Pricing Securities with Multiple Risks: An Empirical Study

Shantaram P. Hegde
University of Connecticut

Tribhuvan Puri
University of Massachusetts Dartmouth

Ravi S. Mateti
University of Massachusetts Dartmouth

Abstract

In this paper we test the empirical performance of the Das and Sundaram model [2006]. This model is for the pricing of securities subject to equity, interest rate and credit risks. The model is embedded on a bivariate recombining lattice in a risk-neutral setting for the joint evolution of equity prices and forward interest rate curve. Assuming that the stock price drops to zero in the event of a default, the evolution of the credit risk process is captured on this bivariate lattice by the possibility of the stock price dropping to zero. The probability of this happening at different nodes of the lattice is estimated from the observed credit default swap spreads. We test this model on a sample of risky non-convertible bonds in the American and European markets. We also extend this model to price convertible bonds which are convertible not into the issuer's stock but the stock of some other company.

This paper is preliminary and incomplete. Please do not quote without the authors' permission.

* Contact author: rmateti@umassd.edu

Pricing Securities with Multiple Risks: An Empirical Study

I. Introduction

In this paper we test the empirical performance of a promising model for pricing securities subject to equity, interest rate and credit risks. This is the Das and Sundaram [2006] model. An example of a security faced with these risks is a defaultable convertible bond. We have chosen this model for an empirical study because of its simplicity and intuitive appeal. Also, it makes good use of the observed spreads in the rapidly growing credit default swap market to estimate default probabilities.

The model we propose to test, Das and Sundaram [2006], is embedded on a bivariate recombining lattice in a risk-neutral setting. One of the variables on this lattice is the risk-free interest rate. The dynamics of this variable is modeled using the Heath-Jarrow-Morton [1990] framework. The second variable on this lattice is the stock price. A generalized form of the Cox – Ross – Rubinstein [1979] model which allows for the firm's default is used to model the stock price. These two variables are combined together on a bivariate lattice in a risk-neutral setting. In other words, the drift term of the forward rates satisfies the well known HJM [1990] condition, and the drift term of the stock price is equal to the risk-free rate. The default probabilities are endogenous to the model and are derived from both equity and the debt markets, rather than from just the debt market alone. This results in a consistent framework as the default probabilities are consistent with the prices in the debt and the equity markets (which probably would not be the case if default is treated as an exogenous process and overlaid on the bivariate lattice) and all the available information, i.e. the prices from both the debt and equity markets, is utilized in the estimation of default probabilities.

In this paper we test the empirical performance of the above model. We price non-convertible bonds in the American and European markets and compare the model's prices with observed market prices. Next we extend the model and price convertible bonds which are convertible not into the issuer's stock but some other company's stock.

II. Literature Review

The literature on credit risk modeling can be divided into two groups – the structural approach and the reduced-form approach. Structural models (e.g., Merton [1974], Black-Cox[1976]) focus on the value of the firm. Equity and debt are treated as contingent claims on this value. The firm defaults when its value hits the "default boundary". The probabilities of default are based on a notion called "distance-to-default". Implementing this class of models poses some problems because the firm value is not directly observable and its estimation involves making some restrictive assumptions. The equity market is usually used to estimate the parameters of the model.

The reduced-form models (e.g., Jarrow and Turnbull [1995], Madan and Unal [1995], [2000], Duffie and Singleton [1999]) treat the time of bankruptcy as an exogenous process and does not explicitly depend on the value of the firm. An exogenous variable is assumed to drive default and the conditional probability of default is captured by what is known as the hazard or the intensity rate. Therefore, the parameters associated with the value of the firm need not be estimated to implement this class of models. The credit risk premium is a function of the probability of default and the recovery rate in the event of default. One set of reduced form models uses the credit ratings of firms to model credit risk. Default is modeled by a change in ratings driven by a Markov transition matrix. In the reduced – form models, the parameters are estimated almost exclusively from the debt market.

The Das and Sundaram [2006] model draws upon the ideas from both the above approaches to modeling credit risk. As in the reduced-form approach, the probability of default is represented by a hazard rate process. And, as in the structural approach, the equity value is assumed to fall to zero in the event of a default. However, there is an important difference. For the implementation of a typical reduced form model, the bond market is solely relied on for the estimation of the hazard rate. Whereas in the present model, both the debt and the equity markets are used for its (the hazard rate) estimation. This is expected to result in an improved estimation of the hazard rate as not only more information is used but also the fact that equity markets are a lot more liquid than the debt markets. Jarrow [2001] also recommends estimating the default rate form both debt and equity markets.

The present model is an extension of various models. Some models such as Schonbucher [1998], [2002], Das and Sundaram [2000] model price of securities subject to only interest rate and default risk and not equity risk. Risky non-convertible bonds are examples of such securities. Others such as Amin and Bodurtha [1995] consider only equity and interest rate risks but no default risk. The present model considers all three risks.

III. The Das and Sundaram Model

We first describe the Das and Sundaram model. The model incorporates interest rate, equity, and credit risks on a recombining bivariate lattice. There is a variable each for interest rate and stock price, but not for credit risk. The credit risk is modeled by the stock price dropping to zero in the event of a default. Below, we describe the Das and Sundaram model.

The Term - Structure Model

The interest rate dynamics, in terms of forward rates, are modeled using the discrete-time version of the HJM model (HJM [1990]). The HJM model gives the dynamics of the forward rates. Below is the discrete time version of the evolution of the forward rates.

$$f(t + h, T) = f(t, T) + \alpha (t, T).h + \sigma (t, T) X_f \sqrt{h}$$

where

$\alpha (t, T)$ is the rate of drift of the forward rate

$\sigma (t, T)$ is the volatility

X_f is a random variable which takes values $\{-1, 1\}$

h is a fixed length of time

Under conditions of no-arbitrage the drift rate cannot be independent of the volatility. As shown in Das and Sundaram [2001], Acharya, Das, Sundaram [2002], the drift terms can be calculated recursively in terms of the volatilities,

$$\sum_{k=t/h+1}^{T/h-1} \alpha\,(t, kh) = \frac{1}{h.h}\ \ln\left(E^t\left[\exp\left\{- \sum_{k=t/h+1}^{T/h-1} \sigma\,(t, kh)\, X_f\, h^{3/2}\right\}\right]\right)$$

Using the above, the risk-neutral drifts can be calculated recursively.

The Equity Model

The discrete time stock price dynamics, under the risk-neutral measure, are assumed to be

$$\ln [\ S(t + h) / S(t)\ ] = r(t).h + \sigma_S\, X_s \sqrt{h}$$

where

r(t) short term risk-free interest rate at time t

X_s is a random variable which takes value $\{- \infty, -1, +1\}$

h is the fixed interval of time

Note that the drift rate has been set equal to the risk-free rate because we are modeling under the risk-neutral measure.

However, the above form of stock price dynamics is not well suited for representing on a lattice. This is because of the stochastic drift rate, r(t). The lattice will not be recombining.

To overcome the above problem, the drift rate is set equal to zero but absorbed in the mean of the diffusion term $\sigma_S X_s \sqrt{h}$. In other words, the probabilities of the 3 different branches of X_S are so changed that the mean of the diffusion term changes from 0 to r(t).h. The stock price dynamics are now given by

$$\ln [S(t + h) / S(t)] = \sigma_S X_s \sqrt{h}$$

where

the mean of X_s is $r(t). \sqrt{h} / \sigma_S$

The probability of the three branches will have to be changed at each node, depending on what r(t) is. This will make the lattice recombining. (Amin and Bodurtha [1995])

The Joint Process

Now the dynamics of the term structure and the stock price are modeled on a bivariate recombining lattice. In order to accomplish this, we need a probability measure over the random shocks $X_f(t)$ and $X_S(t)$ such that the observed correlation between the stock returns and changes in the spot rate is preserved, and the stock price has the required drift, which is the risk free rate. This with the HJM drift condition on the drifts of the forward rates ensures that the stock and bond prices are martingales under the risk-neutral measure.

