Impacts of Jumps and Stochastic Interest Rates on the Fair Costs of Guaranteed Minimum Death Benefit Contracts

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Abstract

The authors offer a new perspective to the domain of guaranteed minimum death benefit contracts. These products have the particular feature to offer investors a guaranteed capital upon death. The authors propose a complete methodology illustrated by a numerical analysis based on Fast Fourier Transform. New results are given in this paper and as a by product, we give the way to price options in a non Gaussian economy with stochastic interest rates. This paper extends Milevsky and Posner (2001). In contrast to their results, the fair costs of the guarantee feature are found to be substantially higher in this more general economy.

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1 Introduction

Life insurance contracts have actuarial and financial components. The first is related to the lifetime of the insured, the second is often linked to financial markets. These contracts generally offer a capital protection and a participation in the performance of the market. Thus they match investors desire to get protection in bear markets and upside participation in bull markets. They also offer shelters from inheritance taxes. The design of the financial part can be quite sophisticated and makes these contracts akin to structured products sold by banks. Under different names such as variable annuities (VA) in the USA, segregated funds in Canada, unit linked in the UK, or other products like equity index annuities or guaranteed annuities options, many kinds of policies bearing these features are offered to investors (see for example Hardy (2003) or Milevsky (2006)). In the present article, the authors study a particular contract, the Guaranteed Minimum Death Benefit (GMDB) issued by insurance companies. It belongs to the class of VA and represent in the USA a multibillion dollar market. This contract is also very similar to Death Protected Mutual Funds.

The GMDB guarantees a specific monetary amount upon the policyholder's death. The contract is associated with a subaccount and the guaranty can take diverse predetermined expressions. The usual one is the maximum of the subaccount value and the initial investment accrued at a guaranteed rate, and this guaranty is only triggered by the insured's death and will be returned to the beneficiary of the policy. Insurance companies also offer other contracts such as capped guaranties or policies with a ratchet clause to lock in a previous gain if any. Lapses and surrenders are possible but with dissuasive penalties. The initial premium is due to the insurance company which can invest it through mutual funds in financial markets. The promised guaranty, only paid on death, is not free but is paid for by the insured by deducting small amounts from her subaccount. In practice, these payments are made on a periodic basis. In this article, the modeling is in continuous time and contractual payments are made instantaneously and are considered equivalent to continuous dividends. These fees are endogenously determined and are related by construction to the guaranteed minimum death benefit. They correspond to the so-called mortality and expense (M&E) risk charge.

In Milevsky and Posner (2001) as in recent works on structured products (see Benet, Giannetti, and Pissaris (2005) or Wilkens and Stoimenov (2007)), the authors found these contracts overpriced, while fees for death protected mutual funds which are not tax-sheltered do not seem so. The main question addressed in this article is: what should be fair costs for GMDB contracts. Here we examine this question under a more general pricing framework than that used by Milevsky and Posner (2001).

Usually pricing is done in a Black and Scholes economy: the subaccount value is assumed to follow a geometric Brownian motion, and the term structure of interest rates is assumed constant. This last assumption, which can be acceptable for short-term options can no longer be justified for medium- or long-term contracts such as life insurance products. In this article we use a one-factor model of a Vasicek (1977) type for stochastic interest rates.

The Gaussian hypothesis for asset returns has been questioned for a long time. It is now widely accepted that many return distributions display asymmetry and fat tails, see Cont (2001). Many alternatives have been suggested; beginning with Mandelbrot's works in the sixties and continuing at the end of the nineties with research modeling financial asset prices by exponentials of Lévy processes. For a survey see Cont and Tankov (2004). Today, it still constitutes an extensive research area. In a seminal article, Carr, Geman, Madan, and Yor (2002) introduce a new type of purely discontinuous process, allowing them to replace a diffusive component by small jumps arriving at infinite rate.

Aït-Sahalia (2004) gave tools to separate the diffusive and jump components of a semi-martingale. In life insurance, Hardy (2003) suggested using regime-switching models introduced by Hamilton (1989), in particular the regime-switching lognormal model. Ballotta (2005) was the first author to analyze the impacts of jumps in valuing participating life insurance contracts, using a jump diffusion process with Gaussian jumps while Kassberger, Kiesel, and Liebmann (2007) made use of Meixner and NIG processes.

In this article, we consider a jump diffusion process whose jumps have mainly a double exponential distribution, which is known as a Kou process. This choice permits a good fit to market data and furthermore is versatile, easy to understand, mathematically tractable and has many interesting properties which can be exploited to obtain quasi-closed-form solutions in options pricing, including barrier options (see Kou (2002) and Kou and Wang (2003)). The jump diffusion model of Merton (1976) with Gaussian jumps is also used for comparison purposes.

The GMDB policy has an embedded option which can be expressed as a kind of European option with a rising exercise price and a stochastic expiry date. A technical part of this paper is devoted to the pricing of this option. This is done in three steps.

First, conditioning on the policyholder's time of death, the option is valued in the context of a Kou process with stochastic interest rates, with the assumption that the financial asset in the investor subaccount is correlated to the interest rates. This is a non trivial problem which, as far as the authors know, has not been solved before. The solution is given here using an adaptation of the Fast Fourier Transform methodology proposed by Boyarchenko and Levendorskiĭ (2002) and developed for Kou processes in Quittard-Pinon and Randrianarivony (2008).

Once this valuation is done, the second step of the pricing is obtained by quadrature using the density of a chosen mortality distribution. We consider three distributions, namely an exponential, a parametrized Makeham, and a Gompertz distribution. For these last two, the authors use estimates from actual data given in Melnikov and Romaniuk's (2006) paper. The solution, although not in closed-form, is fast and accurate. With this model, market risk, jump risk, interest rate risk, and mortality can simultaneously be taken into account.

The third step of the procedure is to compute the fair cost of GMDB. This is done by equating the expected discounted value of the insured's accumulated payments to the GMDB option. In this way, the fair cost is endogenous. The suggested methodology is easy to implement and is illustrated by numerous examples.

The main contribution of this study is twofold. From a theoretical point of view it suggests a general pricing framework where both stochastic interest rates and jumps are taken into account. In this respect, it can be considered as an extension of the seminal Milevsky and Posner (2001) article. From an empirical point of view it gives an answer to the question of how far from fair value costs are the observed Mortality and Expense risk charges?

The remainder of this article is organized as follows. The next section will introduce main notations and the general framework used in the sequel. The pricing model and applications are studied in section 3. Numerical results are shown and various risk factor impacts are discussed in section 4. A last section concludes the paper.

2 General Framework and Main Notations

In this section, the main definitions, and notations are given in a formal way. The general framework of the analysis is set up. In a first subsection mortality and financial risk are considered, then in subsection two, the GMDB under analysis is defined. Subsection three deals with the main equations of this article and the last subsection recalls Milevsky and Posner's (2001) solution.

2.1 Financial risk and mortality

Financial risk is related to market risk firstly because the policyholder's account is linked to a financial asset or an index, and secondly via interest rates. We denote by r the stochastic process modeling the instantaneous risk free rate. The value of the money market account is then:

$$R_t = e^{\int_0^t r_s \, ds}, \tag{1}$$

and the discount factor is:

$$\delta_t = e^{-\int_0^t r_s\, ds}. \tag{2}$$

The policyholder's account value is modeled by the stochastic process S. In that model, ℓ stands for the fees associated with the Mortality and Expense (M&E) risk charge.

As far as mortality is concerned, we use the traditional actuarial notations. The future lifetime of a policyholder aged x is the random variable T_x. For an individual aged x, the probability of death before time $t \geq 0$ is $P(T_x \leq t) = 1 - ({}_tp_x)$. If we introduce λ, the force of mortality, we have

$$P(T_x \leq t) = 1 - \exp\Big(-\int_0^t \lambda(x+s)ds\Big). \tag{3}$$

As usual $F_x(t)$ and $f_x(t)$ are respectively the c.d.f. and the p.d.f. of the random variable T_x. We recall the well-known relationship

$$\lambda(x+t) = \frac{f_x(t)}{1 - F_x(t)}, \tag{4}$$

see for example Gerber (1997) or Bowers, Gerber, Hickman, Jones, and Nesbitt (1997). To ease notation, we generally omit the x for the future lifetime and write T when no confusion is possible. We assume stochastic independence between mortality and financial risks.

