Skewness Correction for Asset Pricing

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Skewness Correction for Asset Pricing

It is shown that, for CRRA agents, the sensitivity of risk correction for any cumulant depends on the cumulant of the next order. This result is then used to derive some interesting approximations for variance and skewness correction. The first corollary is that negative skewness alone leads to higher variance swap rates since the variance swap contract provides insurance against sudden market drops. Thus, high variance swap rates are not necessarily an indication of high variance risk premia. When the results are extended to the multifactor case, we are able to disentangle the swap rate premia into their skewness and stochastic variance premia components. Finally, we contribute to the understanding of option skews by showing that only $1 - u$ percent of excess kyrtosis contributes to negative skewness correction, where u is a newly introduced statistic that normalizes skewness with kyrtosis.

Introduction

There is a long line of literature that establishes the importance of systematic return skewness, as well as co-skewness of individual returns with the market, in the formation of asset prices[1]. The contribution of this paper is to first recover the exact *sensitivity* of risk correction to risk aversion, and second by using this relation to provide linear approximations for variance and skewness risk correction without specifying exact return dynamics. Even though results are approximate, this approach avoids to pre-specify a stochastic process that imposes a particular functional relation between volatility, skewness, and kyrtosis. The results are developed in continuous-time within the framework of the Lévy processes, which are now given considerable attention since they nest the Brownian motion by incorporating jumps that arrive at some, potentially infinite, rate[2]. Lévy models are used to model time

[1]This research started with the early co-skewness models of Rubinstein (1973) and Kraus and Litzenberger (1976). Harvey and Siddique (2000) show that systematic skewness can explain an average 3.6 percent risk premium. Recently, Carr and Wu (2006) find that skewness is also highly variable over time.

[2]Examples of infinite jump activity models are the normal inverse Gaussian by Barndorff-Nielsen (1998), the generalized hyperbolic by Eberlein et al. (1998), the Variance Gamma by Madan and Milne (1991), and the generalization in Carr et al. (2002)

changes (Clark, 1973), capture higher moments, generate flexible volatility surfaces, and discuss market incompleteness[3]. Ait-Sahalia (2004) suggests that disentangling the pure jump from the diffusive component may be at the core of risk management, since the diffusive risks are hedgeable.

In the simplest case, the risk premium of a security, the difference between its expected return and the risk free rate, is driven by variance. Since the risk free rate is equal to the risk neutrally expected return, we may think of variance as the sensitivity of drift correction to changes in risk aversion. Our main theorem generalizes this idea by showing that when agents exhibit constant relative risk aversion, the magnitude of risk correction for the n^{th} cumulant depends on the $(n+1)^{th}$ cumulant.

Widespread interest for direct exposure to variance risk has led to the introduction of variance swaps that provide payoffs driven by the differential between realized variance and an ex-ante swap rate. Since variance swaps can be initiated at zero cost, the no arbitrage condition implies that the variance swap rate equals a *risk neutral* expectation of the realized variance for the underlying security. Variance swap rates (as well as option implied

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[3]Carr, Jin and Madan (2001), Cvitanic, Polimenis and Zapatero (2005).

volatilities squared[4]) tend to be higher than historical variance rates, and it is almost universally suggested that the *entire* rate differential is due to the pricing of stochastic variance risk. The often cited explanation is that if there is no variance risk (or it is not priced), the variance rate under the historical and risk neutral measures should equal. In the first corollary of the main theorem it is shown that this reasoning is not always valid, since when skewness is negative, variance is upwards adjusted. The novel insight is that when SKEW is negative, a long position in the variance swap contract is more valuable as insurance against extreme negative movements in the underlying, and swap rates will be upwards adjusted according to

$$K_{var} \approx \sigma^2 - \gamma \mathrm{SKEW} \sigma^3 \tag{1}$$

When we model the individual stocks as having a beta exposure to the market plus a Gaussian idiosyncratic part, it is shown that the rates at which individual variance swaps may be entered is

$$K_{var}^i \approx \sigma_i^2 - \frac{\gamma}{b_i} \mathrm{SKEW}_i \sigma_i^3 \tag{2}$$

In a related paper, Demeterfi, Derman, Kamal and Zou (1999) approximate the effect of volatility skew, defined as the slope of the implied volatility

[4] Jackwerth and Rubinstein (1996).

curve, on variance swaps.

Nevertheless, variance is stochastically changing, and there is great interested in the academic and practitioner communities in pricing variance risk.[5] For example, by analyzing the gains of delta hedged strategies, Bakshi and Kapadia (2003a,b) find evidence of negative market volatility premia. When the main theorem is later extended to a multifactor setting, we are also able to measure the effect of stochastic variance premia on the variance swap rate. Thus, we *disentangle* the variance correction into separate skewness, and stochastic variance premia components. More exactly, it is shown that it is not the leverage effect alone (i.e. a negative return-volatility correlation) that is responsible for higher swap rates. Rather, *a bias in the strength* of the leverage effect, that makes it more pronounced in negatively moving (falling) markets, is responsible for such high variance swap rates. The intuition is that, due to the biased leverage effect, large negative returns tend to increase volatility more than positive returns tend to decrease it. Thus large payoffs to the long position of a variance swap will tend to arrive at states where volatility has been upwards updated since inception, and thereby provide

[5] Carr and Wu (2004) propose a new method for the estimation of variance risk premia from options data.

insurance against such undesirable volatility increases.

It is known that the Black-Scholes-Merton implied volatility for deeply out of the money put options is higher than that for out of the money calls. Pan (2002) finds that jump and volatility premia are significant for explaining option "smirks". There is an almost unanimous agreement that these volatility smirks are signs of a *strongly negative risk neutral skewness.* Since empirical return skewness is not high enough, risk neutral skewness should then be the result of *skew correction.* This is highlighted in a related paper by Bakshi, Kapadia and Madan (2003) who derive theoretical links from risk aversion, and actual returns' higher moments to risk neutral skewness and option prices.

Even though, it is widely recognized that skewness and kyrtosis (and the jumps that generate them) are significant in pricing non-linear payoffs such as the ones generated by options, the question of whether it is skewness or kyrtosis the most important factor in determining option smiles is still open. The third corollary of the central theorem provides some new insight by defining a new statistic u that normalizes skewness for excess kyrtosis. It is shown that leftward risk correction for market skewness, ΔSKEW, is driven by the $(1-u)$ percent of the excess kyrtosis,

$$\Delta\text{SKEW} \approx -\gamma(1-u)\,(\text{KURT}-3)\,\sigma \tag{3}$$

Thus, a fat-tailed return distribution generates an increasingly negative skewness, only to the extend that *kyrtosis-normalized* skewness u is less than 100%. This result generalizes Theorem 2 in Bakshi, Kapadia and Madan (2003), which argues that the source of the negative risk neutral skewness is total excess kyrtosis, and is valid in the special case of symmetric distributions for which the kyrtosis-normalized skew u is zero. We show that for skewed processes, the u fraction can be quite high, and skewness correction small, or even zero. A counter-intuitive consequence is that, for a given kyrtosis, skewness is more heavily corrected for more symmetric processes.

In section 1, the central result for Lévy cumulants is developed, and we derive the approximate variance swap rate for the index and individual stocks. In section 2, the relation of kyrtosis to skewness correction is developed. In section 3, the results are extended for many risk factors, and the relation of stochastic volatility to variance swap rates is developed. In section 4, the simpler case of stochastic volatility as an independent time change is discussed.

