% of times LRPD is underestimated	58%	37%	49%	49%
Std. deviation	1.075	1.152	0.923	0.926
97.5 prctile	4.752	5.403	4.375	4.382
2.5 prctile	0.616	0.962	0.838	0.844
97.5-2.5 prctile	4.136	4.441	3.537	3.537

Case 3: True LRPD =0.5%; *T* = 25

	$LRPD_{ave}$	$LRPD_{mod}$	$LRPD_{MLE1}$	$LRPD_{MLE2}$
Mean	0.503	0.628	0.527	0.527
Median	0.460	0.581	0.503	0.504
% of times LRPD is underestimated	58%	36%	49%	49%
Std. deviation	0.226	0.252	0.171	0.171
97.5 prctile	1.055	1.240	0.921	0.924
2.5 prctile	0.192	0.274	0.258	0.259
97.5-2.5 prctile	0.863	0.967	0.663	0.665

Case 4: True LRPD =2.0%; *T* = 25

	$LRPD_{ave}$	$LRPD_{mod}$	$LRPD_{MLE1}$	$LRPD_{MLE2}$
Mean	2.012	2.142	2.074	2.074
Median	1.921	2.045	2.013	2.012
% of times LRPD is underestimated	55%	47%	49%	49%
Std. deviation	0.663	0.670	0.559	0.559
97.5 prctile	3.529	3.691	3.324	3.334
2.5 prctile	0.970	1.102	1.146	1.151
97.5-2.5 prctile	2.559	2.589	2.179	2.183

Table 3: Simulated Type 1 Errors

Panel A: Two-Tail Test at 95% Confidence Level

	CI_{ave}	CI_{mod}	CI_{MLE1}	CI_{MLE2}
Case 1: True LRPD=0.5%; $T = 10$	27.7%	17.6%	5.2%	7.7%
Case 2: True LRPD =2.0%; $T = 10$	22.2%	11.4%	5.2%	7.7%
Case 3: True LRPD =0.5%; $T = 25$	19.4%	11.8%	5.4%	7.6%
Case 4: True LRPD =2.0%; $T = 25$	14.8%	10.4%	5.4%	7.6%

Panel B: One-Tail Test at 97.5% Confidence Level

	CI_{ave}	CI_{mod}	CI_{MLE1}	CI_{MLE2}
Case 1: True LRPD =0.5%; $T = 10$	27.5%	12.4%	2.5%	3.7%
Case 2: True LRPD =2.0%; $T = 10$	21.5%	10.2%	2.5%	3.7%
Case 3: True LRPD =0.5%; $T = 25$	19.2%	8.9%	2.8%	3.9%
Case 4: True LRPD =2.0%; $T = 25$	14.1%	9.2%	2.8%	3.9%

Table 4: Internal and External Annual Default Rate Data

	Internal Risk Rating	S&P's Speculative Grade
1981	-	0.62%
1982	-	4.41%
1983	-	2.96%
1984	-	3.29%
1985	-	4.37%
1986	-	5.71%
1987	-	2.80%
1988	-	3.99%
1989	-	4.16%
1990	-	7.87%
1991	-	10.67%
1992	-	5.85%
1993	-	2.20%
1994	-	2.19%
1995	-	3.62%
1996	0.79%	1.83%
1997	0.21%	2.15%
1998	0.62%	3.22%
1999	0.95%	5.16%
2000	1.17%	7.00%
2001	1.17%	10.51%
2002	0.95%	7.12%
2003	0.23%	5.55%
2004	0.01%	2.30%

Table 5: Estimated LRPD of Internal and External Portfolio

	Bank's Internal Risk Rating		**S&P's Speculative Grade**	
	Jointly	**Separately**	**Jointly**	**Separately**
$LRPD_{MLE}$	0.765%	0.841%	4.585%	4.585%
95% CI				
Upper	1.378%	1.682%	5.633%	5.724%
Lower	0.406%	0.395%	3.699%	3.635%