2.2 Contract Payoff

The insurer promises to pay upon the policyholder's death the contractual amount $\max\{S_0 e^{gT}, S_T\}$, where g is a guaranteed rate and S_0 is the insured initial investment and S_T is the subaccount value at time of death T. We can generalize this payoff a bit further: if we consider a contractual expiry date $x + \Theta$, the contract only provides a guarantee on death. If the insured is otherwise still alive after time Θ passes, she will receive the account value by that time. For the sake of simplicity, we keep the first formulation, and we note that:

$$\max\{S_0 e^{gT}, S_T\} = S_T + \left[S_0 e^{gT} - S_T\right]^+ . \tag{5}$$

Written in this way, the contract appears as a long position on the policyholder account plus a long position on a put option written on the insured account. Two remarks are in order: firstly, the policyholder has the same amount as if she invested in the financial market (kept aside the fees), but has the insurance to get more, due to the put option. Secondly because T is a random variable, her option is not a vanilla one but an option whose exercise date is itself random (the policyholder's death). The other difference with the option analogy lies in the fact that in this case there is no upfront payment. Similarly to Milevsky and Posner, we call the option part in (5) the GMDB Option Payoff. In non formal terms we can write

$$\text{Death Payment} = \text{Account Value} + \text{GMDB Option Payoff}.$$

In this contract, the investor pays the guarantee by installments. The paid fees constitute the so-called M&E risk charges. We assume they are continuously deducted from the policyholder account at the contractual proportional rate ℓ. More precisely, we consider that in the time interval $(t, t + dt)$, the life insurance company receives $\ell S_t \, dt$ as instantaneous earnings. We denote by F the cumulative discounted fees. F_τ is the discounted accumulated fees up to time τ which can be a stopping time for the subaccount price process S.

The contract can also be designed in order to cap the guaranteed rate g; in the VA literature, this is known as capping the rising floor. Let M be the maximum amount chosen by the insurer, $M \geq S_0$. In that case, the payoff becomes $[\min[M, S_0 e^{gT}] - S_T]^+$. As Milevsky and Posner (2001) noticed, this can be further simplified by

$$\begin{cases} (S_0 e^{gT} - S_T)^+ & \text{if} \qquad T \leq \ln[M/S_0]/g \\ (M - S_T)^+ & \text{if} \qquad T > \ln[M/S_0]/g. \end{cases}$$

We could also consider lapses or surrenders in the way suggested by Milevsky and Posner. The policyholders, supposing they adopt a purely financial attitude, will lapse or surrender their contracts if their account value at time t is for the first time equal or below a barrier $L > S_0$.

2.3 Main Equations

We are in this article essentially interested in the fair value of the M&E charges. To do so, we consider the fair price is the arbitrage-free price which, according to arbitrage theory, is given by the expectation of the discounted payoff of the contract under an equivalent martingale measure to the physical or historical one. In our paper we do not necessarily have a complete market; however we consider that a risk-neutral measure has been chosen. For practical purposes, this one can be obtained from the market, see for example (Björk 2004). We suppose we have a stochastic basis or a risk neutral universe, given by the quadruplet $(\Omega, \mathcal{A}, \mathcal{F}_{t,t\geq 0}, Q)$, where $(\Omega, \mathcal{A}, Q)$ is a probability space with Q as a risk-neutral probability measure, and $(\mathcal{F}_t, t \geq 0)$ a filtration. Under this assumption, the GMDB option fair price is

$$G(\ell) = E_Q\Big[\delta_T(S_0 e^{gT} - S_T)^+\Big],$$

and upon conditioning on the insured lifetime,

$$G(\ell) = E_Q\Big[E_Q\Big[\delta_T(S_0 e^{gT} - S_T)^+ | T = t\Big]\Big]. \tag{6}$$

If F_T denotes the discounted value of all fees collected up to time T, the fair value of the M&E charges can be written

$$ME(\ell) = E_Q[F_T]$$

which after conditioning, also gives:

$$ME(\ell) = E_Q\Big[E_Q[F_T | T = t]\Big]. \tag{7}$$

Because the protection is only triggered by the policyholder's death, the endogenous equilibrium price of the fees is the solution in ℓ, if any, of the following equation

$$G(\ell) = ME(\ell). \tag{8}$$

This is the key equation of this article. To solve it we have to define the investor account dynamics, make assumptions on the process S, and, of course, on mortality.

2.4 Milevsky and Posner solution

In their paper Milevsky and Posner assume: ℓ is a constant, S a Geometric Brownian motion with a volatility σ, and the mortality is either of an exponential type or of a Gompertz type. With obvious notations the GMDB option price, which they call a Titanic Option, is then given by

$$G(\ell) = \int_0^{\Theta} f_x(t) E_Q\Big[\delta_T (S_0 e^{gT} - S_T)^+ | T = t\Big]\, dt, \tag{9}$$

where the inner conditional expectation is exactly the standard Black Scholes Merton put formula with a strike price of $K = S_0 e^{gt}$.

Assuming an exponential future lifetime they obtain the price in a closed form formula. Although this modeling is not realistic, it has the advantage of providing simple formulas and it gives a kind of benchmark. They also obtained a closed form solution for the present value of fees. For a parametrized version of Gompertz lifetime, numerical methods were necessary. In any case, the equilibrium value of ℓ can only be obtained by a root searching algorithm.

3 Pricing Model

Here we adopt Milevsky and Posner's (2001) approach. The model presented here is a very general one:

- First, the model takes into account stochastic interest rates. A one-factor interest rate model with exponential volatility structure is used;
- Second, the underlying asset process incorporates jumps;
- Third, the impact of the chosen mortality model on the fair cost of the GMDB contract is taken into account.

The zero-coupon bond is assumed to obey the following stochastic differential equation (SDE) in the risk-neutral universe:

$$\frac{dP(t,T)}{P(t,T)} = r_t\,dt + \sigma_P(t,T)\,dW_t, \tag{10}$$

where $P(t,T)$ is the price at time t of a zero-coupon bond maturing at time T, r_t is the instantaneous risk-free rate, and $\sigma_P(t,T)$ describes the volatility structure, and W is a standard Brownian motion.

In order to take into account a dependency between the subaccount and the interest rates, we suggest to introduce a correlation between the diffusive part of the subaccount process and the zero-coupon bond dynamics. The underlying account price process S is supposed to behave according to the following SDE under the chosen equivalent pricing measure Q:

$$\frac{dS_t}{S_{t^-}} = \Big(r_t - \ell\Big)\,dt + \rho\sigma\,dW_t + \sigma\sqrt{1-\rho^2}\,dZ_t + (Y-1)\,d\tilde{N}_t. \tag{11}$$

Again, r_t is the instantaneous interest rate, ℓ represents the fixed proportional insurance risk charge, σ is the asset's volatility, ρ is the correlation between the asset and the interest rate, W and Z are two independent standard Brownian motions, and the last part takes into account the jumps. $\tilde{N}$ is a compensated Poisson process with intensity λ, while Y, a random variable independent from the former processes, represents the price change after a jump. The jump size is defined by $J = \ln(Y)$.

Let us emphasize here that the non-drift part M, defined by $dM_t = \rho\sigma\, dW_t + \sigma\sqrt{1-\rho^2}\, dZ_t + (Y-1)\, d\tilde{N}_t$, is a martingale in the considered risk-neutral universe.