1 Correcting market cumulants

I assume the market index returns X_t are generated by a Lévy process,

$$X_t = \eta w_t + \int_0^t \int_{-\infty}^{\infty} xN(ds, dx) \tag{4}$$

where w_t is a diffusion, and $N(dt, dx)$ is the jump counter with Lévy measure $\pi(dx)$. I further assume that $1 \wedge |x|$ is $\pi-$integrable.[6] In the moment generating function, $\mathcal{M}$(s), of a Lévy process, time is factored out,

$$\mathcal{M}(s) = Ee^{sX_t} = e^{t\mathcal{K}(s)} \tag{5}$$

where $\mathcal{K}$(s) is the cumulant generator of the Lévy process.

For agents who exhibit a constant relative risk aversion γ, the risk neutral index process is an exponentially tilted version[7] of the original process X_t

$$\left(\frac{dQ}{dP}\right)_t = e^{-\gamma X_t - t\mathcal{K}(-\gamma)} \tag{6}$$

Given (6), the risk neutral[8] cumulant function of X_t is a first difference of the actual $\mathcal{K}(s)$,

$$\mathcal{K}^*(s) = \mathcal{K}(s - \gamma) - \mathcal{K}(-\gamma) \tag{7}$$

[6]This is stronger that the general condition $\int(1 \wedge x^2)\pi(dx) < \infty$.

[7]See, for example, Carr and Wu (2004).

[8]A star superscript denotes a risk neutral quantity.

The cumulants of a Lévy process are horizon-scaled derivatives of its cumulant function at zero. From (7), risk-neutral cumulants are recovered by differentiating at $s = -\gamma$. The nth order risk neutral cumulant is thus a *function of risk aversion*,

$$c_n(\gamma) = \frac{\partial^n \mathcal{K}^*(0)}{\partial s^n} = \frac{\partial^n \mathcal{K}(-\gamma)}{\partial s^n} \tag{8}$$

When risk neutral cumulants are explicitly written as functions of γ, we may think of actual cumulants as risk corrected cumulants *for risk neutral agents*,

$$c_n = c_n(0) \tag{9}$$

The central goal of the paper is to provide linear approximations to variance and skewness *risk correction* for Lévy processes. By risk correction for a quantity f we mean the difference between the risk neutral and actual quantities, $\Delta f = f^* - f$. Since economic theory anticipates risk correction due to risk aversion,

$$f^* = f(\gamma) \tag{10}$$

the natural approximation to the risk-adjusted quantity is the linear approximation with respect to the risk aversion parameter (or price of risk) γ,

$$\Delta f \approx \left(\frac{\partial f}{\partial \gamma}\right)_{\gamma=0} \times \gamma$$

The linear approximation will be exact for linear (CAPM-style) risk corrections, but in the general case it will be of the type

$$f(\gamma) = f + \frac{\partial f(0)}{\partial \gamma}\gamma + o(\gamma)$$

where the little o notation shows that only sub-linear terms are discarded[9].

In Merton's benchmark case, the market risk premium $\mu - r$ equals $\gamma\sigma^2$, or equivalently, since the risk neutral drift μ^* equals r

$$\mu^* - \mu = -\gamma\sigma^2 \tag{11}$$

which implies that the sensitivity of drift equals

$$\frac{\partial \mu^*}{\partial \gamma} = -\sigma^2 \tag{12}$$

That is, the drift correction, $\mu^* - \mu$, is driven by the variance (i.e. the next order cumulant). The above discussion is generalized for cumulants of higher order in the central result for this paper:

Theorem 1. *The risk aversion sensitivity of the n^{th} risk corrected market cumulant equals the negative (n+1)th cumulant,*

$$\left(\frac{\partial c_n(\gamma)}{\partial \gamma}\right)_{\gamma=0} = -c_{n+1} \tag{13}$$

[9] $\lim_{\gamma\to 0}\frac{o(\gamma)}{\gamma} = 0$.

Proof: From (8),

$$\frac{\partial c_n(\gamma)}{\partial \gamma} = -\frac{\partial^{n+1}\mathcal{K}(-\gamma)}{\partial s^{n+1}} = -c_{n+1}(\gamma) \tag{14}$$

Since $c_{n+1}(0) = c_{n+1}$, we have

$$\frac{\partial c_n(0)}{\partial \gamma} = -c_{n+1} \tag{15}$$

1.1 The approximate variance swap rate

Variance correction is less straightforward than drift correction because the next cumulant can be positive or negative depending on the sign of skewness. That is, while negative market skewness will increase risk neutral variance, a positive skewness *will lower* variance. This observation has implications for the formation of *variance swap rates.*

One way to take a position in volatility is to have a delta-neutral position on the market. A more direct facility for volatility trading, available to large investors, is a variance swap that pays the difference between a realized estimate of return variability and a fixed variance rate determined at time zero. Since variance swaps have zero initial cost, the rate at which variance swaps can be entered equals the risk neutrally expected value of the future realized quadratic variation.

Lemma 1. *The linear approximation of the variance swap rate for a constant volatility market is*

$$K_{var} \approx \sigma^2 - \gamma \text{SKEW} \sigma^3 \tag{16}$$

Proof: From Theorem 1, $\Delta c_2 \approx -\gamma c_3$, and also SKEW $= c_3/\sigma^3$ $\square$

Informally, it is almost universally argued that variance swap rates are higher than stock variance rates to reflect the stochastic nature of volatility (or variance), that is to capture negative volatility premia[10]. The novel insight here is that the main force behind variance correction that leads to higher swap rates, is not negative volatility premia but negative skewness, since negative SKEW alone generates higher swap rates even for a constant volatility[11]. For example, for $\gamma = 3$, SKEW$= -1.5$ and $\sigma = 20\%$ the SKEW correction 3.6% which is added to the actual variance of 4%. For $\gamma = 3$, SKEW$= -1.5$ and $\sigma = 30\%$ the skewness-related correction is 12.15%, actually bigger than σ^2 which is 9%. The intuition is that when SKEW< 0, a long position on a variance swap is more valuable because the positive payoffs tend to arrive due to extremely negative returns, and thus provide insurance against extreme negative movements in the underlying.

[10]The volatility premia connection to high swap rates is developed at the last section.

[11]I am grateful to a referee for pointing this out.

1.2 Individual stock swap rates.

Individual stocks are assumed to have a beta exposure to systematic risk,

$$X_t^i = a_i t + b_i X_t + \varepsilon_i w_t^i \tag{17}$$

with X_t the market risk and w_t^i an idiosyncratic diffusion orthogonal to the market. In this case, $R_i^2 = \frac{b_i^2 \sigma^2}{\sigma_i^2}$ of the total quadratic variation rate

$$\sigma_i^2 = b_i^2 \eta^2 + b_i^2 \int x^2 \pi(dx) + \varepsilon_i^2 = b_i^2 \sigma^2 + \varepsilon_i^2 \tag{18}$$

of the i^{th} stock is systematic.

The first key observation is that the diffusive idiosyncratic risk *does not enter in higher cumulants*

$$c_n^i = b_i^n c_n \qquad \text{for } n \geq 3 \tag{19}$$

As the next corollary shows, the individual stock in (17) conforms with recent empirical findings (e.g. Bakshi, Kapadia and Madan (2003)) that most individual stocks seem to be less left-skewed than the market:

Corollary 1. *(proof in appendix) For individual stocks in (17)*

$$\text{SKEW}_i = \text{SKEW} \times R_i^3 \tag{20}$$

The fact that the entire market is more left skewed than its individual components seems counter-intuitive at first, but should not surprise since portfolio skewness is *not* a weighted sum of individual skews.