At each node of the tree, there are 6 branches. The table below shows the different combination of values of $X_f(t)$ and $X_S(t)$, each of these is represented by a branch, and the probabilities of each branch. $\lambda(t)$ is the probability of default at each node. When there is a default, the stock price drops to zero.

Table I

Branch	Value of X_f	Value of X_S	Probability
B1	1	1	$1/4.((1 + m_1)(1 – \lambda(t))$
B2	1	-1	$1/4.((1 - m_1)(1 – \lambda(t))$
B3	-1	1	$1/4.((1 + m_2)(1 – \lambda(t))$

B4	-1	-1	$1/4.((1 - m_2)(1 - \lambda(t))$
B5	1	$-\infty$	$\lambda / 2$
B6	-1	$-\infty$	$\lambda / 2$

The unknowns in the above table are $\lambda(t)$, m_1 and m_2. In the next section, we will see how the default probability $\lambda(t)$ is estimated. Given the default probability, the conditions on the drift rate of the stock price, and the correlations between the stock price and the risk-free rate, we should be able to solve for m_1 and m_2. After deriving the probabilities of various branches, set the drift rates of the forward rates according to the HJM condition given above. This would complete the construction of the lattice. Using this lattice, we can price any security with interest rate, equity, and credit risks. Also, by "turning off" risks which are not relevant, we can also price securities subject to just interest rate and equity risks, or interest rate and credit risks.

Credit Risk

We need to estimate the *conditional* probability of default, $\lambda(t)$, at each node of the tree and this completes the model. Rather than adding an extra dimension to the lattice for the default process, $\lambda(t)$, a one- period default probability function is defined at each node of the bivariate lattice. One of the contributions of the original paper is to show how to estimate this function.

If the default intensity is $\xi(t)$, then the conditional probability of default, $\lambda(t)$, during the time interval h is given by

$$\lambda(t) = 1 - \exp(-\xi(t).h)$$

The task now is to estimate a default intensity function from the stock prices and the interest rates. The default intensity function is assumed to have the following functional form.

$\xi(t) = \exp[\ a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t - t_0)\ ]$

The parameters of the above function can be estimated using credit default swap spreads. Given the liquidity of the credit default swap market, this would be a valuable source of information.

To illustrate calibration to credit default swap (CDS) spreads, let us see how we can determine the CDS spreads assuming we have all information on the lattice, including conditional default probabilities. Inverting the problem will help us estimate the default intensity function (and the default probabilities) in terms of the observed CDS spreads.

For ease of exposition, it is assumed that the recovery rate, denoted by φ, is constant. This assumption can easily be relaxed. In the event of default, the security holders are assumed to recover a fraction φ (recovery rate) of the market value of the security just before the default. This is the fractional recovery of market value (RMV) condition of Duffie and Singleton[1999].

The idea is simple. The CDS spreads have to be so set that the expected present value of payments for default insurance (expectations to be taken under the risk-neutral measure) have to equal to the expected loss in value due to default. Let's consider the CDS for a defaultable zero coupon bond (ZCB). Using the RMV condition, the price of the zero coupon bond at time t is given by

$$ZCB(t) = \exp(-r(t).h)\ \{ \sum_{k=1}^{4} q_k(t).\ ZCB_k(t+h) \}\ [1 - \lambda(t)(1 - \varphi(t)\ ], \quad ZCB(T) = 1$$

where

$q_k(t) = p_k / [1 - \lambda(t)]$, k = 1, 2, 3, 4 are the four probabilities of the non-default branches, conditional on no default

Next, conditional on no default having occurred till t, we find the present value at time t of expected compensation for a possible default in the future. Let this be denoted by CDS(t). Then,

$$CDS(t) = \exp(-r(t).h)\ \{\sum_{k=1}^{4} q_k(t).\ CDS_k(t+h)\}\ [1 - \lambda(t)]\} + \lambda(t)\ ZCB(t)\ (1 - \varphi),$$

$$CDS(T) = 0$$

The term in the flower brackets gives the expected losses on the default swap, conditional on no default till time t. The following term gives present value of expected loss due to default at the end of the period.

Finally, we find at time t the present value of expected *future* payments of $1 each period till the time of default, conditional on no default having occurred till time t. Let this be denoted by G(t). Then,

$$G(t) = [\exp(-r(t).h)\ \{\sum_{k=1}^{4} q_k(t).\ G_k(t+h) + 1\}]\ [1 - \lambda(t)], \quad G(T) = 0$$

In order to get the annualized basis points spread (s) for the premium payments, the present value of expected premium payments is equated to the present value of expected loss due to default,

$$s.h.G(0) = CDS(0)$$

⇨ $s = [CDS(0) / h.G(0)] \times 10{,}000$ basis points

Given how the spreads are calculated assuming knowledge of the default intensity function, we can invert the problem and estimate the unknown parameters of the default intensity function in terms of the observed CDS spreads.

With this, the lattice is complete. We have all the information to price any security subject to any combination of the three risks using the usual method of backward induction.

III. Estimating the Inputs to Implement the Das and Sundaram Model

We will to test the empirical performance of this model using non-convertible and convertible price data. The convertible bonds we price have a peculiar feature. These bonds are convertible not into the issuer's stock but into the stock of some other company's stock. For these bonds, the Das and Sundaram model has to be extended. We need to build a trivariate lattice. We do this and price many such convertible bonds.

Data

The data required to implement the DS model are forward rates, volatilities of forward rates, stock price, volatility of stock price, correlation between the changes in term structure and return on the stock, credit default swap spreads, and the prices of securities subject to multiple risks, like convertible bonds. The observable data are the stock price, credit default swap spreads, and prices of securities. All other inputs for the model have to be derived from observable variables. All the observable data is from Bloomberg.

In this section we will describe how to estimate certain critical non-observable variables to implement the DS model. In particular, we describe how to estimate forward rates, two methods of estimating forward rate volatilities, and how to estimate the parameters of the default intensity function.

Forward Rates

Though forward rates are not directly observable, Bloomberg reports these rates based on the swap rates. (Forward rates can also be estimated from the prices of treasuries). Though these rates can be got directly from Bloomberg, let us see how one can estimate the forward rates from swap rates. To estimate the forward rates, we will first need zero coupon bond prices for various maturities, from which the forward rates can be inferred. We will estimate the zero coupon bond curve from the swap rates.