3.1 Valuation of the guarantee feature

Denoting by N_t the Poisson process with intensity λ and applying Itō's lemma, the dynamics of S writes as:

$$S_t = S_0\, e^{\int_0^t r_s\, ds - (\ell + \frac{1}{2}\sigma^2 + \lambda\kappa)\, t + \rho\sigma\, W_t + \sigma\sqrt{1-\rho^2}\, Z_t + \sum_{i=1}^{N_t} \ln\left((Y)_i\right)}, \tag{12}$$

where $\kappa = E(Y-1)$. On the other hand, the zero-coupon bond price process obeys the following two equations:

$$P(t,T) = P(0,T)\, e^{\int_0^t \sigma_P(s,T)\, dW_s - \frac{1}{2}\int_0^t \sigma_P^2(s,T)\, ds + \int_0^t r_s\, ds} \tag{13a}$$

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\, e^{\int_0^t [\sigma_P(s,T) - \sigma_P(s,t)] dW_s + \frac{1}{2}\int_0^t [\sigma_P^2(s,t) - \sigma_P^2(s,T)] ds}. \tag{13b}$$

Dividing equation (12) by equation (13a) gives

$$\frac{S_t}{P(t,T)} = \frac{S_0}{P(0,T)} \exp\Big(-\big(\ell + \tfrac{1}{2}\sigma^2 + \lambda\kappa\big)\, t + \tfrac{1}{2}\int_0^t \sigma_P^2(s,T)\, ds\Big)$$
$$\times \exp\Big(\int_0^t \big(\rho\sigma - \sigma_P(s,T)\big) dW_s + \sigma\sqrt{1-\rho^2}\, Z_t + \sum_{i=1}^{N_t} \ln\big((Y)_i\big)\Big).$$

Plugging (13b) into the latter results in

$$S_t = \frac{S_0}{P(0,t)}\, e^{-(\ell+\frac{1}{2}\sigma^2+\lambda\kappa)\, t+\frac{1}{2}\int_0^t \sigma_P^2(s,t)\, ds+\int_0^t [\rho\sigma-\sigma_P(s,t)] dW_s+\sigma\sqrt{1-\rho^2}\, Z_t+\sum_{i=1}^{N_t} \ln\big((Y)_i\big)}.$$

Let us introduce the T-forward measure Q_T defined by:

$$\frac{dQ_T}{dQ}\Big|_{\mathcal{F}_t} = \frac{\delta_t P(t,T)}{P(0,T)}, \tag{14}$$

where δ_t is the discount factor defined in equation (2). This equation can be expressed as follows, from equation (13a):

$$\frac{dQ_T}{dQ}\Big|_{\mathcal{F}_t} = e^{\int_0^t \sigma_P(s,T)\, dW_s - \frac{1}{2}\int_0^t \sigma_P^2(s,T)\, ds}.$$

Girsanov's theorem states that the stochastic process W^T, defined by $W_t^T = W_t - \int_0^t \sigma_P(s,T)\, ds$, is a standard Brownian motion under Q_T. Hence, the subaccount price process can be derived under the T-forward measure:

$$S_t = \frac{S_0}{P(0,t)} e^{X_t} \tag{15}$$

where X is the process defined by

$$
\begin{aligned}
X_t = &-(\ell + \tfrac{1}{2}\sigma^2 + \lambda\kappa)t + \int_0^t \Big(\sigma_P(s,T)\big(\rho\sigma - \sigma_P(s,t)\big) + \tfrac{1}{2}\sigma_P^2(s,t)\Big) ds \\
&+ \int_0^t \big(\rho\sigma - \sigma_P(s,t)\big) dW_s^T + \sigma\sqrt{1-\rho^2}\, Z_t + \sum_{i=1}^{N_t} \ln\big((Y)_i\big).
\end{aligned}
\tag{16}
$$

Notice we do not have anymore an exponential of a Lévy process here as X is no longer a Lévy process due to the stochastic nature of interest rates.

Let us proceed by computing the inner expectation in equation (6). Conditional on a given remaining lifetime T and using the T-forward measure, the inner expectation I_T becomes:

$$
I_T = E_Q\Big[\delta_T (S_0 e^{gT} - S_T)^+\Big] = P(0,T) E_{Q_T}\Big[(K - S_T)^+\Big],
$$

where a vanilla put payoff can be recognized with a strike of $K = S_0 e^{gT}$. The last expectation can be rewritten from equation (15) as:

$$
E_{Q_T}[(K - S_T)^+] = E_{Q_T}\Big[\Big(K - \tfrac{S_0}{P(0,T)} e^{X_T}\Big)^+\Big].
$$

The payoff functional will be denoted by h, here

$$
h(x) = \left(K - \frac{S_0}{P(0,T)} e^x\right)^+, \tag{17}
$$

so the question is now how to compute $E_{Q_T}[h(X_T)]$.

The answer begins by defining the function ϕ_T as the exponent of the characteristic function of X_T:

$$E_{Q_T}\left[e^{iuX_T}\right] = e^{\phi_T(u)},$$

from which the p.d.f. of X_T can be derived as

$$f_{X_T}(x) = \frac{1}{2\pi}\int e^{-iux+\phi_T(u)}du. \tag{18}$$

Let us also introduce the Fourier transform of the payoff functional:

$$\hat{h}(u) = \int e^{-iux}h(x)\,dx \tag{19}$$

which can be extended to a generalized Fourier transform defined on the line $\Im u = \Delta$, parallel to the real numbers axis in the complex plane, provided $e^{\Delta x}h(x)$ is integrable. Using (17), the computation of $\hat{h}$ can proceed as follows:

$$\begin{aligned}\hat{h}(u) &= \int e^{-iux}\left(K - \frac{S_0}{P(0,T)}e^x\right)^+ dx\\ &= \frac{S_0}{P(0,T)}\left[\frac{KP(0,T)}{S_0}\left[\frac{e^{-iux}}{-iu}\right]_{-\infty}^{\ln\left(\frac{KP(0,T)}{S_0}\right)} - \left[\frac{e^{(1-iu)x}}{1-iu}\right]_{-\infty}^{\ln\left(\frac{KP(0,T)}{S_0}\right)}\right]\end{aligned}$$

under condition $\Delta = \Im u > 0$ to ensure convergence, and eventually

$$\hat{h}(u) = K\,\frac{e^{-iu\ln\left(\frac{KP(0,T)}{S_0}\right)}}{(-iu)(-iu+1)}. \tag{20}$$

Getting back to the previous expectation:

$$\begin{aligned}
E_{Q_T}[h(X_T)] &= \int h(x) f_{X_T}(x)\, dx && \text{by definition,} \\
&= \frac{1}{2\pi} \int h(x) \int e^{-iux+\phi_T(u)} du\, dx && \text{by (18),} \\
&= \frac{1}{2\pi} \int \left(\int e^{-iux} h(x) dx \right) e^{\phi_T(u)} du && \text{changing integration order,} \\
E_{Q_T}[h(X_T)] &= \frac{1}{2\pi} \int \hat{h}(u) e^{\phi_T(u)} du && \text{from (19),} \qquad (21)
\end{aligned}$$

where the last integral is computed along the line $(\Delta) = \mathbb{R} + i\Delta$. Substituting (20) into (21) gives:

$$\begin{aligned}
E_{Q_T}[h(X_T)] &= K\, \frac{1}{2\pi} \int_{(\Delta)} \frac{e^{-iu \ln\left(\frac{KP(0,T)}{S_0}\right)}}{(-iu)(-iu+1)} e^{\phi_T(u)} du \\
&= K\, \frac{1}{2\pi} \int_{(\Delta)} e^{iu \ln(m)}\, \frac{e^{-iu \ln(P(0,T)) + \phi_T(u)}}{(-iu)(-iu+1)} du
\end{aligned}$$

where $m = \frac{S_0}{K}$ is the moneyness,

$$= K\, \frac{1}{2\pi} \int_{(\Delta)} e^{iuv}\, \zeta(u)\, du$$

where v is the log-moneyness and $\zeta(u)$ is the fraction part of the integrand,

$$= K e^{-\Delta v} \times \frac{1}{2\pi} \int_{(\mathbb{R})} e^{iuv}\, \zeta(u + i\Delta)\, du \qquad (22)$$

after the $u \to u + i\Delta$ change of variable.

The rightmost part of (22) is the inverse Fourier transform of $\zeta(u+i\Delta)$. This fact allows the straightforward use of a very efficient numerical algorithm, namely the Fast Fourier Transform (FFT).

