Dynamic Versus Static Optimization of Hedge Fund Portfolios: The Relevance of Performance Measures.

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Abstract

This paper analyzes the relevance of a set of some performance measures for optimal portfolios including hedge funds. Four criteria are considered: the Sharpe Ratio, the Returns on VaR and on CVaR, and the Omega performance measure. The results are illustrated by an allocation on several indices: HFR (Global Hedge Fund Index), JPM Global Bond Index, S&P GSCI, MSCI World and the UBS Global Convertible. Both static and dynamic optimizations are considered. Due to the non-convexity of some of the criteria, we use the "threshold accepting algorithm" to solve numerically the optimization problems. The time period of the analysis is September 1997 to August 2007. Our results suggest that, for the dynamic optimization, the portfolio which maximizes the Omega measure has the more stable performances, in particular when compared to the Return-on-CVaR portfolio. As a by-product, we prove that all the optimal portfolios had to contain hedge funds for the time period 1997-2007.

Keywords: *hedge funds, CVaR, tail risk, Omega measure.*

JEL classifications: *C6, G11, G24, L10.*

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1. Introduction

The seminal Markowitz's analysis is based on the first two moments of portfolio return and ignored the higher ones. By using an improved estimator of the covariance structure of hedge fund index returns, Amenc and Martellini (2002) prove that a portfolio including hedge funds can have a significantly smaller volatility (on an out-of-sample basis), while having almost the same mean return. This result suggests to incorporate hedge funds to get a mean-variance efficient portfolio. As illustrated by Cremers *et al.* (2005), hedge funds returns usually have higher means and lower standard deviations than standard assets, but they also have undesirable higher moment properties. For example, hedge funds returns can be negatively skewed. This is due to for instance to the non-linearity of their payoffs when they are generated by option-like strategies (see Goetzmann *et al.*, 2002, 2003).

Many empirical studies have also proved that the mean-variance approach is no more valid when investors include hedge funds in their portfolios (Amenc and Martellini, 2002; Terhaar *et al.*, 2003; McFall Lamm, 2003; Kat, 2004; Agarwal and Naik, 2004; Alexander and Dimitriu, 2004; Morton *et al.*, 2006).

Empirical studies show that the assumption of normality in return distribution is not justified, in particular when dealing with hedge funds which have significant positive or negative skewness and high kurtosis (Fung and Hsieh, 1997; Ackerman *et al.*, 1999; Brown *et al.*, 1999; Caglayan and Edwards, 2001; Bacmann and Scholz, 2003; Agarwal and Naik, 2004).

One possible extension of the mean-variance analysis is the use of the expected utility theory. For a truncated utility at the order 4, we have to maximize a linear combination of the first four moments of the return. In this more general framework, the hedge fund portfolio is built according also to the skewness and kurtosis. This problem has been studied by Jurczenko and Maillet (2004), Malevergne and Sornette (2005), Davies et *al.* (2004), Anson and *al.* (2007)...Martellini and Ziemann (2007) extend the Black-Litterman Bayesian approach when building a portfolio of hedge funds with non-trivial preferences about higher moments. Their results show that active style allocation provides a significant value in a hedge fund portfolio if we take account of non-normality and parameter uncertainty in hedge fund return distributions.

Since the demise of Long Term Capital Management (LTCM), downside risk measures have been introduced[4] to take more account of the tails of the distributions. In most cases, they are based on the Value-at-Risk (VaR) approach. As proved by Weisman (2002), Goetzman et *al.* (2003), Agarwal and Naik (2004), Davies e*t al.* (2004), the VaR can be high, due to important leverage effect. VaR also is more appropriate than the first four moments to measure extreme risks. VaR optimization has been previously studied by Literman (1997a, 1997b), Lucas and Klaasen, (1998). Additionally, Favre and Galeano (2002) introduce a new value-at-risk method to adjust the volatility risk with the skewness and the kurtosis of the distribution of returns. They minimize the modified VaR at a given confidence level. However, as pointed by Lo (2001), VaR has several shortcomings. For instance, it does not indicate the magnitude of the potential losses below itself. Additionally, it is not a convex risk measure (see Artzner *et al.*, 1999) and its minimization is rather involved.

For these reasons, Acerbi and Tasche (2002) introduce the expected shortfall (ES). This measure, also called CVaR, measures the amount of losses below the VaR, is coherent and leads to portfolio optimization problems that can be solved more easily (see Acerbi, Nordio and Sirtori, 2001).

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[4]Maybe more by academic researchers than by practitioners, as mentioned by Amenc *et al.* (2004).

Indeed, Rockafellar and Uryasev (2000) prove that the CVaR minimization is equivalent to a convex optimization problem. Agarwal and Naik (2004) estimate ES from the HFR hedge fund indices. They show that downside risk is significantly underestimated by the mean-variance analysis, which suggests that mean-ES optimization must be introduced.
Given benchmark related investment objectives, Popova *et al.* (2007) study the optimal allocation to hedge funds, for criteria such as expected shortfall and semi-variance. Using a stochastic programming model based on Monte Carlo simulation, they prove that a 20% allocation to hedge funds is justified. Their optimal portfolios are more skew to the right relative to those of the optimal mean-variance portfolios. Thus, they have higher Sortino ratios.
Liang and Park (2007) examine the risk-return trade-off for the hedge funds. They compare semi-deviation, VaR, ES and Tail Risk (TR) at both the individual fund and the portfolio levels. They show that the cross-sectional variation in hedge fund return is well explained by the left-tail risk captured by ES and TR and not by the other risk measures. They prove that between January 1995 and December 2004, hedge funds with high ES outperform those with low ES (annual return difference of 7%).
As shown by Hubner (2007) for the information ratio and the alpha, the relevance of performance measures heavily depends on the kind of portfolios that is managed by the investor. Impacts of individual constraints are analyzed by Bernd and Xu (2007).
In this paper, we propose also to examine the relevance of different performance measures when determining optimal portfolios including hedge funds, while taking account of portfolio constraints. In particular, we focus on the Sharpe, mean VaR and CVaR ratios. As a by-product, we also introduce the Omega performance measure, defined by Keating and Shadwick (2002) and Cascon *et al.* (2003). This ratio takes account of all the moments of the distribution. It penalizes the returns below a given level and emphasizes the returns beyond this threshold.
Section 2 recalls definitions and main properties of the performance measures that are used in the paper. Section 3 examines the static and dynamic optimization of the four performance measures, using VaR Cornish-Fisher estimates and the *threshold accepting algorithm* for portfolio optimization. The empirical analysis is based on a portfolio that includes five indices: HFR (Global Hedge Fund Index), JPM GBI, S&P GSCI TR, MSCI World and the UBS Global Convertible.
The results highlight the importance of the asset allocation model based on the maximization of the Omega performance measure, with or without specific constraints on portfolio weights. Indeed, this approach takes account of the characteristics of hedge fund portfolios. Some technical analysis and empirical illustrations are gathered in the Appendix.

2. Risk-Adjusted Performance Measures and Optimization

To compare funds with different characteristics of returns and risks, several performance measures have been introduced[5]. Among them, the Sharpe Ratio, the Returns on VaR and on CVaR, and the Omega performance measure.
For each criterion to be maximized, we consider that the investor chooses among n financial assets with returns R= ($R_1,\ldots,R_n$). Let R_f be the return of the riskless asset. Denote e= (1,…,1) Denote also by w=($w_1,\ldots,w_n$) the weighting vector satisfying :

$$w^t.e = \sum_{i=1}^{n} w_i = 1$$

We assume that no shortselling is allowed: $w_i \geq 0$. The portfolio return at maturity with cdf *F* and mean $\overline{R}$ is given by

$$R^w = w^t.R = \sum_{i=1}^{n} w_i R_i$$

2.1. The Sharpe ratio maximization

It is the standard optimization problem, which is yet a benchmark. We have to solve:

$$\text{Max } \frac{w^t.\overline{R} - R_f}{\sqrt{w^t \Sigma w}}, \text{ with } w^t.e = 1 \text{ and w} \geq 0,$$

where Σ is the variance covariance matrix. The mean $\overline{R}$ is computed from data, observed along a given time period.

