Preliminary and Incomplete— Comments Welcome

Asset Pricing with Skewed Payouts

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I examine how investors' preference for skewness impacts the price of limited liability equity. Using the actual probability density function of equity payouts, I obtain equity pricing relations that directly link investors' risk preferences to the skewness induced by the limited liability feature of equity. I show expected equity returns may decline as leverage increases. The declines are most pronounced for firms with high asset systematic risk. For these firms, as leverage increases, the value of limited liability dominates compensation for systematic risk. The results gives insight into the empirical literature on how equity returns vary with leverage or financial distress. Many empirical studies find that equity returns decline as financial leverage or the probability of default increases. I show the declines may be attributed to the value investors place on limited liability protection.

Keywords: Equity returns, skewness, limited liability, call option, Stein's lemma, truncated normal distribution

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1. Introduction

This paper examines how investors value the skewness induced by the limited liability feature of equity. Limited liability protection induces skewness in the ex-ante distribution of equity returns by limiting equity cash-flows at zero; the skewness increases with financial leverage. I derive an equity pricing relation that explicitly links investors' preference for skewness to the equity payout distribution. The value of limited liability increases with financial leverage. It also increases with the systematic risk of firm assets. For firms with high asset systematic risk and pronounced levels of financial leverage, the value of limited liability exceeds compensation for bearing systematic risk. In addition, expected equity returns may decrease as the financial leverage increases. The declines are most pronounced for firms with high asset systematic risk and low idiosyncratic risk. When leverage is high, the benefits of having limited losses offsets the compensation for bearing the systematic risk of the firm's assets.

This result helps explain the puzzling empirical relation between equity returns and financial distress. Several studies show average equity returns may decline as financial leverage or the probability of default increases (Dichev (1998), Griffin and Lemmon (2002), Vassalou and Xing (2004), Campbell and Taksler (2003)). The declines are most pronounced for firms with low book-to-market ratios (Griffin and Lemmon (2002), Vassalou and Xing (2004)). The empirical evidence is puzzling since exposure to systematic risk increases as leverage increases. Thus expected returns should also increase if investors demand compensation for bearing greater systematic risk. Dichev (1998) and Griffin and Lemmon (2002) argue that investors mis-perceive the risk of these firms, and that the negative relation between equity returns and financial distress is evidence of investor irrationality.

I calibrate the equity pricing relation to equity portfolios grouped by book-to-market ratios and debt-equity ratios. Average equity returns decrease as leverage increases. The declines are more pronounced for low book-to-market firms. In addition, asset betas decline as leverage increases. The calibration results suggest that asset risk characteristics and investor preference for skewness drive the negative relation between average equity returns and financial leverage.

The approach used in this paper differs from much of the literature on asset pricing for skewed payouts. Two approaches are commonly used. The first relies on restricting preferences. This approach typically specifies the stochastic discount factor as a quadratic function. For example, Rubinstein (1973) and Kraus and Litzenberger (1976) use a Taylor's series expansion of the utility function, stopping with the cubic form.[1] My approach also differs from many pricing models that treat equity claims as a call option (Black and Scholes (1973) and Geske (1979)). Under certain restrictions, the claim may be valued under the risk-neutral payout density. My approach uses the actual payout density, and produces closed-form pricing relations without placing restrictions on utility.

[1] A quadratic approximation to the stochastic discount factor is only exact if investors have cubic utility over wealth (Levy (1969)).

2. The Price of Limited Liability Equity

Consider an economy of firms worth A_t in the current period and A_{t+1} next period. Firms issue two claims on their assets. The first is a zero-coupon debt claim that pays the face value $D_{i,t+1}$ if the value of firm i exceeds the debt's face value. If firm value is below face value, the equity holders default, and the debt holders get the firm value, $A_{i,t+1}$. The second claim is an equity claim that pays the difference between firm value and the face value of debt, $A_{i,t+1} - D_{i,t+1}$, if firm value is greater than the face value of debt, and pays zero otherwise. I assume firms' investment policies are independent of financing policy and managers do not use financing decisions to signal future firm prospects. Furthermore, firms can costlessly default, have free access to external funds, and pay no taxes. To insure no arbitrage opportunities exist, the price of the equity and debt claims equal the value of the firm, A_t.

A representative investor determines the value of equity. The investor maximizes expected utility of next period wealth, W_{t+1}, by allocating initial wealth between equity and debt claims to firm assets, and a risk-free asset that pays unity at $t+1$. From the first order conditions for expected utility maximization, the price of equity depends on the equity payout and the marginal utility of investor wealth at $t+1$ (Beja (1971)).

$$P_{i,E,t} = \frac{E_t\left(u'(W_{t+1})X_{i,E,t+1}\right)}{E\left(u'(W_{t+1})\right)R_{f,t+1}}, \qquad \forall i. \tag{1}$$

$u'(\bullet)$ is the investor's marginal utility of wealth, $X_{i,E,t+1}$ is the equity payout next period, and $R_{f,t+1}$ is the one-period risk free rate. The properties of the covariance operator imply that the price of equity equals [2]

$$P_{i,E,t} = \frac{E_t(X_{i,E,t+1})}{R_{f,t+1}} + \underbrace{\frac{Cov_t(u'(W_{t+1}), X_{i,E,t+1})}{R_{f,t+1}E(u'(W_{t+1}))}}_{\text{Covariance Risk Adjustment}}. \tag{2}$$

The price of the equity claim depends on its expected cash flow and a covariance risk adjustment that decreases prices for payouts that do poorly when future wealth innovations are low.

The covariance risk adjustment reflects investors' preference for skewness induced by limited liability. To show this, I assume investor wealth and the value of firm assets at $t + 1$ are normally distributed. Thus, investor wealth and the equity payout at t+1 have a joint truncated normal distribution. The covariance risk adjustment is decomposed using the following lemma.