Table 6: Summary Statistics (in percents) of Sample Distributions of LRPD Estimators under Finite Initial Number of Borrowers

n = 50

	T = 10								T = 25							
	True LRPD = 0.5%				True LRPD = 2.0%				True LRPD = 0.5%				True LRPD = 2.0%			
	mean	Med	%under	std	mean	med	%under	std	mean	med	%under	std	mean	med	%under	std
$LRPD_{ave}$	0.497	0.400	63%	0.495	1.967	1.800	62%	1.206	0.527	0.480	58%	0.298	1.987	1.920	58%	0.763
$LRPD_{mod}$	0.745	0.633	43%	0.622	2.474	2.332	41%	1.319	0.666	0.618	33%	0.305	2.146	2.079	50%	0.776
$LRPD_{MLE1}$	0.428	0.292	69%	0.532	2.103	1.874	54%	1.324	0.350	0.289	76%	0.286	1.973	1.882	55%	0.799
$LRPD_{MLE2}$	0.417	0.281	72%	0.523	2.073	1.830	56%	1.316	0.345	0.286	76%	0.283	1.960	1.870	56%	0.796
$LRPD_{MLEf1}$	0.534	0.491	51%	0.489	2.074	1.866	54%	1.162	0.538	0.501	49%	0.277	2.024	1.945	53%	0.712
$LRPD_{MLEf2}$	0.534	0.490	51%	0.489	2.073	1.864	54%	1.162	0.538	0.502	49%	0.277	2.023	1.946	53%	0.711

n = 100

	T = 10								T = 25							
	True LRPD = 0.5%				True LRPD = 2.0%				True LRPD = 0.5%				True LRPD = 2.0%			
	mean	Med	%under	std	mean	med	%under	std	mean	med	%under	std	mean	med	%under	std
$LRPD_{ave}$	0.502	0.400	66%	0.417	2.005	1.800	60%	1.143	0.513	0.480	55%	0.269	2.005	1.920	56%	0.740
$LRPD_{mod}$	0.756	0.633	29%	0.526	2.521	2.332	41%	1.248	0.652	0.618	34%	0.276	2.164	2.079	46%	0.753
$LRPD_{MLE1}$	0.479	0.399	62%	0.448	2.160	2.013	50%	1.169	0.411	0.374	69%	0.261	2.023	1.954	53%	0.729
$LRPD_{MLE2}$	0.469	0.398	65%	0.443	2.135	1.973	51%	1.164	0.407	0.368	69%	0.259	2.013	1.939	54%	0.727
$LRPD_{MLEf1}$	0.553	0.478	51%	0.418	2.129	1.949	51%	1.096	0.527	0.500	50%	0.239	2.051	1.976	52%	0.676
$LRPD_{MLEf2}$	0.553	0.479	51%	0.418	2.127	1.951	51%	1.095	0.526	0.500	50%	0.238	2.049	1.975	52%	0.676

n = 500

	T = 10								T = 25							
	True LRPD = 0.5%				True LRPD = 2.0%				True LRPD = 0.5%				True LRPD = 2.0%			
	mean	Med	%under	std	mean	med	%under	std	mean	med	%under	std	mean	med	%under	std
$LRPD_{ave}$	0.499	0.400	63%	0.361	1.998	1.800	58%	1.050	0.504	0.456	58%	0.247	1.998	1.912	56%	0.683
$LRPD_{mod}$	0.755	0.633	32%	0.458	2.520	2.332	37%	1.149	0.643	0.594	31%	0.253	2.158	2.071	45%	0.695
$LRPD_{MLE1}$	0.541	0.462	54%	0.350	2.167	2.052	48%	0.955	0.493	0.463	58%	0.216	2.031	1.985	51%	0.585
$LRPD_{MLE2}$	0.534	0.459	54%	0.348	2.158	2.034	48%	0.959	0.490	0.458	58%	0.216	2.026	1.981	51%	0.586
$LRPD_{MLEf1}$	0.548	0.480	53%	0.323	2.159	1.990	50%	0.983	0.521	0.497	50%	0.200	2.052	2.000	50%	0.585
$LRPD_{MLEf2}$	0.548	0.480	53%	0.322	2.163	2.010	49%	0.986	0.521	0.498	50%	0.200	2.050	1.998	50%	0.583

Table 7- Simulated Type 1 Errors of Different CIs

$n = 50$

	$T = 10$				$T = 25$			
	$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$		$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$	
	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail
CI_W	23.0%	18.9%	33.6%	25.0%	15.2%	9.3%	30.9%	19.2%
CI_{AC}	7.4%	0%	25.4%	12.0%	12.0%	1.6%	30.6%	15.6%
CI_{CP}	7.4%	0%	23.0%	12.0%	9.6%	1.6%	28.5%	15.6%
CI_B	19.4%	18.9%	20.1%	19.6%	14.6%	14.3%	13.4%	12.3%
CI_{MLE1}	56.2%	47.6%	23.7%	17.3%	51.3%	46.2%	21.6%	15.1%
CI_{MLE2}	58.6%	48.7%	26.3%	18.6%	54.6%	48.6%	26.0%	18.1%
CI_{MLEf1}	18.9%	18.9%	14.1%	14.0%	9.5%	9.3%	8.2%	7.6%
CI_{MLEf2}	18.9%	18.9%	13.9%	13.8%	9.5%	9.3%	8.3%	7.7%