2.2 The Mean-VaR optimization

The VaR measure has been extensively used in risk management. Recall that the VaR is defined as a quantile: For a position *X* and a given probability level α, the value $VaR(\alpha)$ satisfies:

$$VaR(\alpha) = -\min\left\{\gamma \middle| P\left[X \leq \gamma\right] \geq \alpha\right\}.$$

If X has a non negative pdf, then:

$$P\left[X \leq -VaR(\alpha)\right] = \alpha.$$

$VaR(\alpha)$ is the smallest loss at the probability level α. The Ro VaR (or Mean VaR) ratio is defined by:

$$RoVaR(\alpha) = \frac{\overline{R} - R_f}{VaR(\alpha)}$$

[5]See Lhabitant (2004) for more details about the choice of performance measures.

We use the Cornish-Fisher approximation, which allows an approximation of the VaR from the four first moments of the distribution. The maximization problem is given by:

$$\text{Max } \frac{w^t.\overline{R} - R_f}{VaR}, \text{ with } w^t.e = 1 \text{ and } w \geq 0,$$

Despite its popularity, VaR has some undesirable properties, such the non-convexity (see Artzner *et al.* 1999), which may discourage diversification. Additionally, the VaR only takes account of the probability to be under a given threshold and not of the order of magnitude of the losses beyond this level. Therefore, alternative risk measures have been proposed such as the *Expected Shortfall (ES)*, introduced by Acerbi *and al.* (2002, 2004), also called the *Conditional value-at-risk (CVaR)* in Rockafellar and Uryasev (2002) or *TailVaR* in Artzner *et al.* (1999).

2.3 The Mean-CVaR optimization

The CVaR can be defined as the expected value of the portfolio loss below a given threshold, given the fact that this loss is higher that this level (see Acerbi, 2004). The Ro CVaR (or Mean CVaR) ratio is defined by:

$$RoCVaR(\alpha) = \frac{\overline{R} - R_f}{CVaR(\alpha)}$$

Then, the maximization problem is given by:

$$\text{Max } \frac{w^t.\overline{R} - R_f}{CVaR}, \text{ with } w^t.e = 1 \text{ and } w \geq 0.$$

Note that, if we assume that returns $(R_j^w)_{1 \leq j \leq m}$ are independently distributed, the CVaR optimization problem is equivalent to: (notation: $\mathbf{1}_A$ is the indicator function of set A).

$$\text{Max } \frac{(w^t.\overline{R} - R_f)}{\left(\dfrac{\sum_{j=1}^m R_j^w \mathbf{1}_{R_j^w > VaR}}{\sum_{j=1}^m \mathbf{1}_{R_j^w > VaR}}\right)}, \text{with } w^t.e = 1 \text{ and } w \geq 0.$$

Another risk measure can take account of the characteristics of hedge funds returns: the Omega function, introduced by Keating and Shadwick (2002, 2003).

2.4 The Omega optimization

The Omega function $\Omega(L)$ can potentially take account of the whole probability distribution of the returns. The Omega measure introduced by Keating and Shadwick (2002) is based on the stochastic dominance approach. The measure $\Omega(L)$ is equal to:

$$\Omega_F(L) = \frac{\int_L^b (1 - F(x))dx}{\int_a^L F(x)dx},$$

where $F(.)$ is the cdf of the random variable (for example equal to the portfolio return) with support in $[a,b]$. The level L is the threshold chosen by the investor: returns smaller than L are viewed as losses and those higher than L are gains. For a given threshold L, the investor would prefer the portfolio with the highest Omega measure. As shown by Kazemi, Schneeweis and Gupta (2003), the Omega function has the following properties:

The ratio Omega is equal to:

$$\Omega_{F_X}(L) = \frac{E_P\left[\left(X - L\right)^+\right]}{E_P\left[\left(L - X\right)^+\right]}.$$

This is the ratio of the expectations of gains above the given level L upon the expectation of losses below L. Therefore, $\Omega_{F_X}(L)$ can be interpreted as a ratio Call/Put defined on the same underlying asset X, with strike L and computed with respect to the historical probability P. The Put is the risk measure component. It allows the control of the losses below the threshold L.
Kazemi, Schneeweis and Gupta (2003) define the Sharpe Omega by:

$$Sharpe_\Omega\left(L\right) = \frac{E_P\left[X\right] - L}{E_P\left[\left(L - X\right)^+\right]} = \Omega_{F_X}(L) - 1.$$

If $E_P\left[X\right] < L$, the Sharpe Omega measure is negative. Otherwise, it is positive. Suppose that X is the value at maturity T of a given asset S with a Lognormal distribution:

$$X = S_0 exp[(\mu - \sigma^2/2)T + \sigma W_T],$$

where W_T has a Gaussian distribution. Then $E_P\left[X\right] = S_0 \exp[\mu T]$ does not depend on the volatility σ. Thus, if $S_0 \exp[\mu T] < L$, then the Sharpe Omega ratio is an increasing function of the volatility σ whereas, for $S_0 \exp[\mu T] > L$, it is a decreasing function of σ.

Due to the previous properties, the usual values of the level L are such that $L < \overline{X}$. Indeed, it is more convenient that a performance measure is a decreasing function of the standard risk σ than the converse.

The corresponding optimization problem is:

$$\text{Max}\,\Omega(L)(R^w),\ \text{with}\ w^t.e = 1 \text{ and } w \geq 0,$$

which is equivalent to:

$$\text{Max}\ \frac{E\left[\left(R^w - L\right)^+\right]}{E\left[\left(L - R^w\right)^+\right]},\ \text{with}\ w^t.e = 1 \text{ and } w \geq 0.$$

Then, from the observation of independent returns $(R_j^w)_{1\leq j\leq m}$, we have to solve:

$$\text{Max}\ \frac{\sum_{j=1}^m \left(R_j^w - L\right)\mathbf{1}_{R_j^w > L}}{\sum_{j=1}^m \left(L - R_j^w\right)\mathbf{1}_{R_j^w < L}},\ \text{with}\ w^t.e = 1 \text{ and } w \geq 0.$$

Due to the non convexity of some of the previous objective functions, we use the *threshold accepting algorithm* to solve numerically the optimization problems. This algorithm has been introduced by Dueck and Scheuer (1990). It is a refined version of the standard local search procedure (for more details, see Appendix A and Winker, 2001). It is a local search procedure which accepts moves to neighbourhood solutions that improve the objective function value. Dueck and Scheuer (1990) prove that the *Threshold Accepting Algorithm* converge to the optimal portfolio solution when dealing with complex objective functions such as shortfall optimization case. Recently, Gilli and Kellezi (2000) have used this algorithm for CVaR and Omega portfolios optimization.

3. The Empirical Results

In what follows, we search for the optimal portfolio based on indices. The optimal allocation is determined among five main asset classes: Hedge funds, Equities, Bonds, Commodities and Convertibles. Two optimization methods are used:

- A static optimization with only one period (the whole period of observations).
- A dynamic optimization based on the back testing method.

3.1 The data analysis

The portfolio contains the following indices:

- The HFRX Global Hedge Fund USD,
- The UBS Global Convertible,
- The JPM Global GBI LC,
- The MSCI World Free Equity
- and the S&P GSCI.

Table 1 describes these indices.

Table 1

Description of the five indices

The data on hedge funds are obtained from the commonly used Lipper TASS. The time period of the analysis lies between September 1997 and August 2007. The sample contains 121 net-of-fees monthly returns.

Indices	Description
HFRX Global Hedge Fund USD	The index is built as follows: Sep-97 to March-03: The index corresponds to the mean of performances of hedge funds, quoted by the Lipper basis (without survivorship bias *) April-03 to Aug-07: the index is the HFRX Global Hedge Fund USD
UBS Global Convertible	The index represents convertibles
JPMorganGlobal GBI TR LC	The index represents the performance of bonds in local currency.
MSCI World Free TR USD	It is the international equities index
S&P GSCI TR	It is the commodities index

*See Bing Liang (2000) for such correction.

Looking at realized returns for the given time period, the hedge funds and convertibles generally have better performances, as shown by Figure 1. The HFR index performances were smooth and steady despite the high correlation with stock markets (represented by MSCI World Free). The index has shown real capacity to preserve capital through some crisis periods such as 1998 (LTCM collapse) or 2000 (Technology bubble). During the financial crisis 2000-2002, the hedge funds had clearly higher returns (positive returns while those of the equities were significantly negative).