Consider an interest rate swap with a notional principal of $1. An interest rate swap can be treated as an exchange of fixed and floating rate bonds. The swap rate is the fixed rate in the swap which would make the present value of fixed and floating segments of the swap equal. Since the value of the floating rate bond is always equal to the face value ($1) on coupon payment days, finding the swap rate is equivalent to finding the coupon rate of a par coupon bond. If C is the coupon rate of par bond (and also the swap rate for the corresponding maturity), τ is the time interval between coupon payments, and $P(0, t_i)$ is the price of a zero coupon bond maturing at time t_i, then

$$1 = \tau C \Sigma P(0,t_i) + 1.\ P(0, t_n)$$

=>

$$C = [\ 1 - P(0, t_n)\ ] / \tau \Sigma P(0,t_i)$$

Given the swap rates for different maturities from a swap curve, we will use the expression for the coupon rate of par coupon bonds to bootstrap the zero coupon bond prices. Starting with a swap with only one segment, we can derive $P(0, t_1)$ from the swap rate C_1 as follows:

$$1 = \tau C_1 P(0, t_1) + 1.P(0, t_1)$$

=>

$$P(0, t_1) = 1 / [1 + \tau C_1]$$

Having got $P(0, t_1)$, we can get $P(0, t_2)$ as follows:

$1 = \tau C_2 P(0, t_1) + \tau C_2 P(0, t_2) + 1.\ P(0, t_2)$

=>

$P(0, t_2) = [1 - \tau C_2 P(0, t_1)] / [1 + \tau C_2]$

This way we can work recursively and get the prices of zero coupon bond prices, $P(0, t_i)$. Once we have the zero coupon bond prices it is very simple to calculate the forward rates. The discretely compounded forward rate for the period (t_k, t_{k+1}) is

$$f(t_k, t_{k+1}) = [P(0, t_k) / P(0, t_{k+1}) – 1] / \tau$$

and the continuously compounded forward rate is

$$F(t_k, t_{k+1}) = Log[[P(0, t_k) / P(0, t_{k+1})] / \tau$$

Estimating Forward Rate Volatilities to Implement the DS Model

One of the critical inputs for implementing the Das and Sundaram model is the volatilities of forward rates. For a forward rate maturing at time t_i, $f(, t_i)$, we want its volatility to remain unchanged from the current time, t_0, till it matures at time t_i . That is,

Volatility of $f(t, t_i) = k$ (a constant) for all t, $t_0 \leq t \leq t_i$

For example, consider the forward rate maturing after 5 years. We assume

Volatility of $f(t, 5) = k$ (a constant) for $t_0 \leq t \leq 5$

In other words, we want the volatility of the forward rates to depend on the *time **of** maturity* rather than the *time **to** maturity* of the forward rate. In the DS model, as we move forward in time, the volatility of any given forward rate does not change. In other words, forward rates maturing on different dates would have different volatilities, and these volatilities remain fixed as we move forward in time, even though the time to maturity of these forward rates would have reduced. We have this feature to ensure that the lattice recombines. If the volatility of forward rates were made a function of time to maturity rather than time of maturity, the volatilities of the forward rates would change as we move forward in time. Consider again the case of a five year forward rate, f(, 5). At the current time, t_0, this forward rate would have a certain volatility, say vol(t_0, 5). After one year, the time to maturity of this forward rate be 4 years and would have a different volatility, vol(t_1, 5). Though modeling volatilities to depend on time to maturity, rather than time of maturity, seems more intuitive, we cannot have a recombining lattice with such volatilities. In the DS model, we can interpret the time *of* maturity volatilities as an "average" of the time *to* maturity volatilities.

$$f(,t_i) = (1/(t_i – t_0)). \int_{t0 \text{ to } ti} vol(t_i – t)\, dt$$

We can estimate the "time of maturity" forward rate volatilities in several ways. The two broad approaches are using historical data to estimate historical volatilities, and using the current market information to estimate the implied volatilities. Historical volatilities can be estimated from the historical data on Treasury Constant Maturity Rates provided by the Federal Reserve Bank of St. Louis. Data on Treasury Constant Maturity Volatilities is freely available on its website: http://research.stlouisfed.org/fred2/categories/22).

Implied volatilities for implementing the DS Model can be inferred from the prices of interest rate caps. Interest rate caps are over-the-counter contracts and their prices can be obtained from sources like Bloomberg. Prices of interest rate caps are quoted in terms of Black's volatilities, which are derived from a lognormal model. We need to convert these "lognormal volatilities" to "normal volatilities". Details of this conversion are given below.

Estimating Forward Rate Volatilities from Treasury Constant Maturity Rates

The Federal Reserve Bank St. Louis calculates "Treasury Constant Maturity Rates" for 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year, 20-year, and 30-year maturities every business day. The purpose of the constant maturity series is to estimate what the yield on, say, a (hypothetical) 2-year note would be on a given day even if there is no treasury issue with two years to maturity trading on that day. The estimate is made by interpolating yields from "on-the-run" treasury issues. The most recent treasury issue for a particular time to maturity is the on-the run-issue for that maturity. It rolls to "off-the-run" after the next issue for that maturity, which becomes on-the-run. On the day of issue of, say, a 2-year note, it is easy to determine the treasury rate for a 2-year maturity from the price of the 2-year note. However, a day later, if we want the treasury rate for a 2-year maturity, we do not have any treasuries which have 2 years remaining to maturity to determine the rate from. The 2-year note issued a day ago now has one day less to maturity. Cubic splines are used to smooth between on-the-run issues to determine treasury constant maturity rates for the key maturities of 3 months, 6 months, 1, 2, 3, 5, 7, 10, 20, and 30 years. Since on-the-run issues trade close to par, the treasury constant maturity rates, which are determined from these issues, can be treated as *par coupon bond yields*.

In the DS model, if we take the time interval between two successive nodes of the lattice to be a quarter of a year, we need volatilities of forward rates spaced out at quarterly intervals – 3 months, 6 months, 9 months, 1 year, 1 year 3 months, and so on. Consider the treasury constant maturity rates for maturities till, say, 3 years on January 3, 2006.

Table II

Maturity	3 months	6 months	1 year	2 years	3 years
TCMR	4.16%	4.40%	4.38%	4.34%	4.30%

*TMCR: Treasury Constant Maturity Rate

Since we need volatilities at quarterly intervals, we first need to get an estimate of the constant maturity rates for maturities not estimated by the Fed. We can do so either using sophisticated methods like cubic splines or simple linear interpolation. For this illustration let us use simple linear interpolation. Using this method, we have the following estimates.

Table III

Maturity	*3 months*	*6 months*	*9 months*	*1 year*	*1 yr 3months*
TCMR	4.16%	4.40%	4.39%	4.38%	4.37%
Maturity	*1yr 6months*	*1yr 9months*	*2 years*	*2yrs 3month*	*2yrs 6month*
TCMR	4.36%	4.35%	4.34%	4.33%	4.32%
Maturity	*2yrs 9month*	*3 years*			
TCMR	4.31%	4.30%			

*TMCR: Treasury Constant Maturity Rate

Next, we try to infer the zero-coupon bond prices from the constant maturity rates. This is a critical step and we bootstrap the zero coupon curve form constant maturity rates. We again note that the treasury constant maturity rates can be interpreted as par coupon bond yields because these rates are estimated from on-the-run treasury issues, which trade close to par value. The price of the zero coupon maturing after 3 months, P(0, 0.25), can be easily estimated from the 3-month constant maturity rate.

P(0, 0.25) = 1/(1 + 0.25 x 0.0416) = $0.989707

Similarly, the price of zero-coupon bond maturing after 6 months is

P(0, 0.50) = 1/(1 + 0.50 x 0.0440) = $0.978474

The price of zero-coupon bond maturing after 9 months is

P(0, 0.75) = 1/(1 + 0.75 x 0.0439) = $0.968125

The calculation of the above zero-coupon prices was straight forward. This is because these bonds have a maturity of less than a year and therefore do not pay any interest. For the calculation of the 1-year zero-coupon bond price, we have to take coupon payments into account. The 1-year bond is expected to pay interest after 6 months, and when it matures after one year. Assuming the face value of the bond is \$1, when interest is due, an amount equal to half of the treasury constant maturity rate (this is assumed to be the coupon rate) is paid. For the 1-year bond, this amount is \$0.0438/2 = \$0.0219. The 1-year bond, therefore, makes a payment of \$0.0219 after six months, and \$0.0219 plus \$1 (principal repayment) after one year. The present value of these payment has to be \$1, because par coupon bonds trade at par.

$$1 = P(0, 0.50) \times 0.02019 + P(0, 1) \times (0.0219 + 1)$$
$$= 0.978474 \times 0.02019 + P(0, 1) \times 1.0219$$

Solving for P(0, 1), we get the price of zero-coupon bond maturing after one year to be P(0, 1) = \$0.9576

Using the above bootstrapping procedure, we can derive the zero-coupon bond prices for other maturities.