Lemma 1. Suppose firm asset value, $A_{i,t+1}$, and the investor's wealth portfolio, W_{t+1}, are normally distributed. The investor's marginal utility of wealth, $u'\left(W_{t+1}\right)$, is a differentiable, Lebesgue measurable function with $\left|E\left(u''\left(W_{t+1}\right)\right)\right| < \infty$. The covariance between $u'\left(W_{t+1}\right)$ and an equity claim that pays $X_{i,E,t+1} = A_{i,t+1} - D_{i,t+1}$ if $A_{i,t+1} > D_{i,t+1}$, and zero otherwise, is

$$\begin{aligned} Cov_t\left(u'\left(W_{t+1}\right), X_{i,E,t+1}\right) &= \sigma_{WA_i} E\left(u''\left(W_{t+1}\right)\right) N_W Pr\left(A_{i,t+1} > D_{i,t+1}\right) \\ &\quad + \frac{\sigma^2_{WA_i}}{\sigma_{A_i}} E\left(u'''\left(W_{t+1}\right)\right) N_W^2 \phi\left(\frac{E\left(A_{i,t+1}\right) - D_{i,t+1}}{\sigma_{A_i}}\right), \end{aligned} \tag{3}$$

[2] $E_t\left(u'\left(W_{t+1}\right) X_{i,E,t+1}\right) = E\left(u'\left(W_{t+1}\right)\right) E_t\left(X_{i,E,t+1}\right) + Cov_t\left(u'\left(W_{t+1}\right), X_{i,E,t+1}\right)$

where σ_{WA_i} is the covariance between asset value and wealth, σ_{A_i} is the asset value standard deviation, $Pr\left(A_{i,t+1} > D_{it+1}\right)$ is the probability that equity holders avoid default, N_W is the value of wealth invested in the portfolio of risky assets at t, and

$$\phi\left(\frac{E\left(A_{i,t+1}\right)-D_{i,t+1}}{\sigma_{A_i}}\right)=\frac{1}{\sqrt{2\pi}}Exp\left(-\frac{1}{2}\left(\frac{E\left(A_{i,t+1}\right)-D_{i,t+1}}{\sigma_{A_i}}\right)^2\right) \tag{4}$$

Proof. See Appendix 1.

Lemma 1 extends Stein's lemma (Stein (1973), Rubinstein (1973)). Stein's lemma states that if the wealth portfolio, W_{t+1}, and a payout, X_{t+1}, have a joint normal distribution, then the covariance risk adjustment equals the average change in marginal utility over all future wealth outcomes, $E\left(u''\left(W_{t+1}\right)\right)$, multiplied by the covariance between wealth and the payout.

Applying Lemma 1 to Equation 4, the price of the equity claim equals

$$\begin{aligned} P_{i,E,t} &= \frac{E_t(X_{i,E,t+1})}{R_{f,t+1}} + \underbrace{\sigma_{WA_i}\frac{E\left(u''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))}\frac{N_W}{R_{f,t+1}}Pr\left(A_{i,t+1} > D_{i,t+1}\right)}_{\text{Payment for bearing covariance risk}} \\ &\quad \underbrace{+\frac{\sigma^2_{WA_i}}{\sigma_{A_i}}\frac{E\left(u'''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))}\frac{N_W^2}{R_{f,t+1}}\phi\left(\frac{E\left(A_{i,t+1}\right)-D_{i,t+1}}{\sigma_{A_i}}\right)}_{\text{Value of limited liability protection}}. \end{aligned} \tag{5}$$

Equity prices depend on expected equity payouts, compensation for bearing covariance risk (the second term) and on how investors value the limited liability feature of equity (the third term). When the probability of default is zero, investors do not benefit from limited liability protection. Thus, the value of limited liability is zero, and equity prices depend on asset covariance risk and expected firm value relative net of the face value of debt (Hamada (1973), Rubinstein (1973)).

When the probability of default is positive, equity prices also depend on investors skewness preference, $\frac{E(u'''(W_{t+1}))}{E_t(u'(W_{t+1}))}$, and firm risk characteristics. Skewness preference measures the curvature of marginal utility, or how much investors dislike declines in wealth relative to how much they like gains in wealth. Investor prefer positive skewness and dislike negative skewness under certain restrictions on curvature of the utility function. If investors do not decrease their holdings in risky assets as their level of wealth increases, and have positive marginal utility at all levels of wealth, then they will prefer positive skewness and dislike negative skewness (Scott and Horvath (1980)). This implies that $u'''(W_t)$ is greater than zero. For firms with positive asset covariance, the value of limited liability is positive.

Firm risk characteristics, along with the level of financial leverage, determine skewness of the equity payout. Limited liability protection induces skewness in the ex-ante distribution of equity returns by truncating equity cash-flows at zero; the skewness increases with financial leverage. In addition, skewness increases as either asset covariance or asset idiosyncratic risk increases. The next section examines the impact of asset risk on expected equity returns.

3. Firm Risk, Financial Leverage and Expected Equity Returns

This section illustrates how the relation between leverage and expected equity returns depends on firm risk characteristics. Firm asset value and investor wealth are normally distributed at $t+1$, so firm value at t depends on firm exposure to asset systematic risk (Rubinstein (1973)). The expected return of the firm's assets and the investor's wealth portfolio are

$$E\left(R_{A,i,t+1}\right) = 1+\beta_i\left(E\left(R_{W,t+1}\right)-1\right), \quad (6)$$

$$E\left(R_{W,t+1}\right) = 1-\sigma_W^2\frac{E\left(u''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))}. \quad (7)$$

where $\beta_i = \frac{\sigma_{WA_i}}{\sigma_W^2}$ is the asset beta for firm i and σ_W^2 is the variance of the investor's wealth portfolio. The risk-free asset is in zero net supply, so the representative investor invests all wealth in the risky portfolio of assets. I re-normalize prices in terms of the risk-free asset. Firm value equals 1 at time t, so the expected firm value in the following period equals $E\left(R_{A,i,t+1}\right)$. I vary the debt-asset ratio, $\frac{D_{i,t+1}}{E\left(R_{A,i,t+1}\right)}$, from 0 to 0.99, and use expected firm value to identify $D_{i,t+1}$.

In the examples below, absolute risk aversion equals -4, and the annual standard deviation of the wealth portfolio is 14 percent per year. This implies the risk premium for the wealth portfolio corresponds to 7.84 percent per year. The skewness preference parameter, $\frac{E(u'''(W_{t+1}))}{E(u'(W_{t+1}))}$, equals 1.5 times the square of absolute risk aversion.