It remains to compute the function $\phi_T(u)$. Let $\phi_J(u)$ denote the characteristic function of the i.i.d. random variables $J_i = \ln\Big((Y)_i\Big)$. Equation (16) yields

$$\begin{aligned}
E_{Q_T}\Big[e^{iuX_T}\Big] &= e^{\phi_T(u)} \\
&= \exp\Big(-iu\ell T + iu\int_0^T\Big(-\tfrac{1}{2}\sigma^2 + \rho\sigma\sigma_P(s,T) - \tfrac{1}{2}\sigma_P^2(s,T)\Big)ds\Big) \\
&\times \exp\Big(-\tfrac{1}{2}u^2\int_0^T\Big(\rho^2\sigma^2 - 2\rho\sigma\sigma_P(s,T) + \sigma_P^2(s,T) + \sigma^2(1-\rho^2)\Big)ds\Big) \\
&\times \exp\Big(T\Big[\lambda(\phi_J(u)-1) - \lambda\kappa\Big]\Big).
\end{aligned}$$

Defining

$$\Sigma_T^2 = \int_0^T\Big(\sigma^2 - 2\rho\sigma\sigma_P(s,T) + \sigma_P^2(s,T)\Big)ds \tag{23}$$

and noticing that

$$\kappa = E_{Q_T}[Y-1] = E_{Q_T}[e^J - 1] = \phi_J(-i) - 1,$$

the exponent ϕ_T can be deduced as follows:

$$\phi_T(u) = -iu\ell T - \tfrac{iu}{2}\Sigma_T^2 - \tfrac{u^2}{2}\Sigma_T^2 + \lambda T\Big(\phi_J(u) - \phi_J(-i)\Big). \tag{24}$$

3.2 Present value of fees

By definition, F_t is such that

$$dF_t = \delta_t \, \ell \, S_t dt.$$

Using a chain rule, we get:

$$\begin{aligned} d(\delta_t S_t) &= -r_t \delta_t S_t dt + \delta_t dS_t \\ &= -r_t \delta_t S_t dt + \delta_t (r_t - \ell) S_t dt + \delta_t S_{t^-} dM_t \\ &= -dF_t + \delta_t S_{t^-} dM_t, \end{aligned}$$

so we can deduce

$$F_T = \int_0^T dF_t = S_0 - \delta_T S_T + \int_0^T \delta_t S_{t^-} dM_t.$$

Without loss of generality, we will also assume from now on that $S_0 = 1$.

Eventually, as the last term of F_T is an integral with respect to a martingale, whose expectation is zero, we recover Milevsky and Posner's (2001) previous result on the present value of fees:

$$ME(\ell) = E_Q[F_T] = 1 - E_Q[\delta_T S_T]. \tag{25}$$

Using (2) and (12), we get:

$$ME(\ell) = 1 - E_Q\left[e^{-\ell T - \frac{1}{2}\sigma^2 T + \rho\sigma \, W_T + \sigma\sqrt{1-\rho^2} \, Z_T + \sum_{i=1}^{N_T} \ln\left((Y)_i\right) - \lambda\kappa T} \right]$$

$$= 1 - E_Q\Big[E_Q\big[e^{-\ell T}|T = t\big]\Big], \text{ conditioning on the future lifetime } T,$$

$$ME(\ell) = 1 - \int_0^{\infty} e^{-\ell t} f_x(t)dt \tag{26}$$

where f_x is the p.d.f. of the r.v. T. A very interesting fact is that only the mortality model plays a role in the computation of the present value of fees as seen in (26).

Taking into account the time to contract expiry date Θ, we have:

$$\begin{aligned} ME(\ell) &= 1 - E_Q\Big[e^{-\ell \min(T,\Theta)}\Big] \\ &= 1 - E_Q\Big[e^{-\ell T}\, \mathbb{1}_{\{T<\Theta\}} + e^{-\ell\Theta}\, \mathbb{1}_{\{T\geq\Theta\}}\Big] \\ ME(\ell) &= 1 - \int_0^{\Theta} e^{-\ell t} f_x(t)dt - \Big(1 - F_x(\Theta)\Big)e^{-\ell\Theta}. \end{aligned} \tag{27}$$

3.3 Mortality models

Two mortality models are taken into account, namely the Gompertz model and the Makeham model. In the case of the Gompertz mortality model, the force of mortality at age x follows

$$\lambda(x) = B.C^x, \tag{28}$$

where $B > 0$ and $C > 1$. It can also be written as

$$\lambda(x) = \frac{1}{b}\exp\Big(\frac{x-m}{b}\Big),$$

where $m > 0$ is the modal value of the Gompertz distribution and $b > 0$ is a dispersion parameter. Both notations are related by the following equations:

$$B = \frac{1}{b} e^{-\frac{m}{b}} \text{ and } C = e^{-\frac{1}{b}}.$$

From (3) and (4), we get:

$$f_x(t) = \lambda(x) e^{b\lambda(x)} e^{\frac{t}{b}} e^{-b\lambda(x) e^{\frac{t}{b}}}$$

and

$$F_x(t) = 1 - e^{-b\lambda(x)\left(e^{\frac{t}{b}-1}\right)}.$$

The integral part in (27) can now be computed:

$$\begin{aligned}\int_0^{\Theta} e^{-\ell T} f_x(t) dt &= \lambda(x) e^{b\lambda(x)} \int_0^{\Theta} e^{-\ell T} e^{\frac{t}{b}} e^{-b\lambda(x) e^{\frac{t}{b}}} dt \\ &= \lambda(x) e^{b\lambda(x)} \int_1^{e^{\frac{\Theta}{b}}} u^{-\ell b} e^{-b\lambda(x) u} du\end{aligned}$$

after the $t \to b \ln(u)$ change of variable,

$$= e^{b\lambda(x)} e^{(x-m)\ell} \int_{b\lambda(x)}^{b\lambda(x) e^{\frac{\Theta}{b}}} e^{-y} y^{-\ell b} dy$$

after the $u \to \frac{y}{b\lambda(x)}$ change of variable,

$$\int_0^{\Theta} e^{-\ell T} f_x(t) dt = e^{b\lambda(x)} e^{(x-m)\ell} \Big(\Gamma\big(1 - \ell b, b\lambda(x)\big) - \Gamma\big(1 - \ell b, b\lambda(x) e^{\frac{\Theta}{b}}\big) \Big)$$

where $\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1} dt$ is the upper incomplete gamma function where a

must be positive. This condition entails an upper limit on the possible value of the insurance risk charge ℓ:

$$\ell < \frac{1}{b}. \tag{29}$$

The present value of fees in the case of a Gompertz-type mortality model amounts to:

$$\begin{aligned} ME(\ell) = 1 - e^{b\lambda(x)} e^{(x-m)\ell} \Big[\Gamma\Big(1 - \ell b, b\lambda(x)\Big) - \Gamma\Big(1 - \ell b, b\lambda(x) e^{\frac{\Theta}{b}}\Big) \Big] \\ - e^{b\lambda(x)\left(1 - e^{\frac{\Theta}{b}}\right)} e^{-\ell\Theta}. \end{aligned} \tag{30}$$

The Makeham mortality model adds an age-independent component to the Gompertz force of mortality (28) as follows:

$$\lambda(x) = A + B.C^x, \tag{31}$$

where $B > 0, C > 1$ and $A \geq -B$.

In that case, the presence of the constant A does not allow to get closed-form formulas as in the previous case. Hence, a numerical quadrature was used to compute the M&E fees.

4 Empirical study

This section gives a numerical analysis of jumps, stochastic interest rates and mortality effects. For the jumps and interest rates, a numerical analysis is performed in the first two subsections while the last subsection examines all

Table 1: Gompertz distribution parameters – Milevsky and Posner (2001)

	Female		Male	
Age (years)	m	b	m	b
30	88.8379	9.213	84.4409	9.888
40	88.8599	9.160	84.4729	9.831
50	88.8725	9.136	84.4535	9.922
60	88.8261	9.211	84.2693	10.179
65	88.8403	9.183	84.1811	10.282

these risk factors together. Throughout this analysis, all mortality models were fitted from actual data.

4.1 Jumps impact

In this subsection, we will only study the impact of jumps in the underlying account dynamics. The interest rate structure will also remain flat. The contract expiry is set at age 75. The mortality model is the Gompertz mortality model. The Gompertz parameters used in this subsection and the next one are those calibrated to the 1994 Group Annuity Mortality Basic table in Milevsky and Posner (2001). They are recalled in Table 1.

We will see in turn the no-jumps case and two jump diffusion models. The volatility in the no-jumps case is set to 20 %. In the Merton case, given here for comparison purposes, the jump sizes are Gaussian i.i.d. with mean μ_J and standard deviation σ_J. Here, $\mu_J = 0$ and $\sigma_J = 0.25$.