$n = 100$

	$T = 10$				$T = 25$			
	$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$		$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$	
	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail
CI_W	35.9%	28.7%	45.0%	28.4%	32.4%	22.8%	48.5%	29.0%
CI_{AC}	17.8%	6.1%	44.7%	24.5%	23.6%	9.9%	47.5%	24.9%
CI_{CP}	15.2%	6.1%	42.8%	24.5%	21.8%	9.9%	46.0%	24.9%
CI_B	18.4%	17.8%	20.3%	19.0%	15.4%	14.4%	14.9%	13.3%
CI_{MLE1}	37.7%	30.9%	15.0%	10.0%	38.1%	33.5%	15.0%	10.6%
CI_{MLE2}	38.6%	31.0%	19.3%	13.0%	41.7%	35.9%	18.9%	12.5%
CI_{MLEf1}	18.0%	17.8%	12.0%	11.6%	7.7%	7.3%	9.9%	8.7%
CI_{MLEf2}	18.0%	17.8%	12.0%	11.6%	7.8%	7.4%	10.2%	9.1%

$n = 500$

	$T = 10$				$T = 25$			
	$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$		$\Phi(DP) = 0.5\%$		$\Phi(DP) = 2.0\%$	
	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail	Two-Tail	One-Tail
CI_W	59.6%	38.3%	72.0%	44.5%	58.7%	35.9%	70.1%	40.1%
CI_{AC}	59.1%	35.4%	71.0%	42.1%	56.2%	32.5%	70.1%	39.2%
CI_{CP}	58.4%	35.4%	70.5%	42.1%	58.1%	34.4%	69.9%	39.2%
CI_B	26.2%	25.7%	22.5%	21.1%	18.6%	17.6%	16.3%	15.0%
CI_{MLE1}	15.4%	11.7%	7.8%	5.0%	16.1%	11.8%	7.5%	5.0%
CI_{MLE2}	19.4%	13.8%	11.7%	6.8%	20.0%	14.5%	10.4%	6.7%
CI_{MLEf1}	13.0%	12.4%	13.1%	10.5%	9.5%	8.3%	11.2%	9.1%
CI_{MLEf2}	13.6%	12.9%	13.0%	10.2%	10.1%	8.7%	12.1%	9.4%

Figure 1: Plot of the Ratio of the Sample Standard Deviation of $LRPD_{MLE1}$ to that of $LRPD_{ave}$ against T