The second key observation is that since the idiosyncratic risk *is not priced*, its cumulants are independent of γ and thus any cumulant correction is the result of correction on the market risk:

Corollary 2. *The n^{th} risk corrected cumulant for an individual stock equals,*

$$\frac{\partial c_n^i(0)}{\partial \gamma} = -b_i^n c_{n+1} \tag{21}$$

Thus, the risk neutral quadratic variation of the i^{th} stock grows at a rate

$$\sigma_i^2(\gamma) = \sigma_i^2 - \gamma b_i^2 c_3 + o(\gamma) \tag{22}$$

which implies that

Lemma 2. *The swap rate for the i^{th} individual stock follows*

$$K_{var}^i \approx \sigma_i^2 - \frac{\gamma}{b_i} \text{SKEW}_i \sigma_i^3 \tag{23}$$

Demeterfi, Derman, Kamal and Zou (1999) approximate the effect of volatility skew (the slope of the implied volatility curve) on variance swaps. Instead, Lemmas 1 and 2 measure the direct effect of actual market skewness on the swap rates for both the index and individual stocks.

The above results provide an indirect method of estimating market prices (i.e. γ's) without assuming an exact return generating process. After having estimated individual stock betas we may recover γ as the slope of a cross-sectional regression of variance swap rate differentials $K_{var}^i - \sigma_i^2$ on $\frac{\text{SKEW}_i \sigma_i^3}{b_i}$.

2 Skewness correction

The contemporary empirical option pricing literature agrees that the so called volatility smiles are signs of a *strongly negative risk neutral skewness.* Since empirical return skewness is not high enough, risk neutral skewness should then be the result of *risk correction*. It is informally believed that excess kyrtosis is related to the risk neutral skewness implicit in option smiles.

Furthermore, it is already recognized that fat tails are indeed responsible for option smiles.[12] Here we take this analysis one step further, by showing that only a fraction $1-u$ of excess kyrtosis generates skewness correction, where we define the *horizon-independent kyrtosis-normalized skewness* u statistic as follows

$$u = \frac{3}{2}\frac{c_3^2}{c_2 c_4} = \frac{3}{2}\frac{\text{SKEW}^2}{\text{KURT}-3} \tag{24}$$

[12]Theorem 2 (page 109) in Bakshi, Kapadia and Madan (2003).

Lemma 3. (proof in the appendix) *The index skewness is corrected to the left only by the 1-u percent of excess kyrtosis*

$$\Delta\text{SKEW} = -\gamma\,(1-u)\,(\text{KURT}-3)\,\sigma + o(\gamma) \tag{25}$$

2.1 Why only a fraction of excess kyrtosis?

Lemma 3 shows that fat tails are indeed responsible for skew smiles. Since the full variance is responsible for drift correction, and the full skewness (Lemma 1) for variance correction, it is tempting to ask why when it comes to skewness correction only $1-u$ percent of excess kyrtosis participates. The source of the confusion is that while for $n = 1, 2$ the central moment, m_n, follows

$$m_n(\gamma) \approx m_n - \gamma m_{n+1} \tag{26}$$

for $n > 2$ the recursive equation is *only* valid for cumulants, c_n.

Actually, from Theorem 1, the third moment is corrected as follows,

$$\frac{\partial m_3(0)}{\partial \gamma} = \frac{\partial c_3(0)}{\partial \gamma} = -c_4 = -m_4 + 3\sigma^4 \tag{27}$$

So the correct relation becomes

$$\Delta m_3 = -\gamma m_4 \underbrace{+3\sigma^4\gamma}_{\text{not in Eq. (26)}} +o(\gamma) \tag{28}$$

In the special case of symmetric distributions, the u statistic vanishes, and the correction implied by the entire excess kyrtosis still applies. Thus, keeping kyrtosis constant, the more symmetric the market returns are, the smaller u implies a more aggressive skewness correction.

Corollary 3. *When market returns are symmetric, the entire excess kyrtosis generates risk-neutral skewness*

$$\text{SKEW}(\gamma) \approx -\gamma(\text{KURT} - 3)\sigma \tag{29}$$

2.2 Individual stock skews

We have shown in (20) that individual stocks in (17) will have less pronounced skews. From

$$u_i = \frac{3}{2}\frac{(c_3^i)^2}{c_4^i\sigma_i^2}$$

and $c_n^i = b_i^n c_n$ for $n \geq 3$, we have that for the i^{th} stock

$$u_i = u \times R_i^2 \tag{30}$$

which implies that a larger percentage of the kyrtosis is responsible for skew correction than in the index case, $1-u_i > 1-u$:

Lemma 4. *Individual skew corrections are given by*

$$\Delta\mathrm{SKEW}_i = -\gamma\,(1-u_i)\,(\mathrm{KURT}_i-3)\,\frac{\sigma_i}{b_i} + o(\gamma) \tag{31}$$

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Add figure 1 around here

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The fact that a larger *fraction of kyrtosis* will generate skew correction does not imply that individual stocks have more overall skew, because they start with a smaller actual skew (20), and, since idiosyncratic risk is assumed not to have fat-tails

$$\mathrm{KURT}_i - 3 = \frac{c_4^i}{\sigma_i^4} = \frac{b_i^4 c_4}{b_i^4\sigma^4/R_i^4} = (\mathrm{KURT}-3)\times R_i^4 \tag{32}$$

they also have smaller kyrtosis to start with (see Fig.1). Re-write (31) as

$$\Delta\mathrm{SKEW}_i \approx -\gamma\,(\mathrm{KURT}-3)\,\sigma\left(1-uR_i^2\right)R_i^3 \tag{33}$$

and compare to (25), to see that individual stock skews will be corrected more aggressively when $(1-uR_i^2)\,R_i^3 > 1-u$. For small values of the index u, a higher systematic risk R_i^2 implies a steeper skew correction. But for

$u > 60\%$, this is not true anymore; as R_i^2 grows beyond $\frac{60\%}{u}$ the individual stock skewness will receive less correction; i.e. the sensitivity of ΔSKEW_i with respect to R_i^2 becomes positive.

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Add figures 2,3,4,5 around here

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2.3 The u statistic can be large

Since Lemma 3 is counter-intuitive in asserting that symmetry in the actual returns imparts more asymmetry in the risk corrected ones, a natural question is whether the statistic u, which regulates the intensity of this phenomenon, will attain large enough values for the phenomenon to become significant. As is shown here for a simple pure jump process, the broadly used gamma, $u = 100\%$. For $l, v > 0$, the Lévy measure of the pure jump gamma process, $\gamma_t(l, v)$

$$\pi(dx) = \frac{v}{x} e^{-x/l} dx \qquad \text{for} \quad x > 0 \tag{34}$$

generates an infinite arrival rate of small jumps, in the sense that the arrival rate of jumps away from zero for any $\epsilon > 0$ is finite, $\pi(\epsilon, \infty) < \infty$. It is well known that the nth cumulant of the gamma equals

$$c_n = (n-1)! l^n v \tag{35}$$

and it is thus clear that $u = 100\%$. In other words, the significant heavy tails of gamma do generate any skewness correction, ΔSKEW$= 0$.