After deriving zero-coupon bond prices, we need to estimate the forward rates implicit in these zero-coupon bond prices. The annualized, discretely compounded implicit 3-month forward rate maturing in three months would be

$$f(0, 0.25) = [P(0, 0.25) / P(0, 0.50) - 1] \times 4$$
$$= [0.989707 / 0.978474 - 1] \times 4$$
$$= 0.044928$$

The continuously compounded forward rate for the above maturity is given by

$$F(0, 0.25) = \log[P(0, 0.25) / P(0, 0.50)] \times 4$$
$$= \log[0.989707 / 0.978474] \times 4$$
$$= 0.045659$$

Forward rates for other maturities can be found using a similar procedure.

We derive the forward rates for various maturities every day in the sample period. Then create a time series of daily changes in forward rates for each maturity and calculate the standard deviations of these time series. Multiplying standard deviations by $\sqrt{260}$ (assuming 260 trading days in a year) would give us the annualized volatility of the forward rates.

Finally, after estimating the volatilities of forward rates for various maturities, we need to make one final adjustment. Recall that the volatilities in the DS model depend on the time of maturity, rather than the time to maturity, of the forward rates. This is done to ensure the lattice recombines. The volatilities used on the lattice can be interpreted as the "average" volatilities for different maturities. For example, the volatility of the one year forward rate used on the lattice would be the average of the volatilities of forward rates with maturities up to one year. Since we are estimating volatilities at quarterly intervals, the one year forward rate volatility to be used on the lattice would be the average of three month, six month, nine month, and one year volatilities.

Estimating "Normal" Volatilities from Prices of Interest Rate Caplets

An alternative to using historical volatilities is to use implied volatilities. The implied volatilities can be estimated from prices of interest rate caplets.

A caplet is a call option on an interest rate. Assume that the face value of the contract is \$1, the underlying interest rate is the LIBOR, the cap rate (strike rate) is R , the tenor of the caplet (length of time for which interest is to be calculated) is τ, and maturity of the caplet is T_i. If the LIBOR at time T_i for the period (T_i , T_{i+1}) is L(T_i , T_{i+1}), then the payoff on the caplet would be at time T_{i+1}, and not at time T_i , and it is

$C = \tau.\max[L(T_i , T_{i+1}) – R, 0]$

where $T_{i+1} - T_i = \tau$

A series of caplets comprises a cap. The market practice to value caps and caplets is to the use Black's model. Black's model is a slight modification of the Black-Scholes model with the drift of the underlying variable set equal to zero. It is implicit in the Black's formula that the forward rates (the underlying variable for a cap or a caplet) are lognormal. The market prices of caps and caplets are not quoted in dollars but in terms of implied Black volatilities (just like the implied volatilities are derived from stock option prices). Since the Black's formula assumes that forward rates have a lognormal distribution, we will refer to the implied volatilities of the forward rates from the Black's formula as lognormal volatilities.

Black's formula for caplets is as follows.

Price of caplet maturing at time T_i is

$$C(t) = \tau.P(t, T_{i+1})\{L(t, T_i, T_{i+1})N(d_1) – RN(d_2)\}$$

where
$P(t, T_{i+1})$ = time t price of zero-coupon bond maturing at time T_{i+1}
$L(t, T_i, T_{i+1})$= time t forward LIBOR for the period (T_i , T_{i+1})
R = cap rate

$d_1 = 1/\sigma.\sqrt{(T_i\text{-}t)} . [\ln(L(t, T_i, T_{i+1})/R) + 0.5\sigma^2 \sigma.\sqrt{(T_i\text{-}t)}]$
$d2 = d_1 - \sigma.\sqrt{(T_i\text{-}t)}$

Interestingly, a caplet is equivalent to a certain number of put options on a zero-coupon bond. The payoff from the caplet at time T_{i+1} is

$\tau . \max[L(T_i , T_{i+1}) – R, 0]$

At time T_i , the discounted value of the above payoff is

$\tau/(1 + \tau .R) \max(L(T_i , T_{i+1}) – R, 0)$

The time T_i payoff from the caplet above can also be expressed as

$(1 + \tau .R) \max (1/(1 + \tau .R) – 1/(1 + \tau. L(T_i , T_{i+1}), 0)$

The above expression can be interpreted as the payoff from $(1 + \tau .R)$ put options on zero coupon bonds with a strike price of $1/(1 + \tau .R)$ and maturing at time T_i.

$1/(1 + \tau. L(T_i , T_{i+1})$ can be interpreted as the price of a zero coupon bond (face value of $1) at time T_i.

In the Ho-Lee model there is a closed-form solution for pricing call and put options on zero coupon bonds. The price of a put option on a zero coupon bond is given by

$Put(t, T_i, T_{i+1}) = K.P(t, T_i) N(-d_2) – P(t, T_{i+1})N(-d_1)$

where

$Put(t, T_i, T_{i+1})$ = price at time t of a put which matures at time T_i on a zero coupon bond maturing at time T_{i+1}

$P(t, T_i)$ = price at time t of a zero coupon bond maturing at time T_i

K = strike price

$d_1 = \{ \ln[P(t, T_{i+1}) / P(t, T_i)] / \sigma_p \} + \sigma_p/2$

$d_2 = d_1 - \sigma_p$

$\sigma_p = \sigma.(T_{i+1} - T_i)\sqrt{(T_i - t)}$

Finally, we note that HJM model with constant volatility results in the Ho-Lee model. Since in the DS model the dynamics of the forward rates are modeled using the HJM model, and the volatility of each forward rate is constant until it matures (different forward rates have different constant volatilities), it is like fitting a different Ho-Lee model for each forward rate. Thus, we can assume that the forward rate maturing after, say, 1 year is modeled by a Ho-Lee model with a particular volatility, and similarly the forward rate maturing after 2 years is modeled again by a Ho-Lee model but with a different volatility. Given this, we summarize the steps to arrive at the volatilities for different forward rates to be used in the DS model. We will call these volatilities the "normal volatilities" because the forward rates are normally distributed in the Ho-Lee model.

a) Say we want the volatility of the forward rate maturing after one year. Look at the market price of at-the-money(ATM) cap maturing after 1 year. This is given in terms of the Black volatility, which has to be converted to the dollar price by inputting the implied Black volatility in the Black's formula. We have chosen ATM caps because the trading is maximum in these caps, and therefore best reveals the market expectations of the forward rate compared to caps with other strikes.

b) The above price of the 1-year cap can be interpreted as the price of $(1 + \tau .R_{ATM})$ put options with a strike price of $1/ (1 + \tau .R_{ATM})$ on zero coupon bonds. Now we use the formula for pricing put options on a zero coupon bond in the Ho-Lee model. We search for that value of volatility (σ) which best explains the price of the above $(1 + \tau .R_{ATM})$ put options on the zero coupon bond. The optimum σ from the above exercise is chosen as the volatility of the forward rate maturing after one year. We repeat the above procedure for all other maturities. The volatilities we get are the normal volatilities

We used the above procedure to convert lognormal volatilities to normal volatilities using the cap prices reported in Bloomberg on January 3, 2006. The results are as follows.