In the Kou case, the jump sizes $J = \ln(Y)$ are i.i.d. and follow a double exponential law:

$$f_J(y) = p\lambda_1 e^{-\lambda_1 y} 1_{y>0} + q\lambda_2 e^{\lambda_2 y} 1_{y\leq 0} \tag{32}$$

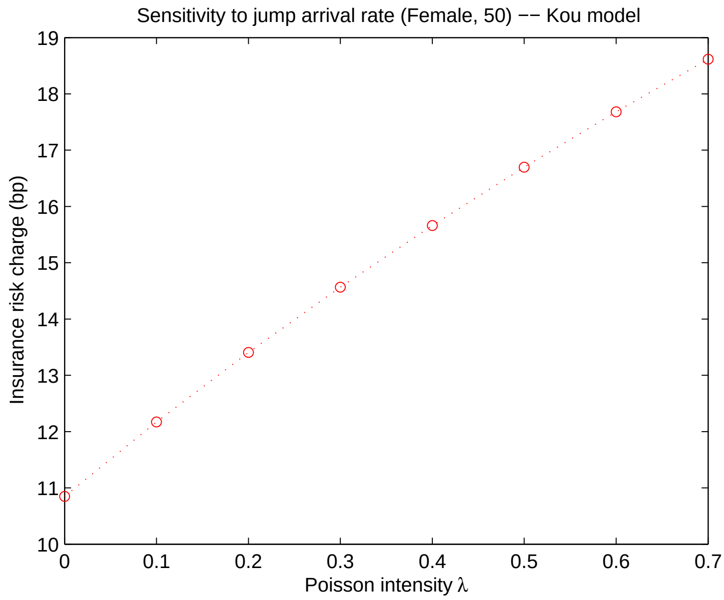

Figure 1: Sensitivity to jump arrival rate – $r = 6\%, g = 5\%$, 200% cap. Gompertz mortality model

with $p \geq 0$, $q \geq 0$, $p + q = 1$, $\lambda_1 > 0$ and $\lambda_2 > 0$. The following parameters for the Kou case serve as reference parameters: $p = 0.4$, $\lambda_1 = 10$ and $\lambda_2 = 5$.

Figure 1 shows the sensitivity of the annual insurance risk charge to the jump arrival rate. The diffusive part is set with a volatility of 20 % and the parameters are those of the reference Kou case described before. It can be seen that the insurance risk charge is increasing with the jump arrival rate. This property stems from the option feature embedded in the GMDB contract, hence a positive sensitivity to a general increase in the total variation of the underlying process.

We continue to explore the impact of jumps on the fair insurance risk charge. Keeping the jump parameters as in the reference Kou case, we keep this time the total quadratic variation constant at 1.5 times the variation of

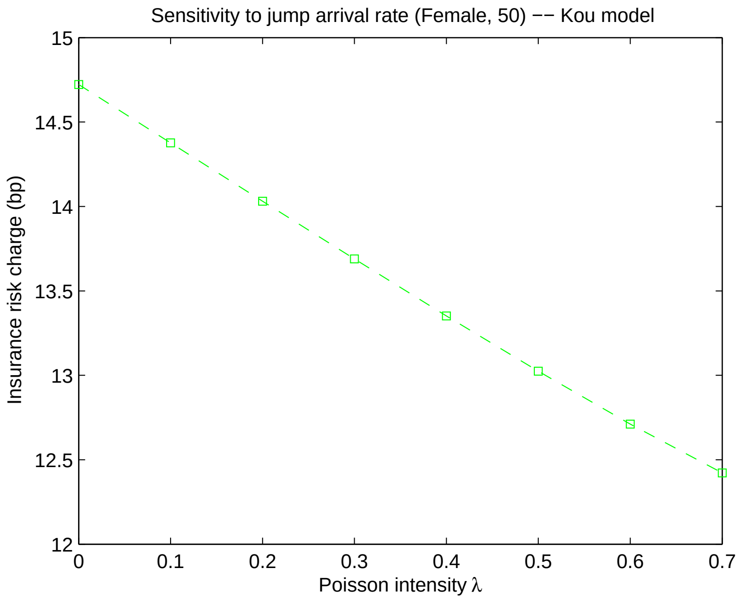

Figure 2: Sensitivity to jump arrival rate – $r = 6\%, g = 5\%$, 200% cap. Gompertz mortality model – Keeping total quadratic variation constant.

the no-jumps case. The diffusive part is then set accordingly. In contrast, Figure 2 shows instead a decrease of the insurance risk charge with respect to the jump arrival rate. The variation gained by the jump component is offset by a decrease in the variation of the diffusive part. Hence, this decrease can be explained by a bigger effect of the variation decrease due to the diffusive part.

From now on, the Poisson intensity is arbitrarily set to $\lambda = 0.5$ in both jump diffusion cases. The diffusive part of both jump diffusion models is such that their overall quadratic variation is 1.5 times the variation of the no-jumps case, unless otherwise stated.

The following tables show the percent of premium versus the annual insurance risk charge in all three cases for a female policyholder (see Table 2)

Table 2: Jumps impact – Female policyholder – $r = 6\%, g = 5\%$, 200% cap. Gompertz mortality model. In each case, the left column displays the relative importance of the M&E charges given by the ratio $ME(\ell)/S_0$. The right column displays the annual insurance risk charge ℓ in basis points (bp).

Purchase age	No jumps case		Merton model		Kou model	
(years)	(%)	(bp)	(%)	(bp)	(%)	(bp)
30	0.76	1.77	1.24	2.89	1.16	2.70
40	1.47	4.45	2.18	6.61	2.04	6.19
50	2.52	10.85	3.41	14.72	3.21	13.86
60	2.99	21.58	3.75	27.24	3.55	25.74
65	2.10	22.56	2.61	28.12	2.47	26.59

Table 3: Jumps impact – Male policyholder – $r = 6\%, g = 5\%$, 200% cap. Gompertz mortality model. In each case, the left column displays the relative importance of the M&E charges given by the ratio $ME(\ell)/S_0$. The right column displays the annual insurance risk charge ℓ in basis points (bp).

Purchase age	No jumps case		Merton model		Kou model	
(years)	(%)	(bp)	(%)	(bp)	(%)	(bp)
30	1.34	3.25	2.15	5.21	2.01	4.86
40	2.52	7.97	3.68	11.73	3.46	10.99
50	4.23	19.22	5.68	26.01	5.35	24.46
60	4.90	37.59	6.14	47.50	5.81	44.82
65	3.48	39.33	4.32	49.05	4.08	46.31

and a male policyholder (see Table 3).

We can notice that both jump diffusion models give roughly the same contractual insurance risk charge fee, having the same overall quadratic variation.

4.2 Stochastic interest rates impact

We will focus in the sequel on the case of the Kou jump diffusion model. Stochastic interest rates are taken into account in this subsection.

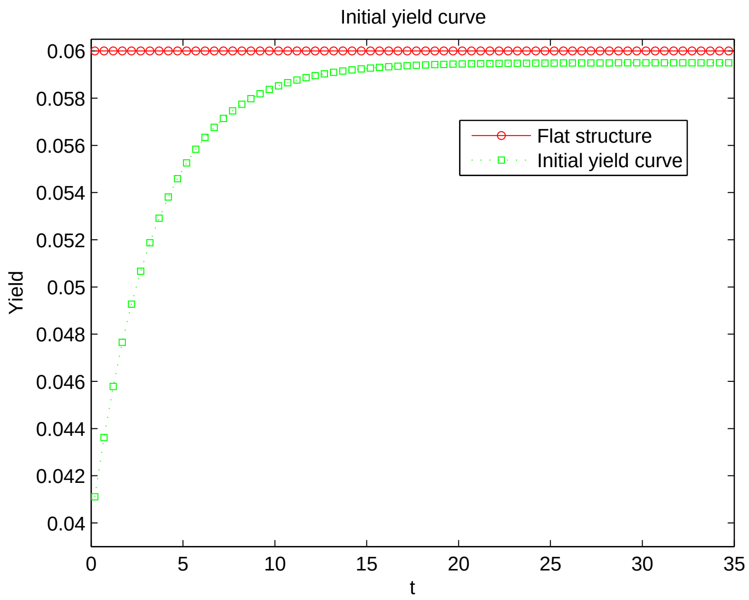

Figure 3: Initial yield curve.