Expected equity returns for firms with high asset betas decline as leverage increases. Expected equity returns are calculated with the pricing relation (Equation 4) and expected equity payouts. The standard deviation of firm assets equals twice the standard deviation of the market portfolio, or 28 percent. Figure 1 shows expected equity returns for different asset betas and leverage ratios. When the asset beta equals 1.8, equity returns decrease for debt-asset ratios above 0.79, from a peak of 38 percent to 16 percent (when the debt asset ratio equals 0.99). A similar dynamic exists for firms with asset beta equal to 1 and 1.4, although the declines are less pronounced. Figure 1 also displays the dynamics of expected equity returns as leverage increases for firms with asset betas of 0.6. For these firms, expected equity returns increase monotonically as leverage increases.

Figure 1 shows that for firms with pronounced asset betas and debt-asset ratios, expected equity returns decline as leverage increases. Figure 2 illustrates how the value of limited liability relative to equity price changes with leverage. As the debt-asset ratio increases, the value of limited liability increases. For firms with higher asset betas, the value of limited liability can exceed fifty percent of the equity price. For example, when the asset beta equals 2, the value of limited liability exceeds fifty percent for debt-asset ratios above 0.92. Assets with higher covariance risk produce payouts that are worse when overall wealth falls. Limited liability protection hedges against these declines.

4. Default Risk, Book-to-Market Ratios and Average Equity Returns

I calibrate the equity pricing relation to stock return portfolios sorted by equity book-to-market ratios and debt-equity ratios. Starting in January, 1974, I partition firms each month into quintiles based on the level of book-to-market and financial leverage. Both ratios use balance sheet information from COMPUSTAT's Annual Industrial file and price and shares outstanding from CRSP Monthly Stock Database. To construct the book-to-market ratio, I use the book value of equity (COMPUSTAT data item 60), and keep firms with positive book value. The market value of equity equals the stock price at the end of the month multiplied by the number of shares outstanding (from CRSP).

The debt-equity ratio uses debt in current liabilities and long-term financial debt (data items 34 and 9 respectively), as well as any accounts payable (data item 70). I lag the balance sheet information by six months to insure that the information is publicly available at the time when portfolio returns are formed. I also lag the market value of equity by one month. For a given month, I drop observations for firms whose prices in CRSP are an average of the end-of-month bid and ask prices.

The sample of equity returns consists of monthly observations for all firms listed on the New York, American, and NASDAQ stock exchanges for the period August, 1974 to December, 2000. I restrict the sample to this period due to data limitations of the debt measures. In January, 1974, a large number of firms were added to the COMPUSTAT files. Restricting the sample to the post-1973 era insures the calibration results do not inherit a bias due to the jump in the number of firms in the sample. For the market portfolio, I use CRSP's value-weighted return index with dividends.

Each month, I group firms into quintiles based on their book-to-market ratio. Within each book-to-market quintile, I partition firms into quintiles based on the debt-equity ratio of the firm. I calculate an equally weighted portfolio return for firms within each ranking category (25 portfolios in total). I use portfolio returns over the sample period to calculate average equity returns, variance and covariance with the index portfolio. The equally weighted average of firm debt-equity ratios is also calculated each month, then averaged over time.

Table 1 illustrates how equity returns vary with financial leverage. For all book-to-market quintiles, equity returns in the lowest quintile of financial leverage are higher than equity returns in the highest quintile. In addition, the decrease in equity returns is more pronounced for lower book-to-market firms. For example, average equity returns in the second quintile of book-to-market drop from 1.46% per month for the lowest leverage quintile to 0.86% in the highest quintile. For the highest quintile of book-to-market, the declines are less pronounced. Average equity returns in the first quintile of leverage are 2.84% per month; in the highest quintile, they are 2.60%.

Calibration of the model requires identification of risk aversion, skewness preference, asset risk characteristics and debt-asset ratios. Risk aversion is estimated with index average returns and variance. From equation 7, absolute risk aversion equals

$$\frac{E\left(u''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))} = -\frac{E\left(R_{W,t+1}\right) - 1}{\sigma_W^2}. \tag{8}$$

For the entire sample, the average net index returns is 1.28 percent per month; the standard deviation of index returns corresponds to 4.59 percent per month, so risk aversion equals -6.09.

The skewness preference parameter is proportional to absolute risk aversion

$$\frac{E\left(u'''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))} = \delta\left(\frac{E\left(u''\left(W_{t+1}\right)\right)}{E(u'\left(W_{t+1}\right))}\right)^2 \tag{9}$$

where δ is a scale factor that varies from 1 to 6. As δ increases, investors preference for skewness also increases. When δ equals one, investors display constant absolute risk aversion (CARA) utility, while for values greater than one, investors have decreasing absolute risk aversion. As mentioned in Section 2, a sufficient condition for investors to prefer positive skewness is when marginal utility is decreasing and positive over all wealth levels. When δ equals or exceeds one, this condition is satisfied.

I set the standard deviation of asset returns to be proportional to index return standard deviation. Asset standard deviation varies from 100 to 600 percent of the standard deviation of the index returns. I generate asset covariance from asset correlation with the index, which ranges from 0 to 1. I use asset covariance, the variance of the market portfolio, and investor risk aversion to generate expected asset returns using Equation 6. I use asset risk measures and skewness preference to generate equity prices and cash-flow moments for debt-asset ratios, $\frac{D_{t+1}}{E\left(R_{A,t+1}\right)}$, that range from 0 to 1. Equity prices and cash-flow moments are used to construct theoretical equity return expected return, variance, covariance with the index and debt-equity ratios.

To calibrate skewness preference, asset risk characteristics and the firm debt-asset ratios, I use equity return moments and debt-equity ratios for the twenty-five portfolios sorted by book-to-market and financial leverage. The calibration consists of finding the skewness preference parameter, asset risk characteristics and debt-asset ratios that minimize the average absolute percent deviation between theoretical and empirical equity moments and debt-equity ratios.