Table IV

Time to Maturity	1 year	2 years	3 years	4 years
Lognormal (Black) Vol.	*0.1189*	*0.1664*	*0.1887*	*0.1997*
Normal Volatility	*0.00566*	*0.00777*	*0.00889*	*0.00945*

Time to Maturity	5 years	6 years	7 years	8 years
Lognormal (Black) Vol.	*0.2037*	*0.2073*	*0.2079*	*0.2076*
Normal Volatility	*0.00962*	*0.00978*	*0.00987*	*0.00991*

Estimating the Parameters of the Default Intensity Function

One of the crucial steps in implementing the model is estimating the parameters of the default intensity function. We will be assume that the default intensity function has the following form,

$$\xi(t) = \exp[\, a_0 + a_1 r(t) - a_2 \ln S(t) + a_3 (t - t_0)\,]$$

In the above function, default intensity depends on three variables – interest rate, stock price, and time. The intensity decreases when the risk-free interest rate increases; it also decreases when the stock price increases. The effect of time on the intensity depends on what the current level of the intensity is. If the intensity is very high currently but the firm manages to survive, then the intensity would be expected to decrease in the future. In this case the intensity decreases with time. The converse will happen if the current level of intensity is low. Given the intensity of default, the probability of default is given by

$$\lambda(t) = 1 - \exp(-\xi(t).h)$$

The task is to estimate the parameters a_0, a_1, a_2, and a_3. We make use of the credit default swap spreads observed in the market to estimate these parameters. Earlier, in the section on credit risk, it was discussed how the credit default spreads could be determined from the model once we know the parameters a_0, a_1, a_2, and a_3. Now the problem is reversed – given the observed spreads we need to determined the values of the parameters which best explain the observed spreads. We have four parameters to estimate, and if we have credit default swap spreads for four different maturities, we can get a good estimate. We undertake a large minimization exercise for this. The objective function is the sum of squared deviations between the observed spreads and the spreads given by the model. Basically, this is an unconstrained minimization problem.

The problem is rather complex and we need a sophisticated algorithm to handle this. We have found that MATLAB's Optimization Toolbox is useful for solving this problem. Of the many functions available in this toolbox, we have found that the "fminsearch" function best handles this problem. The fminsearch function uses the simplex search method. This is a direct search method that does not use numerical or analytical gradients as in other functions in the toolbox, like the function "fminunc". In our problem we have four unknowns, a_0, a_1, a_2, and a_3. The stated algorithm creates a simplex in a 4-dimensional space which is characterized by 5 distinct vectors as its vertices. At each iteration, a new point in or near the current simplex is generated. The value of the function at this new point is compared with the function's values at the old vertices of the simplex and one of them is replaced by the new point, which results in a new simplex. This procedure is repeated until an acceptable solution is found or the number of iterations reaches a predetermined limit.

IV. Empirical Results on Pricing Risky Non-Convertible Bonds

First, we test the model on a sample of risky non-convertible bonds. The objective is twofold. We would like to test how well the model performs, and also investigate how well the information regarding the default risk conveyed by the credit default swap spreads is absorbed in risky bond prices. We recall that an important aspect of the model is using the CDS spreads to estimate the default intensity function. By applying the model to securities subject to only default and interest rate risks, such as risky non-convertible bonds, it would be easy to attribute the results to these risks and get a clearer picture of the linkages between the CDS market and the bond market. In this regard, we would also like to compare the linkages between the CDS and the bond markets in the U.S. and Europe. After this analysis, we will extend the Das and Sundaram model (build a trivariate lattice) to price convertible bonds with the conversion feature mentioned earlier.

To see how well the model prices bonds issued in a particular industry and investigate the linkages mentioned above, we applied the model to a sample of dollar denominated bonds issued by the electric utilities in the U.S. We priced the bonds as on January 3, 2006 if the market prices were available on Bloomberg for this date, and if not, the closest day when the market prices were available. The results are given in the table below.

Table V

Company	Date	Market Price	Model Price	Abs. % dev.
Texas Utilities	Jan-3-2006	1025.3	1027.5	0.2141
American Electric Power	Jan-3-2006	999.94	1019.6	1.9282
Columbus Power	Jan-3-2006	1027.2	1028.4	0.1166
Indiana Michigan Power	Jan-3-2006	1063	1035.5	2.6557
South Western Electric	Jan-3-2006	1032.37	1040.2	0.7527
Ohio Power Company	Jan-3-2006	986.8	985.18	0.1644
Public Service Oklahoma	Jan-3-2006	988.6	983.31	0.5379
Florida Power Corp.	Jan-3-2006	1023.6	1026.4	0.2727
Ohio Edison	Jan-3-2006	970.5	986.89	1.6607
Cleveland Elec. Illum.	Jan-3-2006	1046.4	1060.4	1.3202
TXU Corporation	Jan-3-2006	966	963.26	0.2844
First Energy Corp.	Jan-3-2006	1062.14	1047.1	1.4363
Dominion Resources	Jan-3-2006	982.24	989.16	0.6995
Baltimore Gas	Jan-3-2006	1042.4	1055.5	1.2411
PSEG Power	Jan-3-2006	1090.15	1076.9	1.2303
Duke Energy	Jan-3-2006	976.39	981.65	0.5358
PSI Energy	Jan-3-2006	1036.8	1048	1.0687

Cinergy Global Resources	Jan-3-2006	1040.6	1043	0.2301
Cincinnati Gas & Elec.	Jan-3-2006	1027.83	999.95	2.7881
Florida Power & Light	Jan-4-2006	1029.63	1035.3	0.5476
Exelon Corporation	Jan-4-2006	970.22	962.59	0.7926
Pennsylvania Electricity Co.	Jan-4-2006	1034	1041.9	0.7582
Progress Energy	Jan-4-2006	1076.81	1062.6	1.3372
Virginia Elec.& Power	Jan-5-2006	976.99	951.76	2.6508
Peco Energy	Jan-6-2006	1047.58	1028	1.9046
Constellation Energy	Jan-6-2006	1094.98	1079.6	1.4246
DTE Energy	Jan-6-2006	1044.06	1058.8	1.3921
Detroit Edison	Jan-6-2006	1042.95	1039.9	0.2932
Oncor Electric	Jan-9-2006	999.81	1007.1	0.7238
Appalachian Power	Jan-25-2006	1035.8	1058.4	2.1352
Mean Abs. % Deviation				**1.1033**

We see that there is not too much difference between the market prices and the model prices. We could infer from this that the model performs well in this sample and that the bond prices in this industry seem to absorb well the default risk information contained in the CDS spreads.

Next, we do not confine the sample to any one industry but draw the sample from various industries. The results are as follows.

Table VI

Company	Date	Market Price	Model Price	Abs. % dev.
Baxter International	Jan-3-2006	1005.59	1012.5	0.6871
Wyeth	Jan-3-2006	1011.69	1006	0.5624
Conagra Inc.	Jan-3-2006	1098.83	1101.3	0.2247
Raytheon Co.	Jan-3-2006	996.02	973.86	2.2248
Bristol Myers Squibb	Jan-3-2006	1034.18	1039.8	0.5434
Honeywell International	Jan-3-2006	1097.95	1113.7	1.4344
CVS Corp.	Jan-3-2006	965.49	968.67	0.3293
Marriott International	Jan-3-2006	1037.35	1039.6	0.2168
Southwest Airlines	Jan-3-2006	1068.48	1053.2	1.4300
Lowes Companies	Jan-3-2006	1135	1130.1	0.4317
Wal-Mart Stores	Jan-3-2006	972.49	968.52	0.4082
Newell Rubbermaid	Jan-3-2006	944.79	950.15	0.5673