The initial yield curve $y(0,t)$ is supposed to obey the following parametric equation $y(0,t) = \alpha - \beta e^{-\gamma t}$ where α, β and γ are positive numbers. For comparison purposes, we also take into account a flat interest rate structure set at $r = 0.06$. The yield is then supposed to converge towards r for bigger maturities. The initial yield curve equation is set as follows:

$$y(0,t) = 0.0595 - 0.0195 \exp(-0.2933\, t). \tag{33}$$

As stated earlier, the interest rates volatility structure is supposed to be of exponential form. Technically, it writes as follows:

$$\sigma_P(s,T) = \frac{\sigma_P}{a}\left(1 - e^{-a(T-s)}\right), \tag{34}$$

where $a > 0$. In the sequel, we will take $\sigma_P = 0.033333, a = 1$ and the correlation between the zero-coupon bond and the underlying account will be set at $\rho = 0.35$.

From this expression of the volatility structure, Σ_T^2 can now be completely specified. Indeed, it can be computed by plugging (34) into (23). On the one hand, we have:

$$\int_0^T \sigma_P(s,T)\,ds = \tfrac{\sigma_P}{a}\Big(T - \tfrac{1}{a}\Big(1 - e^{-aT}\Big)\Big)$$

and on the other hand, we have:

$$\int_0^T \sigma_P^2(s,T)\,ds = \tfrac{\sigma_P^2}{a^2}\Big(T - \tfrac{2}{a}\Big(1 - e^{-aT}\Big) + \tfrac{1}{2a}\Big(1 - e^{-2aT}\Big)\Big).$$

Finally, combining these two intermediary results yields:

$$\Sigma_T^2 = \Big(\tfrac{2\rho\sigma\sigma_P}{a^2} - \tfrac{3}{2}\tfrac{\sigma_P^2}{a^3}\Big) + \Big(\sigma^2 + \tfrac{\sigma_P^2}{a^2} - \tfrac{2\rho\sigma\sigma_P}{a}\Big)T + \Big(\tfrac{2\sigma_P^2}{a^3} - \tfrac{2\rho\sigma\sigma_P}{a^2}\Big)e^{-aT} - \tfrac{\sigma_P^2}{2a^3}e^{-2aT}. \quad (35)$$

The results displayed in Tables 4 and 5 show that stochastic interest rates have a tremendous impact on the fair value of the annual insurance risk charge across purchase age and gender. Table 5 shows that a 60 years old male purchaser could be required to pay a risk charge as high as 88.65 bp for the death benefit in a stochastic interest rates environment.

Thus, the stochastic interest rates effect is significantly more pronounced than the jumps effect. Indeed, the longer the time to maturity, the more jumps tend to smooth out hence the lesser impact. On the other hand, the stochastic nature of interest rates are felt deeply for the typical time horizon involved in this kind of insurance contract.

Table 4: Stochastic interest rates impact – Female policyholder – $g = 5\%$, 200% cap. Gompertz mortality model. In each case, the left column displays the relative importance of the M&E charges given by the ratio $ME(\ell)/S_0$. The right column displays the annual insurance risk charge ℓ.

Purchase age (years)	Kou model (flat rate) (%)	(bp)	Kou model (stochastic rates) (%)	(bp)
30	1.16	2.70	5.30	12.63
40	2.04	6.19	6.84	21.29
50	3.21	13.86	8.04	35.63
60	3.55	25.74	6.77	49.93
65	2.47	26.59	4.11	44.61

Table 5: Stochastic interest rates impact – Male policyholder – $g = 5\%$, 200% cap. Gompertz mortality model. In each case, the left column displays the relative importance of the M&E charges given by the ratio $ME(\ell)/S_0$. The right column displays the annual insurance risk charge ℓ.

Purchase age (years)	Kou model (flat rate) (%)	(bp)	Kou model (stochastic rates) (%)	(bp)
30	2.01	4.86	8.87	22.27
40	3.46	10.99	11.38	37.81
50	5.35	24.46	13.38	64.07
60	5.81	44.82	11.14	88.65
65	4.08	46.31	6.82	78.55

In both tables, the annual insurance risk charge decreases after age 60. This decrease after a certain purchase age will be verified again with the figures provided in the next section. Indeed, the approaching contract termination date, set at age 75 as previously, explains this behavior.

4.3 Combined risk factors impact

The impact of mortality models on the fair cost of the GMDB is added in this subsection. Melnikov and Romaniuk's (2006) Gompertz and Makeham

Table 6: Gompertz (G) and Makeham (M) mortality model parameters for the USA (US), Sweden (S) and Japan (J) – Melnikov and Romaniuk (2006)

	A	B	C
G_{US}		6.148×10^{-5}	1.09159
M_{US}	9.566×10^{-4}	5.162×10^{-5}	1.09369
G_S		1.694×10^{-5}	1.10960
M_S	4.393×10^{-4}	1.571×10^{-5}	1.11053
G_J		2.032×10^{-5}	1.10781
M_J	5.139×10^{-4}	1.869×10^{-5}	1.10883

Table 7: Mortality impact on the annual insurance risk charge (bp) – USA – $g = 5\%$, 200% cap

	Gompertz			Makeham
Age	No jumps	Kou (flat)	Kou (stoch.)	Kou (stoch.)
30	4.79	6.99	30.23	32.20
40	11.16	15.15	50.86	52.34
50	24.88	31.50	82.50	83.03
60	44.45	52.97	105.27	104.77
65	45.20	53.18	90.41	89.78

parameters, estimated from the Human mortality database 1959-1999 mortality data, are used in the sequel. As given in Table 6, no more distinction was made between female and male policyholders. Instead, the parameters were estimated across three countries, namely the USA, Sweden and Japan.

In all subsequent figures, the circled curve corresponds to the no-jumps model with a constant interest rate. The crossed curve corresponds to the introduction of Kou jumps but still in a flat term structure of interest rates. The squared curve adds jumps and stochastic interest rates to the no-jumps case. These three curves are built with a Gompertz mortality model. The starred curve takes into account jumps and stochastic interest rates but

Table 8: Mortality impact on the annual insurance risk charge (bp) – Sweden – $g = 5\%$, 200% cap

	Gompertz			Makeham
Age	No jumps (flat)	Kou (flat)	Kou (stoch.)	Kou (stoch.)
30	3.27	4.92	22.83	23.94
40	8.22	11.35	39.42	40.41
50	19.85	25.29	66.73	67.42
60	38.22	45.58	90.53	90.72
65	39.87	46.96	79.83	79.85

Table 9: Mortality impact on the annual insurance risk charge (bp) – Japan – $g = 5\%$, 200% cap

	Gompertz			Makeham
Age	No jumps (flat)	Kou (flat)	Kou (stoch.)	Kou (stoch.)
30	3.58	5.36	24.77	26.04
40	8.94	12.32	42.77	43.91
50	21.45	27.31	72.34	73.08
60	41.04	48.94	97.56	97.70
65	42.71	50.30	85.68	85.63

changes the mortality model to a Makeham one.

Figure 4 displays the annual risk insurance charge with respect to the purchase age in the USA. From 30 years old to around 60 years old, the risk charge is steadily rising across all models. It decreases sharply afterwards as the contract expiry approaches. The same pattern can be observed in Sweden (see figure 5) and Japan (see 6).

The two lower curves in all figures correspond strikingly to the flat term structure of interest rates setting. The jump effect is less pronounced than the stochastic interest rates effect as represented by the two upper curves. The thin band in which lie these upper curves shows that the change of mortality model has also much less impact than the stochastic nature of interest rates.