Table 2 lists the parameters from the calibration. The absolute percent deviations are minimized when δ equals 1. As leverage increases, asset betas decline. The decrease in beta is less pronounced for firms with lower book-to-market ratios. For firms in the lowest book-to-market quintile, asset betas drops from 0.67 for the lowest quintile of leverage to 0.19 for the highest quintile. In contrast, firms in the highest quintile of book-to-market, asset betas drop from 0.72 to 0.05. Asset betas decrease monotonically as leverage increases for all book-to-market quintiles except the lowest quintile. For this quintile, the asset beta first increases, then decreases. The inverted u-shape of asset betas for the lowest book-to-market quintile bears a striking similarity to the dynamics of equity returns.

Table 2 also lists firm asset idiosyncratic risk.[3] Low book-to-market firms with low financial leverage also have less pronounced asset idiosyncratic risk than firms with high financial leverage. In contrast, firms with high book-to-market ratios have a less consistent pattern of how asset idiosyncratic risk differs for firms with low or high financial leverage.

The results in Table 2 give insight into several recent studies that examine the relation between equity returns and financial distress. Dichev (1998) and Griffin and Lemmon (2002) show that equity returns of firms likely to experience financial distress are sometimes smaller than returns of firms with greater likelihood of financial distress. Both studies estimate the likelihood of financial distress using accounting-based measures, such as Altman's (1968) Z-score or Olsen's (1980) O-score. These measures associate higher leverage and lower earnings with higher likelihood of default (Dichev (1998), Shumway (2001), Olsen (1968), Altman (1980)).

Again, we can use the likely properties of low and high book-to-market firms to evaluate the empirical evidence. Dichev (1998) reports that the likelihood of financial distress is positively associated with higher book-to-market ratios; these firms also have lower average monthly returns than firms with low likelihood of distress. Griffin and Lemmon (2002) find high book-to-market firms that are likely to enter into financial distress have greater one-year equity returns than firms with low likelihood of distress. The difference is statistically significant. They also report the converse occurs for low book-to-market firms.

The decrease in equity returns also occurs when the probability of default is inferred equity prices from risk-neutral valuation relations, such as Black and Scholes (1973) and

[3] Asset idiosyncratic risk equals $\sqrt{\sigma_{A_i}^2 - \beta_{A_i}^2 \sigma_W^2}$.

Merton (1974). Vassalou and Xing (2004) estimate the risk-neutral probability of default. They find low book-to-market firms with high likelihood of default have lower one month returns than similar firms that have a low probability of default. The opposite occurs for high book-to-market firms.

Firms with pronounced idiosyncratic risk may have high probabilities of default, but low expected returns. Expected returns will also decline as asset betas decrease. The results in Table 2 suggest low book-to-market firms have higher idiosyncratic risk and lower asset betas, then the empirical results have a risk-based explanation. The results reflect the interaction between the risk characteristics of the firm and how investors value limited liability.

5. Conclusion

This paper presents a equity pricing model that links investors preference to skewness to the limited liability feature of equity. I show that expected equity returns may decline as leverage increases. The decline in equity returns is most pronounced for firms with high asset systematic risk and low asset volatility. This result helps explain the puzzling empirical relation between equity returns and financial distress. Several studies show average equity returns for low book-to-market firms decline as financial leverage increases (Dichev (1998), Griffin and Lemmon (2002), Vassalou and Xing (2004)). I calibrate the model to equity portfolios grouped by book-to-market and debt-equity ratios. Firms with low book-to-market and high leverage ratios have higher asset systematic risk than firms with high book-to-market ratios. The results suggest that negative relation between equity returns and financial leverage is due to the value of limited liability.

Appendix 1

The proof of Lemma 1 follows in two steps. First, Lemma A.1 establishes the covariance between an investor's marginal utility of wealth and firm assets, when firm assets are truncated at the face value of debt outstanding. The second step uses Lemma A.1 to evaluate the covariance between marginal utility and an equity payout.

Lemma A.1. Suppose firm asset value, A, and the investor's wealth portfolio, W, are normally distributed. The investor's marginal utility of wealth, $u'(W)$, is a differentiable, Lebesgue measurable function with $|E(u''(W))| < \infty$. The covariance between $u'(W)|A > D$ and $A|A > D$ is

$$\begin{aligned} Cov\left(u'(W), A|A > D\right) &= \sigma_{WA}\left(E\left(u''(W)\right) + E\left(u'''(W)\right) N_W \frac{1}{\sigma_A}\psi\left(\frac{E(A) - D}{\sigma_A}\right)\right) N_W \\ &\quad -\sigma_A \psi\left(\frac{E(A) - D}{\sigma_A}\right) E\left(u''(W)\right) N_W \frac{\sigma_{WA}}{\sigma_A}\left(E\left(A|A > D\right) - D\right). \end{aligned} \tag{A-1}$$

where D is a scalar constant, σ_{WA} is the covariance between asset value and wealth, σ_A is the asset value standard deviation, N_W is the value of wealth invested invested in the portfolio of risky assets. The Mill's ratio, $\psi\left(\frac{E(A)-D}{\sigma_A}\right)$, equals

$$\psi\left(\frac{E(A) - D}{\sigma_A}\right) = \frac{\phi\left(\frac{E(A)-D}{\sigma_A}\right)}{\Phi\left(\frac{E(A)-D}{\sigma_A}\right)}, \tag{A-2}$$

$$\phi\left(\frac{E(A) - D}{\sigma_A}\right) = \frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(\frac{E(A) - D}{\sigma_A}\right)^2\right), \tag{A-3}$$

$$\Phi\left(\frac{E(A) - D}{\sigma_A}\right) = \int_{-\infty}^{\frac{E(A)-D}{\sigma_A}} \phi(y)\, dy. \tag{A-4}$$

Proof. The covariance between W and A, given that $A > D$ equals

$$Cov\left(u'(W), A|A > D\right) = \int_{-\infty}^{\infty}\int_{D}^{\infty} u'(W)\left(A - E\left(A|A > D\right)\right)\phi\left(W, A|A > D\right) dA dW \tag{A-5}$$

where $\phi(W, A|A > D)$ is the joint normal distribution of W and A, given that A is truncated at D. Using