Pulte Homes Inc.	Jan-3-2006	1090.3	1070	1.8618
Arrow Electronics	Jan-3-2006	1148.8	1165.1	1.4188
Tyson Foods	Jan-3-2006	1126.2	1124.4	0.1598
IPB Inc.	Jan-3-2006	1091.7	1084.8	0.6320
Lennar Corp.	Jan-3-2006	986.8	977.97	0.8948
Cendant Corp.	Jan-3-2006	1035.1	1011.8	2.2509
Caterpillar Inc.	Jan-3-2006	1076.1	1083.2	0.6597
Boeing Co.	Jan-3-2006	1010.45	999.35	1.0985
McDonnell Douglas Corp.	Jan-3-2006	1240.97	1266.8	2.0814
Federated Dept. Stores	Jan-3-2006	1055.12	1063.5	0.7942
Centex Corp.	Jan-3-2006	971.96	956.97	1.5422
Kraft Foods	Jan-3-2006	1041.46	1028.7	1.2252
Nabisco Inc.	Jan-3-2006	1166.2	1151.3	1.2776
Phillip Morris	Jan-3-2006	1059.4	1054.7	0.4436
Safeway Inc.	Jan-3-2006	979.28	956.86	2.2894
May Dept. Stores	Jan-6-2006	1137.9	1112.4	2.2409
Avnet Inc.	Jan-6-2006	1090	1096.9	0.6330
America Home Products	Jan-19-2006	1078.2	1085.5	0.6770
Mean Abs. % Deviation				**1.0414**

Again, we see that the market prices and the model prices are quite close. We could conclude that:

(i) the model is performing well empirically, and

(ii) for those U.S. companies for which CDS quotes are available, the information regarding default risk contained in these quotes seems to be well reflected in the bond prices.

Next, we investigate the same issues for the European market. We again tested the model on a sample on Euro-denominated risky, non-convertible bonds. The results are as follows.

Table VII

Company	Date	Market Price	Model Price	Abs. % dev.
Calyon	Jan-3-2006	962	960.75	0.1299
Vinci	Jan-3-2006	1079.25	1070.2	0.8385
Generali Finance	Jan-3-2006	1098.18	1082.9	1.3913
BAA PLC	Jan-3-2006	1039.9	1031.3	0.8270
BASF AG	Jan-3-2006	998.02	989.68	0.8356
Elec De Portugal	Jan-3-2006	1114.99	1102	1.1650
Mediobanca SPA	Jan-3-2006	997.825	993.15	0.4685
Aegon Inv.	Jan-3-2006	1040.47	1011.7	2.7650
Fortum Oyj	Jan-3-2006	1050.12	1040.1	0.9541
Powergen	Jan-3-2006	1053.09	1054	0.0864
Enel Investment	Jan-3-2006	1029.2	1036.8	0.7384
ENBW Intl. Finance	Jan-3-2006	1132.35	1134.8	0.2163
Banco Sabadell	Jan-3-2006	1023.36	1017.4	0.5823
Union Fenosa	Jan-3-2006	1065.51	1052.8	1.1928
Lyonnai Des Eaux	Jan-3-2006	1089.42	1096.9	0.6866
Intl. Endesa BV	Jan-3-2006	1106.38	1094.4	1.0828
Vattenfall Treasury	Jan-3-2006	1054.25	1052.9	0.1280
RWE AG	Jan-3-2006	1072.1	1074.3	0.2052
Iberdrola	Jan-3-2006	1178.76	1175.3	0.2935
Elec De France	Jan-3-2006	1173.63	1128.5	3.8453
Credit Agricole	Jan-3-2006	1025.6	1030.4	0.4680
Metso Corp.	Jan-3-2006	1038.3	1016.9	2.0610
Rolls-Royce	Jan-3-2006	1048.24	1050.6	0.2251
Lafarge SA	Jan-3-2006	1100.16	1056.2	3.9957
Thales SA	Jan-3-2006	1040.6	1034.9	0.5477
Bouygues	Jan-3-2006	1052.75	1049.4	0.3182
Novartis Secs. Investments	Jan-3-2006	1015	1013.1	0.1871
Intensa Bank	Jan-3-2006	973.3	996.99	2.4339
Mean Abs. % Deviation				**1.0242**

We see that the results for the European market are very close those observed for the U.S. market. From this, we could probably conclude the bond prices in the U.S and European markets reflect equally well the information regarding default risk as contained CDS spreads. Also, the model seems to be performing well.

V. Pricing Convertible Bonds

There are many convertible bonds trading in the market with a particular interesting feature. These bonds are convertible not into the stock of the issuer, but the stock of some other company. For example, consider a convertible bond issued by Morgan Stanley. It is a 1 ½ percent coupon bond, maturing on September 30, 2011 and convertible into 13.1313 shares of Walmart.

The pricing of these convertible bonds is both interesting and challenging. The Das and Sundaram model has be extended to be price these bonds. Rather than a bivariate lattice with six branches emanating from each node, we have a trivariate lattice with eighteen branches emanating from each node. However, all of these branches are not equally important and quite a few can be ignored when writing a program to implement the model. The Das and Sundarm model can be extended as follows.

We now have three state variables – the risk-free interest rate, the stock price of the issuer, and the price of the stock into which the bond is convertible. The interest rate and the stock price of the issuer determine the probability that the issuer will default over the next period on the lattice, which we will take to be a quarter. The interest rate and the price of the stock the bond is convertible into will determine the probability that this second stock will default. When this default does take place, the conversion value of the bond will become zero and the price of the bond will behave like the price of a simple risky, non-convertible bond.

The discrete time, risk- neutral dynamics of the forward rates is given by

$$f(t + h, T) = f(t, T) + \alpha (t, T).h + \sigma (t, T) X_f \sqrt{h}$$

where

$\alpha (t, T)$ is the rate of drift of the forward rate

$\sigma (t, T)$ is the volatility

X_f is a random variable which takes values $\{-1, 1\}$

h is a fixed length of time

Again, we note that that drift of the forward rates is not independent of the volatility in an arbitrage-free market.

The discrete- time, risk- neutral dynamics of the bond issuer's stock price (S_1) is given by

$$\ln [S_1(t + h) / S_1(t)] = r(t).h + \sigma_{S1} X_{S1}\sqrt{h}$$

r(t) short term risk-free interest rate at time t

X_{S1} is a random variable which takes value {- ∞, -1, +1}

h is the fixed interval of time

The drift rate is equal to the risk-free risk in the risk-neutral setup. Again, like in the original Das and Sundaram model, we need to modify the above dynamics of the stock price in order to model it on a recombining lattice. Since the drift is not constant, we have to eliminate it and has to be captured in probabilities of various branches of the lattice. We will model the dynamics of the bond issuer's stock price (S_1) as

$$\ln [S_1(t + h) / S_1(t)] = \sigma_{S1} X_{S1}\sqrt{h}$$

where the mean of X_{S1} is $r(t) \sqrt{h} / \sigma_{S1}$.

Similarly, the dynamics of the price of the stock into which the bond is convertible, let's call it stock S_2 , is modeled as

$$\ln [S_2(t + h) / S_2(t)] = \sigma_{S2} X_{S2}\sqrt{h}$$

where X_{S2} is a random variable which takes value {- ∞, -1, +1} and the mean of X_{S2} is $r(t) \sqrt{h} / \sigma_{S2}$.

Given the way the dynamics of the above three state variables are modeled, we would have a recombining lattice. Each node represents a particular combination of values of the three state variables, the spot interest rate (or the vector of forward rates) and the two stock prices. There are eighteen branches emanating from each node. Each branch would arise for a particular combination of values of X_f , X_{S1}, and X_{S2}. The table below shows this.

Table VIII

	B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18
X_f	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1
X_{S1}	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
X_{S2}	1	-1	$-\infty$	1	-1	$-\infty$	1	-1	$-\infty$	1	-1	$-\infty$	1	-1	$-\infty$	1	-1	$-\infty$

B1 is branch 1, B2 is branch 2, and so on. B13 to B18 are branches relating to the default of the issuer. The lattice does not extend beyond these branches because the bond ceases to exist once the issuer defaults. B3, B6, B9, B12, and B12 are branches relating to default of stock 2. Once this happens, the bond's conversion value is zero and the bond is effectively a non-convertible bond. However, the lattice does extend beyond these branches since the bond continues to exist.