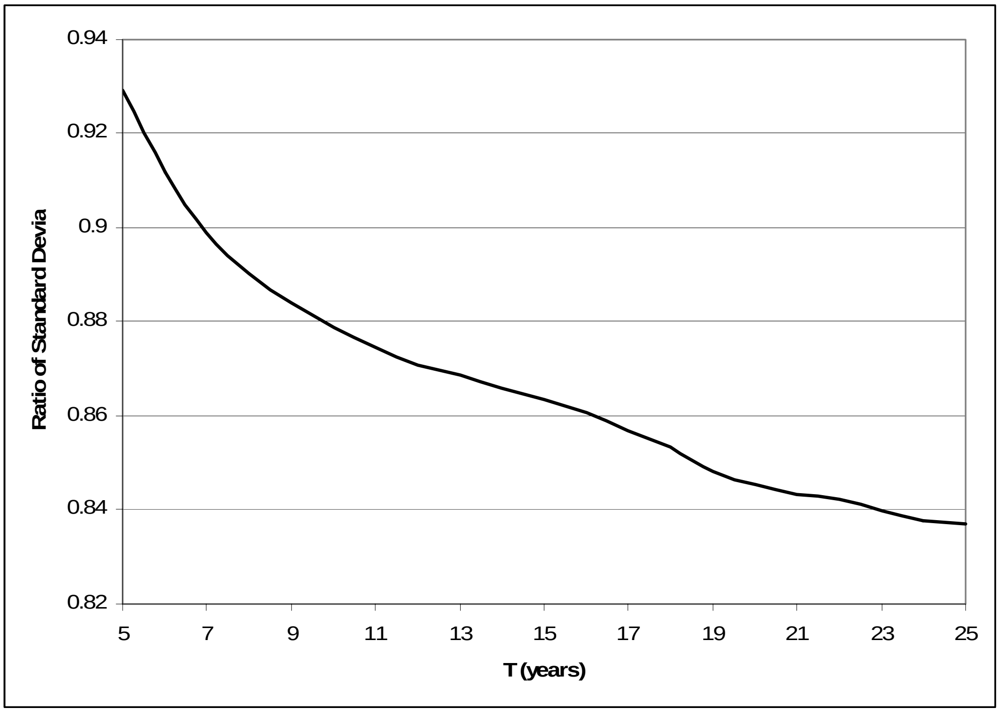

Figure 2: Plot of the Ratio of the Sample Standard Deviation of $LRPD_{MLE1}$ to that of $LRPD_{ave}$ against the True LRPD

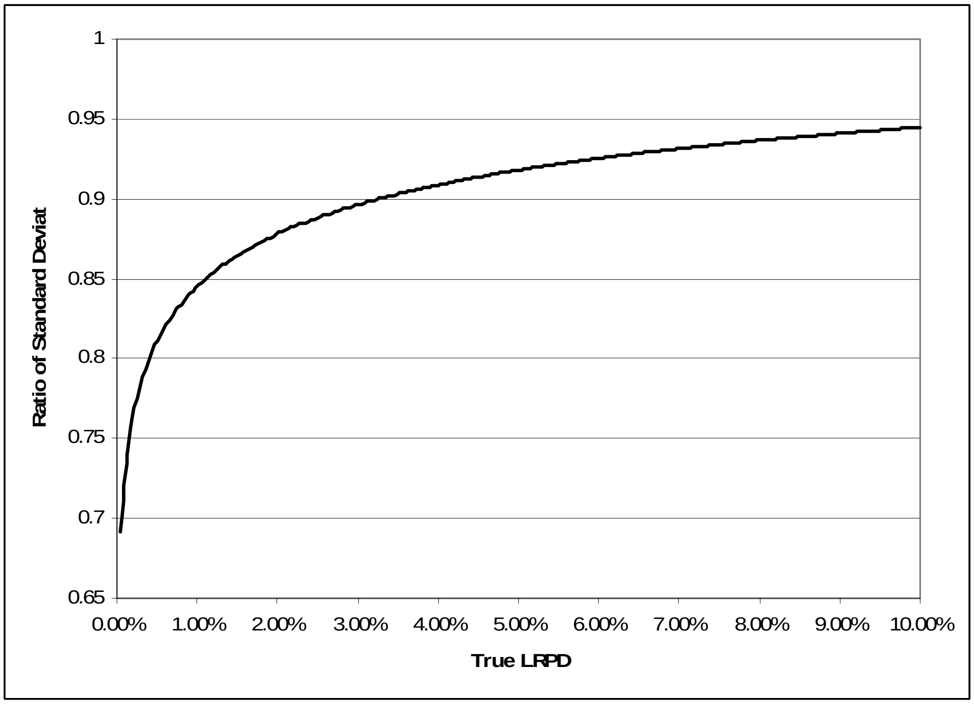

Figure 3: Time-Series Plots of Systematic Factor based on Joint Estimation with Both Internal and External Default Rates