Bayes rule, the covariance equals

$$Cov\left(u'\left(W\right),A|A>D\right) = \int_{-\infty}^{\infty} u'\left(W\right)\int_{D}^{\infty}\left(\left(A-E\left(A|A>D\right)\right)\frac{\phi\left(W,A|A>D\right)}{\phi\left(W|A>D\right)}dA\right)\phi\left(W|A>D\right)dW \tag{A-6}$$

$$= \int_{-\infty}^{\infty} u'\left(W\right)\left(E\left(A|W,A>D\right)-E\left(A|A>D\right)\right)\phi\left(W|A>D\right)dW. \tag{A-7}$$

I evaluate the integral in Equation A-7 using following results from Kotz et al. (2000, Ch. 46). The marginal density function of $W|A>D$ is

$$\phi\left(W|A>D\right)=Pr\left(A>D\right)^{-1}\frac{1}{\sqrt{2\pi\sigma_W^2}}Exp\left(-\frac{1}{2}\left(\frac{W-E\left(W\right)}{\sigma_W}\right)^2\right)\Phi\left(\frac{E\left(A|W\right)-D}{\sigma_{A|W}}\right), \tag{A-8}$$

where

$$E\left(A|W\right) = E\left(A\right)+\frac{\sigma_{WA}}{\sigma_W^2}\left(W-E\left(W\right)\right), \tag{A-9}$$

$$\sigma_{A|W} = \sqrt{\sigma_A^2-\frac{\sigma_{WA}^2}{\sigma_W^2}}. \tag{A-10}$$

The expectation of A given W and $A>D$ equals

$$E(A|W,A>D)=E\left(A\right)+\frac{\sigma_{WA}}{\sigma_W^2}\left(W-E\left(W\right)\right)+\psi\left(\frac{E(A|W)-D}{\sigma_{A|W}}\right), \tag{A-11}$$

while the expectation of A given $A>D$ is

$$E\left(A|A>D\right)=E\left(A\right)+\psi\left(\frac{E\left(A\right)-D}{\sigma_A}\right)\sigma_A, \tag{A-12}$$

where $\psi(y)$ is defined in Equation A-2. Thus, the covariance between W and A given $A > D$ equals

$$Cov\left(u'(W), A|A>D\right) = \int_{-\infty}^{\infty} u'(W) \frac{\sigma_{WA}}{\sigma_W^2}\left(W - E(W)\right)\phi\left(W|A>D\right) dW$$
$$+ \int_{-\infty}^{\infty} u'(W)\left(\psi\left(\frac{E(A|W)-D}{\sigma_{A|W}}\right) - \psi\left(\frac{E(A)-D}{\sigma_A}\right)\sigma_A\right)\phi\left(W|A>D\right) dW, \tag{A-13}$$

where $\phi(W|A>D)$ is defined in Equation A-8. The expression may be evaluated as three separate integrals. The first is

$$\frac{\sigma_{WA}}{\sigma_W^2}\int_{-\infty}^{\infty} u'(W)\frac{\sigma_{WA}}{\sigma_W^2}\left(W-E(W)\right)\phi\left(W|A>D\right) dW \tag{A-14}$$

$$= \frac{\sigma_{WA}}{\sqrt{2\pi\sigma_W^2}} Pr\left(A>D\right)^{-1}\int_{-\infty}^{\infty} u'(W)\,\Phi\left(\frac{E(A|W)-D}{\sigma_{A|W}}\right) dExp\left(-\frac{1}{2}\left(\frac{W-E(W)}{\sigma_W}\right)^2\right), \tag{A-15}$$

$$= \frac{\sigma_{WA}^2 Pr\left(A>D\right)^{-1}}{\sigma_W^3\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_{A|W}^2}} u'(W)\,Exp\left(-\frac{1}{2}\left(\left(\frac{W-E(W)}{\sigma_W}\right)^2 + \left(\frac{E(A|W)-D}{\sigma_{A|W}}\right)^2\right)\right) dW$$
$$+\sigma_{WA}N_W E\left(u''(W)|A>D\right). \tag{A-16}$$

Equation A-16 follows from integration by parts. The second integral equals

$$\frac{Pr\left(A>D\right)^{-1}}{\sqrt{2\pi\sigma_W^2}}\int_{-\infty}^{\infty} u'(W)\frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(\left(\frac{W-E(W)}{\sigma_W}\right)^2 + \left(\frac{E(A|W)-D}{\sigma_{A|W}}\right)^2\right)\right) dW. \tag{A-17}$$

The final integral is

$$-\psi\left(\frac{E(A)-D}{\sigma_A}\right)\sigma_A\int_{-\infty}^{\infty} u'(W)\,\phi\left(W|A>D\right) dW = -\psi\left(\frac{E(A)-D}{\sigma_A}\right)\sigma_A E\left(u'(W)|A>D\right). \tag{A-18}$$

Further algebraic simplification gives

$$Cov\left(u'(W), A|A>D\right) = \sigma_{WA}N_W E\left(u''(W)|A>D\right)$$
$$+\sigma_A\psi\left(\frac{E(A)-D}{\sigma_A}\right)\left(E_n\left(u'(W)\right) - E\left(u'(W)|A>D\right)\right), \tag{A-19}$$

where $E_n\left(u'\left(W\right)\right)$ is formed under a normal density with mean $E\left(W\right)-\frac{\sigma_{WA}}{\sigma_A^2}\left(E\left(A\right)-D\right)$ and variance $\sigma_W^2-\frac{\sigma_{AW}^2}{\sigma_A^2}$. To simplify Equation A-19, I evaluate $E_n\left(u'\left(W\right)\right)$ and $E\left(u'\left(W\right)|A>D\right)$.

$$E_n\left(u'\left(W\right)\right)=E\left(u'\left(W\right)\right)-\frac{Cov\left(u'\left(W\right),A\right)}{\sigma_A^2}\left(E\left(A\right)-D\right). \tag{A-20}$$

W and A are normally distributed, so Stein's (1973) lemma implies that $Cov\left(u'\left(W\right),A\right)=$ $E\left(u''\left(W\right)\right)N_W\sigma_{WA}$. Thus,

$$E_n\left(u'\left(W\right)\right)=E\left(u'\left(W\right)\right)-E\left(u''\left(W\right)\right)N_W\frac{\sigma_{WA}}{\sigma_A^2}\left(E\left(A\right)-D\right). \tag{A-21}$$