We need a trivariate lattice, with each node representing the values of three state variables – short term risk-free interest rate and prices of two stocks. As mentioned earlier, there are eighteen branches emanating from each node of the lattice. Since the lattice is being constructed in a risk-neutral framework, the probabilities of the branches from each node should be so assigned that the drift rate of the two stocks is equal to the risk-free interest rate, and the observed correlations between interest rate and S_1, interest rate and S_2, and S_1 and S_2 are preserved. We can take the probabilities of the branches in the original Das and Sundaram model (Table I, page 7) as a starting point and modify them to suit the needs of our problem. With each of the branches B1, B2, …, and B6 (these are the branches given in Table I and not Table VIII) we will associate three possible movements of stock S_2 . The stock price either goes up, goes down, or defaults.

This would correspond to X_{S2} = 1, -1, or -∞ respectively. For example, let's consider the branch B1. B1 is for an upward movement of both the interest rate and the stock price, S_1 , i.e., $X_f = 1$ and $X_{S1} = 1$. Let the unconditional probability of this branch be denoted by P_1 , and the conditional probability of this branch given no default by p_1. Given the probabilities of branches B1 to B6, our task is to divide each of these branches into three parts (associated with three values of X_{S2}) in a risk neutral setting.

We will make the assumption that the probability of default of S_2 is equally likely given any branch. This may not be a very realistic assumption when the defaults of S_1 and S_2 are correlated. But given the difficulty in determining this correlation, and also the fact that this correlation may not be too high, this is a plausible assumption to make. If λ_2 is the probability of default of S_2, then

$$\text{probability}(B_i \cap X_{S2} = -\infty) = \lambda_2 / 6 \qquad i = 1, 2, \ldots, 6$$

The probabilities of each of the branches of Table I would have to be divided into three parts, each part denoting whether the associated movement of stock price S_2, would be up, down, or drop to zero (that is X_{S2} = +1, -1, or - ∞). Since the probability of $X_{S2} = -\infty$ has already been determined as $\lambda_2 / 6$, this means we will need a variable more for each branch (six variables in all) to get the required three-way division of probabilities in Tabel I. But, we have only three extra conditions for introducing the new stock price, S_2. That is, we have to ensure that

(a) the growth rate of S_2 is equal to the risk-free rate
(b) the correlation between S_2 and the interest rate is preserved
(c) the correlation between S_2 and S_1 is preserved

Given six new variables but only three additional restrictions, we have three degrees of freedom. Since we really are not very concerned with the branches representing the default of the issuer, a good solution to the problem would be to ensure that the above conditions are met, *conditional on no default of the issuer*. Now we have to consider only the four branches on which the firm does not default. This would reduce the number of variables from six to four, which still leaves us with one degree of freedom. We can impose any "innocuous" additional condition on the variables which does not have an undue influence on the results and helps arrive at a solution to the problem.

If p_1 is the conditional probability (given that the firm issuing the bond does not default) of branch B1, let

p_{11} = probability (B1 $\cap$ X_{S2} = 1 / no default of the firm issuing the bond)

p_{12} = probability (B1 $\cap$ X_{S2} = -1 / no default of the firm issuing the bond)

p_{13} = probability (B1 $\cap$ X_{S2} = $-\infty$ / no default of the firm issuing the bond)

We have similar definitions of probabilities for the other three non-default branches. It should be noted here that

$$\text{probability}(X_{S2} = -\infty \cap B_i) = \lambda_2/4 \quad i = 1, 2, 3, 4$$

This is because we are considering conditional probabilities and not unconditional probabilities. We are not taking into account the default branches of the firm issuing the bond.

We now derive the probabilities in the risk-neutral setup. We define

$c = \exp(\sigma_{S2}\sqrt{h})$
$d = \exp(-\sigma_{S2}\sqrt{h})$

The first condition is that the growth rate of the stock has to equal to the risk-free rate. Ignoring the default branches of S_2, we have

$E[S_2(t + h) / S_2(t)] = E[c]$

$= \{p_{11}. c + (p_1 – p_{11} - \lambda_2/4). d\} + \{p_{21}. c + (p_2 – p_{21} - \lambda_2/4).d\} + \{p_{31}. c + (p_3 – p_{31} - \lambda_2/4).d\} + \{p_{41}. c + (p_4 – p_{41} - \lambda_2/4).d\}$

$= \exp(r.h)$

(for definitions of p_1, p_2, p_3, p_4, p_{11}, p_{21}, p_{31}, and p_{41} see above)

=>

$(c – d) p_{11} + (c – d) p_{21} + (c – d) p_{31} + (c – d) p_{41}$

$= 1/(c – d) [\exp(r.h) – d.(p_1 + p_2 + p_3 + p_4 - \lambda_2)]$

$= 1/(c – d) [\exp(r.h) - d.(1 - \lambda_2)]$ … (i)

The second condition is that the correlation between S_2 and interest rate, $\rho_{r,S2}$, has to be preserved.

$Cov[X_f(t), X_{S2}(t)] = \{p_{11} x 1 x 1 + (p_1 – p_{11} - \lambda_2 /4) x 1 x - 1\} + \{p_{21} x 1 x 1 + (p_2 – p_{21} - \lambda_2 /4) x 1 x -1\} + \{p_{31} x -1 x 1 + (p_3 – p_{31} - \lambda_2 /4) x - 1 x -1\} + \{p_{41} x -1 x 1 + (p_4 – p_{41} - \lambda_2 /4) x -1 x -1\}$

$= \rho_{r,S2}$

=>

$p_{11} + p_{21} - p_{31} – p_{41} = 0.5 (\rho_{r,S2} + p_1 + p_2 + p_3 + p_4)$ … (ii)

The third condition is that the correlation between S_1 and S_2, $\rho_{S1,S2}$, has to be preserved.

$Cov[X_{S1}(t), X_{S2}(t)] = \{p_{11} x 1 x 1 + (p_1 – p_{11} - \lambda_2 /4) x 1 x - 1\} + \{p_{21} x -1 x 1 + (p_2 – p_{21} - \lambda_2 /4) x -1 x -1\} + \{p_{31} x 1 x 1 + (p_3 – p_{31} - \lambda_2 /4) x 1 x -1\} + \{p_{41} x -1 x 1 + (p_4 – p_{41} - \lambda_2 /4) x -1 x -1\}$

$= \rho_{S1,S2}$

=>

$p_{11} - p_{21} + p_{31} - p_{41} = 0.5\ (\rho_{S1,S2} + p_1 - p_2 + p_3 - p_4)$ …(iii)

We have three equations and four unknowns. We need an additional condition so that we can arrive at a solution. We would like to equate two probabilities which are small, and because they are small we hope that this equating would not have a significant influence on the results. For now, consider branches B3 and B4 representing the downward movement of stock S_1. The probabilities of upward movement of stock S_2 associated with these branches are equated. That is, we equate probabilities p_{21} and p_{41}. This is just one of the many possible assumptions that can be made. Given this, the system of equations, in matrix notation, can be expressed as

M. K = N

where

M = [1 2 1; 1 0 -1; 1 -2 1]   the semi-colon indicates a new row of the matrix
K = [p_{11}; p_{21}; p_{31}]

N = [1/(c – d) {(exp(r.h) - d.(1 - λ_2))}; 0.5 ($\rho_{r,S2} + p_1 + p_2 - p_3 - p_4$); 0.5 ($\rho_{r,S2} + p_1 + p_2 + p_3 + p_4$); 0.5 ($\rho_{S1,S2} + p_1 - p_2 + p_3 - p_4$)]

So,

K = inv(M). N

This completes determining the probabilities in the risk-neutral setup.