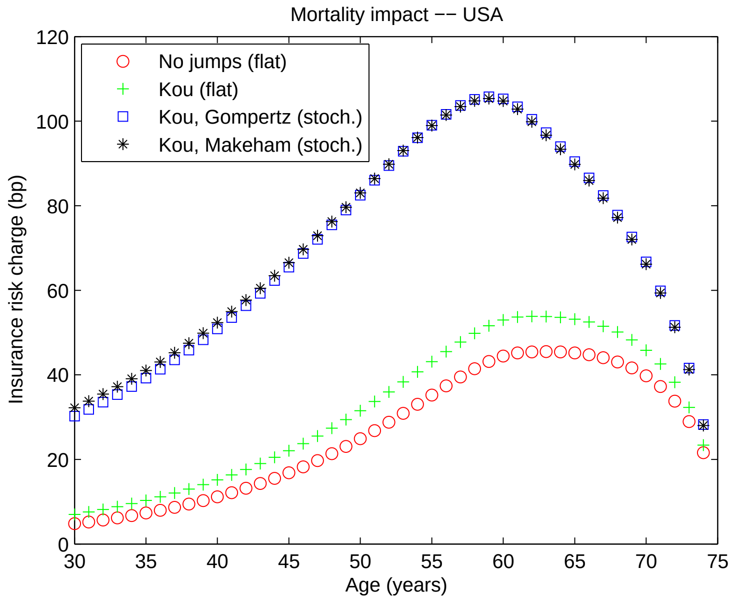

Figure 4: Annual risk insurance charge – USA

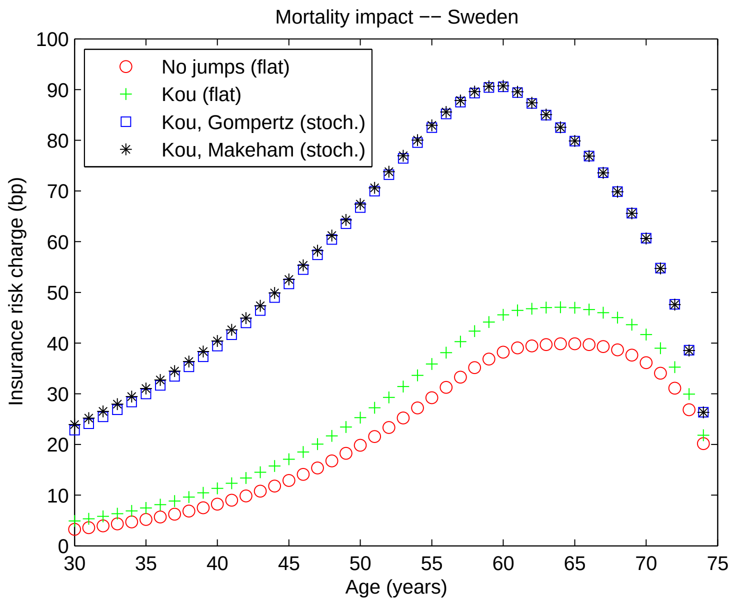

Figure 5: Annual risk insurance charge – Sweden

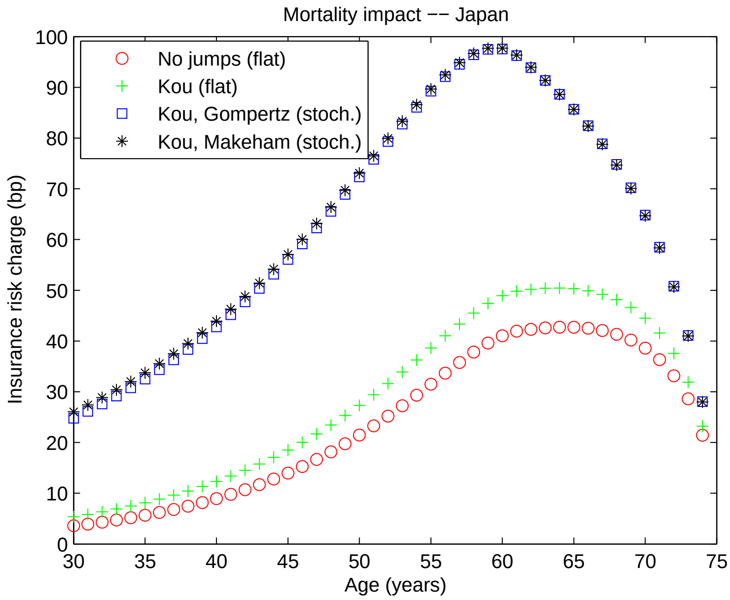

Figure 6: Annual risk insurance charge – Japan

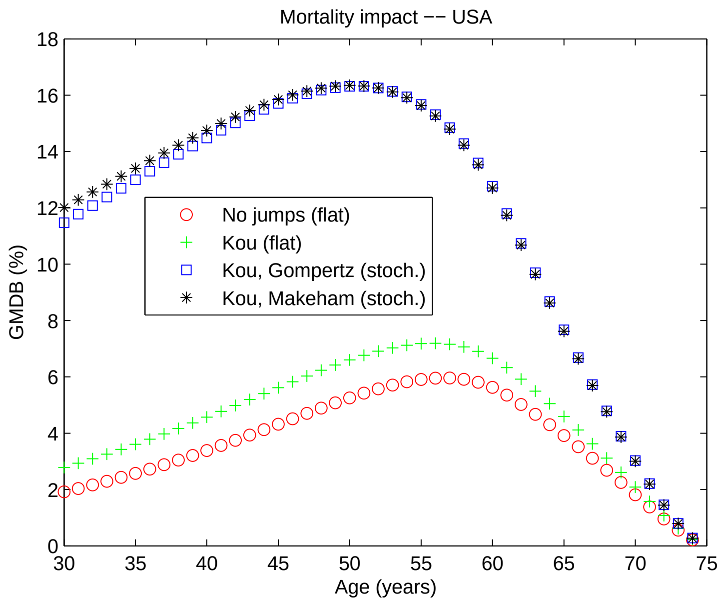

Figure 7: GMDB feature (Titanic option) – USA

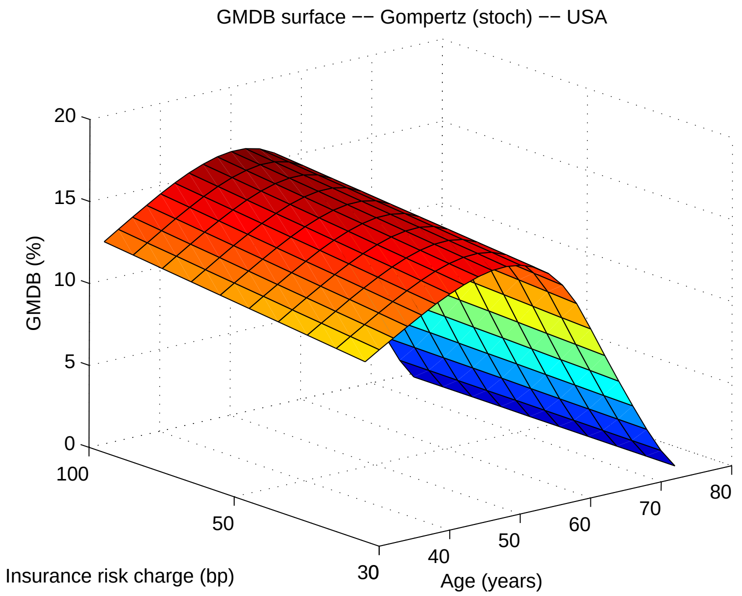

Figure 8: GMDB surface – Stochastic interest rates – USA

The GMDB percentages corresponding to the above fair insurance risk charge were computed and plotted in Figures 7 (USA), 9 (Sweden) and 11 (Japan). They are again rising in all settings until a certain age and fall towards zero as the purchase age nears the contract expiry date. In the no-jumps case, the latter fact stems easily from the integral upper bound time to maturity Θ going to zero in (9). More generally, the guarantee feature provided by the GMDB becomes less and less valuable as the purchase age is near the contract termination date. Indeed, the potential investor has no incentive to buy the GMDB policy if she is almost certain she won't possibly benefit from it in the short time left before contract expiry. Moreover, if the horizon time is short, the uncertainty surrounding the economic outlook is very low and she could possibly profit by investing directly in a government bond.