2.4 Two-sided jumps generate more skew correction

The simple gamma of the previous section is not a good candidate since it won't generate jumps of both signs. We may easily correct this by combining two gammas that generate jumps of opposite signs

$$X_t = \gamma_t^+(l_+, v_+) - \gamma_t^-(l_-, v_-) \quad , \quad \text{where} \;\; l_\pm, v_\pm > 0 \tag{36}$$

When the $v_+ = v_-$, this process is called a Variance Gamma (e.g. Madan, Carr and Chang, 1998). The next lemma is proved in the appendix,

Lemma 5. *For a Variance Gamma process there will always be some skew correction (u<100%).*

Lemma 5 is counter-intuitive, but motivates an important general observation that provides intuition about the skewness correction mechanism: when two-sided jump processes are involved we *always* get leftward skew correction. To understand this general observation we have to consider what happens to the one-sided jump measures $\pi_{\pm}(dx)$ when we correct for risk a *two-sided* process that combines positive and negative jumps,

$$X_t = X_t^+ - X_t^- = \int_0^t \int_0^{+\infty} xN^+(ds, dx) - \int_0^t \int_0^{+\infty} xN^-(ds, dx) \tag{37}$$

The two-sided cumulant function equals

$$\mathcal{K}(s) = \int_{-\infty}^{+\infty} (e^{sx} - 1)\pi(dx) \tag{38}$$

with

$$\pi(dx) = \{^{\pi_+(dx) \text{ for } x>0}_{\pi_-(dx) \text{ for } x<0} \tag{39}$$

From (7) we have that

$$\mathcal{K}^*(s) = \int_{-\infty}^{+\infty} (e^{(s-\gamma)x} - 1)\pi(dx) - \int_{-\infty}^{+\infty} (e^{-\gamma x} - 1)\pi(dx) \tag{40}$$

$$= \int_{-\infty}^{+\infty} (e^{sx} - 1)e^{-\gamma x}\pi(dx) \tag{41}$$

which implies that

Lemma 6. *The corrected Lévy measure equals*

$$\pi^*(dx) = e^{-\gamma x}\pi(dx) \tag{42}$$

Lemma 6 implies that while the positive jumps are arriving at slower rates under the risk neutral measure,

$$\text{for } x > 0: \ \pi_+^*(dx) = \pi^*(dx) = e^{-\gamma x}\pi(dx) < \pi(dx) = \pi_+(dx) \tag{43}$$

negative jumps are accelerated,

$$\text{for } x < 0: \; \pi^*_-(dx) = \pi^*(dx) = e^{-\gamma x}\pi(dx) > \pi(dx) = \pi_-(dx) \tag{44}$$

This asymmetry on the treatment of opposite signed jumps, i.e. the acceleration of negative jump arrivals combined with the deceleration of the positive jumps, generates left skew correction.

3 The multifactor case

Since in practical applications, index returns may be exposed to multiple risks, it is useful to extend Theorem 1 to the multifactor case, where index returns are exposed to a multitude of risk factors X^i_t

$$R_t = \delta t + \beta X_t \tag{45}$$

β_i the i^{th} component of the row vector $\beta = (\beta_1, \beta_2 ... \beta_N)$ is exposure to risk X^i, and each factor potentially includes diffusive and jump components

$$X^i_t = \eta^i W_t + \int_0^t \underbrace{\iint \cdots \int}_{N} z_i N(ds, dz) \tag{46}$$

where the inner N-dimensional integral extends over the entire jump support region, and, where W_t is an N-dimensional vector of independent diffusions.

For each factor, i, the row vector $\eta^i = (\eta^i_1, \eta^i_2, ...\eta^i_N)$ describes the exposure to the diffusive risks, while z_i is the jump to this factor due to a global jump $z = (z_1, z_2, ...)$.

Correlation between the factors may arise from the diffusive, as well as, the jump parts and no restriction is placed here. Observe that the above representation of the vector process X is the most general in this context, since any prior correlation among the diffusions may be subsummed into the vectors η by properly transforming the diffusions. Unlike a diffusion vector, which when properly rotated to supress correlations also results in independent diffusions, rotating a jump vector z to produce uncorrelated jumps does not simplify the framework, since uncorrelated jumps are not necessarily independent.[13] Furthermore, the rotations for the diffusive and jump parts will in general be different; thus, we may choose to either work with uncorrelated (and independent) diffusions, or with uncorrelated (but still dependent) jumps.

[13] While uncorrelated Gaussian variables are also independent, in the jump case, rotating the jumps to supress the correlation between them will not supress their non-zero higher order cross-cumulants.

The total covariance[14] between factors i and j is

$$c^{i,j} = \sum_k \eta_k^i \eta_k^j + \underbrace{\iint \cdots \int}_{N} z_i z_j \pi(dz) \tag{47}$$

with an N-dimensional integral over the entire jump support capturing the jump-induced covariance, and where as before, $\pi(dz)$ is the Lévy measure of the jump process. With a pricing kernel of the form $e^{-\gamma R}$, the change of measure is

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\gamma\beta X_t - t\mathcal{K}(-\gamma\beta)\right) \tag{48}$$

where, for simplicity, I assume that the jump components of X are of finite variation, that is for row vectors $s = (s_1, s_2 ...)$

$$\mathcal{K}(s) = \frac{1}{2} s H s^T + \iint ... \int \left(e^{sz} - 1\right) \pi(dz) \tag{49}$$

It is then straightforward to observe that under the risk neutral measure the cumulant generator of the X process becomes

$$\mathcal{K}^*(s) = \mathcal{K}(s - \gamma\beta) - \mathcal{K}(-\gamma\beta) \tag{50}$$

[14] To facilitate the multifactor analysis, here the superscript notation $c^{i_1,i_2,\dots i_N}$ will be used to denote the N-order cross cumulant. In this notation, indices need not take distinct values, and are fully interchangeable. For example, $c^{2,3,3} = c^{3,3,2}$ is a cross-cumulant of 3^{rd} order, where the second factor enters once and the third factor twice.

Following the same reasoning as in Theorem 1, we anticipate that the risk corrected cross-cumulant[15] of n^{th} order with respect to $X^i, X^j, X^l...$, is a function of γ

$$c^{i,j,l...}(\gamma) = \frac{\partial^n \mathcal{K}(-\gamma\beta)}{\partial s_i \partial s_j \partial s_l ...} \tag{51}$$

and, that the sensitivity of the cumulant with respect to risk aversion depends on the cumulants of higher order as follows

$$\frac{\partial c^{i,j,l...}(\gamma)}{\partial \gamma} = -\sum_k \beta_k c^{k,i,j,l...}(\gamma) \tag{52}$$

which is also valid for $\gamma = 0$ (actual cumulants). Expanding and keeping only terms linear in γ, we get the risk correction to the cumulant as a linear combination of the cross-cumulants of the next order

$$c^{i,j,l,...}(\gamma) = c^{i,j,l,...} - \gamma \sum_k \beta_k c^{k,i,j,l...} + o(\gamma) \tag{53}$$

[15] Higher order single factor cumulants are special cases of cross-cumulants with repeated indices.

3.1 Volatiity as a simple time change

A simple volatility[16] factor may be introduced if we specialize the multifactor theory of the previous section to a model where returns are exposed to two factors, X_t and Y_t, with the factor Y representing a "volatility" factor, and X the market returns. Such a factor could be any proxy for volatility, for example the volatility index VIX, or a measure of realized quadratic variation (RQV). It is known that although raw returns are clearly skewed and leptokurtic, returns conditioned by realized volatilities are approximately Gaussian. To capture this fact here, index returns are generated by a diffusion process evaluated at random stopping times

$$X(dt) = W^x(Y(dt)) \tag{54}$$

Here volatility acts as a simple time change but is devoid of any "leverage effects", which will be introduced in the next subsection. Thus, even though X depends on Y, Y offers no directional information on X, $c^{x,y} = 0$. Later, we will be more specific, but for now it is best to think of Y as some generalized volatility factor with σ_y the "volatility of volatility".