In addition, application of Stein's lemma to $E\left(u'\left(W\right)|A>D\right)$ implies

$$E\left(u'\left(W\right)|A>D\right)=E\left(u'\left(W\right)\right)+E\left(u''\left(W\right)\right)N_W\frac{\sigma_{WA}}{\sigma_A}\psi\left(\frac{E\left(A\right)-D}{\sigma_A}\right). \tag{A-22}$$

Similarly,

$$E\left(u''\left(W\right)|A>D\right)=E\left(u''\left(W\right)\right)+E\left(u'''\left(W\right)\right)N_W\frac{\sigma_{WA}}{\sigma_A}\psi\left(\frac{E\left(A\right)-D}{\sigma_A}\right). \tag{A-23}$$

Substituting Equations A-21 to A-23 into Equation A-19 leads to the result in Lemma A.1

$$\begin{aligned} Cov\left(u'\left(W\right),A|A>D\right) \quad = \quad & \sigma_{WA}\left(E\left(u''\left(W\right)\right)+E\left(u'''\left(W\right)\right)N_W\frac{1}{\sigma_A}\psi\left(\frac{E\left(A\right)-D}{\sigma_A}\right)\right)N_W \\ & -\sigma_A\psi\left(\frac{E\left(A\right)-D}{\sigma_A}\right)E\left(u''\left(W\right)\right)N_W\frac{\sigma_{WA}}{\sigma_A}\left(E\left(A|A>D\right)-D\right). \end{aligned} \tag{A-24}$$

where $E\left(A|A>D\right)$ is defined in Equation A-12. Q.E.D.

Proof of Lemma 1

Lemma 1. Suppose firm asset value, A, and the investor's wealth portfolio, W, are normally distributed. The investor's marginal utility of wealth, $u'(W)$, is a differentiable, Lebesgue measurable function with $|E(u''(W))| < \infty$. The covariance between $u'(W)$ and an equity claim that pays $X = A - D$ if $A > D$, and zero otherwise, is

$$\begin{aligned} Cov\left(u'\left(W_{t+1}\right), X\right) &= \sigma_{WA} E\left(u''\left(W\right)\right) N_W Pr\left(A > D\right) \\ &+ \frac{\sigma_{WA}^2}{\sigma_A} E\left(u'''\left(W\right)\right) N_W^2 \phi\left(\frac{E\left(A\right) - D}{\sigma_A}\right), \end{aligned} \tag{A-25}$$

where σ_{WA} is the covariance between asset value and wealth, σ_A is the asset value standard deviation, $Pr\left(A > D\right)$ is the probability that equity holders avoid default, N_W is the value of wealth invested invested in the portfolio of risky assets, and

$$\phi\left(\frac{E\left(A\right) - D}{\sigma_A}\right) = \frac{1}{\sqrt{2\pi}} Exp\left(-\frac{1}{2}\left(\frac{E\left(A\right) - D}{\sigma_A}\right)^2\right). \tag{A-26}$$

Proof. The covariance between investor's marginal utility of wealth and an equity payout equals

$$Cov\left(u'\left(W\right), \left(A - D\right)\mathbf{1}_{A>D}\right) = Cov\left(u'\left(W\right), A\mathbf{1}_{A>D}\right) - Cov\left(u'\left(W\right), \mathbf{1}_{A>D}\right) D, \tag{A-27}$$

where $\mathbf{1}_{A>D}$ is an indicator function that equals 1 if $A > D$, and zero otherwise. The first covariance term in Equation A-27 equals

$$Cov\left(u'\left(W\right), A\mathbf{1}_{A>D}\right) = E\left(u'\left(W\right) A\mathbf{1}_{A>D}\right) - E\left(u'\left(W\right)\right) E\left(A\mathbf{1}_{A>D}\right), \tag{A-28}$$

$$= Pr\left(A > D\right)\left(E\left(u'\left(W\right) A | A > D\right) - E\left(A | A > D\right)\right). \tag{A-29}$$

Therefore,

$$\frac{Cov\left(u'\left(W\right), A\mathbf{1}_{A>D}\right)}{Pr(A > D)} = Cov\left(u'\left(W\right), A | A > D\right) + E\left(A | A > D\right)\left(E\left(u'\left(W\right) | A > D\right) - E\left(u'\left(W\right)\right)\right). \tag{A-30}$$

Equation A-30 is derived using the following relations

$$E\left(A\mathbf{1}_{A>D}\right)=Pr\left(A>D\right)E\left(A|A>D\right), \tag{A-31}$$

$$E\left(u'\left(W\right)A\mathbf{1}_{A>D}\right)=Pr\left(A>D\right)E\left(u'\left(W\right)A|A>D\right). \tag{A-32}$$

The second covariance term in Equation A-27 equals

$$Cov\left(u'\left(W\right),D\mathbf{1}_{A>D}\right) = \left(E\left(u'\left(W\right)\mathbf{1}_{A>D}\right)-E\left(u'\left(W\right)\right)E\left(\mathbf{1}_{A>D}\right)\right)D, \tag{A-33}$$

$$= Pr\left(A>D\right)\left(E\left(u'\left(W\right)|A>D\right)-E\left(u'\left(W\right)\right)\right) \tag{A-34}$$

Equation A-34 uses the identity

$$E\left(\mathbf{1}_{A>D}\right)=Pr\left(A>D\right). \tag{A-35}$$

To prove Lemma 1, substitute Equation A-22 into Equation A-34, and Equations A-12 and A-22, along with Lemma A.1, into Equation 30, and simplify.

$$\begin{aligned} Cov\left(u'\left(W\right),\left(A-D\right)\mathbf{1}_{A>D}\right) &= \sigma_{WA}E\left(u''\left(W\right)\right)N_W Pr\left(A>D\right) \\ &\quad +\frac{\sigma_{WA}^2}{\sigma_A}E\left(u'''\left(W\right)\right)N_W^2\phi\left(\frac{E\left(A\right)-D}{\sigma_A}\right) \end{aligned} \tag{A-36}$$

Q.E.D.