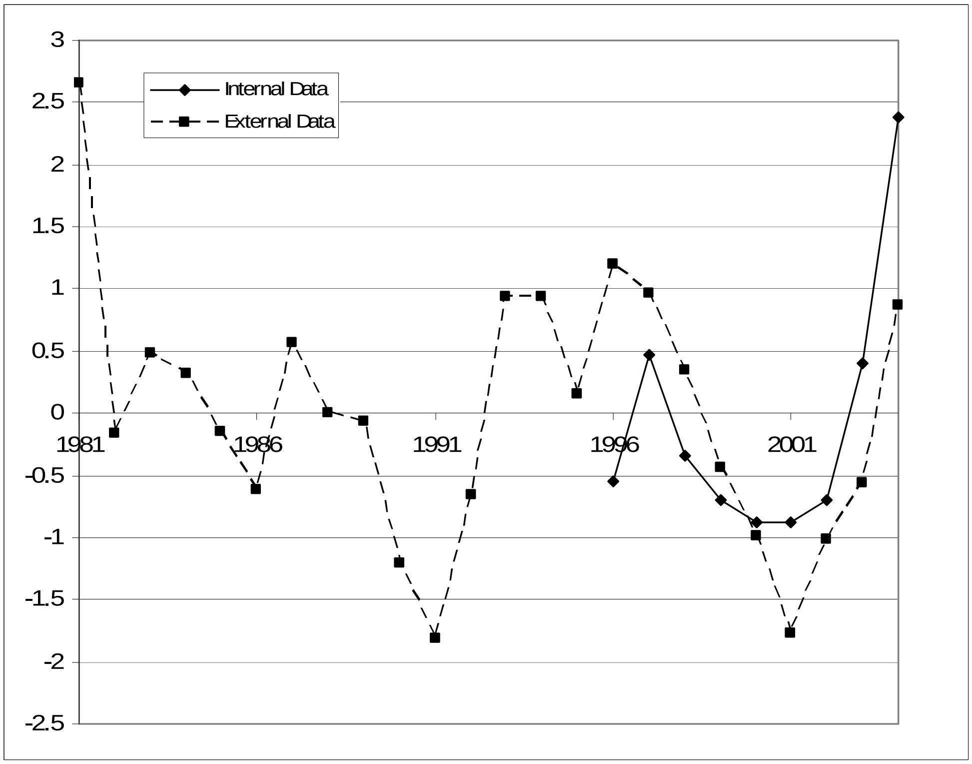

Figure 4: Excess Type I Errors of Two-Tail Test

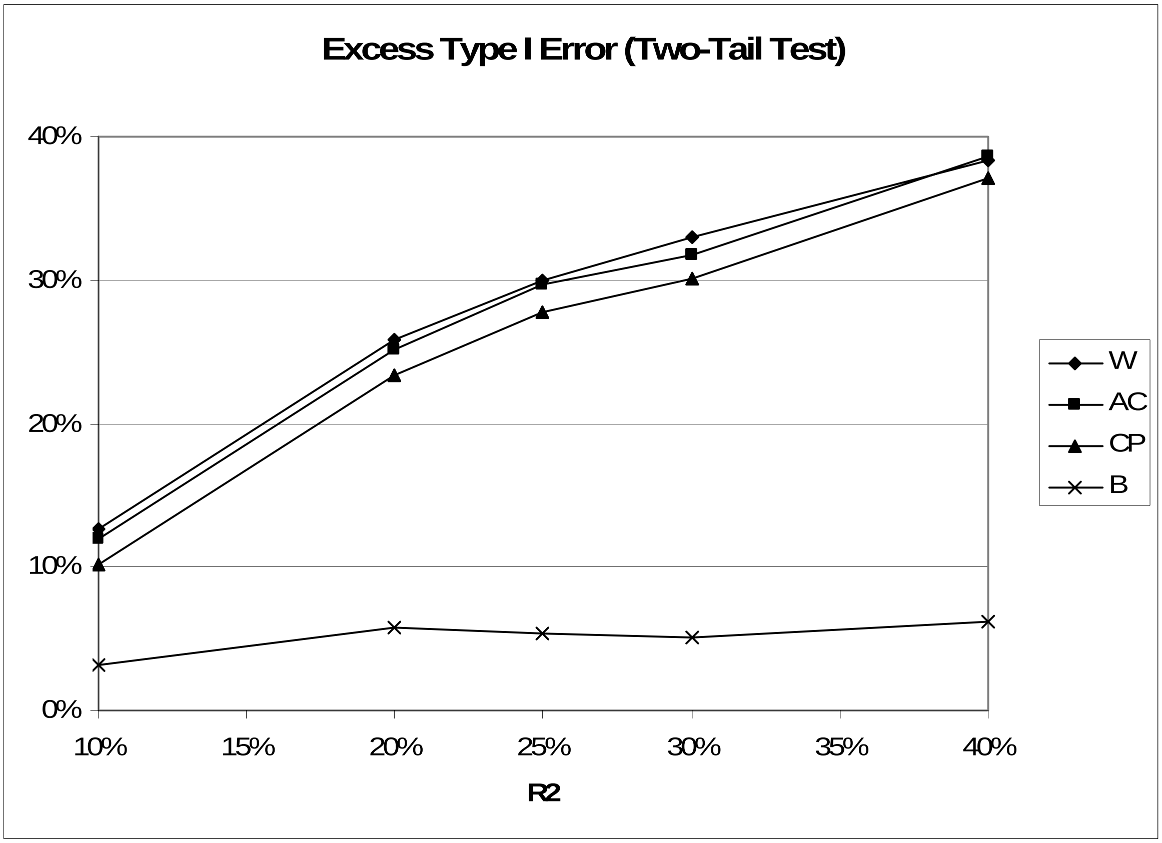

Figure 5: Excess Type I Errors of One-Tail Test