[16]Here, I use the generic terms "volatility" and "activity" interchangeably and in reference to variances and standard deviations. The precise meaning of these terms will be clear from the context.

3.1.1 Swap rates with leverage effect

In reality, (54) alone cannot explain return dynamics since it is well known that the volatility factor Y does offer directional information for returns, and more specifically that changes in volatility have a strong negative correlation to market returns. The model in (54) may be enhanced to capture this so called "leverage" effect if we differentiate between X, the market returns "sans leverage effect", and R in

$$R = X - bY \tag{55}$$

the total market return "cum leverage".[17] The negative sign on b (with $b > 0$) denotes a volatility factor contravariant to the market (leverage).

In this case (53) directly explains the formation of the variance swap rates for swaps written on the factors. Take, for example, what the variance swap rates would be with no leverage, that is the risk neutral rate of variance for X, $K^x_{var} = c^{x,x}(\gamma)$

$$K^x_{var} = \sigma_x^2 - \gamma \text{SKEW}_x \sigma_x^3 + \gamma b \text{co-SKEW}_{xxy} \sigma_x^2 \sigma_y + o(\gamma) \tag{56}$$

[17]Alternatively, one may think of $X = R + bY$, as a hedged portfolio of the market and a volatility index, which is immune to leverage effect.

where, all the terms are as in Lemma 1, except for the new term

$$\text{co-SKEW}_{xxy} = \frac{c^{x,x,y}}{\sigma_x^2 \sigma_y} \tag{57}$$

More specifically, since

$$c^{x,x,y} = \frac{\partial^3 \mathcal{K}(0)}{\partial s_x^2 \partial s_y} \tag{58}$$

and the diffusive part enters (49) only quadratically, the continuous path dynamics of X_t and Y_t cannot survive in $c^{x,x,y}$. Thus, the $c^{x,x,y}$ cumulant is only driven by the pure jump components in X_t and Y_t, and in the appendix it is shown that

$$c^{x,x,y} = \iint x^2 y \pi(dx, dy) = \sigma_y^2 > 0 \tag{59}$$

There are two distinct effects that push swap rates high in (56); the pure skewness effect of Lemma 1, and the novel co-SKEW effect. Even though the X factor has no skewness (see the appendix)

$$c^{x,x,x} = 0 \tag{60}$$

the variance swap rates are still higher

$$K_{var}^x = \sigma_x^2 + \gamma b \sigma_y^2 + o(\gamma) \tag{61}$$

due to leverage. The intuition is that since Y captures a volatility factor, relatively large x jumps of both signs will tend to arrive together with positive y jumps. Intuitively, large payoffs to a variance swap on the X factor -large absolute returns to the first factor-, tend to coincide with positive Y changes. Even though the two factors are uncorrelated, $c^{x,y} = 0$, the leverage effect induces negative correlation between returns and volatility

$$c^{R,y} = -b\sigma_y^2 \tag{62}$$

which means that positive Y changes tend to coincide with market drops. This variance swap is a hedging instrument and it will be sold at a premium -high swap rate.

To analyze the swap rates for the entire market, we may treat R as a single factor and apply Lemma 1, $c^{R,R}(\gamma) = \sigma_R^2 - \gamma c^{R,R,R} + o(\gamma)$, with

$$c^{R,R,R} = \sum_{i,j,k} \beta_i \beta_j \beta_k c^{i,j,k} = c^{x,x,x} - 3bc^{x,x,y} + 3b^2 c^{x,y,y} - b^3 c^{y,y,y}$$

Unlike covariance cumulants, coskewness cumulants are fundamentally asymmetric, in the sense that (even though $c^{x,x,y} > 0$)

$$c^{x,y,y} = \iint xy^2 \pi(dx, dy) = 0 \tag{63}$$

since large volatility movements have no directional information for X (even though they do carry directional information for R).[18] Combining, (60), (57) and (63) we find that the market has negative skewness[19]

$$c^{R,R,R} = -3b\sigma_y^2 - b^3 c^{y,y,y} \tag{64}$$

and arrive at the following swap rate for the market

$$K_{var}^R = \sigma_R^2 + 3\gamma b\sigma_y^2 + \gamma b^3 c^{y,y,y} + o(\gamma) \tag{65}$$

4 Many factors with independent risk prices

In reality different risks may be priced differently. Returns are again assumed to follow (45) and (46), but now risks carry different prices γ_i, some even potentially negative. In this case, the change of measure is

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\gamma X_t - t\mathcal{K}(-\gamma)\right) \tag{66}$$

with $\gamma = (\gamma_1, \gamma_2, ...)$ the row vector of the individual gammas. Under the risk neutral measure the cumulant generator becomes

$$\mathcal{K}^*(s) = \mathcal{K}(s-\gamma) - \mathcal{K}(-\gamma) \tag{67}$$

[18]See the appendix.

[19]Volatility is well known to exhibit a strong positive skewness, $c^{y,y,y} > 0$.

and the cross-cumulant is a function of the risk prices

$$c^{i,j,l\cdots}(\gamma) = \frac{\partial^n \mathcal{K}(-\gamma)}{\partial s_i \partial s_j \partial s_l \ldots} \tag{68}$$

In this case, the derivative of this cumulant with respect to the sensitivity to the k^{th} risk price depends on a single cumulant of higher order

$$\frac{\partial c^{i,j,l\cdots}(\gamma)}{\partial \gamma_k} = -c^{k,i,j,l\cdots}(\gamma) \tag{69}$$

Expanding around 0, and keeping only terms linear in the gammas, we get

$$c^{i,j,l,\cdots}(\gamma) = c^{i,j,l,\cdots} - \sum_k \gamma_k c^{k,i,j,l\cdots} + o(\gamma) \tag{70}$$

4.1 Priced Volatility

In (61), volatility risk is priced in the pricing kernel $e^{-\gamma R} = e^{-\gamma X + \gamma b Y}$, but its price is not determined independently of the market risk. Here we continue on the volatility factor (54) and (55) of the previous section, but we now allow the volatility factor to carry a negative risk price[20] with a pricing kernel that allows for an independent price of volatility risk ($\gamma_y = -\gamma b - \delta < 0$)

$$e^{-\gamma R + \delta Y} = e^{-\gamma X + (\gamma b + \delta) Y} \tag{71}$$

[20]Bakshi and Kapadia (2003) suggest that volatility premia are negative.

The study of how the incorporation of such a factor impacts the pricing of assets, is now a question of growing interest in the practitioner and academic communities. The extra premia for holding securities exposed to such a risk are sometimes called *variance risk premia*.

According to (70), variance swap rates "sans leverage", $K^x_{var} = c^{x,x}(\gamma_x, \gamma_y)$, expand as $K^x_{var} = \sigma_x^2 - \gamma_x c^{x,x,x} - \gamma_y c^{x,x,y} + o(\gamma)$, and from (57) and (60) we have

$$K^x_{var} = \sigma_x^2 + (\gamma b + \delta)\sigma_y^2 + o(\gamma) \tag{72}$$

The total variance correction can be separated into a leverage component $\gamma b \sigma_y^2$ as in (61), and the $\delta\sigma_y^2$ due to the pricing of variance risk.

4.2 Realized volatility as a simple time change

A high realized activity produces a lot of uncertainty and is known to be negatively correlated to returns. The methodology developed in the previous section allows the study of this phenomenon, where the business activity risk is explicitly priced. To provide further insights to the approximation theory, we treat this important case here with a specific jump structure, and are thus able to recover exact risk correction formulas. Yet, to remain loyal to the non-parametric spirit of the paper, we do not assume any persistent dynamics for the stochastic volatility, but we only model the non-persistent component so that the results here apply to a broad set of stochastic volatility processes.