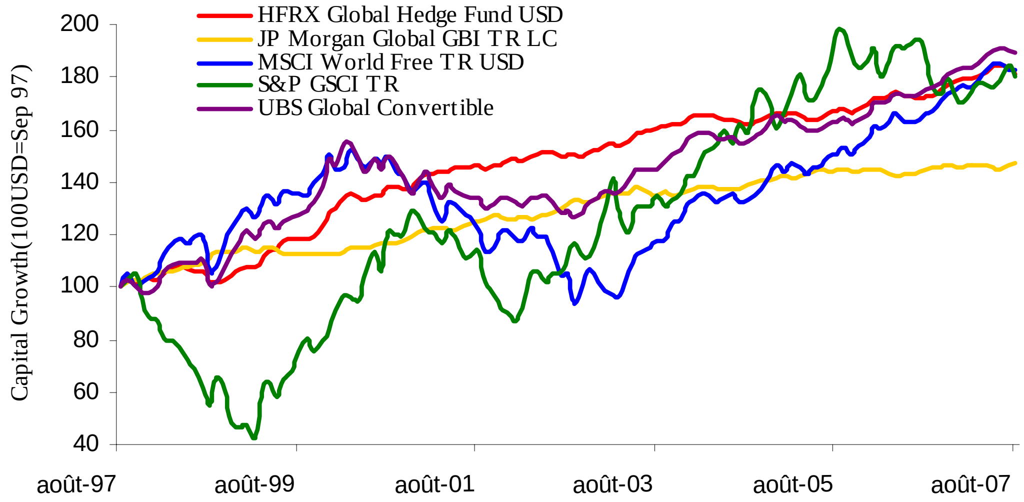


Fig. 1 Cumulative monthly returns

The S&P GSCI TR has the worst performance until 2002, while growing up and having the best performance from January 2005 to January 2007.

The Global Hedge Fund index has regularly grown up. In particular, it has not fallen from 2000 to 2002, contrary to the MSCI World and has provided capital protection during this time period. Looking at cumulative returns, the UBS Convertible index has risen during two periods: from Sep97 to Sep 00 and from Jan-03 to Sep 07, while it has fallen between these two periods, which can be explained by a falling stock market and spreads widening.

Looking at correlations, for most of the cases, the equity index has the highest correlation with the other indices, in particular with the Convertible and Hedge indices (see Table 2).
The Convertibles also have a high correlation with the hedge funds.

Table 2

Correlations between the indices

The correlation matrix is based on the Pearson Correlation measure. Hedge (1) refers to HFRX Global Hedge Fund USD index. Bond (2) is the JP Morgan Global GB Index. Equity (3) represents the MSCI World Free TR USD index. Commodity (4) represents the S&P GSCI. The last index (5) refers to the UBS Global Convertible.

Indices		1	2	3	4	5
Hedge	1	1.00				
Bond	2	-0.12	1.00			
Equity	3	0.57	-0.27	1.00		
Commodity	4	0.20	0.03	0.01	1.00	
Convertible	5	0.73	-0.11	0.81	0.13	1.00

As shown by the QQ-Plots in Figure 9 (see Appendix B), the probability distributions are not Gaussian. The main statistical properties of the five indices are provided in Table 3.

Table 3

Statistical characteristics of the indices (monthly)

Summary statistics for the five indices returns over the period of the analysis. They include the first four moments: Mean, Standard Deviation, Skewness, and Kurtosis. Other statistics are provided such as Median, Maximum, Minimum and the risk measure Value-at-Risk (VaR) which is calculated at the confidence level (1-p=5%).

	Hedge	Bond	Equity	Commodity	Convertible
Mean	0.68	0.40	0.69	0.67	0.74
Median	0.60	0.46	1.23	0.42	0.95
Maximum	5.69	2.19	9.07	16.89	11.21
Minimum	-3.53	-1.59	-13.32	-14.41	-9.34
Std. Deviation	1.40	0.84	4.08	6.35	3.05
Skewness	0.42	-0.32	-0.64	0.10	-0.21
Kurtosis	4.25	2.64	3.76	2.67	4.24
VaR (p=5%)	1.62	1.25	8.42	10.33	4.52

The convertibles have the highest mean, but with a rather high standard deviation. The equity index has similar properties as the convertible index but with a more negative skewness. The hedge index has a weak standard deviation and a positive skewness. Its mean is similar to those of the equity and commodity indices. The highest VaR and CVaR at the level 5%, have the same order of magnitude as the commodity index. Those of the convertibles are higher than those of the equity index.

The Omega ratios of the five indices are plotted in next Figure 2, as function of the threshold L.

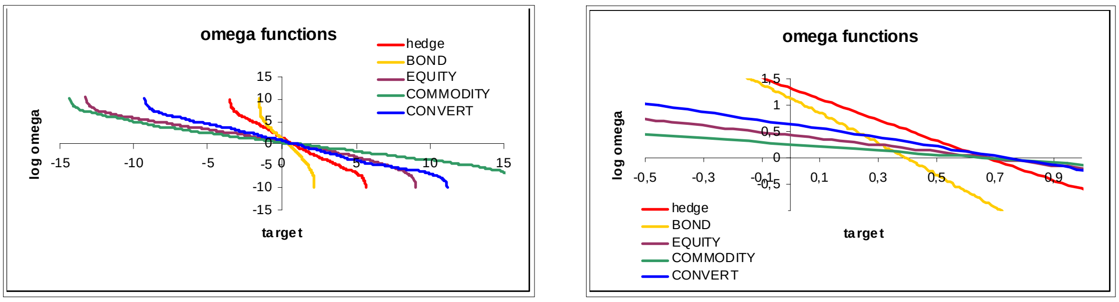


Fig 2. Omega measures of the five indices

In particular, we get the following ranking for usual thresholds.

Table 4

Omega ranking of the five indices (monthly returns)

The target level *L* varies from 0% to 0.65%. The indices are ranked from the best (1) to the worst (5).

	Hedge	Bond	Equity	Commodity	Convertible
Target					
0	1	2	4	5	3
0,2	1	2	4	5	3
0,25	1	2	4	5	3
0,3	1	2	4	5	3
0,35	1	4	3	4	2
0,4	1	5	3	4	2
0,45	1	5	3	4	2
0,5	1	5	3	4	2
0,55	1	5	4	3	2
0,6	1	5	3	4	2
0,65	1	5	3	4	2

The hedge fund index dominates the other indices on the set of values $L \in [0\%; 0.65\%]$. The bond index dominates the equity, commodity and convertible indices for thresholds $L \in [0\%; 0.02\%]$ (which are usual value of L). Most of the time, the commodity index is dominated by the other ones. During the period 1997-2007, the commodities have not provided a true diversification for the four static optimal portfolios. This property is usually verified. This is due to the left tail of its distribution, as shown for example by its VaR at the level 5% which is equal to 10.33.

3.2 The static framework

In what follows, we examine the static optimal allocations. Our base value for the Omega threshold L is equal to 0%. Table 5 indicates the optimal allocations (percentage) which correspond to each optimization criterion.

Table 5

Static optimal weights

This table shows the weights of the five indices for each of the four optimal portfolios, in the static case. Four approaches are considered. The first one (Sharpe) is based on the Sharpe ratio maximization. The second approach is the Mean-VaR optimization at the (1-p=5%) confidence level. Omega refers to the portfolio obtained under Omega optimization ratio. The last one represents the Mean-CVaR optimization (CVaR) at the $(1-p=5\%)$ confidence level. Note that short selling is not allowed and the weighting vector satisfies $w^t.e=1$

Portfolio	Hedge	Bond	Equity	Commodity	Convertible
Sharpe	0.29	0.66	0.00	0.00	0.05
VaR	0.45	0.55	0.00	0.00	0.00
Omega	0.43	0.37	0.06	0.02	0.12
CVaR (p=5%)	0.41	0.02	0.54	0.03	0.00

Only the hedge fund index is a significant component of all optimal portfolios. This is due to the positive skewness of the hedge index (sk=0.42) while having a mean (E=0.68) similar to the other indices (except the convertibles). For example, for the equity index (sk=-0.64, E=0.69), the Sharpe Omega is weaker than for the hedge index. Indeed, the expectation of the losses $E[(-X)^+]$ is higher since the skewness is negative, contrary to the hedge index. The Omega portfolio is the most diversified: all its weights are different from zero.