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Table 1
Equity Characteristics for Portfolios Sorted by Book-to-Market and Financial Leverage

Table 1 contains portfolio return statistics and average debt-equity ratios. Each month, I sort firms into quintiles based on the level of equity book-to-market ratios (BM). For each quintile of BM, I group firms into quintiles based on the level of debt-equity ratio (LEV). Section 4 of the text discusses the construction of both financial ratios. Quintile 1 contains firms with the lowest ratio. Quintile 5 contains firms with the highest ratio.

Monthly equally weighted portfolio returns use equity returns from CRSP Monthly Stock file, and consists of firms from the New York Stock Exchange, the American Stock Exchange, and NASDAQ. The sample includes data from January, 1974 to December, 2000. Equity betas are constructed using the covariance of each BM-LEV sorted portfolio with the CRSP value-weighted return index and the variance of the index.

I calculate average debt-equity ratios by first calculating the average debt-equity ratio for firms in each BM-LEV ranking within each month, then averaging the debt-equity ratio over time.

Table 1a: Average Equity Returns

B/M	Lev 1	2	3	4	5
1	0.0089	0.0100	0.0098	0.0092	0.0067
2	0.0147	0.0136	0.0120	0.0104	0.0086
3	0.0178	0.0153	0.0144	0.0112	0.0130
4	0.0200	0.0182	0.0175	0.0151	0.0178
5	0.0284	0.0260	0.0254	0.0238	0.0260

Table 1b: Equity Beta

B/M	Lev 1	2	3	4	5
1	1.39	1.39	1.38	1.31	1.24
2	1.27	1.24	1.17	1.14	1.11
3	1.15	1.09	1.04	0.96	1.03
4	1.06	1.04	0.98	0.87	0.98
5	0.99	1.04	1.06	1.09	1.15

Table 1c: Equity Return Standard Deviation

B/M	Lev 1	2	3	4	5
1	0.0787	0.0781	0.0775	0.0706	0.0680
2	0.0714	0.0665	0.0607	0.0582	0.0594
3	0.0653	0.0575	0.0542	0.0507	0.0567
4	0.0610	0.0575	0.0549	0.0509	0.0588
5	0.0639	0.0637	0.0659	0.0708	0.0776

Table 1d: Average Debt-Equity Ratios

B/M	Lev 1	2	3	4	5
1	0.0234	0.0774	0.1638	0.3303	1.7538
2	0.0768	0.2118	0.3767	0.6612	2.8311
3	0.1427	0.3865	0.6717	1.1275	5.3752
4	0.2066	0.5763	1.0027	1.7006	9.1721
5	0.3708	1.0155	1.8562	3.7888	157.1916

Table 2
Firm Characteristics for Portfolios Sorted by Book-to-Market and Financial Leverage

Table 2 contains asset risk characteristics for portfolios of firms sorted by equity book-to-market (BM) and debt-equity (LEV) ratios. Quintile 1 contains firms with the lowest ratio. Quintile 5 contains firms with the highest ratio. I calibrate theoretical equity return moments and debt-equity ratios to the empirical moments listed in Table 1. Section 4 describes the method for calibrating the model.

Table 2a: Expected Equity Returns

B/M \ Lev	1	2	3	4	5
1	0.0089	0.0100	0.0098	0.0092	0.0067
2	0.0147	0.0137	0.0120	0.0104	0.0086
3	0.0147	0.0139	0.0133	0.0112	0.0129
4	0.0135	0.0134	0.0125	0.0111	0.0127
5	0.0127	0.0133	0.0136	0.0138	0.0157

Table 2b: Asset Beta

B/M \ Lev	1	2	3	4	5
1	0.68	0.73	0.66	0.54	0.19
2	1.07	0.88	0.69	0.49	0.18
3	1.00	0.79	0.63	0.41	0.17
4	0.88	0.66	0.49	0.32	0.13
5	0.73	0.53	0.37	0.26	0.05

Table 2c: Asset Idiosyncratic Risk

B/M \ Lev	1	2	3	4	5
1	0.0634	0.0664	0.0598	0.0440	0.1084
2	0.0444	0.0376	0.0490	0.0498	0.0498
3	0.0613	0.0379	0.0374	0.0448	0.1140
4	0.0388	0.0426	0.0487	0.0473	0.1214
5	0.0379	0.2401	0.0445	0.1689	0.0458

Table 2d: Debt-Asset Ratio

B/M \ Lev	1	2	3	4	5
1	0.02	0.07	0.14	0.25	0.64
2	0.07	0.18	0.27	0.4	0.74
3	0.13	0.28	0.4	0.53	0.85
4	0.17	0.37	0.5	0.63	0.92
5	0.27	0.51	0.65	0.8	0.99