Empirical Results on Pricing Convertible Bonds

We applied the above model to price convertible bonds which could be converted into the stock of either the issuer or some other company. We have tried to price the bonds as on January 03, 2006. If price of a particular bond is not available for this day on Bloomberg, we have priced bond on the day closest to the above day for which the price is available.

Table IX

Company	**Stock convertible to**	**Date**	**Market Price**	**Model Price**	**% Absolute Deviation**
AIG	AIG	1/03/06	942.5	959.4	1.9
Computer Associates	Computer Associates	1/03/06	1417.5	1417.4	0
Amgen	Amgen	2/16/06	1026.85	978.7	4.7
Amgen	Amgen	2/16/06	1028.09	982.59	4.6
Bear Stearns	Fifth Third Bancorp	1/18/06	843.3	883.16	4.7
Bank of America	NASDAQ-100	3/10/06	1032	1046.6	1.4
Goldman Sachs	WYE	11/28/05	972.7	972.9	0
Goldman Sachs	BJ Services	1/18/06	1040.88	1017.3	2.3
Goldman Sachs	Whirlpool	2/09/06	960.8	989.67	3.0

Goldman Sachs	Cendant Corp.	1/25/06	924.5	929.85	0.6
Salomon Holdings	Pfizer	1/25/06	945	950.77	0.6
GATX Corp.	GATX Corp.	1/06/06	1176.25	1136.70	3.4
Morgan Stanley	3M	1/03/06	871.25	916.96	5.2
Morgan Stanley	Walmart	1/05/06	858.75	850.44	1.0
Morgan Stanley	CISCO	1/19/06	961.33	984.08	2.4
Morgan Stanley	General Electric	1/31/06	896.25	904.47	1.0
Morgan Stanley	CA	4/28/06	937.5	967.0	3.1
Wachovia	Corning	2/21/06	1005.35	953.64	5.1
Wachovia	Johnson & Johnson	2/21/06	1006.7	976.65	3.0
Merrill Lynch	Coca Cola	1/04/06	885.0	902.16	1.9
Merrill Lynch	McDonald’s	1/11/06	970.0	990.57	2.1
Merrill Lynch	Berkshire Hathaway	1/05/06	952.1	928.6	2.5
Lehman Bros.	Deere	5/03/06	1221.5	1229.7	0.7
Lehman Bros.	Amgen	4/28/06	877.5	926.07	5.5

Lehman Bros.	Bristol Meyers	4/28/06	880.0	925.37	5.2
Lehman Bros.	Micorsoft	5/02/06	1001.5	982.0	1.9
Lehman Bros.	Cendant Corp.	12/02/05	875.0	884.74	1.1
Providian Financial	Providian Financial	1/03/06	1405	1369.48	2.5
Medtronic	Medtronic	4/13/06	988.53	957.66	3.12
Medtronic	Medtronic	4/13/06	987.78	953.33	3.49
			Mean Absolute % Deviation		**2.60**

.

VI. Conclusion

The empirical results of the model seem good. The Das and Sundaram model can be used to price not just bonds but many other securities with equity, interest rate and credit risks. We wish to enlarge our sample and compare the results of this model with those of a few other models for pricing risky non-convertible and convertible bonds.

References

Amin, Kaushik., and Jim Bodurtha (1995). "Discrete time Valuation of American Options with Stochastic Interest Rates," Review of Financial Studies, v8, 193-234.

Black, Fischer, and Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, v81(3), 637-654.

Carayannopoulos, Peter., and Madhu Kalimipalli (2003). "Convertible Bonds and Pricing Biases," Journal of Fixed Income, v13(3), 64-73.

Cox, John., and Stephen Ross (1976). "A Survey of some New Results in Financial Option Pricing Theory," Journal of Finance, v31(2), 383-402.

Cox, John., and Stephen Ross (1976). "The Valuation of Options for Alternative Stochastic Processes," Journal of Financial Economics, v3(1-2), 145-166.

Cox, John., Stephen Ross, and Mark Rubinstein (1979). "Option Pricing: A Simplified Approach," Journal of Financial Economics, v7, 229-263.

Das, Sanjiv., and Rangarajan Sundaram (1999). "Of Smiles and Smirks: A Term Structure Perspective," Journal of Financial and Quantitative Analysis, v34(2), 211-240.

Das, Sanjiv., and Rangarajan Sundaram (2001). "A Discrete-Time Approach to Arbitrage-Free Pricing of Credit Derivatives," Management Science, v46(1), 46-62.

Das, Sanjiv, and Peter Tufano (1996). "Pricing Credit Sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic," The Journal of Financial Engineering, v5(2), 161-198.

Das, Sanjiv, and Rangarajan Sundaram (2006). "A Simple Model for Pricing Securities with Equity, Interest-Rate, and Default Risk," working paper, Santa Clara University.

Davis, Mark., and Fabian R. Lischka (1999). "Convertible Bonds with Market Risk and Credit Risk," working paper, Tokyo-Mitsubishi Intl, PLC.

Duffie, Darrell (1999). "Credit Swap Valuation," Financial Analysts Journal, January-February, 73-87.

Duffie, Darrell (2001). Dynamic Asset Pricing Theory, Third English Edition, Princeton University Press, 2001.

Duffie, D., and K. Singleton (1999) "Modeling Term Structures of Defaultable Bonds," Review of Financial Studies, v12, pp. 687-720.

Heath, David., Robert A. Jarrow, and Andrew Morton (1990) “Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation,” Journal of Financial and Quantitative Analysis, v25(4), 419-440.

Ho, Thomas, and Sang-Bin Lee (1986). “Term Structure Movements and Pricing Interest Rate Contingent Claims,” The Journal of Finance, Vol. 41, No.5, 1011-1029

Janosi, T., R. Jarrow, and Y. Yildirim (2001). “Estimating Default Probabilities Implicit in Equity Prices,” working paper, Cornell University.

Jarrow, Robert (2001). “Default Parameter Estimation using Market Prices,” Financial Analysts Journal, Sep/Oct, 75-92.

Jarrow, Robert., and Philip Protter (2004). “Structural vs Reduced Form Models: A New Information Based Perspective,” Journal of Investment Management, v2(2), 34-43.

Jarrow, R.A., and S.M. Turnbull (1995): “Pricing Derivatives on Financial Securities Subject to Credit Risk”, Journal of Finance, 50(1), 53-85.

Leland, Hayne., and Klaus Toft (1996). “Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads,” Journal of Finance, v51(3), 987-1019.

Longstaff, Francis. A., Sanjay Mithal, and Eric Neis (2003). “Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market,” Journal of Finance 55(5), 2213-2253.

Madan, Dilip., and Haluk Unal (1995). “Pricing the Risks of Default,” Review of Derivatives Research, v2, 121-160.

Madan, Dilip., and Haluk Unal (2000). “A Two-Factor Hazard-Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads,” Journal of Financial and Quantitative Analysis, v35, 43-65.

Merton, R. (1974). “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” Journal of Finance, v29, 449-470.

Merton, R., (1976). “Option Pricing when Underlying Stock Returns are Discontinuous,” Journal of Financial Economics, v4(1), 125-144.

Sch¨onbucher, P. (1998). “Term Structure Modeling of Defaultable Bonds,” Review of Derivatives Research, v2, 161-192.

Sch¨onbucher, P. (2002). “A Tree Implementation of a Credit Spread Model for Credit Derivatives,” Journal of Computational Finance, v6(2), 1-38.

Takahashi, Akihiko, Takao Kobayashi, and Naruhisa Nakagawa (2001). “Pricing Convertible Bonds with Default Risk,” Journal of Fixed Income, v11(3), 20-29.