The MeanCvaR and Omega optimal portfolios contain much more hedge funds than convertibles. Indeed, despite its highest mean, the convertible index has high VaR and CVaR. However, the weights on equities and bonds are quite different for the MeanCvaR and Omega optimal portfolios. In fact, the Omega performance measure is more sensitive to the probability of a drawdown than the MeanCVaR criterion. This property is explained in what follows. Consider a given threshold L and a random return R. The EroCVaR (or Excess Mean CVaR) ratio is defined by:

$$ERoCVaR(\alpha, L) = \frac{\overline{R} - L}{CVaR(\alpha)},$$

where $\overline{R}$ denotes the expectation of the rate of return $E_P[R]$. Since we have:

$$CVaR(\alpha) = E_P[-R \mid -R \geq VaR(\alpha)]$$

$$= E_p\left[L - R\,1_{\text{-R}\geq\text{VaR}(\alpha)}\right] / E_p\left[-R \geq VaR(\alpha)\right],$$

we deduce that, for $L = 0$, $ERoCVaR(\alpha, 0) = RoCVaR(\alpha)$ and also that

$$RoCVaR(\alpha) = \frac{\overline{R}P[-R \geq VaR(\alpha)]}{E_P\left[-R1_{-R\geq VaR(\alpha)}\right]}, \text{ and } Sharpe_\Omega(0) = \frac{\overline{R}}{E_P[-R1_{-R\geq 0}]}.$$

Then, in that case, maximizing $RoCVaR(\alpha)$ does not penalize $P[-R \geq VaR(\alpha)]$ as far as the ratio $\overline{R} / E_P[-R1_{-R\geq VaR(\alpha)}]$ and the Sharpe Ω ratio. For $L = -VaR(\alpha)$, we get:

$$RoCVaR(\alpha, L) = \frac{\left[\overline{R} + VaR(\alpha)\right]P[-R \geq VaR(\alpha)]}{E_P\left[-R1_{-R\geq VaR(\alpha)}\right]},$$

and

$$Sharpe\Omega(L) = \frac{\overline{R} + VaR(\alpha)}{E_P\left[-R1_{-R\geq VaR(\alpha)}\right] - VaR(\alpha)E_P[-R \geq VaR(\alpha)]}.$$

Thus, both $RoCVaR(\alpha)$ and Sharpe Ω maximizations do not penalize the probability $P[-R \geq VaR(\alpha)]$ in the same manner. Note that both criteria do not penalize the probability to bear losses beyond $VaR(\alpha)$ as far as the ratio

$$\frac{\overline{R} + VaR(\alpha)}{E_P[-R1_{-R\geq VaR(\alpha)}]}.$$

Let us examine the cumulative returns of the four optimal portfolios[6].

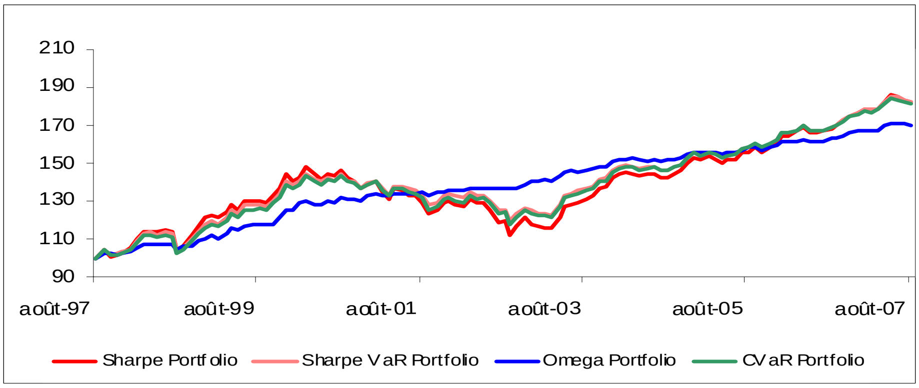


Fig 3. Cumulative returns of the four optimal portfolios

We note that the Omega optimal portfolio is the more stable one, while the other static portfolios have quite similar cumulative returns. This is line with the purpose of such portfolio which is to limit the downside risk and potential losses for $L = 0\%$. The main characteristics of the returns are provided in Table 6. The Omega portfolio has the smaller mean but only this portfolio has a positive skewness. Additionally, it has the best kurtosis.

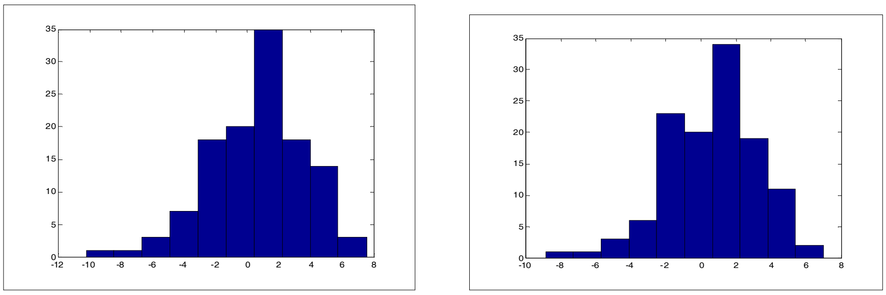
Table 6

Optimal portfolio characteristics

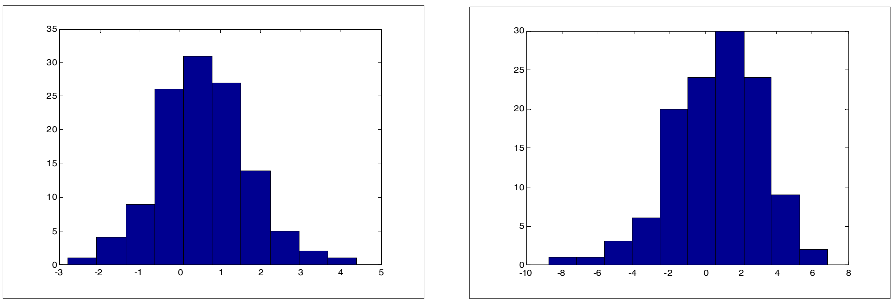
	Sharpe	VaR	Omega	CVaR
Mean	0.70	0.68	0.58	0.68
Median	1.11	1.04	0.50	1.01
Maximum	7.27	7.01	4.39	6.85
Minimum	-9.74	-8.87	-2.78	-8.77
Std. Deviation	0.97	2.64	1.10	2.58
Skewness	-0.60	-0.55	0.26	-0.58
Kurtosis	0.82	0.76	0.97	0.83
VaR (p=5%)	7.42	6.16	1.69	3.20
CVaR (p=5%)	1.27	1.00	0.10	5.18
Sharpe ratio (p=5%)	0.72	0.26	0.53	0.26

Figure 4 illustrates the cdf of the four optimal portfolios. For the threshold *L=0%*, the Omega portfolio return is almost Gaussian. This is in particular due to the diversification effect, as illustrated in Table 5 and due to standard statistical property.

[6]These returns are determined from the optimal allocations and the observed index returns on the given time period 1997-2007.



Sharpe optimal portfolio   Mean-VaR optimal portfolio



Omega optimal portfolio   Mean CVaR optimal portfolio

Fig 4. The four optimal returns (histograms)

The Log Omega ratios of the four optimal portfolios are displayed, in next Figure 5.