It is already known (see for example Carr and Wu, JFE 2004) that volatilities that vary stochastically over time can be treated as a stochastic time change

$$Y_t = \int_0^t v_{u-} du + \int_0^t \int_0^{+\infty} y \left(N(du, dy) - \pi(du, dy) \right) \tag{73}$$

with $\int_0^t \int_0^{+\infty} y \left(N(du, dy) - \pi(du, dy) \right)$ being the martingale component. The literature focus thus far has been in modelling the locally deterministic time change $\int_0^t v_{u-} du$ It is also known that stochastic volatilities exhibit clustering, which in most cases is captured by the fact that the stochastic volatility rate process (instantaneous activity rate) v_t is a mean reverting solution to an Ornstein-Uhlenbeck equation. Such processes are now called Background Driving Lévy Processes (see Barndorff-Nielsen and Shephard 2001 in J of Royal Stat Soc. and Andersen, Bollerslev, Diebold and Ebens 2001 JFE).

The model here is motivated by the empirical observation that the standard volatility models used for capturing long horizon (daily or weekly) volatility level dynamics fail to explain the volatility information in intraday data. Similarly models that fit high frequency data well produce unrealistic daily or weekly dynamics (see Andersen, Bollerslev, Diebold and

Labys, Econometrica 2003). In order to help produce more realistic stochastic volatility models, we provide here a specification analogous to the one introduced in Carr and Wu (JFE 2004) and Carr, Geman, Madan, Yor (MF 2003), but in order to accomodate separate modelling of long horizon volatility dynamics from the intraday realized volatility estimates, we deviate from the literature by allowing an instantaneous non-negative *business activity rate* v_t, that captures the persistent component of market volatility, to only be the locally deterministic part of the intraday activity so that the realized quadratic variation (RQV) becomes

$$Y_t = \int_0^t v_{u-} du + \int_0^t \int_0^{+\infty} y \left(N(du, dy) - \pi(du, dy) \right) \tag{74}$$

with an expected rate

$$\int_0^\infty y\pi(dt, dy) = v_t \tag{75}$$

Over time, the availability of data for increasingly shorter return horizons has allowed the focus to shift from modeling at quarterly, monthly and daily horizons to improving forecasting performance with the incorporation of high frequency data. In this context, it is useful to think of v as the persistent component of market wide volatility[21], while the positive y realization is a

[21]Maybe a volatility index such as VIX.

non-persistent component that is nevertheless correlated to individual stock returns. A high realization of y implies a day of unusually high activity given the current level of volatility v (for example, VIX). This implies that there are three increasing levels of information: $\mathcal{F}_t$ is information based on knowledge of the persistent volatility, $\mathcal{G}_t$ is the information including the realization of quadratic variation, and $\mathcal{H}_t$ is the entire information including the stock return.

To allow the model to capture a leverage effect due to the non-persistent realization y, returns are generated as[22] in (55) and (54)

$$R(dt) = (\mu + \eta\sigma_{r,t}^2)dt + X(dt) - bY(dt) \tag{76}$$

where the total return variance σ_t^2 is included in $\mathcal{F}_t$. Total stock variance is decomposed in a "pure" stock variance $\sigma_{x,t}^2$ and variance due to "leverage" effect of the not yet (given $\mathcal{F}_t$) realized quadratic variation, and is given by

$$\sigma_{R,t}^2 = \sigma_{x,t}^2 + b^2\sigma_{y,t}^2 \tag{77}$$

In an effort to focus on the intraday dynamics, and be as general as possible, we don't specify the dynamics for the evolution of the positive and

[22] Observe the subtle but important difference between (76) and the more commonly used $X_t = W^x(Y_t) - bv_t$. This type of leverage is, for example, used in Barndorff-Nielsen and Shephard (2001), and in Carr, Geman, Madan and Yor (MF 2003).

mean reverting persistent component, v_t, and volatility of volatility, $\sigma^2_{y,t}$, and their evolution is conditioned out.[23] The only assumption is that a generalized *Sharpe ratio for Variance Swap contracts*, c^2, a quantity that turns out to be of great interest, because it captures the hedging value of variance swaps, is assumed to be constant

$$c^2 = \frac{v_t}{\sigma^2_{y,t}} \tag{78}$$

The importance of keeping c^2 constant is similar to keeping constant the return Sharpe ratio in the context of optimal portfolio allocation.

We directly model the dynamics of RQV, by modelling a stopped diffusion process with a drift parameter, $l_t = \frac{1}{v_t}$, and a diffusion parameter, $\theta_t = \frac{\sigma_{y,t}}{v_t^{3/2}}$. Even though in the above definitions the parameters of the time passage process, l and θ, vary stochastically, during a short interval of time dt, they can be treated as constant. The RQV y (clock change) between times t and $t + dt$ is given as the stopping point

$$Y(dt) = \arg\min\left\{y : l_t y + \theta_t\left(W^y(Y_t + y) - W^y(Y_t)\right) = dt\right\} \tag{79}$$

[23]Typical processes used in the literature are the mean-reverting geometric Gaussian, and the constant elasticity of volatility. More recently the literature is shifting to pure jump mean-reverting processes with the jumps driven by a so-called Background Driving Lévy Process (BDLP).

Observe that even though dt in (79) is infinitesimal, y will not necessarily be small. It is actually the unbounded positive jump in

$$Y(dt) = \int_0^\infty yN(dt, dy) \tag{80}$$

where $N(dt, dy)$ given $\mathcal{F}_t$ is a Poisson random measure.

The RQV y during dt is equivalently given by the stopping point

$$Y(dt) = \arg\min \left\{ y : cy + W^y(Y_t + y) - W^y(Y_t) = \frac{dt}{\theta_t} \right\} \tag{81}$$

which is the inverse Gaussian law with a drift c equal to the variance swap Sharpe ratio (78).

Formally, if $(\Omega, \mathcal{H}, \mathcal{P})$ is the probability space, *given the filtration $\mathcal{F}_t$ generated by the process $(v_t, \sigma^2_{y,t})$*, Y_t is a non-decreasing pure-jump subordinator process with independent increments and a Lévy-Khintchine representation (see Bertoin, 1996) of its cumulant generating function of the form

$$\mathcal{K}_{y,t}(q; h) = \log E e^{q(Y(t+h)-Y(t))} | \mathcal{F}_{t+h} = \int_t^{t+h} du \int (e^{qy} - 1)\, \pi_u(dy) \tag{82}$$

where $\pi_t(dy)$, the conditional marginal Lévy measure for the RQV, given $\mathcal{F}_t$, that satisfies

$$\int (y \wedge 1) \pi_t(dy) < \infty \tag{83}$$

Using (81) we find that conditionally[24] on $\mathcal{F}_{t+h}$, the process Y_t has a cumulant generator

$$\mathcal{K}_{y,t}(q;h) = \left(c - \sqrt{c^2 - 2q}\right) \int_t^{t+h} \frac{du}{\theta_u} \tag{84}$$

Conditionally, on the evolution of $\mathcal{F}_{t+h}$, the expected RQV over the next h-interval is indeed given by

$$E \int_t^{t+h} \int yN(ds, dy)|\mathcal{F}_{t+h} = \frac{\partial \mathcal{K}_{y,t}(0;h)}{\partial q} = \int_t^{t+h} v_u du \tag{85}$$

while the variance of RQV, conditional on the $\mathcal{F}_{t+h}$ information, during the same interval is

$$\int_t^{t+h} \int y^2 \pi_u(dy) = \int_t^{t+h} \sigma_{y,u}^2 du \tag{86}$$

Under this specification, by iterated expectations, the marginal returns' Lévy-Khintchine representation of the conditional cumulant generating func-