Figure 1: Expected Equity Returns

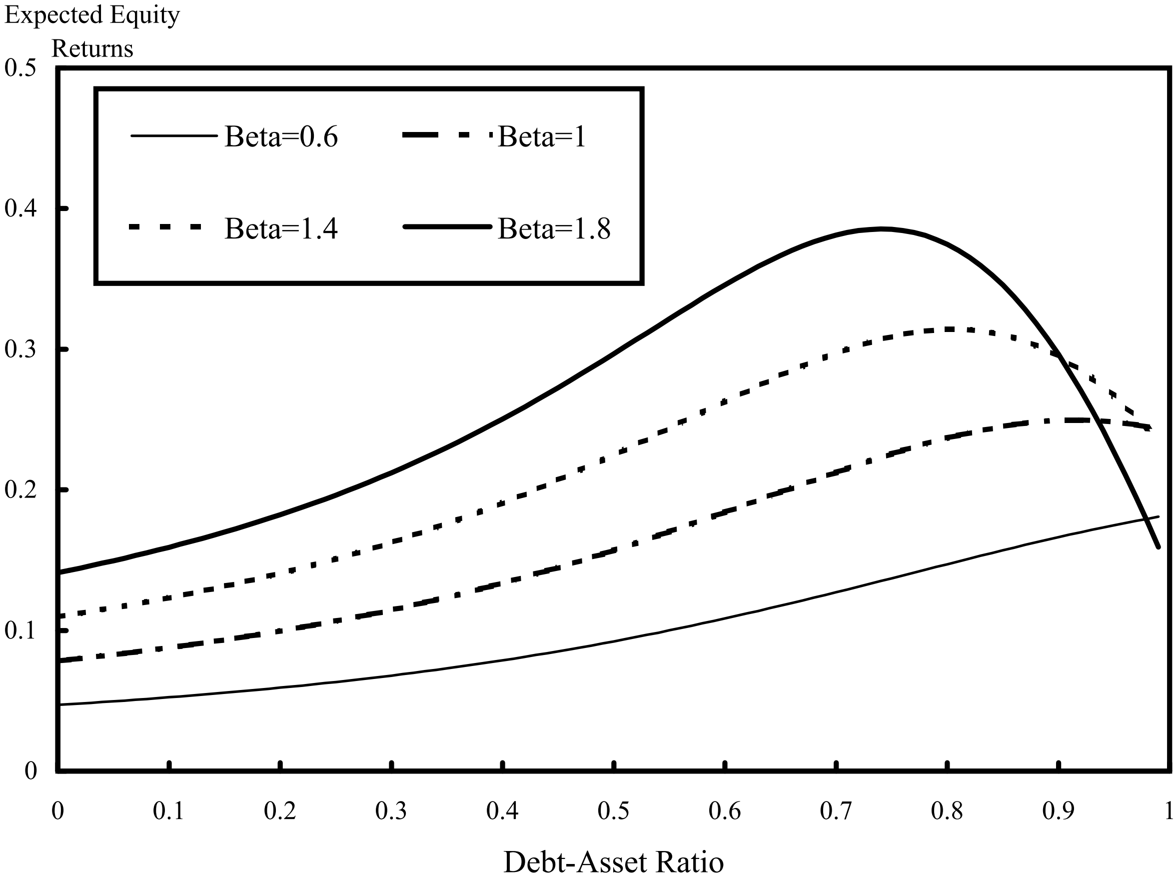

Figure 1 displays expected equity returns for different debt-asset ratios. The debt-asset ratio equals the face value of debt divided by the expected asset return (Equation 6). Expected equity returns are calculated using Equation 5, and expected equity cash-flow. Section 3 discusses the identification of the asset parameters and investor risk preferences.

Figure 2: Value of Limited Liability/Equity Price

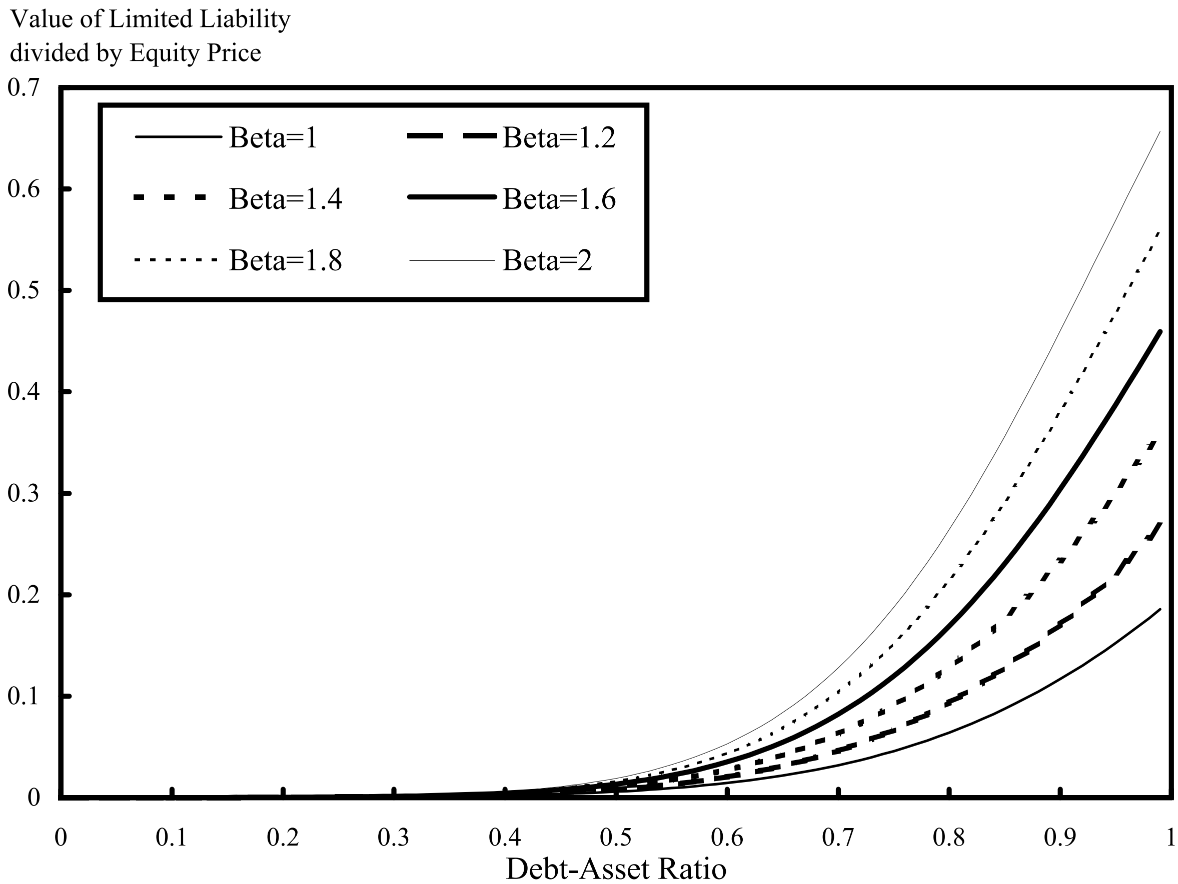

Figure 2 displays the value of limited liability divided by the equity price for different debt-asset ratios. The debt-asset ratio equals the face value of debt divided by the expected asset return (Equation 6). The value of limited liability and equity price are calculated using Equation 5. Section 3 discusses the identification of the asset parameters and investor risk preferences.