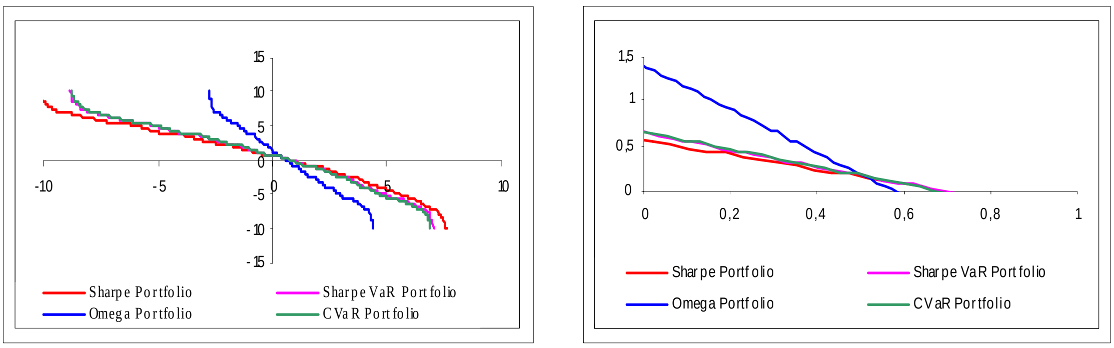


Fig. 5 Omega ratios of the four optimal portfolios, as functions of the threshold L

As expected, the Omega optimal portfolio has the highest Omega value, for thresholds L below 0.6%. Note that this value corresponds (approximately) to the means of the four portfolios.
The other portfolios have quite similar Omega performances for L smaller than 0.6%. The previous results show that the four optimal portfolios exhibit similar absolute performances (about 0.6% per month). Additionally, they suppose that parameters of interest are quite anticipated. In practice, this assumption is rather strong.
In what follows, we propose a more realistic framework: the dynamic allocation, as used by fund managers.

3.3 The dynamic allocation

Now, we consider dynamic allocations. The data correspond to monthly observations from 30/09/1997 to 31/08/2007 (Currency: US Dollar).

The dynamic allocation principle and main results.

Dynamic allocations are based on two time periods: the analysis period and the managing period. The simulation is carried through two steps: First, the estimation of the optimal portfolio weights using the data in the analysis window and, second, the calculation of the optimal portfolio performances over the projection window on a monthly basis.
Next Table 7 provides statistical characteristics of the four optimal portfolios according to various values of both the analysis window length and the projection window length.

Table 7

Statistical properties of the four optimal portfolios

We use the following notations: Analysis window length: Aw; Projection window length: Pw. The table reports summary statistics of the four optimal portfolios. For each one, we compute the statistics of the dynamic allocations, according to the choice of the analysis window length (A_w) and the projection window length (P_w). Six cases are considered: A_w can take the values (36, 24,or 12 months). According to the choice of the A_w, we suppose that optimal portfolios could be obtained with 3 or 6 months rebalancing. For the Omega portfolio, we suppose that the threshold (L=th) is equal to (0%).

Omega th=0	Aw=36 Pw=6	Aw=24 Pw=6	Aw=12 Pw=6	Aw=36 Pw=3	Aw=24 Pw=3	Aw=12 Pw=3
Mean	0,54	0,59	0,69	0,55	0,60	0,72
Median	0,47	0,51	0,56	0,54	0,54	0,63
Max	3,28	3,46	5,11	2,59	4,68	5,32
Min	-2,45	-3,24	-2,86	-2,13	-2,70	-3,82
Std .Dev	1,09	1,22	1,31	0,93	1,37	1,67
Skweness	-0,22	-0,23	0,28	-0,22	0,23	0,67
Kurtosis	0,67	0,35	1,13	0,30	0,37	2,54
Sharpe ratio	0,49	0,48	0,53	0,59	0,44	0,43
VaR 5%	1,14	1,55	0,69	0,94	1,62	0,86
CVaR	1,85	2,11	2,02	1,50	2,13	2,53
observations	84	96	108	84	96	108

Sharpe	Aw=36 Pw=6	Aw=24 Pw=6	Aw=12 Pw=6	Aw=36 Pw=3	Aw=24 Pw=3	Aw=12 Pw=3
Mean	0,46	0,56	0,74	0,47	0,54	0,70
Median	0,49	0,46	0,68	0,50	0,45	0,63
Max	2,29	2,79	6,82	2,29	2,79	6,84
Min	-1,79	-1,75	-2,34	-1,79	-1,75	-2,62
Std .Dev	0,67	0,87	1,32	0,58	0,88	1,38
Skweness	-0,32	0,11	1,47	-0,32	0,21	1,39
Kurtosis	0,42	-0,06	5,41	0,52	0,11	4,96
Sharpe ratio	0,69	0,65	0,56	0,81	0,61	0,51
VaR 5%	0,74	1,28	0,88	0,17	0,07	1,30
CVaR	1,18	1,14	1,62	1,15	1,26	1,76
observations	84	96	108	84	96	108

VaR p=0,05	Aw=36 Pw=6	Aw=24 Pw=6	Aw=12 Pw=6	Aw=36 Pw=3	Aw=24 Pw=3	Aw=12 Pw=3
Mean	0,46	0,56	0,71	0,47	0,56	0,66
Median	0,50	0,51	0,66	0,50	0,44	0,63
Max	2,27	2,56	5,84	2,28	2,57	6,18
Min	-1,74	-1,99	-2,84	-1,74	-1,99	-2,67
Std .Dev	0,73	0,87	1,29	0,72	0,88	1,33
Skweness	-0,31	0,02	0,69	-0,32	0,17	1,14
Kurtosis	0,61	0,05	2,94	0,72	0,18	4,23
Sharpe ratio	0,63	0,64	0,55	0,65	0,64	0,50
VaR 5%	0,64	1,30	1,28	0,42	0,03	0,96
CVaR	1,18	1,23	1,98	1,13	1,18	1,92
observations	84	96	108	84	96	108

CVaR	Aw=36 Pw=6	Aw=24 Pw=6	Aw=12 Pw=6	Aw=36 Pw=3	Aw=24 Pw=3	Aw=12 Pw=3
Mean	0,38	0,75	0,61	0,47	0,60	0,86
Median	0,48	0,74	0,83	0,54	0,87	0,95
Max	6,04	6,90	5,32	2,35	12,72	6,44
Min	-8,76	-6,16	-7,75	-1,63	-10,71	-4,33
Std .Dev	2,82	2,26	2,33	0,70	3,28	2,22
Skweness	-0,73	-0,22	-0,75	-0,29	-0,26	-0,05
Kurtosis	1,26	1,23	1,27	0,84	2,77	-0,17
Sharpe ratio	0,14	0,33	0,26	0,68	0,18	0,39
VaR 5%	4,91	0,13	2,17	0,59	6,78	2,08
CVaR	6,69	4,67	5,08	1,08	7,56	3,54
observations	84	96	108	84	96	108

Figure 10 (see Appendix) allows the comparison of the dynamic allocations of the four optimal portfolios, according to the choice of A_w and P_w. The first column corresponds to $P_w = 6$ and the second one to $P_w=3$ As it can be seen, the performances of the four portfolios are more sensitive to the value A_w than to the value P_w .Indeed, for example, for $A_w=36$ and $A_w=12$, the Omega portfolio has better performances than the MeanCvaR one whereas for $A_w = 24$, it is the converse. Note that the MeanCVaR portfolio is the most sensitive to the choice of A_w, while the Omega is the less sensitive. For the three other portfolios, the performances are relatively stable with respect to the choices of both A_w and P_w. Figure 11 (see Appendix) describes the evolution of the four portfolio allocations. The parameter values are: $A_w = 36; P_w = 3$, which are the most used in practice (36 months correspond to the average life time of hedge funds; 3 months is the standard rebalancing time). The Omega portfolio is relatively different from the three other ones. For example, it is the only one which contains convertibles and still contains hedge funds from 2006. The Sharpe, MeanVaR and MeanCVaR portfolios have similar weighting evolutions.

The Omega portfolio properties

The main statistical characteristics of the Omega portfolio for $L = 0$, $A_w = 36$ and $P_w = 3$ are displayed in next Table 8, jointly with those of the five indices. Note that the Omega portfolio has the highest Sharpe ratio (equal to 0.59) and the smallest kurtosis.

Table 8

Characteristics of the Omega portfolios and the indices

Table 8 represents the main characteristics of the omega portfolio versus the five indices. For the Omega portfolio we suppose that the threshold is fixed at 0 %. The optimal portfolio is constructed with 3 months rebalancing ($Pw = 3$) and the analysis window length (Aw) is equal to 36 months.