[24]Actually we only need to condition on the v_t path, since by (78), the filtration generated by $\sigma_{x,t}^2$ and $\sigma_{y,t}^2$ is the same as the one generated by v_t.

tion is

$$\mathcal{K}_{x,t}(s;h) = \log E e^{s(X(t+h)-X(t))} | \mathcal{F}_{t+h} = \log E \left(E e^{s(X(t+h)-X(t))} | \mathcal{G}_{t+h} \right) | \mathcal{F}_{t+h} \tag{87}$$

$$= \int_t^{t+h} du \int \left(e^{\frac{1}{2}s^2 y} - 1 \right) \pi_u(dy)$$

$$= \left(c - \sqrt{c^2 - s^2} \right) \int_t^{t+h} \frac{du}{\theta_u}$$

with $\pi_t(dx)$ the conditional marginal Lévy measure for the return process.[25]

Following an analogous reasoning, we find the joint Lévy-Khintchine representation for the two factors X and Y (as always conditional on the information $\mathcal{F}_{t+h}$)

$$\mathcal{K}_t(s,q;h) = \int_t^{t+h} du \iint \left(e^{sx+qy} - 1 \right) \pi_u(dx,dy) \tag{88}$$

$$= \int_t^{t+h} du \int \left(e^{(q+\frac{1}{2}s^2)y} - 1 \right) \pi_u(dy) \tag{89}$$

$$= \left(c - \sqrt{c^2 - s^2 - 2q} \right) \int_t^{t+h} \frac{du}{\theta_u}$$

with $\pi_t(dx,dy)$ the joint Lévy measure for returns and realized quadratic variation conditional on $\mathcal{F}_t$.

The total return for the market is given as the $(1,-b)$ linear combination

[25]We overload notation since the specific jump will be clear by context.

of the X and Y factors and the total conditional expected return is

$$\int_t^{t+h} \left(\mu + \eta\sigma_{r,u}^2 - bv_u\right) du \tag{90}$$

which makes sense since b measures the negative sensitivity of returns to RQV, while v_t measures the conditional expected RQV. The $\mathcal{F}_{t+h}-$conditional return variance for the X factor (without the leverage) is

$$\sigma_{x,t}^2(h) = \int_t^{t+h} du \int x^2\pi_u(dx) = \int_t^{t+h} v_u du \tag{91}$$

and when we account for the leverage effect

$$\sigma_{r,t}^2(h) = \int_t^{t+h} du \int r^2\pi_u(dr) = \int_t^{t+h} \left(v_u + b^2\sigma_{y,u}^2\right) du = a^2 \int_t^{t+h} \sigma_{y,u}^2 du \tag{92}$$

with

$$a^2 = b^2 + c^2 \tag{93}$$

capturing the relation between the expected rate of quadratic variation for stock returns and a generalized concept of business activity for the entire market

$$\sigma_{r,t} = a\sigma_{y,t} \tag{94}$$

Furthermore, (93) is the decomposition of the total $\mathcal{F}_t$-conditional quadratic variation in the $\mathcal{F}_t$-conditional variance of the $\mathcal{G}_t$-expected return, $b^2\sigma^2_{y,t}$, and the $\mathcal{F}_t$-conditional expectation of the $\mathcal{G}_t$-conditional variance, $v = c^2\sigma^2_{y,t}$.

When the leverage effect is not accounted for, there is no correlation between the factors, $\sigma_{xy} = 0$, but as anticipated by the $-b$ term in (76), the true conditional covariance between returns and RQV over an h-year horizon is

$$c_t^{r,y}(h) = \int_t^{t+h} du \iint ry\pi_u(dx, dy) = -b \int_t^{t+h} \sigma^2_{y,u} du \tag{95}$$

As was discussed in the previous section, the negative covariance is not enough to describe risk premia, and we have to analyze the skewness and coskew terms that capture the sign of one factor movements when the other factor experiences large jumps. Even though returns (sans leverage) are symmetric $c^{x,x,x} = 0$, with leverage conditional return skewness is negative due to the positive skewness of RQV y

$$c_t^{r,r,r}(h) = \int_t^{t+h} du \iint r^3\pi_u(dx, dy) = -\frac{3b}{c^2}\left(1 + \frac{b^2}{c^2}\right) \int_t^{t+h} v_u du \tag{96}$$

Furthermore, as we discussed before, it is natural to expect that a succesful model will predict that the biggest (in absolute size) returns will tend to occur during a period of a larger than expected realization of quadratic variation (or business activity). This is captured by a positive cross coskewness of realized quadratic variation to returns, which is the case of our model here

$$c_t^{x,x,y}(h) = \int_t^{t+h} du \iint x^2 y \pi_u(dx, dy) = \int_t^{t+h} \sigma_{y,u}^2 du \tag{97}$$

On the other hand, as discussed in the previous section, this phenomenon is assymetric, in the sense that periods of large realized quadratic variation will not imply any sign for the unleveraged returns $c^{x,y,y} = 0$ except when leverage is accounted for, such periods tend to coincide with negative returns

$$c_t^{r,y,y}(h) = \iint r y^2 \pi_t(dx, dy) = -\frac{3b}{c^2} \int_t^{t+h} \sigma_{y,u}^2 du \tag{98}$$

4.3 Pricing the risk of realized quadratic variation

The critical observation here is that we may factor out the conditioning information v_t in (88), (**??**) and (84) and thus recover the time invariant factor $c_n^{i,j,\cdots}$ in the $\mathcal{F}_{t+h}-$conditional cumulant

$$c_{n,t}^{i,j,k,\cdots}(h) = c_n^{i,j,k,\cdots} \int_t^{t+h} \frac{du}{\theta_u} \tag{99}$$

where following the notation in the previous section, the indices i, j, k take (possibly repeting) values in the set $\{X, Y\}$.

We assume that realized quadratic variation risk is priced, and variance premia are negative in the sense of a pricing kernel of the type

$$e^{-\gamma R+\delta Y} \tag{100}$$

This implies that agents seek protection in states of high RQV (and states of low stock returns). Observe that we may put the pricing kernel in the form (66) as

$$e^{-\gamma_x X-\gamma_y Y} \tag{101}$$

with $\gamma_y = -b\gamma - \delta$ which implies that states of high RQV need hedging not only due to the higher volatility but also due to leverage effect. In this case, conditionally on the information in $\mathcal{F}_t$, the joint Lévy measure is

$$\pi_t^*(dx, dy) = e^{-\gamma R+\delta y}\pi_t(dx, dy) \tag{102}$$

Lemma 7. *The risk neutral process is the same type of process, but with parameters connected to their actual counterparts as follows*

$$\theta_{*,t} = \theta_t \tag{103}$$

$$a_*^2 = a^2 - 2\delta < a^2 \tag{104}$$

$$b_* = b + \gamma > b \tag{105}$$

$$c_*^2 = a_*^2 - b_*^2 < c^2 \tag{106}$$

Proof: Using the analysis of the previous section, and (7), we may identify the process through its cumulant generator

$$\mathcal{K}_t^*(s,q;h) = \mathcal{K}_t(s-\gamma, q+b\gamma+\delta; h) - \mathcal{K}_t(-\gamma, b\gamma+\delta; h) =$$
$$\left(\sqrt{c^2-\gamma^2-2b\gamma-2\delta} - \sqrt{c^2-(s-\gamma)^2-2q-2b\gamma-2\delta}\right)\int_t^{t+h}\frac{du}{\theta_u}$$

□

A SIMPLE CALIBRATION EXERCISE

Assume that the value for the persistent volatility currently is at 20%, or equivalently $v = 4\%$. Assume also that the volatility of the RQV is $\sigma_y = 5\%$. This implies that $c = 20\%/5\% = 4$. Finally if we assume a correlation of returns with their RQV equal to $\rho = -60\%$, from (95) we find that $\frac{b}{a} = 60\%$ and $\frac{c}{a} = 80\%$. Thus under these assumptions we have $a = 5$ and $b = 3$. From (92) we have that the total return variance equals $\sigma_x^2 = 6.25\%$ ($\sigma_x = a\sigma_y = 25\%$) of which 4% (or 64% of the total) is the expected persistent part v, and 2.25% due to the variance of the non-persistent realization $b^2\sigma_y^2$. Also note that in this case from (96) we have an annual return skewness equal to $\text{SKEW}_x = \frac{3\rho}{c\sqrt{v}} = -2.25$.