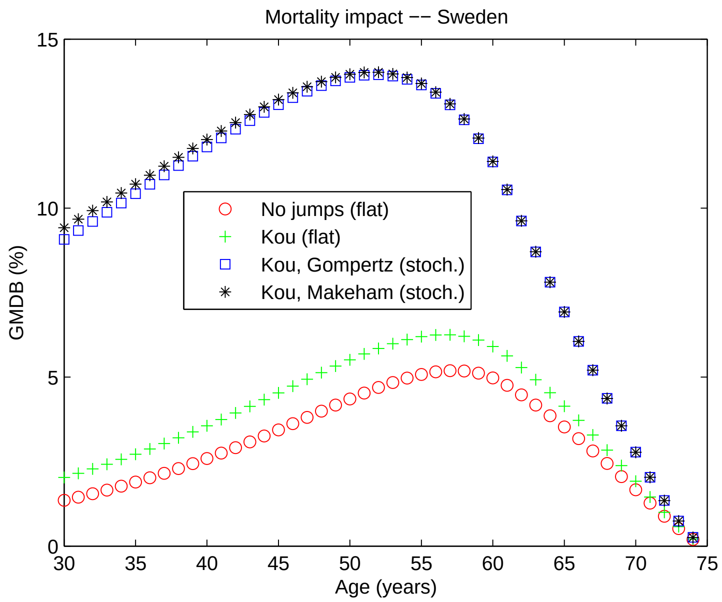

Figure 9: GMDB feature (Titanic option) – Sweden

Figures 8 (USA), 10 (Sweden), 12 (Japan) present the GMDB percentages across purchase age and across levels of annual insurance risk charge. The setting incorporates Kou jumps effect and stochastic interest rates alongside a Gompertz mortality model. The falling-to-zero effect is again present at high purchase ages. For low purchase ages, the GMDB percentage increases with the level of insurance risk charge.

This section is the most complete one because it takes into account jumps, stochastic interest rates, and two standard mortality models estimated in the aforementioned three developed countries. As it is reported in Tables 7, 8, 9, and displayed in Figures 4, 5, and 6, the behavior of the insurance risk charge with respect to age is of the same type whatever the considered model. However, within this type, differences can be noticed. First, the jump effect alone does not change the fees very much but there are more

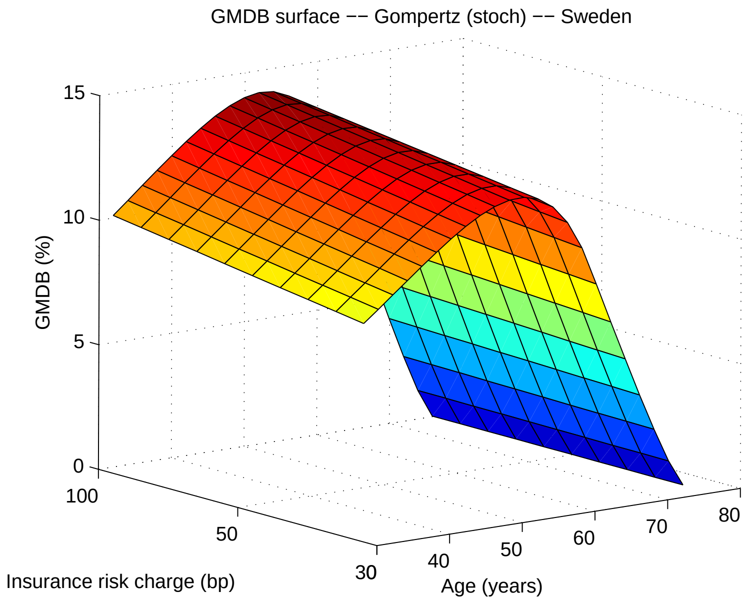

Figure 10: GMDB surface – Stochastic interest rates – Sweden

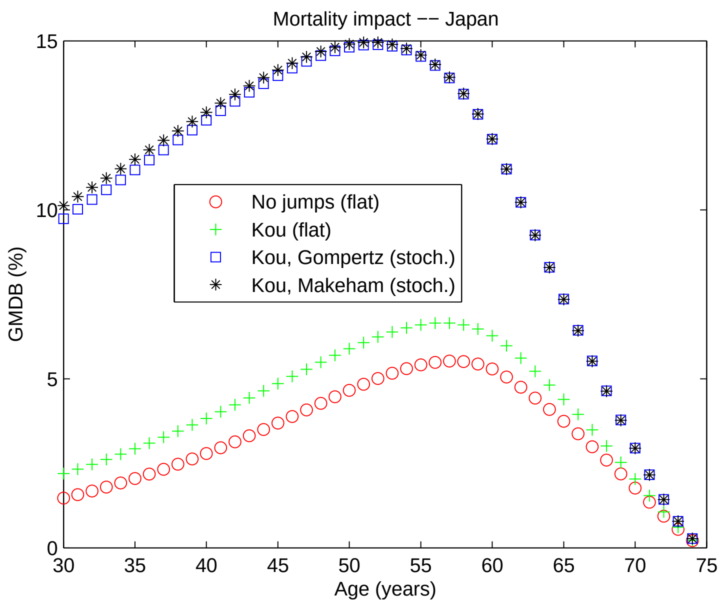

Figure 11: GMDB feature (Titanic option) – Japan

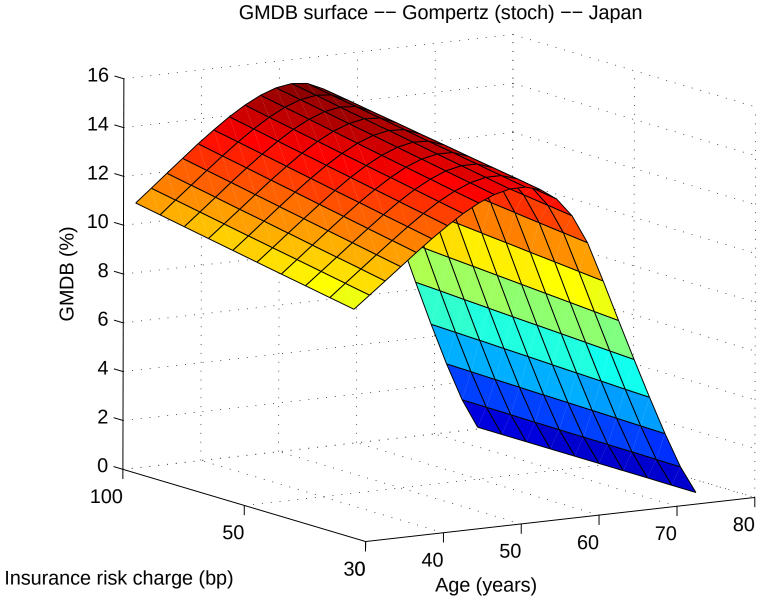

Figure 12: GMDB surface – Stochastic interest rates – Japan

differences when stochastic interest rates are introduced. In this case, fees are notably higher. Second, the choice of mortality model does not have a significant impact. Another point to emphasize is the fact that there is a kind of hierarchy between countries. In increasing order, Sweden is cheapest, then comes Japan and finally, the USA. This observed fact corresponds to the drops of the mortality indices already mentioned by Melnikov and Romaniuk (2006). It could also be inferred from the parameter estimates recorded in Table 6, especially from the B parameter.

5 Conclusion

In this paper, we consider the Guaranteed Minimum Death Benefit contract (GMDB). In short, this contract offers to beneficiaries upon death the maximum of the policyholder initial capital accrued at a minimum guaranteed rate and the value of the insured account linked to a financial market. Milevsky and Posner named this contract Titanic option. From a purely financial point of view, the pricing is done via contingent claim analysis. Up until now, the fair contract price was obtained in a Black and Scholes context as in Milevsky and Posner (2001) or in a regime-switching lognormal context as in Hardy (2003).

In this paper, the Black and Scholes framework is extended in a more general case allowing for stochastic interest rates and jumps. Specifically, the market value of the policyholder's account is assumed to follow a geometric Lévy process. We examine the cases of jump diffusions. The mortality is of a Gompertz or a Makeham type. We propose a complete methodology illustrated by a numerical analysis based on Fast Fourier Transform. New results are given in this paper and as a by-product, we give the way to price options in a non Gaussian economy with stochastic interest rates.

For the typical maturities involved in this kind of contract, we found that introducing jumps while keeping constant the overall quadratic variation in a flat interest rate setting doesn't change the fair costs of the GMDB that much. On the other hand, introducing a stochastic interest rate setting raises substantially these fair costs. The important result stemming from this paper is that, in contrast to Milevsky and Posner (2001), significantly higher insurance risk charges are found. However, these fair prices are still below the fees claimed by life insurance companies but not by much, especially for policyholders aged around 60.

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