	Omega	Hedge	Bond	Equity	Commodity	Convertible
Mean	0.55	0.68	0.40	0.69	0.67	0.74
Median	0.54	0.60	0.46	1.23	0.42	0.95
Maximum	2.59	5.69	2.19	9.07	16.89	11.21
Minimum	-2.13	-3.53	-1.59	-13.32	-14.41	-9.34
Std. Deviation	0.93	1.40	0.84	4.08	6.35	3.05
Skewness	-0.22	0.42	-0.32	-0.64	0.10	-0.21
Kurtosis	0.30	4.25	2.64	3.76	3.67	4.24
Sharpe Ratio	0.59	0.48	0.47	0.16	0.01	0.24

Figure 12 (see Appendix) shows how the optimal portfolio depends on the upper bound a imposed on the weights. A small upper bound induces more diversification. Note for example that for an upper bound about 30% to 40% on all weights, the Omega portfolio must include convertibles whereas for higher upper bound values, it does not contain convertibles. Next figure provides the optimal Omega allocations for May 2007, according to various upper bound constraints ($A_w = 36, P_w = 3$). For $a = 0.20$, the optimal weighting leads necessarily to weights that are all equal to 20% . The higher the upper bound, the smaller the diversification.

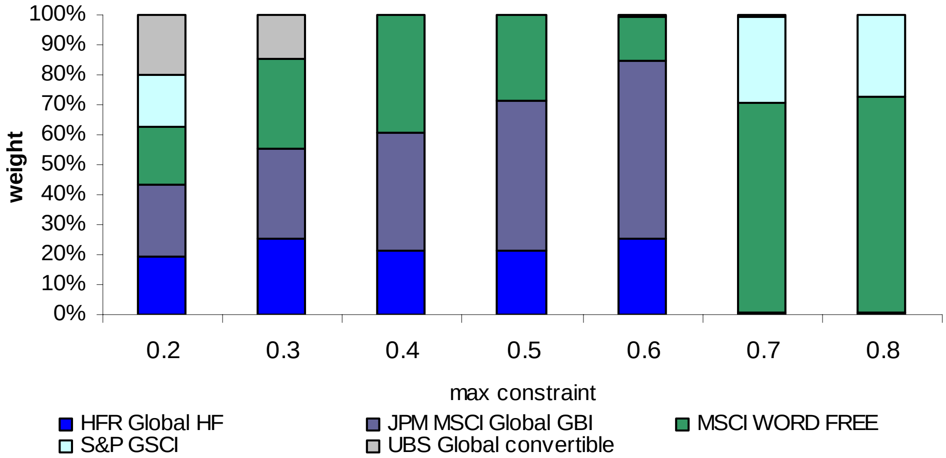


Fig. 6. Omega portfolio weights according to the maximal constraint

In what follows, we examine the properties of the Omega optimal portfolio according to the threshold L which is varying from 0% to 0.8% by a step equal to 0.05%. The choice of the upper and lower target L is justified by the characteristic of the index portfolio. In fact, the mean of the distribution for all assets in the index portfolio is 0.66%.

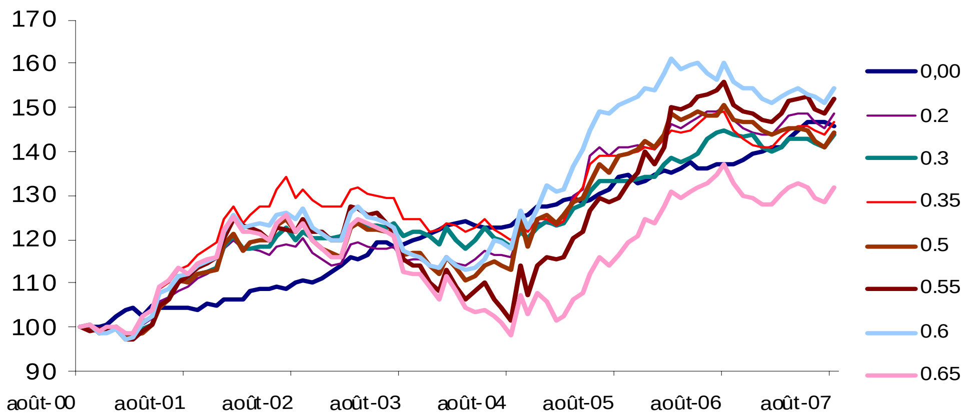


Fig. 7. Cumulative returns of Omega portfolio according to the level L

From August 2004 to August 2007, the Omega portfolio associated to the threshold $L = 0.6\%$ provides the best cumulative returns. Note that the Omega portfolio corresponding to the value $L = 0\%$ has the more stable returns along the whole period (as for the static case).

Table 9 and Figure 8 illustrate the influence of the threshold on the four first moments and the extreme values.

Table 9

Characteristics of the Omega portfolios for different target thresholds *L*

Table 9 examines the main statistical properties of the optimal Omega portfolio according to the target threshold: mean, minimum, maximum, standard deviation, skewness and kurtosis. The threshold *L* varies from 0% to 0.80%.

	Threshold																
	0,00	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80
Mean	0,55	0,52	0,57	0,47	0,58	0,52	0,52	0,56	0,58	0,36	0,52	0,62	0,65	0,38	0,54	0,84	0,55
Max	2,59	4,20	4,55	4,38	7,41	4,96	4,96	5,89	6,57	6,77	11,50	12,35	8,86	9,15	9,83	9,27	8,00
Min	-2,13	-4,31	-2,42	-4,57	-3,36	-3,35	-3,35	-4,54	-6,87	-6,62	-5,88	-6,70	-5,41	-8,07	-6,45	-7,67	-6,06
Std .Dev	0,93	1,33	1,47	1,62	2,05	1,76	1,76	2,10	2,20	2,35	2,68	3,14	2,63	2,90	2,90	2,89	2,78
Skewness	-0,22	-0,27	0,31	0,02	0,92	0,10	0,10	0,04	0,15	0,02	0,79	0,68	0,52	0,24	0,30	0,09	0,24
excess-Kurtosis	0,30	1,50	-0,25	0,49	1,61	0,04	0,04	0,48	1,30	0,45	2,77	2,02	0,43	0,67	0,61	0,91	0,31

The highest Sharpe ratio is reached for the threshold equal to 0%. However, the Sharpe ratio is not a decreasing function of the threshold (see for example the values for $L = 0.15$ and $L = 0.25$). Note also that none of the standard statistical characteristics is a monotonic function of the threshold L. For values of the threshold above 0.10%, the skewness is positive since increasing the threshold corresponds to the search of higher right tail of the distribution.

4. Conclusion

We have examined the relevance of four performance measures when they are used to determine optimal portfolios including hedge funds. Both CVaR and Omega measures are more appropriate, especially when the Cornish-Fisher expansion is introduced to calculate the CVaR. In the static optimization framework, the Omega provides more stable results whereas, for high volatility, the CVaR portfolio seems to perform better. In the dynamic case, corresponding to more actual portfolio management of hedge funds, the difference between the two methods is due to the different penalization of a given drawdown. We have also provided the analysis of the Omega performance measure when upper bounds on the weights are considered or when the associated threshold varies. As a by-product, we have shown that all the optimal portfolios had to contain hedge funds, for the time period 1997-2007. All these results are in line with those that can be deduced, for example when dealing only with purely hedge funds portfolios such as the *commodities trading advisers* (CTA).