Figure 6 is generated to provide a comparison between the parametric risk corrected return volatility $\sigma_x(\gamma,\delta)$ and the risk neutral volatility and rom the fact that $\theta = \frac{1}{vc}$ remains unchanged under both measures (103) and the ratio of persistent volatilities is $\frac{v(\gamma,\delta)}{v} = 1.1^2 = 1.21$, we have that $c(\gamma, \delta) = 3.30$. From $\theta = \frac{\sigma_y}{v^{3/2}}$ we also find that $\sigma_y(\gamma, \delta) = 1.21^{1.5}\sigma_y = 6.655\%$. Now observe that if we do not assume any other risk correction (besides $v(\gamma, \delta)$) we cannot recover both risk prices γ and δ, but we have a relation that they have to satisfy from (106)

$$2\delta + (\gamma + 3)^2 = 21.70 \tag{107}$$

This implies Assume that for a large enough period the persistent components v and σ_y are constant. $a^2 = 1$. From CBOE website we get the long term stock volatility during 2005 as $v = 10\%$. At the end of 2005, the VIX index (a risk neutral volatility) was around $v_* = 12\%$. This implies a 2% differential between risk neutral and historical volatility values. Furthermore, the reported correlation between S&P 500 and its volatility index between Jan 1990 and June 2004, was around -60%. The volatility of volatility was around $\sigma_y = 83\%$.

From (92) we have that

$$\sigma^2 = a^2\sigma_y^2 \Rightarrow a = \tag{108}$$

5 Conclusion

The first result of the paper is that, when agents exhibit a constant degree of relative risk aversion, risk correction for any cumulant depends on the next order cumulant. This result is then used to develop some general results for skewness correction. Firstly, it is shown that negative return skewness implies an increase in the variance swap rates. It is then shown that only $1 - u$ percent of the excess kyrtosis generates skewness correction, where u is a new kyrtosis-normalized skew measure. Finally, the results are extended for stochastic volatility and multi-factor risks. In the last application, we are able to disentangle the variance swap rate differential into a skewness component, and a second component due to negative volatility premia.

A Appendix

Proof of (20). The diffusive idiosyncratic risk does not enter in higher cumulants

$$\text{SKEW}_i = \frac{c_3^i}{\sigma_i^3} = \frac{b_i^3 c_3}{b_i^3 \sigma^3 / R_i^3} = \text{SKEW} R_i^3$$

Proof of Lemma 3. Using (13), the derivative of the corrected skewness with respect to risk aversion at zero equals

$$\left(\frac{\partial \text{SKEW}(\gamma)}{\partial \gamma}\right)_{\gamma=0} = \frac{\partial}{\partial \gamma}\left(\frac{c_3(\gamma)}{c_2^{3/2}(\gamma)}\right)_{\gamma=0}$$

$$= \left(\frac{\partial c_3(\gamma)}{\partial \gamma} c_2^{-3/2}(\gamma) - \frac{3}{2} c_3(\gamma) c_2^{-5/2}(\gamma) \frac{\partial c_2(\gamma)}{\partial \gamma}\right)_{\gamma=0}$$

$$= -\left(c_4 \sigma^{-3} - \frac{3}{2} c_3^2 \sigma^{-5}\right) = -\frac{c_4}{\sigma^4}\left(1 - \frac{3}{2}\frac{c_3^2}{c_4 c_2}\right)\sigma$$

Finally expanding SKEW(γ) around zero and using the above value for $\frac{\partial \text{SKEW}}{\partial \gamma}\Big|_{\gamma=0}$ we recover (25).

Proof of Lemma 4. The proof follows the same steps as above, with the added observation that for individual stocks $\frac{\partial c_n^i(0)}{\partial \gamma} = -b_i^n c_{n+1}$

$$\left(\frac{\partial \text{SKEW}_i(\gamma)}{\partial \gamma}\right)_{\gamma=0} = \frac{\partial}{\partial \gamma}\left(\frac{c_3^i(\gamma)}{(c_2^i(\gamma))^{3/2}}\right)_{\gamma=0}$$

$$= \left(\frac{\partial c_3^i(\gamma)}{\partial \gamma}(c_2^i(\gamma))^{-3/2} - \frac{3}{2} c_3^i(\gamma)(c_2^i(\gamma))^{-5/2}\frac{\partial c_2^i(\gamma)}{\partial \gamma}\right)_{\gamma=0}$$

$$= -\left(b_i^3 c_4 \sigma_i^{-3} - \frac{3}{2} c_3^i \sigma_i^{-5} b_i^2 c_3\right) = -\frac{c_4^i}{\sigma_i^4}\left(1 - \frac{3}{2}\frac{(c_3^i)^2}{c_4^i c_2^i}\right)\frac{\sigma_i}{b_i}$$

Proof of Lemma 5. From (35) we can recover the cumulants of the VG process

$$c_2 = (l_+^2 + l_-^2)v$$

$$c_3 = 2(l_+^3 - l_-^3)v$$

$$c_4 = 6(l_+^4 + l_-^4)v \tag{109}$$

Even though for the one-sided u statistics

$$u_\pm = \frac{3}{2}\frac{c_3^{\pm 2}}{c_4^\pm c_2^\pm} = 100\% \tag{110}$$

the two-sided u, which depends on total cumulants, is smaller

$$u = \frac{3}{2}\frac{(c_3^+ - c_3^-)^2}{(c_4^+ + c_4^-)(c_2^+ + c_2^-)} = \frac{(l_+^3 - l_-^3)^2}{(l_+^4 + l_-^4)(l_+^2 + l_-^2)} < 100\% \tag{111}$$

as simple algebra shows. $\square$

Proof of (60) (57) and (63). From (54) we have that the joint cumulant kernel equals $\mathcal{K}(s,q) = \mathcal{K}_y(u)$ for $u = \frac{1}{2}s^2 + q$. Firstly, $c^{x,x,x} = \frac{\partial^3\mathcal{K}(0,0)}{\partial s^3} = 3s\frac{\partial^2\mathcal{K}_y(0)}{\partial u^2} + s^3\frac{\partial^3\mathcal{K}_y(0)}{\partial u^3} = 0$. Then, $c^{x,x,y} = \frac{\partial^3\mathcal{K}(0,0)}{\partial s^2\partial q} = \frac{\partial^2\mathcal{K}_y(0)}{\partial u^2} = \sigma_y^2$. Using the same algebra, we have $c^{x,y,y} = \frac{\partial^3\mathcal{K}(0,0)}{\partial s\partial q^2} = 0$. $\square$