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Appendix

A.The Threshold Accepting Algorithm

The optimization problem is the following (see Winker, 2001):
Let $f : X \to R$ be a function where X is a finite set:

$$X_{\min} = \{\chi \in X \mid f(\chi) = f_{opt}\}, \text{with } f_{opt} = \min_{\chi \in X} f(\chi).$$

A standard problem consists in searching for a local optimum local of the function f. Let χ_c be an approximated solution and χ_n a new approximated solution in the neighbourhood $N(\chi_c)$ of χ_c. The solution χ_n must satisfy:

$$f(\chi_n) < f(\chi_c).$$

Then, consider the following algorithm 1:

Table 11

Algorithm 1 (local search for a minimum)

1:	generate an approximated solution χ_c
2:	while criterion to stop if the condition is not satisfied do
3:	select $\chi_n \in N(\chi_c)$ (neighborough $N(\chi_c)$ of χ_c)
4:	if $f(\chi_n) < f(\chi_c)$ then $\chi_c = \chi_n$
5:	end while

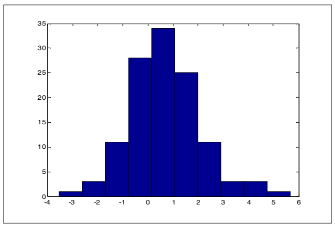
The criterion to stop the algorithm often is the number of iterations. Different methods are proposed to choose this criterion and the acceptation of the neighbourhood.
For this latter one, for each iteration r, the acceptance of a neighbor $\chi_n \in N(\chi_c)$ is only based on an auxiliary function $r(\chi_c, \chi_{c_0})$ and a threshold T_r $:\chi_{c_0}$ is accepted if and only if $r(\chi_c, \chi_{c_0}) < T_r$:The sequence of thresholds Tr is non increasing: $T_1 > T_2 > ... > 0$ and $Tr \to 0$.
Therefore, the algorithm is defined as follows:

Table 12

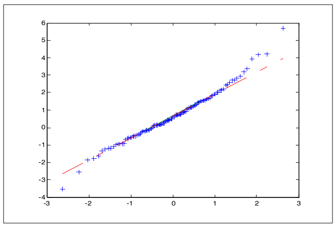
Algorithm 2: pseudo code of the Threshold Accepting Algorithm

Initialisation	Choose a threshold sequence $T_i, i = 0, \ldots, I_{max}$, set $i = 0$ and generate an intial x^c
Step 1	Choose $x^n \in N(x^c)$
Step 2	Calculate $\Delta f = f(x^n) - f(x^c)$
Step 3	If $\Delta f \le T_i$, set $x^c = x^n$
Step 4	If $i < I_{\max}$, set $i = i + 1$ and go to step 1 Otherwise, x^c is the output of the algorithm

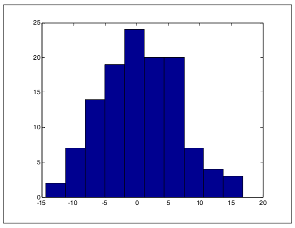
B. The pdf of the five indices



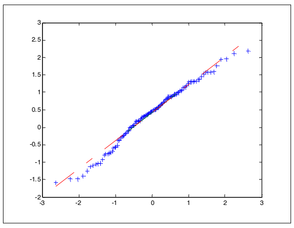
Hedge Funds



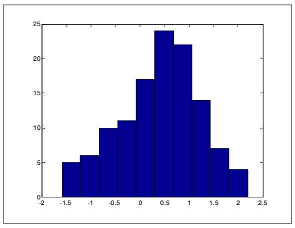
QQ-Plot Hedge funds



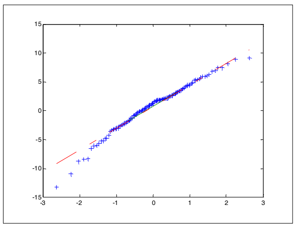
Bond



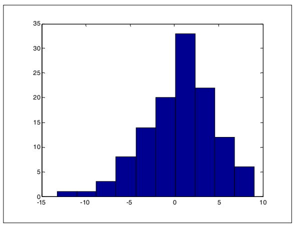
QQ-Plot Bonds



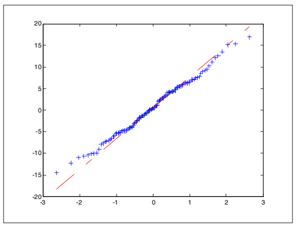
Equity



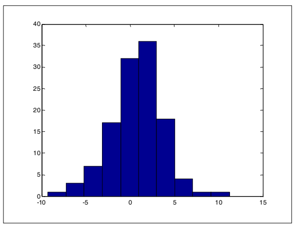
QQ-Plot Equities



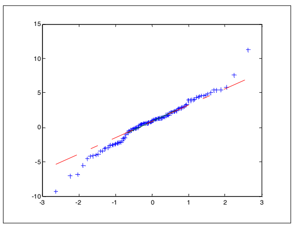
Commodity



QQ-Plot Commodities



Convertible



QQ-Plot Convertibles

Fig. 9**.** The pdf of the five indices (histograms).

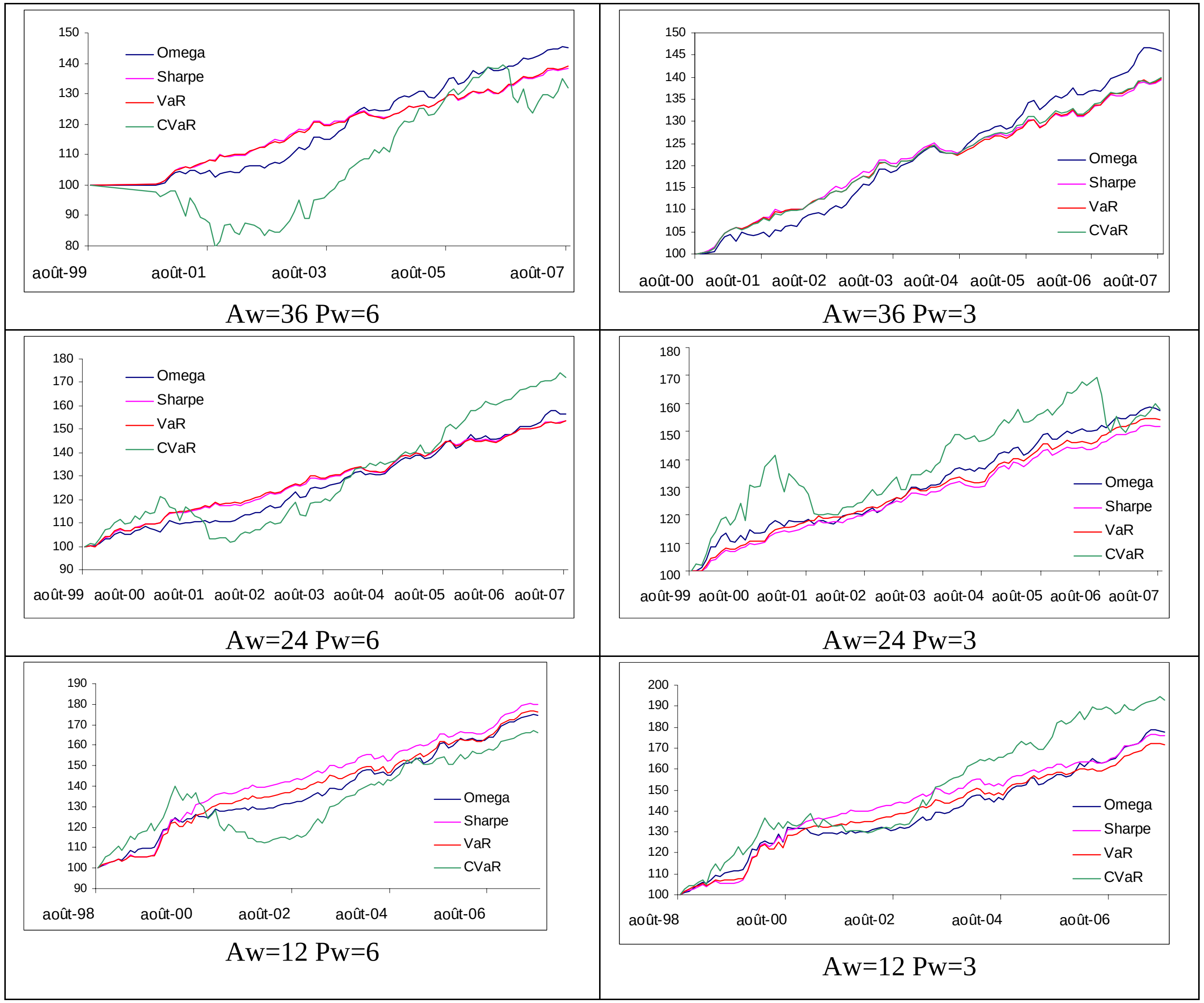
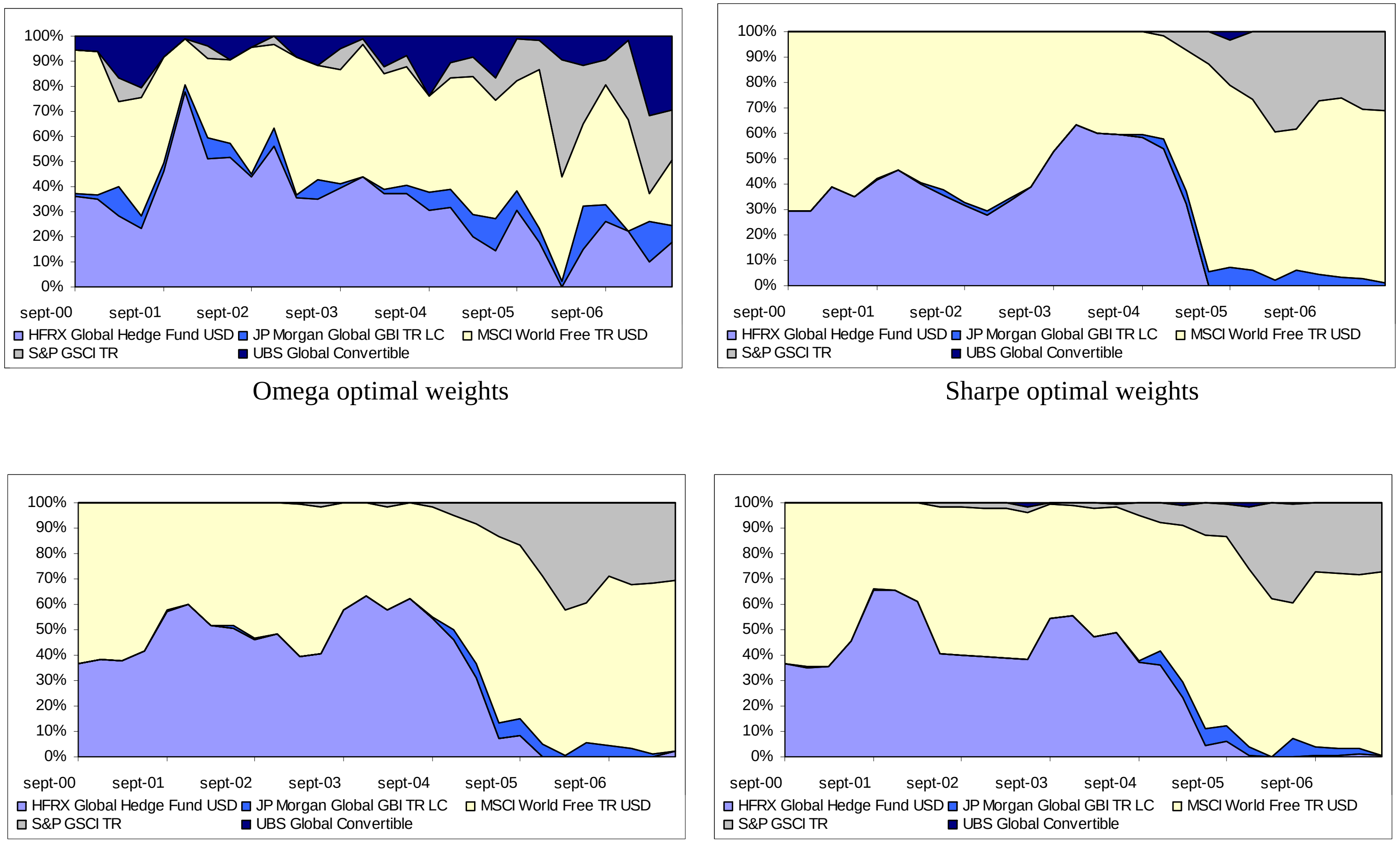


Fig. 10. Cumulative returns according to the analysis and projection windows



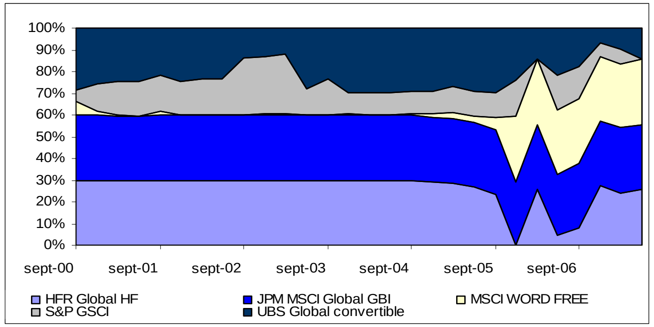
Omega optimal weights

Sharpe optimal weights

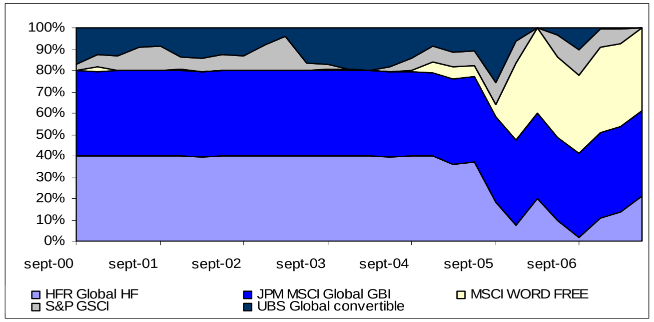
VaR optimal weights

CVaR optimal weights

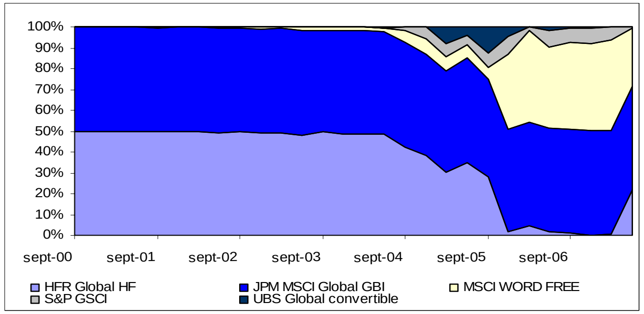
Fig. 11. Optimal allocation for the four criteria (A_w=36; P_w=3)



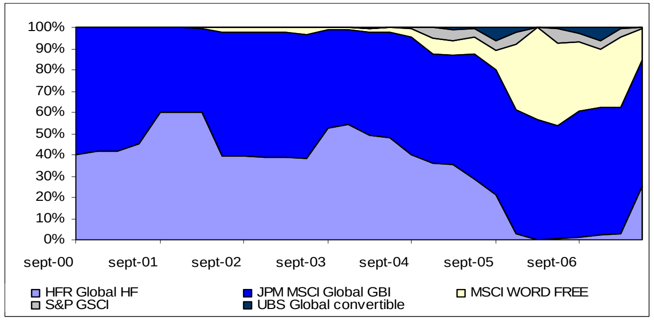
Omega optimal allocation (max 30%)



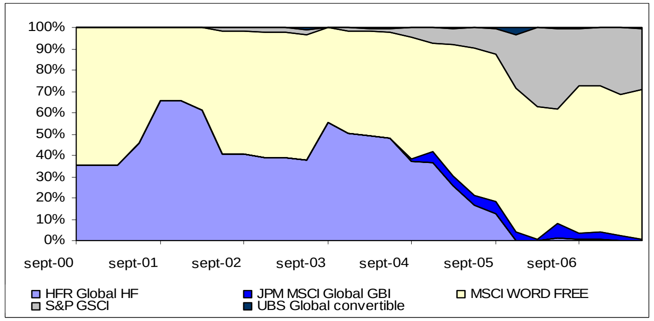
Omega optimal allocation (max 40%)



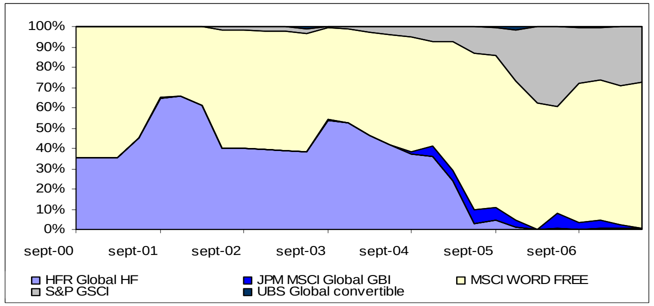
Omega optimal allocation (max 50%)



Omega optimal allocation (max 50%)



Omega optimal allocation (max 60%)



Omega optimal allocation (max 70%)

Fig. 12. Omega optimal allocation according to the upper bound